A note on the product

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A NOTE ON THE PRODUCT

\( r^{-k} \cdot \nabla(\Delta r^2 - m) \)

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The difficulties inherent in defining products, powers or nonlinear operations of generalized functions have not prevented their appearance in the literature (see e.g. [1, 2, 3]). In [4], a definition for a product of distributions is given using a \( \delta \)-sequence. However, they show that \( \delta^2 \) does not exist. Extending definitions of products from one-dimensional space to \( m \)-dimensional by using appropriate delta-sequences has recently been an interesting topic in distribution theory. From a single variable function \( \rho(s) \) (see [5]) defined on \( \mathbb{R}^+ \), Li and Fisher introduce a 'workable' delta sequence in \( \mathbb{R}^m \) to deduce a non-commutative neutrix product of \( r^{-k} \) and \( \Delta \delta \) (\( \Delta \) denotes the Laplacian) for any positive integer \( k \) between 1 and \( m - 1 \) inclusive. In [6], Li provides a modified \( \delta \)-sequence and defines a 'bridge' distribution \( \frac{d^k}{dr^k} \delta(x) \), which can be used to obtain the more general product of \( r^{-k} \) and \( \Delta \delta \). The object of this paper is to utilise the fact that \( E = \frac{\Gamma(m/2)}{(2-m)2\pi^{m/2}} r^{2-m} \) is an elementary solution of Laplacian equation \( \Delta u(x) = \delta(x) \) and to apply a much simpler version of the main result in [7] to derive the product \( r^{-k} \cdot \nabla(\Delta r^2 - m) \) where \( m \) is any dimension greater than 2. Finally, we also compute the products \( r^{-k} \cdot \Delta \ln r \) and \( r^{-k} \cdot \nabla(\Delta \ln r) \) based on the equation \( \frac{1}{2\pi} \Delta \ln r = \delta \) for \( m = 2 \).

KEY WORDS: Pizetti's formula, delta sequence, neutrix limit, neutrix product and distribution

MSC (2000): 46F10

1. THE DISTRIBUTION \( r^\lambda \)

Let \( r = (x_1^2 + \ldots + x_m^2)^{1/2} \) and consider the functional \( r^\lambda \) (see [8]) defined by

\[
(r^\lambda, \phi) = \int_{\mathbb{R}^m} r^\lambda \phi(x) \, dx,
\]

(1)

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where $\text{Re} \lambda > -m$ and $\phi(x) \in \mathcal{D}_m$, the space of infinitely differentiable functions of the variable $x = (x_1, x_2, \ldots, x_m)$ with compact support. Because the derivative

$$\frac{\partial}{\partial \lambda} (r^\lambda, \phi) = \int r^\lambda \ln r \, \phi(x) \, dx$$

exists, the functional $r^\lambda$ is an analytic function of $\lambda$ for $\text{Re} \lambda > -m$.

For $\text{Re} \lambda \leq -m$, we should use the following identity (2) to define its analytic continuation. For $\text{Re} \lambda > 0$, we could deduce

$$\Delta \left( r^{\lambda+2} \right) = (\lambda + 2)(\lambda + m) r^\lambda$$

simply by calculating the left-hand side. By iteration we find for any integer $k$ that

$$r^\lambda = \frac{\Delta^k r^\lambda+2k}{(\lambda + 2) \cdots (\lambda + 2k)(\lambda + m) \cdots (\lambda + m + 2k - 2)} \quad (2)$$

On making substitution of spherical coordinates in (1), we come to

$$\int \phi(r \omega) \, d\omega = \infty \int_0^\infty \int \phi(r \omega) \, d\omega \left\{ \int_1^\infty r^{m-1} \, dr \right\} dr$$

where $d\omega$ is the hypersurface element on the unit sphere. The integral appearing in the above integrand can be written in the form

$$\int_1^\infty r^{m-1} \, dr = \Omega_m S_\phi(r), \quad (3)$$

where $\Omega_m = \frac{2\pi^m}{\Gamma(m/2)}$ is the hypersurface area of the unit sphere imbedded in Euclidean space of $m$ dimensions, and $S_\phi$ is the mean value of $\phi$ on the sphere of radius $r$.

It was proved in [8] that $S_\phi(r)$ is infinitely differentiable for $r \geq 0$ and has bounded support, and that

$$S_\phi(r) = \phi(0) + a_1 r^2 + a_2 r^4 + \ldots + a_k r^{2k} + o(r^{2k})$$

for any positive integer $k$ (and $a_k$ to be determined). From (3) and (4), we obtain

$$(r^\lambda, \phi) = \Omega_m \int_0^\infty r^{\lambda+m-1} S_\phi(r) \, dr$$

which indicates the application of $\Omega_m x_+^\mu$ with $\mu = \lambda + m - 1$ to the testing function $S_\phi(r)$. Using the following Laurent series for $x_+^\lambda$ about $\lambda = -k$

$$x_+^\lambda = \frac{(-1)^{k-1}\delta^{k-1}(x)}{(k-1)! (\lambda + k)} + x_+^{-k} + (\lambda + k) x_+^{-k} \ln x + \ldots$$

we could show that the residue of $(r^\lambda, \phi(x))$ at $\lambda = -m - 2k$ for nonnegative integer $k$ is given by
A NOTE ON THE PRODUCT $r^{-k} \cdot \nabla (\Delta r^{2-m})$

\[ \Omega_m \left( \frac{\delta^{(2k)}}{(2k)!} \phi(x) \right) = \Omega_m \frac{S^{(2k)}(0)}{(2k)!} \]

On the other hand, the residue of the function $r^\lambda$ of (2) for the same value of $\lambda$ is

\[ \frac{\Omega_m \Delta^k \delta(x)}{2^k k! m(m+2) \cdots (m+2k-2)} \]

Therefore we get

\[ \frac{S^{(2k)}(0)}{(2k)!} \Delta^k \phi(0) \]

This result can be used to write out the Taylor's series for $S\phi(r)$, namely

\[ S\phi(r) = \phi(0) + \frac{1}{2!} S''\phi(0) r^2 + \cdots + \frac{1}{(2k)!} S^{(2k)}(0) r^{2k} + \cdots \]

\[ = \sum_{k=0}^{\infty} \frac{\Delta^k \phi(0) r^{2k}}{2^k k! m(m+2) \cdots (m+2k-2)} \]

which is the well-known Pizetti's formula.

2. THE PRODUCT $r^{-k} \cdot \nabla \delta$

Let $\rho(x)$ be a fixed infinitely differentiable function defined on $R$ with the following properties:

(i) $\rho(x) \geq 0$,
(ii) $\rho(x) = 0$ for $|x| \geq 1$,
(iii) $\rho(x) = \rho(-x)$,
(iv) $\int_{-1}^{1} \rho(x) dx = 1$.

The function $\delta_n(x)$ is defined by $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots$. It follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$.

Now let $D$ be the space of infinitely differentiable functions of a single variable with compact support and let $D'$ be the space of distributions defined on $D$. Then if $f$ is an arbitrary distribution in $D'$, we define

\[ f_n(x) = (f \ast \delta_n)(x) = (f(t), \delta_n(x-t)) \quad \text{for} \quad n = 1, 2, \ldots \]

It follows that $\{f_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $f(x)$ in $D'$.

The following definition for the non-commutative neutrix product $f \cdot g$ of two distributions $f$ and $g$ in $D'$ was given by Fisher in [9].

**Definition 1.** Let $f$ and $g$ be distributions in $D'$ and Let $g_n = g \ast \delta_n$. We say that the neutrix product $f \cdot g$ of $f$ and $g$ exists and is equal to $h$ if
for all functions $\phi$ in $\mathcal{D}$, where $N$ is the neutrix (see [10]) having domain $N' = \{1, 2, \ldots\}$ and range $N''$ the real numbers, with negligible functions that are finite linear sums of the functions $n^\lambda \ln^{r-1} n$, $\ln^r n$ ($\lambda > 0$, $r = 1, 2, \ldots$) and all functions of $n$ which converge to zero in the normal sense as $n$ tends to infinity.

The product of Definition 1 is not symmetric and hence $f \cdot g \neq g \cdot f$ in general.

In order to give a definition for a neutrix product $f \cdot g$ of two distributions in $\mathcal{D}'_m$, the space of distributions defined $\mathcal{D}_m$. We may attempt to define a $\delta$-sequence in $\mathcal{D}_m$ by simply putting (see [12])

$$\delta_n(x_1, \ldots, x_m) = \delta_n(x_1) \cdots \delta_n(x_m),$$

where $\delta_n$ is defined as above. However, this definition is very difficult to use for distributions in $\mathcal{D}_m$ which are functions of $r$. We therefore consider the following approach (see [7]).

Let $\rho(s)$ be a fixed infinitely differentiable function defined on $R^+ = [0, \infty)$ having the properties:

(i) $\rho(s) \geq 0$;
(ii) $\rho(s) = 0$ for $s \geq 1$;
(iii) $\int_{R^m} \delta_n(x) \; dx = 1$,

where $\delta_n(x) = c_m n^m \rho(n^2 r^2)$ and $c_m$ is the constant satisfying (iii). It follows that \{\delta_n(x)\} is a regular $\delta$-sequence of infinitely differentiable functions converging to $\delta(x)$ in $\mathcal{D}_m$.

**Definition 2.** Let $f$ and $g$ be distributions in $\mathcal{D}_m$ and let

$$g_n(x) = (g * \delta_n)(x) = (g(x - t), \delta_n(t)),$$

where $t = (t_1, t_2, \ldots, t_m)$. We say that the neutrix product $f \cdot g$ of $f$ and $g$ exists and is equal to $h$ if

$$N - \lim_{n \to \infty} (fg_n, \phi) = (h, \phi),$$

where $\phi \in \mathcal{D}_m$ and the $N$-limit is defined as above.

With Definition 2 and the normalization procedure of $\mu(x) x^\lambda$, Li (also in [7]) shows that the non-commutative neutrix product $r^{-k} \cdot \nabla \delta$, ($\nabla$ is the gradient operator, that is, $\nabla = \sum_{i=1}^{m} \frac{\partial}{\partial x_i}$) exists and

$$r^{-2k} \cdot \nabla \delta = -\frac{1}{2k+1(k+1)! (m+2) \cdots (m+2k)} \sum_{i=1}^{m} (x_i \Delta^{k+1} \delta)$$

and

$$r^{1-2k} \cdot \nabla \delta = 0,$$

where $k$ is a positive integer.
3. THE DISTRIBUTION $\Delta r^{2-m}$

We note that $r^{2-m}$ for $m \geq 3$ is a harmonic function in any region which does not contain the origin, i.e., that $\Delta r^{2-m}$ vanishes in the ordinary sense for all $r \neq 0$. For the case of generalized functions we have

$$(\Delta r^{2-m}, \phi) = (r^{2-m}, \Delta \phi) = \lim_{\epsilon \to 0^+} \int_{r \geq \epsilon} \frac{\Delta \phi}{r^{m-2}} \, dv$$

Now applying Green's theorem to this integral, we have

$$\int_{r \geq \epsilon} \frac{\Delta \phi}{r^{m-2}} \, dv = \int_{r \geq \epsilon} \phi \Delta r^{2-m} \, dv - \int_{r = \epsilon} \frac{\partial \phi}{\partial r} r^{2-m} \, ds + \int_{r = \epsilon} \phi \frac{\partial}{\partial r} r^{2-m} \, ds,$$

where $ds$ is the element of area on the sphere $r = \epsilon$. Now

$$\int_{r \geq \epsilon} \phi \Delta r^{2-m} \, dv = 0$$

since outside the ball $r < \epsilon$ the function $r^{2-m}$ is harmonic. As for the second term,

$$\int_{r = \epsilon} \frac{\partial \phi}{\partial r} r^{2-m} \, ds = \epsilon^{2-m} \int_{r = \epsilon} \frac{\partial \phi}{\partial r} \, ds = O(\epsilon)$$

by noting that $\partial \phi / \partial r$ is bounded near the origin. On the other hand,

$$\int_{r = \epsilon} \phi \frac{\partial}{\partial r} r^{2-m} \, ds = (2 - m) \epsilon^{1-m} \int_{r = \epsilon} \phi \, ds = (2 - m) \Omega_m S_\epsilon(\phi),$$

where $S_\epsilon(\phi)$ is the mean value of $\phi(x)$ on the sphere of radius $\epsilon$. In the limit as $\epsilon \to 0^+$, of course, $S_\epsilon(\phi) \to \phi(0)$, so that

$$(\Delta r^{2-m}, \phi) = \lim_{\epsilon \to 0^+} \int_{r \geq \epsilon} \frac{\Delta \phi}{r^{m-2}} \, dv = (2 - m) \Omega_m \phi(0).$$

Hence we may write

$$\frac{1}{(2 - m) \Omega_m} \Delta r^{2-m} = \delta(x)$$

and it immediately follows that

$$E = \frac{\Gamma(m/2)}{(2 - m) 2\pi^{m/2}} r^{2-m}$$

is an elementary solution of Laplacian equation $\Delta u(x) = \delta(x)$.

A similar calculation for dimension $m = 2$ leads to the result

$$\frac{1}{2\pi} \Delta \ln r = \delta(x)$$
4. SOME PRODUCTS

In this section, we will provide a nicer version of the main theorem in [7] as well as the results obtained in [6] to provide the products outlined in the abstract.

It obviously follows from equations (5) and (7) that

Theorem 1. The non-commutative neutrix product \( r^{-k} \cdot \nabla(\Delta r^{2-m}) \) exists for \( m \geq 3 \). Furthermore

\[
 r^{-k} \cdot \nabla(\Delta r^{2-m}) = \frac{(m-2) \pi^{m/2}}{2^k(k+1)! (m+2) \cdots (m+2k) \Gamma(m/2)} \sum_{i=1}^{m} (x_i \Delta^{k+1} \delta) \tag{9}
\]

and

\[
 r^{1-2k} \cdot \nabla(\Delta r^{2-m}) = 0, \tag{10}
\]

where \( k \) is a positive integer.

Remark. The product of \( x_i \) and \( \Delta^{k+1} \delta \) in our theorem is well defined since

\[
(x_i \Delta^{k+1} \delta, \phi) = (\delta, \Delta^{k+1} (x_i \phi)) \tag{11}
\]

The following lemma will play an important role in simplifying the above theorem.

Lemma 1.

\[
\sum_{i=1}^{m} \Delta^{k+1} (x_i \phi) = 2(k+1) \nabla(\Delta^k \phi) + \left( \sum_{i=1}^{m} x_i \right) \Delta^{k+1} \phi \quad \text{for} \quad k \geq 0. \tag{12}
\]

Proof. We use an inductive method to show the lemma. It is obviously true for \( k = 0 \). When \( k = 1 \), we have \( \Delta^{2}(x_i \phi) = 4 \frac{\partial}{\partial x_i} \Delta \phi + x_i \Delta^{2} \phi \) simply by calculating the left-hand side. Then,

\[
\sum_{i=1}^{m} \Delta^{2}(x_i \phi) = 4 \nabla(\Delta \phi) + \left( \sum_{i=1}^{m} x_i \right) \Delta^{2} \phi.
\]

Assume, equation (12) holds for the case of \( k = 1 \), that is

\[
\sum_{i=1}^{m} \Delta^{k}(x_i \phi) = 2k \nabla(\Delta^{k-1} \phi) + \left( \sum_{i=1}^{m} x_i \right) \Delta^{k} \phi
\]

Hence it follows that

\[
\sum_{i=1}^{m} \Delta^{k+1}(x_i \phi) = \Delta \sum_{i=1}^{m} \Delta^{k}(x_i \phi) = \Delta \left\{ 2k \nabla(\Delta^{k-1} \phi) + \left( \sum_{i=1}^{m} x_i \right) \Delta^{k} \phi \right\}
\]

\[
= 2k \nabla(\Delta^{k} \phi) + \sum_{i=1}^{m} \Delta(x_i \Delta^{k} \phi) = 2k \nabla(\Delta^{k} \phi) + \sum_{i=1}^{m} \left\{ 2 \frac{\partial}{\partial x_i} \Delta^{k} \phi + x_i \Delta^{k+1} \phi \right\}
\]

\[
= 2k \nabla(\Delta^{k} \phi) + 2 \nabla(\Delta^{k} \phi) + \left( \sum_{i=1}^{m} x_i \right) \Delta^{k+1} \phi = 2(k+1) \nabla(\Delta^{k} \phi) + \left( \sum_{i=1}^{m} x_i \right) \Delta^{k+1} \phi
\]

This completes the proof of Lemma 1.
Lemma 2. \[ \sum_{i=1}^{m} (x_i \Delta^{k+1} \delta) = -2(k + 1) \nabla(\Delta^k \delta), \quad \text{where} \quad k \geq 0. \quad (13) \]

**Proof.** Applying equation (12), we have
\[
\left( \sum_{i=1}^{m} (x_i \Delta^{k+1} \delta), \phi \right) = \left( \delta, \sum_{i=1}^{m} \Delta^{k+1} (x_i \phi) \right) = \left( \delta, 2(k + 1) \nabla(\Delta^k \phi) + \left( \sum_{i=1}^{m} x_i \right) \Delta^{k+1} \phi \right) = 2(k + 1) \nabla(\Delta^k \phi(0)) = -2(k + 1) \nabla(\Delta^k \delta), \phi \]

Therefore, we have reached our conclusion in Lemma 2.

Now, using equations (9) and (13), we can prove the following theorem.

**Theorem 2.** The non-commutative neutrix product \( r^{-k} \cdot \nabla(\Delta r^{2-m}) \) exists for \( m \geq 3 \). Further
\[
r^{-2k} \cdot \nabla(\Delta r^{2-m}) = \frac{(2 - m) \pi^{m/2}}{2^{k-1} k! (m + 2) \cdots (m + 2k) \Gamma(m/2)} \nabla(\Delta^k \delta) \quad (14)
\]
and
\[
r^{1-2k} \cdot \nabla(\Delta r^{2-m}) = 0, \quad (15)
\]
where \( k \) is a positive integer.

In particular, we obtain
\[
(x_1^2 + x_2^2 + x_3^2)^{-1} \cdot \nabla \left( \Delta \left( x_1^2 + x_2^2 + x_3^2 \right)^{-1/2} \right) = -\frac{2\pi}{5} \nabla(\Delta \delta)
\]
\[
(x_1^2 + x_2^2 + x_3^2)^{-1/2} \cdot \nabla \left( \Delta \left( x_1^2 + x_2^2 + x_3^2 \right)^{-1/2} \right) = 0
\]
\[
(x_1^2 + x_2^2 + x_3^2 + x_4^2)^{-2} \cdot \nabla \left( \Delta \left( x_1^2 + x_2^2 + x_3^2 + x_4^2 \right)^{-1} \right) = -\frac{\pi^2}{96} \nabla(\Delta^2 \delta)
\]
by noting that \( \Gamma(3/2) = \sqrt{\pi}/2 \).

The following result can be found in [6].

**Theorem 3.** The non-commutative neutrix product \( r^{-k} \cdot \delta \) exists. Furthermore
\[
r^{-2k} \cdot \delta = \frac{\Delta^k \delta}{2^k k! m(m + 2) \cdots (m + 2k - 2)} \quad (16)
\]
and
\[
r^{1-2k} \cdot \delta = 0 \quad (17)
\]
where \( k \) is a positive integer.

**Theorem 4.** The non-commutative neutrix products \( r^{-k} \cdot \Delta \ln r \) and \( r^{-k} \cdot \nabla(\Delta \ln r) \) exist for \( m = 2 \), and
\[
r^{-2k} \cdot \Delta \ln r = \frac{\pi \Delta^k \delta}{2^{2k-1} (k!)^2}, \quad r^{-2k} \cdot \nabla(\Delta \ln r) = \frac{\pi \nabla(\Delta^k \delta)}{2^{2k-1} k! (k + 1)!},
\]
where \( k \) is a positive integer.
Proof. It immediately follows from equations (5), (8), (13) and (16).
In particular for \( k = 1 \), we get
\[
\left( x_1^2 + x_2^2 \right)^{-1} \cdot \Delta \ln \left( x_1^2 + x_2^2 \right)^{1/2} = \frac{\pi \Delta \delta}{2}
\]
\[
\left( x_1^2 + x_2^2 \right)^{-1} \cdot \nabla \left( \Delta \ln \left( x_1^2 + x_2^2 \right)^{12} \right) = \frac{\pi \nabla (\Delta \delta)}{4}
\]

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REFERENCES

5. C.K. Li and B. Fisher, Examples of the neutrix product of distributions on \( R^m \), Rad. Mat., 6 (1990), 129–137.
7. C.K. Li, The product of \( r^{-k} \) and \( \nabla \delta \) on \( R^m \), Int. J. of Mathematics and Math. Sci., 24 (2000), No 6, 361–369.