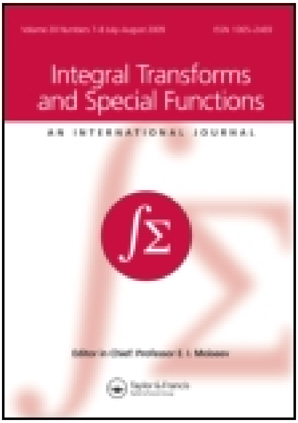

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A note on the product

C.K. Li [a]

[a] Department of Mathematics and Computer Science, Augustana University College, 4901-46 Avenue, Camrose, Alberta, T4V 2R3, Canada

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A NOTE ON THE PRODUCT* $r^{-k} \cdot \nabla(\Delta r^{2-m})$

C.K. LI

*Department of Mathematics and Computer Science,
Augustana University College, 4901–46 Avenue, Camrose, Alberta
Canada T4V 2R3*

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The difficulties inherent in defining products, powers or nonlinear operations of generalized functions have not prevented their appearance in the literature (see e.g. [1, 2, 3]). In [4], a definition for a product of distributions is given using a δ-sequence. However, they show that δ^2 does not exist. Extending definitions of products from one-dimensional space to m-dimensional by using appropriate delta-sequences has recently been an interesting topic in distribution theory. From a single variable function $\rho(s)$ (see [5]) defined on R^+, Li and Fisher introduce a 'workable' delta sequence in R^m to deduce a non-commutative neutrix product of r^{-k} and $\Delta\delta$ (Δ denotes the Laplacian) for any positive integer k between 1 and $m-1$ inclusive. In [6], Li provides a modified δ-sequence and defines a 'bridge' distribution $\frac{d^k}{dr^k}\delta(x)$, which can be used to obtain the more general product of r^{-k} and $\Delta^l\delta$. The object of this paper is to utilize the fact that $E = \frac{\Gamma(m/2)}{(2-m)2\pi^{m/2}} r^{2-m}$ is an elementary solution of Laplacian equation $\Delta u(x) = \delta(x)$ and to apply a much simpler version of the main result in [7] to derive the product $r^{-k} \cdot \nabla(\Delta r^{2-m})$ where m is any dimension greater than 2. Finally, we also compute the products $r^{-k} \cdot \Delta \ln r$ and $r^{-k} \cdot \nabla(\Delta \ln r)$ based on the equation $\frac{1}{2\pi} \Delta \ln r = \delta$ for $m = 2$.

KEY WORDS: Pizetti's formula, delta sequence, neutrix limit, neutrix product and distribution

MSC (2000): 46F10

1. THE DISTRIBUTION r^λ

Let $r = (x_1^2 + \ldots + x_m^2)^{1/2}$ and consider the functional r^λ (see [8]) defined by

$$(r^\lambda, \phi) = \int_{R^m} r^\lambda \phi(x)\, dx, \tag{1}$$

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where $\operatorname{Re}\lambda > -m$ and $\phi(x) \in \mathcal{D}_m$, the space of infinitely differentiable functions of the variable $x = (x_1, x_2, \ldots, x_m)$ with compact support. Because the derivative

$$\frac{\partial}{\partial\lambda}(r^\lambda, \phi) = \int r^\lambda \ln r\, \phi(x)\, dx$$

exists, the functional r^λ is an analytic function of λ for $\operatorname{Re}\lambda > -m$.

For $\operatorname{Re}\lambda \le -m$, we should use the following identity (2) to define its analytic continuation. For $\operatorname{Re}\lambda > 0$, we could deduce

$$\Delta\left(r^{\lambda+2}\right) = (\lambda+2)(\lambda+m)\, r^\lambda$$

simply by calculating the left-hand side. By iteration we find for any integer k that

$$r^\lambda = \frac{\Delta^k r^{\lambda+2k}}{(\lambda+2)\cdots(\lambda+2k)(\lambda+m)\cdots(\lambda+m+2k-2)} \tag{2}$$

On making substitution of spherical coordinates in (1), we come to

$$(r^\lambda, \phi) = \int_0^\infty r^\lambda \left\{ \int_{r=1} \phi(r\omega)\, d\omega \right\} r^{m-1}\, dr \tag{3}$$

where $d\omega$ is the hypersurface element on the unit sphere. The integral appearing in the above integrand can be written in the form

$$\int_{r=1} \phi(r\omega)\, d\omega = \Omega_m S_\phi(r), \tag{4}$$

where $\Omega_m = \frac{2\pi^{m/2}}{\Gamma(m/2)}$ is the hypersurface area of the unit sphere imbedded in Euclidean space of m dimensions, and S_ϕ is the mean value of ϕ on the sphere of radius r.

It was proved in [8] that $S_\phi(r)$ is infinitely differentiable for $r \ge 0$ and has bounded support, and that

$$S_\phi(r) = \phi(0) + a_1 r^2 + a_2 r^4 + \ldots + a_k r^{2k} + o(r^{2k})$$

for any positive integer k (and a_k to be determined). From (3) and (4), we obtain

$$(r^\lambda, \phi) = \Omega_m \int_0^\infty r^{\lambda+m-1} S_\phi(r)\, dr$$

which indicates the application of $\Omega_m x_+^\mu$ with $\mu = \lambda + m - 1$ to the testing function $S_\phi(r)$. Using the following Laurent series for x_+^λ about $\lambda = -k$

$$x_+^\lambda = \frac{(-1)^{k-1}\delta^{(k-1)}(x)}{(k-1)!\,(\lambda+k)} + x_+^{-k} + (\lambda+k)\, x_+^{-k} \ln x + \ldots$$

we could show that the residue of $(r^\lambda, \phi(x))$ at $\lambda = -m - 2k$ for nonnegative integer k is given by

$$\Omega_m\,\frac{(\delta^{(2k)},\phi(x))}{(2k)!}=\Omega_m\,\frac{S_\phi^{(2k)}(0)}{(2k)!}$$

On the other hand, the residue of the function r^λ of (2) for the same value of λ is

$$\frac{\Omega_m\Delta^k\delta(x)}{2^k k!\,m(m+2)\cdots(m+2k-2)}$$

Therefore we get

$$S_\phi^{(2k)}(0)=\frac{(2k)!\,\Delta^k\phi(0)}{2^k k!\,m(m+2)\cdots(m+2k-2)}$$

This result can be used to write out the Taylor's series for $S_\phi(r)$, namely

$$\begin{aligned}S_\phi(r)&=\phi(0)+\frac{1}{2!}\,S_\phi''(0)\,r^2+\ldots+\frac{1}{(2k)!}\,S_\phi^{(2k)}(0)\,r^{2k}+\ldots\\&=\sum_{k=0}^{\infty}\frac{\Delta^k\phi(0)\,r^{2k}}{2^k k!\,m(m+2)\cdots(m+2k-2)}\end{aligned}$$

which is the well-known Pizetti's formula.

2. THE PRODUCT $r^{-k}\cdot\nabla\delta$

Let $\rho(x)$ be a fixed infinitely differentiable function defined on R with the following properties:

(i) $\rho(x)\geq 0$,

(ii) $\rho(x)=0$ for $|x|\geq 1$,

(iii) $\rho(x)=\rho(-x)$,

(iv) $\int_{-1}^{1}\rho(x)\,dx=1$.

The function $\delta_n(x)$ is defined by $\delta_n(x)=n\rho(nx)$ for $n=1,2,\ldots$. It follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$.

Now let $\mathcal{D}$ be the space of infinitely differentiable functions of a single variable with compact support and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. Then if f is an arbitrary distribution in $\mathcal{D}'$, we define

$$f_n(x)=(f*\delta_n)(x)=(f(t),\delta_n(x-t))\qquad\text{for}\qquad n=1,2,\ldots.$$

It follows that $\{f_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $f(x)$ in $\mathcal{D}'$.

The following definition for the non-commutative neutrix product $f\cdot g$ of two distributions f and g in $\mathcal{D}'$ was given by Fisher in [9].

Definition 1. Let f and g be distributions in $\mathcal{D}'$ and Let $g_n=g*\delta_n$. We say that the neutrix product $f\cdot g$ of f and g exists and is equal to h if

$$N - \lim_{n\to\infty} (fg_n, \phi) = (h, \phi)$$

for all functions ϕ in $\mathcal{D}$, where N is the neutrix (see [10]) having domain $N' = \{1, 2, \ldots\}$ and range N'' the real numbers, with negligible functions that are finite linear sums of the functions $n^\lambda \ln^{r-1} n$, $\ln^r n$ $(\lambda > 0,\ r = 1, 2, \ldots)$ and all functions of n which converge to zero in the normal sense as n tends to infinity.

The product of Definition 1 is not symmetric and hence $f \cdot g \neq g \cdot f$ in general.

In order to give a definition for a neutrix product $f \cdot g$ of two distributions in $\mathcal{D}'_m$, the space of distributions defined $\mathcal{D}_m$. We may attempt to define a δ-sequence in $\mathcal{D}_m$ by simply putting (see [12])

$$\delta_n(x_1, \ldots, x_m) = \delta_n(x_1) \cdots \delta_n(x_m),$$

where δ_n is defined as above. However, this definition is very difficult to use for distributions in $\mathcal{D}'_m$ which are functions of r. We therefore consider the following approach (see [7]).

Let $\rho(s)$ be a fixed infinitely differentiable function defined on $R^+ = [0,\ \infty)$ having the properties:

(i) $\rho(s) \geq 0$;

(ii) $\rho(s) = 0$ for $s \geq 1$;

(iii) $\displaystyle\int_{R^m} \delta_n(x)\, dx = 1$,

where $\delta_n(x) = c_m n^m \rho(n^2 r^2)$ and c_m is the constant satisfying (iii). It follows that $\{\delta_n(x)\}$ is a regular δ-sequence of infinitely differentiable functions converging to $\delta(x)$ in $\mathcal{D}'_m$.

Definition 2. Let f and g be distributions in $\mathcal{D}'_m$ and let

$$g_n(x) = (g * \delta_n)(x) = (g(x-t), \delta_n(t)), \qquad \text{where} \qquad t = (t_1, t_2, \ldots, t_m).$$

We say that the neutrix product $f \cdot g$ of f and g exists and is equal to h if

$$N - \lim_{n\to\infty} (fg_n,\ \phi) = (h, \phi),$$

where $\phi \in \mathcal{D}_m$ and the N-limit is defined as above.

With Definition 2 and the normalization procedure of $\mu(x)\, x_+^\lambda$, Li (also in [7]) shows that the non-commutative neutrix product $r^{-k} \cdot \nabla\delta$, $\left(\nabla \text{ is the gradient operator, that is, } \nabla = \sum_{i=1}^m \frac{\partial}{\partial x_i}\right)$ exists and

$$r^{-2k} \cdot \nabla\delta = -\frac{1}{2^{k+1}(k+1)!\,(m+2)\cdots(m+2k)} \sum_{i=1}^m (x_i \Delta^{k+1}\delta) \tag{5}$$

and

$$r^{1-2k} \cdot \nabla\delta = 0, \tag{6}$$

where k is a positive integer.

3. THE DISTRIBUTION Δr^{2-m}

We note that r^{2-m} for $m \geq 3$ is a harmonic function in any region which does not contain the origin, i.e., that Δr^{2-m} vanishes in the ordinary sense for all $r \neq 0$. For the case of generalized functions we have

$$(\Delta r^{2-m}, \phi) = (r^{2-m}, \Delta\phi) = \lim_{\epsilon \to 0+} \int_{r \geq \epsilon} \frac{\Delta\phi}{r^{m-2}}\, dv$$

Now applying Green's theorem to this integral, we have

$$\int_{r \geq \epsilon} \frac{\Delta\phi}{r^{m-2}}\, dv = \int_{r \geq \epsilon} \phi\, \Delta r^{2-m}\, dv - \int_{r=\epsilon} \frac{\partial\phi}{\partial r}\, r^{2-m}\, ds + \int_{r=\epsilon} \phi\, \frac{\partial}{\partial r}\, r^{2-m}\, ds,$$

where ds is the element of area on the sphere $r = \epsilon$. Now

$$\int_{r \geq \epsilon} \phi \Delta r^{2-m}\, dv = 0$$

since outside the ball $r < \epsilon$ the function r^{2-m} is harmonic. As for the second term,

$$\int_{r=\epsilon} \frac{\partial\phi}{\partial r}\, r^{2-m}\, ds = \epsilon^{2-m} \int_{r=\epsilon} \frac{\partial\phi}{\partial r}\, ds = O(\epsilon)$$

by noting that $\partial\phi/\partial r$ is bounded near the origin. On the other hand,

$$\int_{r=\epsilon} \phi\, \frac{\partial}{\partial r}\, r^{2-m}\, ds = (2-m)\, \epsilon^{1-m} \int_{r=\epsilon} \phi\, ds = (2-m)\, \Omega_m S_\epsilon(\phi),$$

where $S_\epsilon(\phi)$ is the mean value of $\phi(x)$ on the sphere of radius ϵ. In the limit as $\epsilon \to 0^+$, of course, $S_\epsilon(\phi) \to \phi(0)$, so that

$$(\Delta r^{2-m}, \phi) = \lim_{\epsilon \to 0+} \int_{r \geq \epsilon} \frac{\Delta\phi}{r^{m-2}}\, dv = (2-m)\, \Omega_m \phi(0).$$

Hence we may write

$$\frac{1}{(2-m)\,\Omega_m}\, \Delta r^{2-m} = \delta(x) \tag{7}$$

and it immediately follows that

$$E = \frac{\Gamma(m/2)}{(2-m)\, 2\pi^{m/2}}\, r^{2-m}$$

is an elementary solution of Laplacian equation $\Delta u(x) = \delta(x)$.

A similar calculation for dimension $m = 2$ leads to the result

$$\frac{1}{2\pi}\, \Delta \ln r = \delta(x) \tag{8}$$

4. SOME PRODUCTS

In this section, we will provide a nicer version of the main theorem in [7] as well as the results obtained in [6] to provide the products outlined in the abstract.

It obviously follows from equations (5) and (7) that

Theorem 1. *The non-commutative neutrix product* $r^{-k} \cdot \nabla(\Delta r^{2-m})$ *exists for* $m \geq 3$. *Furthermore*

$$r^{-2k} \cdot \nabla(\Delta r^{2-m}) = \frac{(m-2)\,\pi^{m/2}}{2^k (k+1)!\,(m+2)\cdots(m+2k)\,\Gamma(m/2)} \sum_{i=1}^{m} (x_i \Delta^{k+1}\delta) \quad (9)$$

and

$$r^{1-2k} \cdot \nabla(\Delta r^{2-m}) = 0, \quad (10)$$

where k *is a positive integer.*

Remark. The product of x_i and $\Delta^{k+1}\delta$ in our theorem is well defined since

$$(x_i \Delta^{k+1}\delta, \phi) = (\delta, \Delta^{k+1}(x_i\phi)) \quad (11)$$

The following lemma will play an important role in simplifying the above theorem.

Lemma 1.

$$\sum_{i=1}^{m} \Delta^{k+1}(x_i\phi) = 2(k+1)\,\nabla(\Delta^k\phi) + \left(\sum_{i=1}^{m} x_i\right) \Delta^{k+1}\phi \qquad \textit{for} \qquad k \geq 0. \quad (12)$$

Proof. We use an inductive method to show the lemma. It is obviously true for $k = 0$. When $k = 1$, we have $\Delta^2(x_i\phi) = 4\,\frac{\partial}{\partial x_i}\,\Delta\phi + x_i\Delta^2\phi$ simply by calculating the left-hand side. Then,

$$\sum_{i=1}^{m} \Delta^2(x_i\phi) = 4\nabla(\Delta\phi) + \left(\sum_{i=1}^{m} x_i\right) \Delta^2\phi.$$

Assume, equation (12) holds for the case of $k-1$, that is

$$\sum_{i=1}^{m} \Delta^k(x_i\phi) = 2k\nabla(\Delta^{k-1}\phi) + \left(\sum_{i=1}^{m} x_i\right) \Delta^k\phi$$

Hence it follows that

$$\begin{aligned}
\sum_{i=1}^{m} \Delta^{k+1}(x_i\phi) &= \Delta \sum_{i=1}^{m} \Delta^k(x_i\phi) = \Delta\left\{2k\nabla(\Delta^{k-1}\phi) + \left(\sum_{i=1}^{m} x_i\right)\Delta^k\phi\right\} \\
&= 2k\nabla(\Delta^k\phi) + \sum_{i=1}^{m} \Delta(x_i\Delta^k\phi) = 2k\nabla(\Delta^k\phi) + \sum_{i=1}^{m}\left\{2\,\frac{\partial}{\partial x_i}\,\Delta^k\phi + x_i\Delta^{k+1}\phi\right\} \\
&= 2k\nabla(\Delta^k\phi) + 2\nabla(\Delta^k\phi) + \left(\sum_{i=1}^{m} x_i\right)\Delta^{k+1}\phi = 2(k+1)\,\nabla(\Delta^k\phi) + \left(\sum_{i=1}^{m} x_i\right)\Delta^{k+1}\phi
\end{aligned}$$

This completes the proof of Lemma 1.

Lemma 2. $$\sum_{i=1}^{m}(x_i\Delta^{k+1}\delta) = -2(k+1)\,\nabla(\Delta^k\delta), \quad \textit{where} \quad k\geq 0. \tag{13}$$

Proof. Applying equation (12), we have

$$\begin{aligned}\left(\sum_{i=1}^{m}(x_i\Delta^{k+1}\delta),\phi\right) &= \left(\delta,\sum_{i=1}^{m}\Delta^{k+1}(x_i\phi)\right)\\ &= \left(\delta,\left\{2(k+1)\,\nabla(\Delta^k\phi)+\left(\sum_{i=1}^{m}x_i\right)\Delta^{k+1}\phi\right\}\right)\\ &= 2(k+1)\,\nabla(\Delta^k\phi(0)) = (-2(k+1)\,\nabla(\Delta^k\delta),\phi)\end{aligned}$$

Therefore, we have reached our conclusion in Lemma 2.

Now, using equations (9) and (13), we can prove the following theorem.

Theorem 2. *The non-commutative neutrix product $r^{-k}\cdot\nabla(\Delta r^{2-m})$ exists for $m\geq 3$. Further*

$$r^{-2k}\cdot\nabla(\Delta r^{2-m}) = \frac{(2-m)\,\pi^{m/2}}{2^{k-1}k!\,(m+2)\cdots(m+2k)\,\Gamma(m/2)}\,\nabla(\Delta^k\delta) \tag{14}$$

and

$$r^{1-2k}\cdot\nabla(\Delta r^{2-m}) = 0, \tag{15}$$

where k is a positive integer.

In particular, we obtain

$$\begin{aligned}\left(x_1^2+x_2^2+x_3^2\right)^{-1}\cdot\nabla\left(\Delta\left(x_1^2+x_2^2+x_3^2\right)^{-1/2}\right) &= -\frac{2\pi}{5}\,\nabla(\Delta\delta)\\ \left(x_1^2+x_2^2+x_3^2\right)^{-1/2}\cdot\nabla\left(\Delta\left(x_1^2+x_2^2+x_3^2\right)^{-1/2}\right) &= 0\\ \left(x_1^2+x_2^2+x_3^2+x_4^2\right)^{-2}\cdot\nabla\left(\Delta\left(x_1^2+x_2^2+x_3^2+x_4^2\right)^{-1}\right) &= -\frac{\pi^2}{96}\,\nabla(\Delta^2\delta)\end{aligned}$$

by noting that $\Gamma(3/2)=\sqrt{\pi}/2$.

The following result can be found in [6].

Theorem 3. *The non-commutative neutrix product $r^{-k}\cdot\delta$ exists. Furthermore*

$$r^{-2k}\cdot\delta = \frac{\Delta^k\delta}{2^k\,k!\,m(m+2)\cdots(m+2k-2)} \tag{16}$$

and

$$r^{1-2k}\cdot\delta = 0 \tag{17}$$

where k is a positive integer.

Theorem 4. *The non-commutative neutrix products $r^{-k}\cdot\Delta\ln r$ and $r^{-k}\cdot\nabla(\Delta\ln r)$ exist for $m=2$, and*

$$r^{-2k}\cdot\Delta\ln r = \frac{\pi\Delta^k\delta}{2^{2k-1}(k!)^2}, \qquad r^{-2k}\cdot\nabla(\Delta\ln r) = \frac{\pi\nabla(\Delta^k\delta)}{2^{2k-1}k!\,(k+1)!},$$

where k is a positive integer.

Proof. It immediately follows from equations (5), (8), (13) and (16).

In particular for $k = 1$, we get

$$\left(x_1^2 + x_2^2\right)^{-1} \cdot \Delta \ln \left(x_1^2 + x_2^2\right)^{1/2} = \frac{\pi \Delta \delta}{2}$$

$$\left(x_1^2 + x_2^2\right)^{-1} \cdot \nabla \left(\Delta \ln \left(x_1^2 + x_2^2\right)^{12}\right) = \frac{\pi \nabla (\Delta \delta)}{4}$$

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