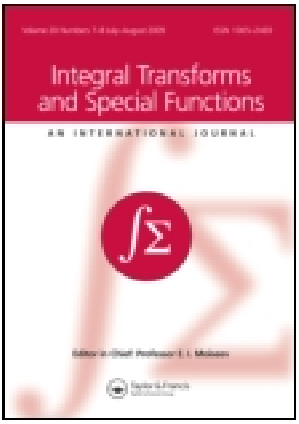

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An approach for distributional products on R^m

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An approach for distributional products on R^m

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One of the main problems in the theory of generalized functions is the lack of definitions for products and powers of distributions. Antosik, Mikusiński and Sikorski in 1972 introduced a definition for a product of distributions using a δ-sequence. However, δ^2 as a product of δ with itself was shown not to exist. Later, Koh and Li in 1992 chose a fixed δ-sequence without compact support and used the concept of the neutrix limit of van der Corput to define δ^k and $(\delta')^k$ for some values of k [Koh, E.L. and Li, C.K., 1992, On the distributions δ^k and $(\delta')^k$. *Mathematische Nachrichten*, **157**, 243–248]. To extend the sequence approach from one-dimensional to m-dimensional, Li and Fisher [Li, C. K. and Fisher, B. (1990). Examples of the neutrix product of distributions on R^m. *Rad. Mat.*, **6**, 129–137.] constructed a 'useful' δ-sequence δ_n on R^m to deduce a non-commutative neutrix product $r^{-k} \cdot \triangle\delta$ for any positive integer k between 1 and $m-1$ inclusive. Their method of completing such a product is totally based on the fact that $\triangle\delta_n$ is computable. However, it seems impossible to deal with more general products involving $\triangle^l\delta$ along the same line because of difficulties in evaluating $\triangle^l\delta_n$, where l is a positive integer. The objective of this paper is to provide a modified δ-sequence and define a new 'bridge' distribution $(\mathrm{d}^k/\mathrm{d}r^k)\delta(x)$, which is used to compute $\triangle^l\delta$. By applying the normalization procedure of distribution x_+^{-n} given by Gel'fand and Shilov [Gel'fand, I. M. and Shilov, G. E. (1964). *Generalized Functions*, Vol. I. Academic Press.] and two identities of δ distribution, we derive an interesting distributional product $(\sum_{i=1}^m x_i/r^k) \cdot \triangle^l\delta$ (hence $\nabla r^{-k} \cdot \triangle^l\delta$ as well).

Keywords: Pizetti's formula; δ-Sequences; Neutrix limit and distributions

2000 Mathematics Subject Classification: Primary: 46F10

1. Introduction

Physicists have long been using the so-called singular functions such as δ, although these cannot be properly defined within the framework of classical function theory. In elementary particle physics [see ref. 3], the expression δ^2 is used when calculating the transition rates of certain particle interactions. Similarly, one finds in the scientific literature such mathematical objects as $\log\delta$ and $\sqrt{\delta}$. Many attempts have been made to define multiplication of distributions including the major works of Rosinger (Generalized Solutions of Nonlinear Partial Differential Equations, North-Holland, 1987) and Columbeau (New Generalized Functions and Multiplications of Distributions, North-Holland, Amsterdam, 1984) as well as Antosik, Mikusiński, Sikorski, Jones, and Fisher's sequence methods.

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Let $\rho(x)$ be a fixed infinitely differentiable function with the following properties:

(i) $\rho(x) \geq 0$,

(ii) $\rho(x) = 0$ for $|x| \geq 1$,

(iii) $\rho(x) = \rho(-x)$,

(iv) $\int_{-1}^{1} \rho(x)\,\mathrm{d}x = 1$.

The function $\delta_n(x)$ is defined by $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots$. It follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$ in the distributional sense.

Now, let $\mathcal{D}$ be the testing function space of infinitely differentiable functions of a single variable with compact support and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. Then if f is an arbitrary distribution in $\mathcal{D}'$, we define

$$f_n(x) = (f * \delta_n)(x) = (f(t), \delta_n(x - t)),$$

for $n = 1, 2, \ldots$. It follows that $\{f_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $f(x)$ in $\mathcal{D}'$.

The following definition for the non-commutative neutrix product $f \cdot g$ of two distributions f and g in $\mathcal{D}'$ was given by Fisher in ref. [4].

DEFINITION 1 *Let f and g be distributions in $\mathcal{D}'$ and let $g_n = g * \delta_n$. We say that the neutrix product $f \cdot g$ of f and g exists and is equal to h if*

$$N - \lim_{n\to\infty} (f g_n, \phi) = (h, \phi)$$

for all functions ϕ in $\mathcal{D}$, where N is the neutrix [*see ref.* 5] *having domain $N' = \{1, 2, \ldots\}$ and range N'', the real numbers, with negligible functions that are finite linear sums of the functions*

$$n^{\lambda} \ln^{r-1} n, \quad \ln^{r} n \quad (\lambda > 0, r = 1, 2, \ldots),$$

and all functions of n which converge to zero in the normal sense as n tends to infinity.

The product of Definition 1 is not symmetric and hence $f \cdot g \neq g \cdot f$ in general.

Extending definitions of products from a one-dimensional space R to an m-dimensional space R^m by using appropriate δ-sequences has recently been an interesting topic in distribution theory. The following work on the non-commutative neutrix product of distributions on R^m can be found in refs. [1, 8].

Let $\rho(s)$ be a fixed infinitely differentiable function defined on $R^+ = [0, \infty)$ having the properties:

(i) $\rho(s) \geq 0$,

(ii) $\rho(s) = 0$ for $s \geq 1$,

(iii) $\int_{R^m} \delta_n(x)\,\mathrm{d}x = 1$,

where $\delta_n(x) = c_m n^m \rho(n^2 r^2)$ and c_m is the constant satisfying (iii).

It follows that $\{\delta_n(x)\}$ is a regular δ-sequence of infinitely differentiable functions converging to $\delta(x)$ in $\mathcal{D}'_m$.

DEFINITION 2 *Let f and g be distributions in $\mathcal{D}'_m$ (an m-dimensional space of distributions) and let*

$$g_n(x) = (g * \delta_n)(x) = (g(x - t), \delta_n(t)),$$

where $t = (t_1, t_2, \ldots, t_m)$. We say that the neutrix product $f \cdot g$ of f and g exists and is equal to h if

$$N - \lim_{n\to\infty} (f g_n, \phi) = (h, \phi),$$

where $\phi \in \mathcal{D}_m$ (an m-dimensional Schwartz space) and the N-limit is defined as mentioned earlier.

With Definition 2, Li and Fisher [see also ref. 1] show that the non-commutative neutrix product $r^{-k} \cdot \triangle\delta$ ($\triangle$ is the Laplacian) exists and

$$r^{-2k} \cdot \triangle\delta = \frac{\triangle^{k+1}\delta}{2^k(k+1)!(m+2)(m+4)\cdots(m+2k)}, \tag{1}$$

for $k = 1, 2, \ldots, \lfloor (m-1)/2 \rfloor$ (the greatest integer function) and

$$r^{1-2k} \cdot \triangle\delta = 0, \tag{2}$$

for $k = 1, 2, \ldots, \lfloor m/2 \rfloor$.

The following work on the commutative neutrix product of distributions on R^m can be found in ref. [2].

Let $\rho(s)$, for $s \in R$, be a fixed infinitely differentiable function having the properties:

(i) $\rho(s) \geq 0$,

(ii) $\rho(s) = 0$ for $|s| \geq 1$,

(iii) $\rho(s) = \rho(-s)$,

(iv) $\int_{|x|\leq 1} \rho(|x|^2)\, dx = 1, \quad x \in R^m$.

The property (iv) in the spherical coordinates is represented as

(v) $\Omega_m \int_0^1 \rho(s^2) s^{m-1}\, ds = 1$,

where Ω_m is the surface area of the unit sphere in R^m. Putting $\delta_\epsilon(x) = \epsilon^{-m}\rho(|\epsilon^{-1}x|^2)$, where $\epsilon > 0$, it follows that ϵ-net $\{\delta_\epsilon(x)\}$ converges to the Dirac delta-function $\delta(x)$.

DEFINITION 3 *Let f and g be arbitrary distributions in $\mathcal{D}'_m$ and let*

$$f_\epsilon = f * \delta_\epsilon, \quad g_\epsilon = g * \delta_\epsilon,$$

we say that the neutrix product $f \cdot g$ of f and g exists and is equal to h on the open domain $\Omega \subseteq R^m$ if the neutrix limit

$$N - \lim_{\epsilon\to 0^+} \frac{1}{2}\{(f \cdot g_\epsilon, \phi) + (g \cdot f_\epsilon, \phi)\} = (h, \phi),$$

for all test functions ϕ with compact support contained in the domain Ω, where N is the neutrix having domain $N' = R^+$, the positive numbers, and range $N'' = R$, the real numbers, with negligible functions that are linear sums of the functions

$$\epsilon^{-\lambda} \ln^{r-1} \epsilon, \quad \ln^r \epsilon,$$

for $\lambda > 0$ and $r = 1, 2, \ldots$, and all functions of ϵ which converge to zero as ϵ tends to zero.

Using Definition 3 and the normalization procedure of $\mu(x)x_+^\lambda$, Cheng and Li [2] prove that the commutative neutrix product $r^{-p}\cdot\delta$ exists and

$$r^{-p}\cdot\delta(x) = \begin{cases} z(m, p, \delta) & p = 2, 4, 6, \ldots \\ 0 & \text{else,} \end{cases}$$

where $x \in R^m$, and

$$z(m, p, \delta) = \frac{\triangle^{p/2}\delta(x)}{2^{1+p/2}(p/2)!m(m+2)\cdots(m+p-2)}.$$

In order to consider a more general product $r^{-k}\cdot\triangle^l\delta$ for any positive integer, we need the following modified δ-sequence so that a new distribution $\mathrm{d}^k/\mathrm{d}r^k\delta(x)$ can be introduced. As we will see, it plays an important role in section 4.

Again let $\rho(s)$ be a fixed infinitely differentiable function defined on $R^+ = [0, \infty)$ having the properties:

(i) $\rho(s) \geq 0$,

(ii) $\rho(s) = 0$ for $s \geq 1$,

(iii) $\int_{R^m} \tilde{\delta}_n(x)\,\mathrm{d}x = 1$,

where $\tilde{\delta}_n(x) = C_m n^m \rho(nr)$ is the modified δ-sequence on R^m and C_m is the constant satisfying (iii).

It obviously follows that $\tilde{\delta}_n(x)$ is not equal to $\delta_n(x)$ defined in Definition 2 and that $\tilde{\delta}_n(x)$ is a δ-sequence because of the three above mentioned conditions. The following definition will be used to evaluate our general product in section 4.

Definition 4 *Let f and g be distributions in $\mathcal{D}'(m)$ and let*

$$g_n(x) = (g * \tilde{\delta}_n)(x) = (g(x-t), \tilde{\delta}_n(t)),$$

where $t = (t_1, t_2, \ldots, t_m)$. We say that the non-commutative neutrix product $f\cdot g$ of f and g exists and is equal to h if

$$N - \lim_{n\to\infty}(fg_n, \phi) = (h, \phi),$$

where $\phi \in \mathcal{D}_m$.

2. The distributions r^λ and $\mu(x)x_+^\lambda$

Let $r = (x_1^2 + \cdots + x_m^2)^{1/2}$ and consider the functional r^λ [see ref. 7] defined by

$$(r^\lambda, \phi) = \int_{R^m} r^\lambda \phi(x)\,\mathrm{d}x, \tag{3}$$

where $\operatorname{Re}\lambda > -m$ and $\phi(x) \in \mathcal{D}_m$. Because the derivative

$$\frac{\partial}{\partial\lambda}(r^\lambda, \phi) = \int r^\lambda \,\ln r\,\phi(x)\,\mathrm{d}x$$

exists, the functional r^λ is an analytic function of λ for $\operatorname{Re}\lambda > -m$.

For Re $\lambda \leq -m$, we should use the following identity (4) to define its analytic continuation. For Re $\lambda > 0$, we could deduce

$$\triangle(r^{\lambda+2}) = (\lambda+2)(\lambda+m)r^{\lambda}$$

simply by calculating the left-hand side, where $\triangle$ is the Laplacian operator. By iteration, we find for any integer k that

$$r^{\lambda} = \frac{\triangle^k r^{\lambda+2k}}{(\lambda+2)\cdots(\lambda+2k)(\lambda+m)\cdots(\lambda+m+2k-2)}. \tag{4}$$

On making substitution of spherical coordinates in equation (3), we come to

$$(r^{\lambda}, \phi) = \int_0^{\infty} r^{\lambda} \left\{ \int_{r=1} \phi(r\omega)\,\mathrm{d}\omega \right\} r^{m-1}\,\mathrm{d}r, \tag{5}$$

where $\mathrm{d}\omega$ is the hypersurface element on the unit sphere. The integral appearing in the previous integrand can be written in the form

$$\int_{r=1} \phi(r\omega)\,\mathrm{d}\omega = \Omega_m S_{\phi}(r), \tag{6}$$

where Ω_m is the hypersurface area of the unit sphere imbedded in Euclidean space of m dimensions, and S_{ϕ} is the mean value of ϕ on the sphere of radius r.

It was proved in ref. [7] that $S_{\phi}(r)$ is infinitely differentiable for $r \geq 0$, has bounded support and

$$S_{\phi}(r) = \phi(0) + a_1 r^2 + a_2 r^4 + \cdots + a_k r^{2k} + \mathrm{o}(r^{2k}),$$

for any positive integer k. From equations (5) and (6), we obtain

$$(r^{\lambda}, \phi) = \Omega_m \int_0^{\infty} r^{\lambda+m-1} S_{\phi}(r)\,\mathrm{d}r,$$

which indicates the application of $\Omega_m x_+^{\mu}$ with $\mu = \lambda + m - 1$ to the testing function $S_{\phi}(r)$. Using the following Laurent series for x_+^{λ} about $\lambda = -k$

$$x_+^{\lambda} = \frac{(-1)^{k-1}\delta^{(k-1)}(x)}{(k-1)!(\lambda+k)} + x_+^{-k} + (\lambda+k)x_+^{-k}\ln x + \cdots,$$

we could show that the residue of $(r^{\lambda}, \phi(x))$ at $\lambda = -m-2k$ for non-negative integer k is given by

$$\Omega_m \frac{(\delta^{(2k)}, S_{\phi}(x))}{(2k)!} = \Omega_m \frac{S_{\phi}^{(2k)}(0)}{(2k)!}.$$

On the other hand, the residue of the function r^{λ} of equation (4) for the same value of λ is

$$\frac{\Omega_m \triangle^k \delta(x)}{2^k k! m(m+2)\cdots(m+2k-2)}$$

[ref. 7]. Therefore, we get

$$S_{\phi}^{(2k)}(0) = \frac{(2k)!\triangle^k \phi(0)}{2^k k! m(m+2)\cdots(m+2k-2)}.$$

This result can be used to write out the Taylor's series for $S_\phi(r)$, namely,

$$S_\phi(r) = \phi(0) + \frac{1}{2!}S''_\phi(0)r^2 + \cdots + \frac{1}{(2k)!}S^{(2k)}_\phi(0)r^{2k} + \cdots$$
$$= \sum_{k=0}^{\infty} \frac{\triangle^k\phi(0)r^{2k}}{2^k k! m(m+2)\cdots(m+2k-2)},$$

which is the well-known Pizetti's formula.

Let $\mu(x)$ be an infinitely differentiable function on R^+ having properties:

(i) $\mu(x) \geq 0$,

(ii) $\mu(0) \neq 0$,

(iii) $\mu(x) = 0$ for $x \geq 1$.

Let $\phi(x)$ be a testing function. Then, the functional

$$(\mu(x)x_+^\lambda, \phi) = \int_0^1 \mu(x)x^\lambda\phi(x)\,\mathrm{d}x,$$

is regular for $\mathrm{Re}\,\lambda > -1$. It can be extended to the domain $\mathrm{Re}\,\lambda > -n-1$ ($\lambda \neq -1, -2, \ldots$) by analytic continuation as shown by Gel'fand and Shilov [ref. 2]:

$$(\mu(x)x_+^\lambda, \phi) = \int_0^1 \mu(x)x^\lambda\phi(x)\,\mathrm{d}x = \sum_{k=1}^{n} \frac{\phi^{(k-1)}(0)\mu(\theta_{k-1})}{(k-1)!(\lambda+k)}$$
$$+ \int_0^1 \mu(x)x^\lambda\left[\phi(x) - \phi(0) - x\phi'(0) - \cdots - \frac{x^{n-1}}{(n-1)!}\phi^{(n-1)}(0)\right]\mathrm{d}x,$$

by applying the mean value theorem with $0 < \theta_{k-1} < 1$ for $1 \leq k \leq n$. This means that the generalized function $\mu(x)x_+^\lambda$ is well defined for $\lambda \neq -1, -2, \ldots$.

We thus normalize the value of the functional $(\mu(x)x_+^\lambda, \phi)$ at $-n$ by

$$(\mu(x)x_+^{-n}, \phi) = \sum_{k=1}^{n-1} \frac{\phi^{(k-1)}(0)\mu(\theta_{k-1})}{(k-1)!(-n+k)} + \int_0^1 \mu(x)x^{-n}$$
$$\times \left[\phi(x) - \phi(0) - x\phi'(0) - \cdots - \frac{x^{n-1}}{(n-1)!}\phi^{(n-1)}(0)\right]\mathrm{d}x. \tag{7}$$

3. The distribution $\mathrm{d}^k/\mathrm{d}r^k\delta(x)$

As $\tilde{\delta}_n(x) = C_m n^m \rho(nr)$, we obtain

$$\frac{\mathrm{d}^k}{\mathrm{d}r^k}\tilde{\delta}_n(x) = C_m n^{m+k}\rho^{(k)}(nr), \tag{8}$$

where k is any positive integer.

We define

$$\left(\frac{\mathrm{d}^k}{\mathrm{d}r^k}\delta(x), \phi(x)\right) = N - \lim_{n\to\infty}\left(\frac{\mathrm{d}^k}{\mathrm{d}r^k}\tilde{\delta}_n(x), \phi(x)\right)$$
$$= N - \lim_{n\to\infty} C_m n^{m+k}\int_{R^m}\rho^{(k)}(nr)\phi(x)\,\mathrm{d}x$$
$$\triangleq N - \lim_{n\to\infty} I.$$

On changing to spherical polar coordinates and then making the substitution $t = nr$, we arrive at

$$I = C_m n^{m+k}\Omega_m\int_0^{1/n}\rho^{(k)}(nr)r^{m-1}S_\phi(r)\,\mathrm{d}r$$
$$= C_m\Omega_m n^k\int_0^1\rho^{(k)}(t)t^{m-1}S_\phi\left(\frac{t}{n}\right)\mathrm{d}t,$$

where $S_\phi(r)$ is defined in equation (6).

By Taylor's formula, we obtain

$$S_\phi(r) = \sum_{j=0}^{k-1}\frac{S_\phi^{(j)}(0)}{j!}r^j + \frac{S_\phi^{(k)}(0)}{k!}r^k + \frac{S_\phi^{(k+1)}(\theta r)}{(k+1)!}r^{k+1},$$

where $0 < \theta < 1$.

Hence,

$$I = C_m\Omega_m\sum_{j=0}^{k-1}\frac{S_\phi^{(j)}(0)}{j!}n^{k-j}\int_0^1\rho^{(k)}(t)t^{m+j-1}\,\mathrm{d}t + C_m\Omega_m\frac{S_\phi^{(k)}(0)}{k!}\int_0^1\rho^{(k)}(t)t^{m+k-1}\,\mathrm{d}t$$
$$+ \frac{C_m\Omega_m}{n}\int_0^1\rho^{(k)}(t)t^{m+k}S_\phi^{(k+1)}\left(\frac{\theta t}{n}\right)\mathrm{d}t$$
$$\triangleq I_1 + I_2 + I_3,$$

respectively.

It obviously follows from $k - j \geq 1$ that

$$N - \lim_{n\to\infty} I_1 = 0.$$

Putting

$$M = \sup\left\{\left|S_\phi^{(k+1)}(r)\right| : \quad r \in R^+\right\},$$

we see that

$$|I_3| \leq \frac{C_m\Omega_m}{n}M\int_0^1|\rho^{(k)}(t)|t^{m+k}\,\mathrm{d}t \longrightarrow 0,$$

as $n \to \infty$.

Integrating by parts, we have

$$C_m\Omega_m\int_0^1 \rho^{(k)}(t)t^{m+k-1}\,\mathrm{d}t = (-1)^k(m+k-1)(m+k-2)\cdots mC_m\Omega_m\int_0^1 \rho(t)t^{m-1}\,\mathrm{d}t$$
$$= (-1)^k(m+k-1)(m+k-2)\cdots m. \tag{9}$$

Hence, it follows from equation (9) that

$$N-\lim_{n\to\infty} I = I_2 = \frac{(-1)^k(m+k-1)(m+k-2)\cdots m}{k!}S_\phi^{(k)}(0).$$

Using Pizetti's formula, we get

$$S_\phi^{(k)}(0) = \begin{cases} \dfrac{(2l)!\triangle^l\phi(0)}{2^l l! m(m+2)\cdots(m+2l-2)} & \text{if } k=2l \\ 0 & \text{if } k=2l-1, \end{cases}$$

where $l = 1, 2, \ldots$.

Therefore, we have reached

$$\frac{\mathrm{d}^{2l-1}}{\mathrm{d}r^{2l-1}}\delta(x) = 0$$
$$\frac{\mathrm{d}^{2l}}{\mathrm{d}r^{2l}}\delta(x) = \frac{(m+1)(m+3)\cdots(m+2l-1)}{2^l l!}\triangle^l\delta(x). \tag{10}$$

In particular,

$$\frac{\mathrm{d}}{\mathrm{d}r}\delta(x) = 0 \quad \frac{\mathrm{d}^2}{\mathrm{d}r^2}\delta(x) = \frac{m+1}{2}\triangle\delta(x) \quad \frac{\mathrm{d}^4}{\mathrm{d}r^4}\delta(x) = \frac{(m+1)(m+3)}{8}\triangle^2\delta(x).$$

4. The main results

In this section, we utilize the distribution $\mathrm{d}^k/\mathrm{d}r^k\delta(x)$ obtained in previous section as a 'bridge' to derive a distributional product $(\sum_{i=1}^m x_i/r^k)\cdot\triangle^l\delta$. To proceed, we would like to present the following lemmas, which will be used to simplify our main result.

LEMMA 1

$$\sum_{i=1}^m \triangle^{k+1}(x_i\phi) = 2(k+1)\nabla(\triangle^k\phi) + \left(\sum_{i=1}^m x_i\right)\triangle^{k+1}\phi, \tag{11}$$

for $k \geq 0$.

Proof We use an inductive method to prove the lemma. It is obviously true for $k = 0$. Assuming $k = 1$, we have

$$\triangle^2(x_i\phi) = 4\frac{\partial}{\partial x_i}\triangle\phi + x_i\triangle^2\phi,$$

simply by calculating the left-hand side. Hence,

$$\sum_{i=1}^{m}\triangle^2(x_i\phi) = 4\nabla(\triangle\phi) + \left(\sum_{i=1}^{m}x_i\right)\triangle^2\phi.$$

By the hypothesis, equation (11) holds for the case of $k-1$, that is,

$$\sum_{i=1}^{m}\triangle^k(x_i\phi) = 2k\nabla(\triangle^{k-1}\phi) + \left(\sum_{i=1}^{m}x_i\right)\triangle^k\phi.$$

Hence, it follows that

$$\begin{aligned}
\sum_{i=1}^{m}\triangle^{k+1}(x_i\phi) &= \triangle\sum_{i=1}^{m}\triangle^k(x_i\phi) = \triangle\left\{2k\nabla(\triangle^{k-1}\phi) + \left(\sum_{i=1}^{m}x_i\right)\triangle^k\phi\right\}\\
&= 2k\nabla(\triangle^k\phi) + \sum_{i=1}^{m}\triangle(x_i\triangle^k\phi)\\
&= 2k\nabla(\triangle^k\phi) + \sum_{i=1}^{m}\left\{2\frac{\partial}{\partial x_i}\triangle^k\phi + x_i\triangle^{k+1}\phi\right\}\\
&= 2k\nabla(\triangle^k\phi) + 2\nabla(\triangle^k\phi) + \left(\sum_{i=1}^{m}x_i\right)\triangle^{k+1}\phi\\
&= 2(k+1)\nabla(\triangle^k\phi) + \left(\sum_{i=1}^{m}x_i\right)\triangle^{k+1}\phi.
\end{aligned}$$

This completes the proof of Lemma 1. Note that this Lemma still holds when $k = -1$, *i.e.*, we have

$$\sum_{i=1}^{m}\triangle^k(x_i\phi) = 2k\nabla(\triangle^{k-1}\phi) + \left(\sum_{i=1}^{m}x_i\right)\triangle^k\phi,$$

for $k \geq 0$. ■

LEMMA 2

$$\sum_{i=1}^{m}(x_i\triangle^k\delta) = -2k\nabla(\triangle^{k-1}\delta), \tag{12}$$

where $k \geq 0$.

Proof Obviously $\sum_{i=0}^{m} x_i \triangle^k \delta = \sum_{i=0}^{m} x_i \delta = 0$ for $k = 0$. Applying equation (11) with $k > 0$, we have

$$\begin{aligned}
\left(\sum_{i=1}^{m}(x_i \triangle^k \delta), \phi\right) &= \left(\delta, \sum_{i=1}^{m} \triangle^k (x_i \phi)\right) \\
&= \left(\delta, \left\{2k\nabla(\triangle^{k-1}\phi) + \left(\sum_{i=1}^{m} x_i\right)\triangle^k \phi\right\}\right) \\
&= 2k\nabla(\triangle^k \phi(0)) \\
&= (-2k\nabla(\triangle^{k-1}\delta), \phi).
\end{aligned}$$

Therefore, we have reached our conclusion in Lemma 2. ■

THEOREM *The non-commutative neutrix product* $\left(\sum_{i=1}^{m} x_i/r^k\right) \cdot \triangle^l \delta$ *exists. Furthermore*

$$\left(\sum_{i=1}^{m} x_i/r^{2k}\right) \cdot \triangle^l \delta = \frac{-l!}{2^{k-1}(l+k-1)!(m+2l)\cdots(m+2l+2k-2)} \nabla(\triangle^{l+k-1}\delta)$$

and

$$\left(\sum_{i=1}^{m} x_i/r^{2k-1}\right) \cdot \triangle^l \delta = 0,$$

where k and l are positive integers.

Proof From equation (10), we have

$$\begin{aligned}
\triangle^l \delta(x) &= \frac{2^l l!}{(m+1)(m+3)\cdots(m+2l-1)} \frac{\mathrm{d}^{2l}}{\mathrm{d}r^{2l}} \delta(x) \\
&= f(m,l) \frac{\mathrm{d}^{2l}}{\mathrm{d}r^{2l}} \delta(x),
\end{aligned}$$

where $f(m,l)$ is the constant depending on m and l.

We note that $\sum_{i=1}^{m} x_i/r^k$ is a locally summable function on R^m for $k = 1, 2, \ldots, m$. It follows from Definition 4 and equation (8) for any testing function ψ that

$$\begin{aligned}
\left(\sum_{i=1}^{m} x_i/r^k\right) \cdot \triangle^l \delta, \psi) &= f(m,l) \left(\left(\sum_{i=1}^{m} x_i/r^k\right) \cdot \frac{\mathrm{d}^{2l}}{\mathrm{d}r^{2l}} \delta, \psi\right) \\
&= N - \lim_{n\to\infty} \sum_{i=1}^{m} f(m,l) \int_{R^m} r^{-k} \frac{\mathrm{d}^{2l}}{\mathrm{d}r^{2l}} \tilde{\delta}_n(x) x_i \psi(x)\,\mathrm{d}x \\
&= N - \lim_{n\to\infty} \sum_{i=1}^{m} f(m,l) C_m n^{m+2l} \int_{R^m} r^{-k} \rho^{(2l)}(nr) \phi_i(x)\,\mathrm{d}x \\
&\stackrel{\Delta}{=} N - \lim_{n\to\infty} I,
\end{aligned}$$

where $\phi_i = x_i\psi$. Making the two substitutions as before, we obtain

$$I = \sum_{i=1}^{m} f(m,l) C_m \Omega_m n^{m+2l} \int_0^{1/n} r^{m-k-1} \rho^{(2l)}(nr) S_{\phi_i}(r)\, \mathrm{d}r$$
$$= \sum_{i=1}^{m} f(m,l) C_m \Omega_m n^{2l+k} \int_0^{1} t^{m-k-1} \rho^{(2l)}(t) S_{\phi_i}\left(\frac{t}{n}\right) \mathrm{d}t. \tag{13}$$

Using Taylor's formula, we obtain

$$S_{\phi_i}(r) = \sum_{j=0}^{2l+k-1} \frac{S_{\phi_i}^{(j)}(0)}{j!} r^j + \frac{S_{\phi_i}^{(2l+k)}(0)}{(2l+k)!} r^{2l+k} + \frac{S_{\phi_i}^{(2l+k+1)}(\theta_i r)}{(2l+k+1)!} r^{2l+k+1}$$

where $0 < \theta_i < 1$.

Following similar techniques of section 3, we can prove

$$N - \lim_{n\to\infty} I = \sum_{i=1}^{m} f(m,l) \frac{S_{\phi_i}^{(2l+k)}(0)}{(2l+k)!} C_m \Omega_m \int_0^1 t^{2l+m-1} \rho^{(2l)}(t)\, \mathrm{d}t.$$

Applying equation (9) with $k = 2l$, we get

$$C_m \Omega_m \int_0^1 t^{2l+m-1} \rho^{(2l)}(t)\, \mathrm{d}t = (m+2l-1)(m+2l-2)\cdots m.$$

It follows from the previous equation that

$$N - \lim_{n\to\infty} I = \sum_{i=1}^{m} f(m,l) \frac{S_{\phi_i}^{(2l+k)}(0)}{(2l+k)!} (m+2l-1)(m+2l-2)\cdots m$$
$$= \frac{2^l l!(m+2l-1)(m+2l-2)\cdots m}{(m+1)(m+3)\cdots(m+2l-1)(2l+k)!} \sum_{i=1}^{m} S_{\phi_i}^{(2l+k)}(0).$$

Using Pizetti's formula, we have

$$S_{\phi_i}^{(2l+2k)}(0) = \frac{(2l+2k)!}{2^{l+k}(l+k)!m(m+2)\cdots(m+2l+2k-2)} \triangle^{l+k}\phi_i(0)$$
$$S_{\phi_i}^{(2l+2k-1)}(0) = 0.$$

By Lemma 2, we have

$$\sum_{i=1}^{m} \triangle^{l+k}\phi_i(0) = \left(\sum_{i=1}^{m} x_i \triangle^{l+k}\delta, \psi\right) = (-2(l+k)\nabla(\triangle^{l+k-1}\delta), \psi),$$

and the result follows for $k = 1, 2, \ldots, m$.

We now turn our attention to the product $\left(\sum_{i=1}^m x_i/r^k\right) \cdot \triangle^l \delta$ for $k \geq m+1$. Note that, in this case, the functional $\sum_{i=1}^m x_i/r^k$ is not locally summable. We assume $k = m+q+1$ for $q = 0, 1, 2, \ldots$ and apply the regularization in equation (7) to I of equation (13) to deduce

$$\begin{aligned} I &= \sum_{i=1}^m f(m,l) C_m \Omega_m n^{2l+k} \left\{ \sum_{j=1}^{q=k-m-1} \frac{S_{\phi_i}^{(j-1)}(0) \rho^{(2l)}(\theta_{i,\,j-1})}{(j-1)!(m-k+j)} \quad (= I_1) \right. \\ &\quad \left. + \sum_{i=1}^m \int_0^1 \rho^{(2l)}(t) t^{m-k-1} \left[S_{\phi_i}\left(\frac{t}{n}\right) - S_{\phi_i}(0) - \cdots - \frac{t^q}{n^q q!} S_{\phi_i}^{(q)}(0) \right] \mathrm{d}t \right\} \quad (= I_2) \\ &= I_1 + I_2, \end{aligned}$$

respectively.

Clearly,

$$\mathrm{N} - \lim_{n\to\infty} I_1 = 0.$$

Applying Taylor's theorem, we obtain

$$\begin{aligned} I_2 &= \sum_{i=1}^m f(m,l) C_m \Omega_m n^{2l+k} \int_0^1 \rho^{(2l)}(t) t^{m-k-1} \left[\frac{t^{q+1}}{n^{q+1}(q+1)!} S_{\phi_i}^{(q+1)}(0) + \cdots \right. \\ &\quad + \frac{t^{q+2l+m}}{n^{q+2l+m}(q+2l+m)!} S_{\phi_i}^{(q+2l+m)}(0) + \frac{t^{q+2l+m+1}}{n^{q+2l+m+1}(q+2l+m+1)!} \\ &\quad \left. \times S_{\phi_i}^{(q+2l+m+1)}\left(\frac{\theta_i t}{n}\right) \right] \mathrm{d}t, \end{aligned}$$

where $0 < \theta_i < 1$.

Similarly, we can prove

$$\begin{aligned} \mathrm{N} - \lim_{n\to\infty} I_2 &= \sum_{i=1}^m \frac{f(m,l) C_m \Omega_m \; S_{\phi_i}^{(q+2l+m)}(0)}{(q+2l+m)!} \int_0^1 \rho^{(2l)}(t) t^{2l+m-1}\, \mathrm{d}t \\ &= \sum_{i=1}^m \frac{f(m,l)\; S_{\phi_i}^{(2l+k)}(0)}{(2l+k)!} C_m \Omega_m \int_0^1 \rho^{(2l)}(t) t^{2l+m-1}\, \mathrm{d}t \\ &= \sum_{i=1}^m \frac{f(m,l) S_{\phi_i}^{(2l+k)}(0)}{(2l+k)!} (m+2l-1)(m+2l-2)\cdots m. \end{aligned}$$

This implies our assertion for $k = m+1, m+2, \ldots$.

Obviously, we have the following

$$\left(\sum_{i=1}^m \frac{x_i}{r^{2k}}\right) \cdot \Delta\delta = -\frac{\nabla(\Delta^k \delta)}{2^{k-1} k!(m+2)(m+4)\cdots(m+2k)}$$

$$\left(\sum_{i=1}^m \frac{x_i}{r^{2k-1}}\right) \cdot \Delta\delta = 0$$

by setting $l = 1$ in the theorem. ■

From our result, we can easily derive the product $\nabla r^{-k} \cdot \triangle^l \delta$ where $\nabla = \sum_{i=1}^m \partial/\partial x_i$. The author leaves this for interested readers.

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