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The products on the unit sphere and even-dimension spaces ☆

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Abstract

The distribution $\delta^{(k)}(r-1)$ focused on the unit sphere Ω of R^m is defined by

$$\left\langle \delta^{(k)}(r-1), \phi \right\rangle = (-1)^k \int\limits_{\Omega} \frac{\partial^k}{\partial r^k} \left(\phi r^{m-1} \right) d\omega,$$

where ϕ is Schwartz testing function. We apply the expansion formula

$$\int\limits_{\Omega} \frac{\partial^k}{\partial r^k} \phi(r\omega)\, d\omega = (-1)^k \left\langle \sum_{i=0}^{k} \binom{k}{i} C(m,i) \delta^{(k-i)}(r-1), \phi(x) \right\rangle$$

to evaluate the product of $f(r)$ and $\delta^{(k)}(r-1)$ on Ω. Furthermore, utilizing the Laurent series of r^λ and the residue of $\langle r^\lambda, \phi \rangle$ at the singular point $\lambda = -m-2k$, we derive that $\delta^2(x) = 0$ on even-dimension space. Finally, we are able to imply $\Delta^k(r^{2k-m} \ln r) \cdot \delta(x) = 0$ based on the fact that $r^{2k-m} \ln r$ is an elementary solution of partial differential equation $\Delta^k E = \delta(x)$ by using the generalized Fourier transform.

Keywords: Laurent series; δ-Function; Residue and the Hilbert transform

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1. Introduction

The sequential method (Antosik, Mikusiński, and Sikorski, 1972) and complex analysis approach (Bremermann, 1965) have been the main tools in dealing with products, powers and convolutions of distributions, such as δ^2, which is needed when calculating the transition rates of certain particle interactions in physics [1]. Fisher [2] has actively used Jones' δ-sequence $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots$, where $\rho(x)$ is a fixed infinitely differentiable function on R with the following properties:

(i) $\rho(x) \geqslant 0$,
(ii) $\rho(x) = 0$ for $|x| \geqslant 1$,
(iii) $\rho(x) = \rho(-x)$,
(iv) $\int_{-1}^{1} \rho(x)\,dx = 1$

and the concept of neutrix limit of van der Corput to deduce numerous products, powers, convolutions, and compositions of distributions on R since 1969. The technique of neglecting appropriately defined infinite quantities and resulting finite value extracted from the divergent integral is usually referred to as the Hadamard finite part. In fact, Fisher's method in the computation can be regarded as a particular application of the neutrix calculus. This is a general principle for the discarding of unwanted infinite quantities from asymptotic expansions and has been exploited in context of distribution by Fisher in connection with the problem of distributional multiplication, convolution and composition. To extend such an approach from one-dimensional to m-dimensional, Li [3,4] constructed several workable δ-sequences on R^m for non-commutative neutrix products such as $r^{-k} \cdot \nabla\delta$ as well as $r^{-k} \cdot \Delta^l \delta$, where Δ denotes the Laplacian. Aguirre [5] uses the Laurent series expansion of r^λ and derives a more general product $r^{-k} \cdot \nabla(\Delta^l \delta)$ by calculating the residue of r^λ. His approach is another interesting example of using complex analysis to obtain products of distributions on R^m.

The problem of defining products of distribution on a manifold (unit sphere as a particular example) has been a serious challenge since Gel'fand introduced special types of generalized functions, such as P_+^λ and $\delta^{(k)}(P)$. Aguirre [6] employs the Taylor expansion of distribution $\delta^{(k-1)}(m^2 + P)$ and gives a meaning of the product $\delta^{(k-1)}(m^2 + P) \cdot \delta^{(l-1)}(m^2 + P)$. As outlined in the abstract, the goal of this work is to attempt on a regular product $f(r) \cdot \delta^{(k)}(r - 1)$ on Ω where $f(x)$ is a differentiable function at $x = 1$, as well as to compute several new products related to $\delta(x)$ on even-dimension space by complex analysis method, where $r = (x_1^2 + x_2^2 + \cdots + x_m^2)^{1/2}$. As a note, we also use the Hilbert transform

$$\phi(z) = \frac{1}{\pi i} \int_{-\infty}^{+\infty} \frac{\phi(t)}{t - z}\,dt,$$

where $I_m z > 0$ to verify $\delta^2(x) = 0$ for $x \in R$.

2. The expansion of $\int_{\Omega} \frac{\partial^k}{\partial r^k} \phi(r\omega)\, d\omega$

Let us consider the functional r^λ (see [7]) defined by

$$\left(r^\lambda, \phi\right) = \int_{R^m} r^\lambda \phi(x)\, dx, \tag{1}$$

where $R_e(\lambda) > -m$ and $\phi(x) \in \mathcal{D}_m$ (m-dimensional Schwartz testing function space). Because the derivative

$$\frac{\partial}{\partial \lambda}\left(r^\lambda, \phi\right) = \int r^\lambda \ln r \phi(x)\, dx$$

exists, the functional r^λ is an analytic function of λ for $R_e(\lambda) > -m$.

For $R_e(\lambda) \leqslant -m$, we should use the following identity (2) to define its analytic continuation. For $R_e(\lambda) > 0$, we could deduce

$$\Delta\left(r^{\lambda+2}\right) = (\lambda+2)(\lambda+m)r^\lambda$$

simply by calculating the left-hand side. By iteration, we find for any integer k that

$$r^\lambda = \frac{\Delta^k r^{\lambda+2k}}{(\lambda+2)\cdots(\lambda+2k)(\lambda+m)\cdots(\lambda+m+2k-2)}. \tag{2}$$

On making following substitution of spherical coordinates in (1),

$$\begin{aligned}
x_1 &= r\cos\theta_1,\\
x_2 &= r\sin\theta_1\cos\theta_2,\\
x_3 &= r\sin\theta_1\sin\theta_2\cos\theta_3,\\
&\cdots\\
x_{m-1} &= r\sin\theta_1\sin\theta_2\cdots\sin\theta_{m-2}\cos\theta_{m-1},\\
x_m &= r\sin\theta_1\sin\theta_2\cdots\sin\theta_{m-2}\sin\theta_{m-1},
\end{aligned}$$

we come to

$$\left(r^\lambda, \phi\right) = \int_0^\infty r^\lambda \left\{ \int_{r=1} \phi(r\omega)\, d\omega \right\} r^{m-1}\, dr, \tag{3}$$

where $d\omega$ is the hypersurface element on the unit sphere. The integral appearing in the above integrand can be written in the form

$$\int_{r=1} \phi(r\omega)\, d\omega = \int_{\Omega} \phi(r\omega)\, d\omega = \Omega_m S_\phi(r), \tag{4}$$

where Ω_m is the hypersurface area of the unit sphere imbedded in Euclidean space of m dimensions, and $S_\phi(r)$ is the mean value of ϕ on the sphere of radius r.

In order to compute the product $f(r) \cdot \delta^{(k)}(r-1)$ (such as $(r-1)^n \cdot \delta^{(k)}(r-1)$ and $\sin r \cdot \delta^{(k)}(r-1)$) on Ω, where $f(r)$ is differentiable at $r=1$, we need to express $\int_{\Omega} \frac{\partial^k}{\partial r^k} \phi(r\omega)\, d\omega$ in terms of a linear combination of $\delta^{(i)}(r-1)$, which applies to ϕ.

Lemma 2.1. *For $k \leqslant m-1$ and $\phi(x) \in \mathcal{D}_m$,*

$$\int_{\Omega} \frac{\partial^k}{\partial r^k} \phi(r\omega)\, d\omega = (-1)^k \left\langle \sum_{i=0}^{k} \binom{k}{i} C(m,i) \delta^{(k-i)}(r-1), \phi(x) \right\rangle,$$

where

$$C(m,i) = \begin{cases} 1, & \text{if } i = 0, \\ (m-1)m \cdots (m+i-2), & \text{if } i > 0. \end{cases}$$

Proof. We use an inductive method to show the lemma. It is obviously true for $k = 0$ as

$$\int_{\Omega} \phi(r\omega)\, d\omega = \int_{\Omega} \phi(\omega)\, d\omega = \langle \delta(r-1), \phi \rangle$$

by noting that $r = 1$ on Ω. When $k = 1$, we have

$$\begin{aligned} \langle \delta'(r-1), \phi \rangle &= (-1)^1 \int_{\Omega} \frac{\partial}{\partial r} \left(\phi r^{m-1} \right) d\omega \\ &= (-1)^1 \int_{\Omega} \frac{\partial}{\partial r} \phi(r\omega)\, d\omega - (m-1) \int_{\Omega} \phi(\omega)\, d\omega \\ &= (-1)^1 \int_{\Omega} \frac{\partial}{\partial r} \phi(r\omega)\, d\omega - (m-1) \langle \delta(r-1), \phi(x) \rangle \end{aligned}$$

and hence it holds in this case.

We assume that Lemma 2.1 is true for the case k and we need to consider

$$\begin{aligned} &\langle \delta^{(k+1)}(r-1), \phi \rangle \\ &\quad = (-1)^{k+1} \int_{\Omega} \frac{\partial^{k+1}}{\partial r^{k+1}} \left(\phi r^{m-1} \right) d\omega \\ &\quad = (-1)^{k+1} \left\{ \int_{\Omega} \frac{\partial^{k+1}}{\partial r^{k+1}} \phi(r\omega)\, d\omega + \binom{k+1}{1} \int_{\Omega} \frac{\partial^k}{\partial r^k} \phi(r\omega)(m-1)\, d\omega \right. \\ &\qquad + \binom{k+1}{2} \int_{\Omega} \frac{\partial^{k-1}}{\partial r^{k-1}} \phi(r\omega)(m-1)(m-2)\, d\omega \\ &\qquad + \binom{k+1}{3} \int_{\Omega} \frac{\partial^{k-2}}{\partial r^{k-2}} \phi(r\omega)(m-1)(m-2)(m-3)\, d\omega + \cdots \\ &\qquad \left. + \binom{k+1}{k+1} \int_{\Omega} \phi(r\omega) \frac{d^{k+1}}{dr^{k+1}} r^{m-1}\, d\omega \right\}, \end{aligned}$$

and note that

$$\binom{k+1}{1}\binom{k}{1}(m-1)^2-\binom{k+1}{2}(m-1)(m-2)=\binom{k+1}{2}(m-1)m,$$
$$\binom{k+1}{1}\binom{k}{2}(m-1)^2m-\binom{k+1}{2}\binom{k-1}{1}(m-1)^2(m-2)$$
$$+\binom{k+1}{3}(m-1)(m-2)(m-3)=\binom{k+1}{3}(m-1)m(m+1),$$
$$\ldots.$$

It follows from our hypothesis that

$$\begin{aligned}&(-1)^{k+1}\int\limits_{\Omega}\frac{\partial^{k+1}}{\partial r^{k+1}}\phi(r\omega)\,d\omega\\&\quad=\biggl\langle\delta^{(k+1)}(r-1)+\binom{k+1}{1}(m-1)\delta^{(k)}(r-1)\\&\qquad+\binom{k+1}{2}(m-1)m\delta^{(k-1)}(r-1)\\&\qquad+\binom{k+1}{3}(m-1)m(m+1)\delta^{(k-2)}(r-1)+\cdots\\&\qquad+\binom{k+1}{k+1}(m-1)m(m+1)\cdots(m+k+1-2)\delta(r-1),\phi\biggr\rangle.\end{aligned}$$

This completes the proof of Lemma 2.1. □

Here we would like to indicate that following the same step of Lemma 2.1 it is easy to obtain an expansion of

$$\int\limits_{\Omega}\frac{\partial^k}{\partial r^k}\phi(r\omega)\,d\omega\quad\text{for } k\geqslant m,$$

in which the end term should be $\delta^{(k-m+1)}(r-1)$, rather than $\delta(r-1)$ if we ignore coefficient difference.

It was proven in [7] that $S_\phi(r)$ is infinitely differentiable for $r\geqslant 0$, bounded support, and

$$S_\phi(r)=\phi(0)+a_1r^2+a_2r^4+\cdots+a_kr^{2k}+o\bigl(r^{2k}\bigr)$$

for any positive integer k. From (3) and (4), we obtain

$$\bigl(r^\lambda,\phi\bigr)=\Omega_m\int\limits_0^\infty r^{\lambda+m-1}S_\phi(r)\,dr,$$

which indicates the application of $\Omega_m x_+^\mu$ with $\mu=\lambda+m-1$ to the testing function $S_\phi(r)$. Using the following Laurent series for x_+^λ about $\lambda=-k$,

$$x_+^\lambda=\frac{(-1)^{k-1}\delta^{(k-1)}(x)}{(k-1)!(\lambda+k)}+x_+^{-k}+(\lambda+k)x_+^{-k}\ln x+\cdots,$$

we can write out the Taylor's series for $S_\phi(r)$, namely

$$\begin{aligned} S_\phi(r) &= \phi(0) + \frac{1}{2!}S''_\phi(0)r^2 + \cdots + \frac{1}{(2k)!}S^{(2k)}_\phi(0)r^{2k} + \cdots \\ &= \sum_{k=0}^{\infty} \frac{\Delta^k\phi(0)r^{2k}}{2^k k!m(m+2)\cdots(m+2k-2)}, \end{aligned}$$

which is the well-known Pizetti's formula and it plays an important role in previous work of Li [3,4,8,9].

In 1991, Aguirre expressed distribution $\delta^{(k)}(r-c)$ in terms of an infinite sum of linear combinations of $\Delta^l\delta$. Please refer to reference [10] for detail.

3. The regular products on Ω

Let $f(x)$ be a differentiable function at $x=1$, we are going to compute $f(r)\cdot\delta^{(k)}(r-1)$ directly on Ω by Lemma 2.1. Subsequently, we choose concrete functions as $f(r)$ to obtain expressions for $(r-1)^n\cdot\delta^{(k)}(r-1)$ and $\sin r\cdot\delta^{(k)}(r-1)$.

Theorem 3.1. *For $k \leqslant m-1$,*

$$\begin{aligned} f(r)\cdot\delta^{(k)}(r-1) = \sum_{j=0}^{k}\sum_{i=0}^{j}\sum_{s=0}^{k-j}(-1)^j \frac{k!}{i!s!(j-i)!(k-j-s)!} f^{(i)}(1) \\ \cdot\chi(m,i,j)C(m,s)\delta^{(k-j-s)}(r-1), \end{aligned}$$

where $C(m,s)$ is given in Lemma 2.1 and

$$\chi(m,i,j) = \begin{cases} 1, & \text{if } i=j, \\ (m-1)m\cdots(m-j+i), & \text{if } i<j. \end{cases}$$

Proof. For $\phi(x)\in\mathcal{D}_m$, we need to consider

$$\begin{aligned} &\left\langle f(r)\cdot\delta^{(k)}(r-1),\phi\right\rangle \\ &\quad= (-1)^k\int_\Omega \frac{\partial^k}{\partial r^k}\left(\phi(r\omega)f(r)r^{m-1}\right)d\omega \\ &\quad= (-1)^k\sum_{j=0}^{k}\binom{k}{j}\int_\Omega \frac{\partial^{k-j}}{\partial r^{k-j}}\phi(r\omega)\frac{d^j}{dr^j}\left(f(r)r^{m-1}\right)d\omega \\ &\quad= (-1)^k\sum_{j=0}^{k}\binom{k}{j}\sum_{i=0}^{j}\binom{j}{i}\int_\Omega \frac{\partial^{k-j}}{\partial r^{k-j}}\phi(r\omega)f^{(i)}(r)\frac{d^{j-i}}{dr^{j-i}}r^{m-1}\,d\omega \\ &\quad= (-1)^k\sum_{j=0}^{k}\binom{k}{j}\sum_{i=0}^{j}\binom{j}{i}f^{(i)}(1)\chi(m,i,j)\int_\Omega \frac{\partial^{k-j}}{\partial r^{k-j}}\phi(r\omega)\,d\omega. \end{aligned}$$

The theorem follows from Lemma 2.1,

$$\int_{\Omega} \frac{\partial^{k-j}}{\partial r^{k-j}} \phi(r\omega)\, d\omega = (-1)^{k-j} \left\langle \sum_{s=0}^{k-j} \binom{k-j}{s} C(m,s) \delta^{(k-j-s)}(r-1), \phi(x) \right\rangle$$

and

$$\binom{k}{j}\binom{j}{i}\binom{k-j}{s} = \frac{k!}{i!s!(j-i)!(k-j-s)!}.$$

This completes the proof of Theorem 3.1. □

In particular, for $k = 0, 1$, we have

$$f(r) \cdot \delta(r-1) = f(1)\delta(r-1) \quad \text{and}$$
$$f(r) \cdot \delta'(r-1) = f(1)\delta'(r-1) - f'(1)\delta(r-1),$$

respectively.

Let us choose $f(x) = (x-1)^n$ where n is a positive integer. Then simple calculation implies

$$f^{(i)}(1) = \begin{cases} n!, & \text{if } i = n, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, for $n \leqslant k \leqslant m-1$,

$$(r-1)^n \cdot \delta^{(k)}(r-1) = \sum_{j=n}^{k} \sum_{s=0}^{k-j} (-1)^j \frac{k!}{s!(j-n)!(k-j-s)!} \cdot \chi(m,n,j) C(m,s) \delta^{(k-j-s)}(r-1).$$

Obviously

$$(r-1)^n \cdot \delta^{(k)}(r-1) = 0$$

for $k < n$.

Clearly we have, on the other hand,

$$\frac{1}{r} \cdot \delta^{(k)}(r-1) = \sum_{j=0}^{k} \sum_{i=0}^{j} \sum_{s=0}^{k-j} (-1)^{j+i} \frac{k!}{s!(j-i)!(k-j-s)!} \cdot \chi(m,i,j) C(m,s) \delta^{(k-j-s)}(r-1)$$

and

$$\sin r \cdot \delta^{(k)}(r-1) = \sum_{j=0}^{k} \sum_{i=0}^{j} \sum_{s=0}^{k-j} (-1)^j \frac{k! \sin(1 + \frac{\pi}{2} i)}{i!s!(j-i)!(k-j-s)!} \cdot \chi(m,i,j) C(m,s) \delta^{(k-j-s)}(r-1).$$

To end this section, we would like to point out that following a similar approach of Theorem 3.1, one can carry out the product of $f(r)$ and $\delta^{(k)}(r^2-1)$, where

$$\big\langle \delta^{(k)}\big(r^2-1\big), \phi\big\rangle = \frac{(-1)^k}{2} \int\limits_{\Omega} \left(\frac{\partial}{2r\partial r}\right)^k \big(\phi r^{m-2}\big)\, d\omega.$$

However, if we let $P = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2$ (where $p+q=m$) and define

$$\big\langle \delta^{(k)}(P), \phi\big\rangle = (-1)^k \int\limits_0^\infty \left[\left(\frac{\partial}{2r\partial r}\right)^k \left\{ r^{p-2}\frac{\psi(r,s)}{2}\right\}\right]_{r=s} s^{q-1}\, ds,$$

where $\psi(r,s) = \int \phi\, d\omega_p\, d\omega_q$, $d\omega_p$ and $d\omega_q$ are the elements of surface area on the unit sphere in R_p and R_q, respectively. A challenge problem is how to deduce the singular product $P^n \cdot \delta^{(k)}(P)$ at $x=0$. The author welcomes and appreciates any discussion from interested readers.

4. On even-dimension space

Using the Laurent expansion of r^λ at $\lambda = -m-2j$,

$$r^\lambda = \frac{\Omega_m}{(2j)!(\lambda+m+2j)} \delta^{(2j)}(r) + \Omega_m r^{-2j-m} + \Omega_m(\lambda+m+2j) r^{-2j-m} \ln r + \cdots,$$

Aguirre [5] derived the following identity:

$$\begin{aligned}\delta^{(2j)}(r) &= \frac{(2j)!}{\Omega_m} \lim_{\lambda\to -m-2j} (\lambda+m+2j) r^\lambda \\ &= \frac{(2j)!}{\Omega_m} \operatorname{Res}_{\lambda=-m-2j} r^\lambda = \frac{(2j)!\Gamma(m/2)}{2^{2j} j!\Gamma(m/2+j)} \Delta^j \delta(x)\end{aligned} \tag{5}$$

from the below fact

$$\operatorname{Res}_{\lambda=-m-2j} \big\langle r^\lambda, \phi\big\rangle = \frac{\Omega_m \Gamma(m/2)}{2^{2j} j!\Gamma(m/2+j)} \big\langle \Delta^j \delta, \phi\big\rangle$$

in [7].

Lemma 4.1. *The power $\delta^2(x) = 0$ in space of even dimension.*

Proof. It follows from identity (5) that $\delta(r) = \delta(x)$ by setting $j=0$. Since m is even, there exists a positive integer j such that $m = 2j$. Thus

$$\begin{aligned}\delta^2(x) = \delta(r)\cdot\delta(r) &= \frac{1}{\Omega_m} \lim_{\lambda\to -m} (\lambda+m) r^\lambda \cdot \frac{1}{\Omega_m} \lim_{\lambda\to -m} (\lambda+m) r^\lambda \\ &= \frac{1}{\Omega_m^2} \lim_{\lambda\to -m} (\lambda+m)^2 r^{2\lambda}\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Omega_m^2} \lim_{s\to -m-m} \left(\frac{s}{2}+m\right)^2 r^s \quad \text{set } s = 2\lambda \\
&= \frac{1}{4\Omega_m^2} \lim_{s\to -m-m} (s+m+m)^2 r^s \\
&= \frac{1}{4\Omega_m^2} \lim_{s\to -m-2j} (s+m+2j)^2 r^s \\
&= \frac{1}{4\Omega_m^2} \lim_{s\to -m-2j} (s+m+2j) \lim_{s\to -m-2j} (s+m+2j) r^s \\
&= \frac{1}{4\Omega_m} \lim_{s\to -m-2j} (s+m+2j) \frac{\delta^{(m)}(r)}{m!} = 0.
\end{aligned}
$$

On the other hand, we can follow a different approach to show that $\delta^2(x) = 0$ for $x \in R$ by applying the Hilbert transform

$$\phi(z) = \frac{1}{\pi i} \int_{-\infty}^{+\infty} \frac{\phi(t)}{t-z}\, dt, \quad \text{where } \phi \in \mathcal{D}(R),$$

where $I_m z > 0$.

Indeed from Cauchy's representation of distribution, we have

$$
\begin{aligned}
\left\langle \delta^2(x), \phi(x) \right\rangle &= \lim_{\varepsilon\to 0^+} R_e \left\langle \delta^2(z - i\varepsilon), \phi(z) \right\rangle \\
&\stackrel{\Delta}{=} \lim_{\varepsilon\to 0^+} R_e \frac{1}{(2\pi i)^2} \oint_{|z-i\varepsilon|=\varepsilon/2} \frac{\phi(z)}{(z-i\varepsilon)^2}\, dz.
\end{aligned}
$$

By Cauchy's integral formula, we come to

$$\left\langle \delta^2(x), \phi(x) \right\rangle = \lim_{\varepsilon\to 0^+} R_e \frac{1}{2\pi i} \frac{\phi'(i\varepsilon)}{(2-1)!} = R_e \frac{1}{2\pi i} \phi'(0) = 0.$$

Therefore $\delta^2(x) = 0$. $\square$

Theorem 4.1. *The product* $\Delta^k (r^{2k-m} \ln r) \cdot \delta(x) = 0$ *in space of even dimension if* $2k > m$.

Proof. We complete it by applying Lemma 4.1 if we are able to show that $r^{2k-m} \ln r$ is an elementary solution of partial differential equation

$$\Delta^k E = \delta(x). \tag{6}$$

Taking the Fourier transform of (6), we have

$$(-1)^k \rho^{2k} V = 1, \quad \text{where } \rho^2 = \sum \sigma_j^2, \tag{7}$$

where V denotes the Fourier transform of E.

From above, we may write the Laurent expansion of ρ^λ at the singular point $\lambda = -m - 2k$ as

$$\rho^{\lambda} = \frac{a_{-1}}{\lambda + m + 2k} + a_0 + a_1(\lambda + m + 2k) + \cdots, \tag{8}$$

where $a_{-1}, a_0, a_1, \ldots$ are given by

$$a_{-1} = \Omega_m \frac{\delta^{(2k)}(\rho)}{(2k)!}, \qquad a_0 = \Omega_m \rho^{-2k-m}, \qquad a_1 = \Omega_m \rho^{-2k-m} \ln \rho, \quad \ldots.$$

We multiply Eq. (8) term by term by ρ^{2k+m} and then let $\lambda \to -2k - m$. As above, the left-hand side converges to unity. On the right-hand side all the terms higher than the second vanish in the limit and the second term $\rho^{2k+m} a_0$ remains constant, and if we assume that $\rho^{2k+m} a_{-1} \neq 0$, the first term increases without bound. But this would contradict the limit equation in which all the other terms are finite, so we could conclude that $\rho^{2k+m} a_{-1} = 0$, and therefore that

$$\rho^{2k+m} a_0 = 1.$$

Using the following identity

$$F\left[r^{\lambda_0} \ln r\right] = C' \rho^{-\lambda_0 - m} + C \rho^{-\lambda_0 - m} \ln \rho,$$

we derive the solution of Eq. (6) is

$$A r^{2k-m} \ln r + B r^{2k-m}.$$

However, the second term may be dropped since it is annihilated by the operator Δ^k. Thus $r^{2k-m} \ln r$ is an elementary solution of (6). This completes the proof. □

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