An approach for distributional products on $\mathbb{R}^m$

C. K. Li

Department of Mathematics and Computer Science, Brandon University, Brandon, Manitoba, Canada, R7A 6A9
Published online: 26 Jan 2007.

To cite this article: C. K. Li (2005) An approach for distributional products on $\mathbb{R}^m$, Integral Transforms and Special Functions, 16:2, 139-151, DOI: 10.1080/1065246042000272117

To link to this article: http://dx.doi.org/10.1080/1065246042000272117
An approach for distributional products on $R^n$

C. K. LI*

Department of Mathematics and Computer Science, Brandon University, Brandon, Manitoba, Canada R7A 6A9

(Received 3 June 2003)

One of the main problems in the theory of generalized functions is the lack of definitions for products and powers of distributions. Antosik, Mikusiński and Sikorski in 1972 introduced a definition for a product of distributions using a $\delta$-sequence. However, $\delta^2$ as a product of $\delta$ with itself was shown not to exist. Later, Koh and Li in 1992 chose a fixed $\delta$-sequence without compact support and used the concept of the neutrix limit of van der Corput to define $\delta^k$ and $(\delta')^k$ for some values of $k$ [Koh, E.L. and Li, C.K., 1992, On the distributions $\delta^k$ and $(\delta')^k$. Mathematische Nachrichten, 157, 243–248]. To extend the sequence approach from one-dimensional to $m$-dimensional, Li and Fisher [Li, C. K. and Fisher, B. (1990). Examples of the neutrix product of distributions on $R^n$. Rad. Mat., 6, 129–137.] constructed a ‘useful’ $\delta$-sequence $\delta_n$ on $R^m$ to deduce a non-commutative neutrix product $r^{-k} \cdot \Delta \delta$ for any positive integer $k$ between 1 and $m-1$ inclusive. Their method of completing such a product is totally based on the fact that $\Delta \delta_n$ is computable. However, it seems impossible to deal with more general products involving $\Delta l \delta$ along the same line because of difficulties in evaluating $\Delta l \delta_n$, where $l$ is a positive integer. The objective of this paper is to provide a modified $\delta$-sequence and define a new ‘bridge’ distribution $(d_k/d r^k)\delta(x)$, which is used to compute $\Delta l \delta$. By applying the normalization procedure of distribution $x^{-n}$ given by Gel’fand and Shilov [Gel’fand, I. M. and Shilov, G. E. (1964). Generalized Functions, Vol. I. Academic Press.] and two identities of $\delta$ distribution, we derive an interesting distributional product $(\sum_{i=1}^{m} x_i/r^k) \cdot \Delta l \delta$ (hence $\nabla r^{-k} \cdot \Delta l \delta$ as well).

Keywords: Pizetti’s formula; $\delta$-Sequences; Neutrix limit and distributions

2000 Mathematics Subject Classification: Primary: 46F10

1. Introduction

Physicists have long been using the so-called singular functions such as $\delta$, although these cannot be properly defined within the framework of classical function theory. In elementary particle physics [see ref. 3], the expression $\delta^2$ is used when calculating the transition rates of certain particle interactions. Similarly, one finds in the scientific literature such mathematical objects as $\log \delta$ and $\sqrt{\delta}$. Many attempts have been made to define multiplication of distributions including the major works of Rosinger (Generalized Solutions of Nonlinear Partial Differential Equations, North-Holland, 1987) and Columbeau (New Generalized Functions and Multiplications of Distributions, North-Holland, Amsterdam, 1984) as well as Antosik, Mikusiński, Sikorski, Jones, and Fisher’s sequence methods.

*Email: lic@brandonu.ca
Let \( \rho(x) \) be a fixed infinitely differentiable function with the following properties:

(i) \( \rho(x) \geq 0 \),
(ii) \( \rho(x) = 0 \) for \( |x| \geq 1 \),
(iii) \( \rho(x) = \rho(-x) \),
(iv) \( \int_{-1}^{1} \rho(x) \, dx = 1 \).

The function \( \delta_n(x) \) is defined by \( \delta_n(x) = n \rho(nx) \) for \( n = 1, 2, \ldots \). It follows that \( \{ \delta_n(x) \} \) is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function \( \delta(x) \) in the distributional sense.

Now, let \( \mathcal{D} \) be the testing function space of infinitely differentiable functions of a single variable with compact support and let \( \mathcal{D}' \) be the space of distributions defined on \( \mathcal{D} \). Then if \( f \) is an arbitrary distribution in \( \mathcal{D}' \), we define

\[
\begin{align*}
f_n(x) &= (f * \delta_n)(x) = (f(t), \delta_n(x-t)),
\end{align*}
\]

for \( n = 1, 2, \ldots \). It follows that \( \{f_n(x)\} \) is a regular sequence of infinitely differentiable functions converging to the distribution \( f(x) \) in \( \mathcal{D}' \).

The following definition for the non-commutative neutrix product \( f \cdot g \) of two distributions \( f \) and \( g \) in \( \mathcal{D}' \) was given by Fisher in ref. [4].

**Definition 1** Let \( f \) and \( g \) be distributions in \( \mathcal{D}' \) and let \( g_n = g * \delta_n \). We say that the neutrix product \( f \cdot g \) of \( f \) and \( g \) exists and is equal to \( h \) if

\[
N - \lim_{n \to \infty} (fg_n, \phi) = (h, \phi)
\]

for all functions \( \phi \) in \( \mathcal{D} \), where \( N \) is the neutrix [see ref. 5] having domain \( N' = \{1, 2, \ldots\} \) and range \( N'' \), the real numbers, with negligible functions that are finite linear sums of the functions

\[
n^\lambda \ln^r n, \quad \ln^r n \quad (\lambda > 0, r = 1, 2, \ldots),
\]

and all functions of \( n \) which converge to zero in the normal sense as \( n \) tends to infinity.

The product of Definition 1 is not symmetric and hence \( f \cdot g \neq g \cdot f \) in general.

Extending definitions of products from a one-dimensional space \( R \) to an \( m \)-dimensional space \( R^m \) by using appropriate \( \delta \)-sequences has recently been an interesting topic in distribution theory. The following work on the non-commutative neutrix product of distributions on \( R^m \) can be found in refs. [1, 8].

Let \( \rho(s) \) be a fixed infinitely differentiable function defined on \( R^+ = [0, \infty) \) having the properties:

(i) \( \rho(s) \geq 0 \),
(ii) \( \rho(s) = 0 \) for \( s \geq 1 \),
(iii) \( \int_{R^+} \delta_n(x) \, dx = 1 \),

where \( \delta_n(x) = c_m n^m \rho(n^2 r^2) \) and \( c_m \) is the constant satisfying (iii).

It follows that \( \{\delta_n(x)\} \) is a regular \( \delta \)-sequence of infinitely differentiable functions converging to \( \delta(x) \) in \( \mathcal{D}'_m \).
DEFINITION 2  Let $f$ and $g$ be distributions in $\mathcal{D}'_m$ (an $m$-dimensional space of distributions) and let
\[ g_n(x) = (g \ast \delta_n)(x) = (g(x - t), \delta_n(t)), \]
where $t = (t_1, t_2, \ldots, t_m)$. We say that the neutrix product $f \cdot g$ of $f$ and $g$ exists and is equal to $h$ if
\[ N - \lim_{n \to \infty} (fg_n, \phi) = (h, \phi), \]
where $\phi \in \mathcal{D}_m$ (an $m$-dimensional Schwartz space) and the $N$-limit is defined as mentioned earlier.

With Definition 2, Li and Fisher [see also ref. 1] show that the non-commutative neutrix product $r^{-k} \cdot \Delta \delta$ ($\Delta$ is the Laplacian) exists and
\[
\begin{align*}
\frac{\Delta^{k+1} \delta}{2^k (k + 1)! (m + 2) (m + 4) \cdots (m + 2k)}, \\
\end{align*}
\]
for $k = 1, 2, \ldots, \lfloor (m - 1)/2 \rfloor$ (the greatest integer function) and
\[
\begin{align*}
r^{1-2k} \cdot \Delta \delta = 0, \\
\end{align*}
\]
for $k = 1, 2, \ldots, \lfloor m/2 \rfloor$.

The following work on the commutative neutrix product of distributions on $\mathbb{R}^m$ can be found in ref. [2].

Let $\rho(s)$, for $s \in \mathbb{R}$, be a fixed infinitely differentiable function having the properties:
\begin{enumerate}
  \item $\rho(s) \geq 0$,
  \item $\rho(s) = 0$ for $|s| \geq 1$,
  \item $\rho(s) = \rho(-s)$,
  \item $\int_{|x| \leq 1} \rho(|x|^2) \, dx = 1$, $x \in \mathbb{R}^m$.
\end{enumerate}
The property (iv) in the spherical coordinates is represented as
\[
\begin{align*}
\Omega_m \int_0^1 \rho(s^2) s^{m-1} \, ds = 1,
\end{align*}
\]
where $\Omega_m$ is the surface area of the unit sphere in $\mathbb{R}^m$. Putting $\delta_\epsilon(x) = \epsilon^{-m} \rho(|\epsilon^{-1}x|^2)$, where $\epsilon > 0$, it follows that $\epsilon$-net $\{\delta_\epsilon(x)\}$ converges to the Dirac delta-function $\delta(x)$.

DEFINITION 3  Let $f$ and $g$ be arbitrary distributions in $\mathcal{D}'_m$ and let
\[ f_\epsilon = f \ast \delta_\epsilon, \quad g_\epsilon = g \ast \delta_\epsilon, \]
we say that the neutrix product $f \cdot g$ of $f$ and $g$ exists and is equal to $h$ on the open domain $\Omega \subseteq \mathbb{R}^m$ if the neutrix limit
\[ N - \lim_{\epsilon \to 0^+} \frac{1}{2} \{(f \cdot g_\epsilon, \phi) + (g \cdot f_\epsilon, \phi)\} = (h, \phi), \]
for all test functions $\phi$ with compact support contained in the domain $\Omega$, where $N$ is the neutrix having domain $N' = \mathbb{R}^+$, the positive numbers, and range $N'' = \mathbb{R}$, the real numbers, with negligible functions that are linear sums of the functions
\[ \epsilon^{-\lambda} \ln^{-1} \epsilon, \quad \ln^r \epsilon, \]
for $\lambda > 0$ and $r = 1, 2, \ldots$, and all functions of $\epsilon$ which converge to zero as $\epsilon$ tends to zero.
Using Definition 3 and the normalization procedure of $\mu(x)x_+^\lambda$, Cheng and Li [2] prove that the commutative neutrix product $r^{-p} \cdot \delta$ exists and

$$r^{-p} \cdot \delta(x) = \begin{cases} z(m, p, \delta) & \text{for } p = 2, 4, 6, \ldots \\ 0 & \text{else}, \end{cases}$$

where $x \in \mathbb{R}^m$, and

$$z(m, p, \delta) = \frac{\Delta^{p/2} \delta(x)}{2^{1+p/2}(p/2)! m(m + 2) \cdots (m + p - 2)}.$$

In order to consider a more general product $r^{-k} \cdot \Delta^l \delta$ for any positive integer, we need the following modified $\delta$-sequence so that a new distribution $d^k/d^k \delta(x)$ can be introduced. As we will see, it plays an important role in section 4.

Again let $\rho(s)$ be a fixed infinitely differentiable function defined on $\mathbb{R}^+ = [0, \infty)$ having the properties:

(i) $\rho(s) \geq 0$,
(ii) $\rho(s) = 0$ for $s \geq 1$,
(iii) $\int_{\mathbb{R}^m} \tilde{\delta}_n(x) \, dx = 1$,

where $\tilde{\delta}_n(x) = C_m n^m \rho(nr)$ is the modified $\delta$-sequence on $\mathbb{R}^m$ and $C_m$ is the constant satisfying (iii).

It obviously follows that $\tilde{\delta}_n(x)$ is not equal to $\delta_n(x)$ defined in Definition 2 and that $\tilde{\delta}_n(x)$ is a $\delta$-sequence because of the three above mentioned conditions. The following definition will be used to evaluate our general product in section 4.

**Definition 4** Let $f$ and $g$ be distributions in $\mathcal{D}'(m)$ and let

$$g_n(x) = (g * \tilde{\delta}_n)(x) = (g(x - t), \tilde{\delta}_n(t)),$$

where $t = (t_1, t_2, \ldots, t_m)$. We say that the non-commutative neutrix product $f \cdot g$ of $f$ and $g$ exists and is equal to $h$ if

$$N - \lim_{n \to \infty} (fg_n, \phi) = (h, \phi),$$

where $\phi \in \mathcal{D}_m$.

2. The distributions $r^\lambda$ and $\mu(x)x_+^\lambda$

Let $r = (x_1^2 + \cdots + x_m^2)^{1/2}$ and consider the functional $r^\lambda$ [see ref. 7] defined by

$$(r^\lambda, \phi) = \int_{\mathbb{R}^m} r^\lambda \phi(x) \, dx,$$  \hspace{1cm} (3)$$

where $\text{Re } \lambda > -m$ and $\phi(x) \in \mathcal{D}_m$. Because the derivative

$$\frac{\partial}{\partial \lambda} (r^\lambda, \phi) = \int r^\lambda \ln r \phi(x) \, dx$$

exists, the functional $r^\lambda$ is an analytic function of $\lambda$ for $\text{Re } \lambda > -m$. 

For $\text{Re} \lambda \leq -m$, we should use the following identity (4) to define its analytic continuation. For $\text{Re} \lambda > 0$, we could deduce

$$\triangle (r^{\lambda+2}) = (\lambda + 2)(\lambda + m)r^{\lambda}$$

simply by calculating the left-hand side, where $\triangle$ is the Laplacian operator. By iteration, we find for any integer $k$ that

$$r^{\lambda} = \frac{\Delta^k r^{\lambda+2k}}{(\lambda + 2) \cdots (\lambda + 2k)(\lambda + m) \cdots (\lambda + m + 2k - 2)}. \tag{4}$$

On making substitution of spherical coordinates in equation (3), we come to

$$(r^\lambda, \phi) = \int_0^{\infty} r^\lambda \left\{ \int_{r=1} r \phi (r \omega) \, d\omega \right\} r^{m-1} \, dr, \tag{5}$$

where $d\omega$ is the hypersurface element on the unit sphere. The integral appearing in the previous integrand can be written in the form

$$\int_{r=1} \phi (r \omega) \, d\omega = \Omega_m S_\phi (r), \tag{6}$$

where $\Omega_m$ is the hypersurface area of the unit sphere imbedded in Euclidean space of $m$ dimensions, and $S_\phi$ is the mean value of $\phi$ on the sphere of radius $r$.

It was proved in ref. [7] that $S_\phi (r)$ is infinitely differentiable for $r \geq 0$, has bounded support and

$$S_\phi (r) = \phi (0) + a_1 r^2 + a_2 r^4 + \cdots + a_k r^{2k} + o(r^{2k}),$$

for any positive integer $k$. From equations (5) and (6), we obtain

$$(r^\lambda, \phi) = \Omega_m \int_0^{\infty} r^{\lambda+m-1} S_\phi (r) \, dr,$$

which indicates the application of $\Omega_m x^\mu$ with $\mu = \lambda + m - 1$ to the testing function $S_\phi (r)$. Using the following Laurent series for $x^\lambda_+$ about $\lambda = -k$

$$x^\lambda_+ = \frac{(-1)^{k-1} \delta (k-1) (x)}{(k-1)! (\lambda + k)} + x^{-k} + (\lambda + k) x^{-k} \ln x + \cdots,$$

we could show that the residue of $(r^\lambda, \phi (x))$ at $\lambda = -m - 2k$ for non-negative integer $k$ is given by

$$\Omega_m \frac{(\delta^{(2k)} , S_\phi (x))}{(2k)!} = \frac{S_\phi^{(2k)} (0)}{(2k)!} \cdot$$

On the other hand, the residue of the function $r^\lambda$ of equation (4) for the same value of $\lambda$ is

$$\frac{\Omega_m \Delta^k \delta (x)}{2^k k! m(m + 2) \cdots (m + 2k - 2)}$$

[ref. 7]. Therefore, we get

$$S_\phi^{(2k)} (0) = \frac{(2k)! \Delta^k \phi (0)}{2^k k! m(m + 2) \cdots (m + 2k - 2)}.$$
This result can be used to write out the Taylor’s series for $S_\phi(r)$, namely,

$$S_\phi(r) = \phi(0) + \frac{1}{2!} S''_\phi(0) r^2 + \cdots + \frac{1}{(2k)!} S^{(2k)}_\phi(0) r^{2k} + \cdots = \sum_{k=0}^{\infty} \frac{\Delta^k \phi(0) r^{2k}}{2^k k! m(m+2) \cdots (m+2k-2)},$$

which is the well-known Pizetti’s formula.

Let $\mu(x)$ be an infinitely differentiable function on $\mathbb{R}^+$ having properties:

(i) $\mu(x) \geq 0,$
(ii) $\mu(0) \neq 0,$
(iii) $\mu(x) = 0$ for $x \geq 1.$

Let $\phi(x)$ be a testing function. Then, the functional

$$(\mu(x)x_+^\lambda, \phi) = \int_0^1 \mu(x)x_+^\lambda \phi(x) \, dx,$$

is regular for $\text{Re} \lambda > -1.$ It can be extended to the domain $\text{Re} \lambda > -n - 1$ ($\lambda \neq -1, -2, \ldots$) by analytic continuation as shown by Gel’fand and Shilov [ref. 2]:

$$(\mu(x)x_+^\lambda, \phi) = \int_0^1 \mu(x)x_+^\lambda \phi(x) \, dx = \sum_{k=1}^{n} \frac{\phi^{(k-1)}(0) \mu(\theta_{k-1})}{(k-1)! (\lambda + k)} + \int_0^1 \mu(x)x_+^\lambda \left[ \phi(x) - \phi(0) - x\phi'(0) - \cdots - \frac{x^{n-1}}{(n-1)!} \phi^{(n-1)}(0) \right] \, dx,$$

by applying the mean value theorem with $0 < \theta_{k-1} < 1$ for $1 \leq k \leq n.$ This means that the generalized function $\mu(x)x_+^\lambda$ is well defined for $\lambda \neq -1, -2, \ldots.$

We thus normalize the value of the functional $(\mu(x)x_+^\lambda, \phi)$ at $-n$ by

$$\left(\mu(x)x_+^{-n}, \phi\right) = \sum_{k=1}^{n-1} \frac{\phi^{(k-1)}(0) \mu(\theta_{k-1})}{(k-1)! (-n + k)} + \int_0^1 \mu(x)x_+^{-n} \times \left[ \phi(x) - \phi(0) - x\phi'(0) - \cdots - \frac{x^{n-1}}{(n-1)!} \phi^{(n-1)}(0) \right] \, dx.$$

(7)

3. **The distribution $d^k/dr^k \delta(x)$**

As $\tilde{\delta}_n(x) = C_m n^m \rho(nr)$, we obtain

$$\frac{d^k}{dr^k} \tilde{\delta}_n(x) = C_m n^{m+k} \rho^{(k)}(nr),$$

(8)

where $k$ is any positive integer.
We define

\[
\left( \frac{d^k}{dr^k} \delta(x), \phi(x) \right) = N - \lim_{n \to \infty} \left( \frac{d^k}{dr^k} \tilde{\delta}_n(x), \phi(x) \right)
\]

\[
= N - \lim_{n \to \infty} C_m n^{m+k} \int_{R^m} \rho^{(k)}(nr) \phi(x) \, dx
\]

\[
\triangleq N - \lim_{n \to \infty} I.
\]

On changing to spherical polar coordinates and then making the substitution \( t = nr \), we arrive at

\[
I = C_m n^{m+k} \Omega_m \int_0^{1/n} \rho^{(k)}(t) r^{m-1} S_{\phi}(r) \, dr
\]

\[
= C_m \Omega_m n^{k} \int_0^1 \rho^{(k)}(t) t^{m-1} S_{\phi} \left( \frac{t}{n} \right) \, dt,
\]

where \( S_{\phi}(r) \) is defined in equation (6).

By Taylor’s formula, we obtain

\[
S_{\phi}(r) = \sum_{j=0}^{k-1} \frac{S^{(j)}(0)}{j!} r^j + \frac{S^{(k)}(0)}{k!} r^k + \frac{S^{(k+1)}(\theta r)}{(k+1)!} r^{k+1},
\]

where \( 0 < \theta < 1 \).

Hence,

\[
I = C_m \Omega_m \sum_{j=0}^{k-1} \frac{S^{(j)}(0)}{j!} n^{k-j} \int_0^1 \rho^{(k)}(t) t^{m+j-1} \, dt + C_m \Omega_m \frac{S^{(k)}(0)}{k!} \int_0^1 \rho^{(k)}(t) t^{m+k-1} \, dt
\]

\[
+ \frac{C_m \Omega_m}{n} \int_0^1 \rho^{(k)}(t) t^{m+k} S^{(k+1)}_{\phi} \left( \frac{\theta t}{n} \right) \, dt
\]

\[
\triangleq I_1 + I_2 + I_3,
\]

respectively.

It obviously follows from \( k - j \geq 1 \) that

\[
N - \lim_{n \to \infty} I_1 = 0.
\]

Putting

\[
M = \sup \left\{ \left| S^{(k+1)}_{\phi}(r) \right| : \ r \in R^+ \right\},
\]

we see that

\[
|I_3| \leq \frac{C_m \Omega_m}{n} \int_0^1 |\rho^{(k)}(t)| t^{m+k} \, dt \to 0,
\]

as \( n \to \infty \).
Integrating by parts, we have
\[ C_m \Omega_m \int_0^1 \rho^{(k)}(t)t^{m+k-1} dt = (-1)^k (m + k - 1)(m + k - 2) \cdots m C_m \Omega_m \int_0^1 \rho(t)t^{m-1} dt \]
\[ = (-1)^k (m + k - 1)(m + k - 2) \cdots m. \]  \hspace{1cm} (9)

Hence, it follows from equation (9) that
\[ N - \lim_{n \to \infty} I = I_2 = \frac{(-1)^k (m + k - 1)(m + k - 2) \cdots m}{k!} S^{(k)}(0). \]

Using Pizetti’s formula, we get
\[ S^{(k)}(0) = \begin{cases} \frac{(2l)!\Delta^l \phi(0)}{2^l l! (m + 2) \cdots (m + 2l - 2)} & \text{if } k = 2l \\ 0 & \text{if } k = 2l - 1, \end{cases} \]

where \( l = 1, 2, \ldots \)

Therefore, we have reached
\[ \frac{d^{2l-1}}{dr^{2l-1}} \delta(x) = 0 \]
\[ \frac{d^{2l}}{dr^{2l}} \delta(x) = \frac{(m + 1)(m + 3) \cdots (m + 2l - 1)}{2^l l!} \Delta^l \delta(x). \]  \hspace{1cm} (10)

In particular,
\[ \frac{d}{dr} \delta(x) = 0 \quad \frac{d^2}{dr^2} \delta(x) = \frac{m + 1}{2} \Delta \delta(x) \quad \frac{d^4}{dr^4} \delta(x) = \frac{(m + 1)(m + 3)}{8} \Delta^2 \delta(x). \]

4. The main results

In this section, we utilize the distribution \( d^k/dx^k \delta(x) \) obtained in previous section as a ‘bridge’ to derive a distributional product \( (\sum_{i=1}^m x_i/r^k) \cdot \Delta^l \delta(x) \). To proceed, we would like to present the following lemmas, which will be used to simplify our main result.

**Lemma 1**

\[ \sum_{i=1}^m \Delta^{k+1}(x_i \phi) = 2(k + 1)\nabla(\Delta^k \phi) + \left( \sum_{i=1}^m x_i \right) \Delta^{k+1} \phi, \]  \hspace{1cm} (11)

for \( k \geq 0 \).
Proof We use an inductive method to prove the lemma. It is obviously true for $k = 0$. Assuming $k = 1$, we have

$$\Delta^2(x_i \phi) = 4 \frac{\partial}{\partial x_i} \Delta \phi + x_i \Delta^2 \phi,$$

simply by calculating the left-hand side. Hence,

$$\sum_{i=1}^{m} \Delta^2(x_i \phi) = 4 \nabla(\Delta \phi) + \left( \sum_{i=1}^{m} x_i \right) \Delta^2 \phi.$$

By the hypothesis, equation (11) holds for the case of $k − 1$, that is,

$$\sum_{i=1}^{m} \Delta^k(x_i \phi) = 2k \nabla(\Delta^{k−1} \phi) + \left( \sum_{i=1}^{m} x_i \right) \Delta^k \phi.$$

Hence, it follows that

$$\sum_{i=1}^{m} \Delta^{k+1}(x_i \phi) = \Delta \sum_{i=1}^{m} \Delta^k(x_i \phi) = \Delta \left\{ 2k \nabla(\Delta^{k−1} \phi) + \left( \sum_{i=1}^{m} x_i \right) \Delta^k \phi \right\}$$

$$= 2k \nabla(\Delta^k \phi) + \sum_{i=1}^{m} \Delta(x_i \Delta^k \phi)$$

$$= 2k \nabla(\Delta^k \phi) + \sum_{i=1}^{m} \left\{ 2 \frac{\partial}{\partial x_i} \Delta^k \phi + x_i \Delta^{k+1} \phi \right\}$$

$$= 2k \nabla(\Delta^k \phi) + 2 \nabla(\Delta^k \phi) + \left( \sum_{i=1}^{m} x_i \right) \Delta^{k+1} \phi$$

$$= 2(k + 1) \nabla(\Delta^k \phi) + \left( \sum_{i=1}^{m} x_i \right) \Delta^{k+1} \phi.$$

This completes the proof of Lemma 1. Note that this Lemma still holds when $k = −1$, i.e., we have

$$\sum_{i=1}^{m} \Delta^k(x_i \phi) = 2k \nabla(\Delta^{k−1} \phi) + \left( \sum_{i=1}^{m} x_i \right) \Delta^k \phi,$$

for $k \geq 0$.

Lemma 2

$$\sum_{i=1}^{m} (x_i \Delta^k \delta) = -2k \nabla(\Delta^{k−1} \delta),$$

where $k \geq 0$. ■
Proof. Obviously, $\sum_{i=0}^{m} x_i \Delta^k \delta = \sum_{i=0}^{m} x_i \delta = 0$ for $k = 0$. Applying equation (11) with $k > 0$, we have

$$\left( \sum_{i=1}^{m} (x_i \Delta^k \delta), \phi \right) = \left( \delta, \sum_{i=1}^{m} \Delta^k (x_i \phi) \right)$$

$$= \left( \delta, \left\{ 2k \nabla (\Delta^{k-1} \phi) + \left( \sum_{i=1}^{m} x_i \right) \Delta^k \phi \right\} \right)$$

$$= 2k \nabla (\Delta^k \phi(0))$$

$$= (-2k \nabla (\Delta^k \delta), \phi).$$

Therefore, we have reached our conclusion in Lemma 2. \(\blacksquare\)

Theorem. The non-commutative neutrix product $\left( \sum_{i=1}^{m} x_i / r^{k} \right) \cdot \Delta^l \delta$ exists. Furthermore

$$\left( \sum_{i=1}^{m} x_i / r^{2k} \right) \cdot \Delta^l \delta = \frac{-l!}{2^{k-1} (l + k - 1)! (m + 2l) \cdots (m + 2l + 2k - 2)} \nabla (\Delta^l + k - 1 \delta)$$

and

$$\left( \sum_{i=1}^{m} x_i / r^{2k-1} \right) \cdot \Delta^l \delta = 0,$$

where $k$ and $l$ are positive integers.

Proof. From equation (10), we have

$$\Delta^l \delta(x) = \frac{2^l l!}{(m + 1)(m + 3) \cdots (m + 2l - 1)} \frac{d^{2l}}{dr^{2l}} \delta(x)$$

$$= f(m, l) \frac{d^{2l}}{dr^{2l}} \delta(x),$$

where $f(m, l)$ is the constant depending on $m$ and $l$.

We note that $\sum_{i=1}^{m} x_i / r^k$ is a locally summable function on $R^m$ for $k = 1, 2, \ldots, m$. It follows from Definition 4 and equation (8) for any testing function $\psi$ that

$$\left( \sum_{i=1}^{m} x_i / r^{k} \right) \cdot \Delta^l \delta, \psi) = f(m, l) \left( \left( \sum_{i=1}^{m} x_i / r^k \right) \cdot \frac{d^{2l}}{dr^{2l}} \delta, \psi \right)$$

$$= N - \lim_{n \to \infty} \sum_{i=1}^{m} f(m, l) \int_{R^m} r^{-k} \frac{d^{2l}}{dr^{2l}} \delta_n(x) x_i \psi(x) \, dx$$

$$= N - \lim_{n \to \infty} \sum_{i=1}^{m} f(m, l) C_m n^{m+2l} \int_{R^m} r^{-k} \rho^{(2l)}(nr) \phi_i(x) \, dx$$

$$\triangleq N - \lim_{n \to \infty} I.$$
where \( \phi_i = x_i \psi \). Making the two substitutions as before, we obtain

\[
I = \sum_{i=1}^{m} f(m, l)C_{m, \Omega_m}n^{m+2l} \int_{0}^{1/n} r^{m-k-1} \rho^{(2l)}(nr) S_{\phi_i}(r) \, dr
\]

\[
= \sum_{i=1}^{m} f(m, l)C_{m, \Omega_m}n^{2l+k} \int_{0}^{1} t^{m-k-1} \rho^{(2l)}(t) S_{\phi_i} \left( \frac{t}{n} \right) \, dt.
\]

(13)

Using Taylor’s formula, we obtain

\[
S_{\phi_i}(r) = 2l + k \sum_{j=0}^{m} \frac{S^{(j)}(0)}{j!} r^j + \frac{S^{(2l+k)}(0)}{(2l+k)!} r^{2l+k} + \frac{S^{(2l+k+1)}(\theta_i r)}{(2l+k+1)!} r^{2l+k+1}
\]

where \( 0 < \theta_i < 1 \).

Following similar techniques of section 3, we can prove

\[
N - \lim_{n \to \infty} I = \sum_{i=1}^{m} f(m, l) \frac{S^{(2l+k)}(0)}{(2l+k)!} C_{m, \Omega_m} \int_{0}^{1} t^{2l+m-1} \rho^{(2l)}(t) \, dt.
\]

Applying equation (9) with \( k = 2l \), we get

\[
C_{m, \Omega_m} \int_{0}^{1} t^{2l+m-1} \rho^{(2l)}(t) \, dt = (m + 2l - 1)(m + 2l - 2) \cdots m.
\]

It follows from the previous equation that

\[
N - \lim_{n \to \infty} I = \sum_{i=1}^{m} f(m, l) \frac{S^{(2l+k)}(0)}{(2l+k)!} (m + 2l - 1)(m + 2l - 2) \cdots m
\]

\[
= \frac{2l!}{(m+1)(m+3) \cdots (m+2l-1)(2l+k)!} \sum_{i=1}^{m} S^{(2l+k)}(0).
\]

Using Pizetti’s formula, we have

\[
S^{(2l+2k)}(0) = \frac{(2l+2k)!}{2l+k!(l+k)!m(m+2) \cdots (m+2l+2k-2)} \Delta^{l+k} \phi_i(0)
\]

\[
S^{(2l+2k-1)}(0) = 0.
\]

By Lemma 2, we have

\[
\sum_{i=1}^{m} \Delta^{l+k} \phi_i(0) = \left( \sum_{i=1}^{m} x_i \Delta^{l+k} \delta, \psi \right) = (-2(l+k)) \nabla (\Delta^{l+k-1} \delta, \psi),
\]

and the result follows for \( k = 1, 2, \ldots, m \).
We now turn our attention to the product \( \left( \sum_{i=1}^{m} x_i/r^k \right) \cdot \Delta^l \delta \) for \( k \geq m + 1 \). Note that, in this case, the functional \( \sum_{i=1}^{m} x_i/r^k \) is not locally summable. We assume \( k = m + q + 1 \) for \( q = 0, 1, 2, \ldots \) and apply the regularization in equation (7) to \( I \) of equation (13) to deduce

\[
I = \sum_{i=1}^{m} f(m, l) C_m \Omega_m n^{2l+k} \left\{ \sum_{j=1}^{q=m-1} \frac{S_{\phi_i}^{(j-1)}(0) \rho^{(2l)}(\theta_i, j-1)}{(j-1)!(m-k+j)} \right\} (I_1)
\]

\[
+ \sum_{i=1}^{m} \int_{0}^{1} \rho^{(2l)}(t) t^{m-k-1} \left[ S_{\phi_i} \left( \frac{t}{n} \right) - S_{\phi_i}(0) - \cdots - \frac{t^q}{n^q q!} S_{\phi_i}^{(q)}(0) \right] dt \right\} (I_2)
\]

\[
= I_1 + I_2.
\]

respectively.

Clearly,

\[
N - \lim_{n \to \infty} I_1 = 0.
\]

Applying Taylor’s theorem, we obtain

\[
I_2 = \sum_{i=1}^{m} f(m, l) C_m \Omega_m n^{2l+k} \int_{0}^{1} \rho^{(2l)}(t) t^{m-k-1} \left[ \frac{t^{q+1}}{n^{q+1}(q+1)!} S_{\phi_i}^{(q+1)}(0) + \cdots \right.
\]

\[
+ \frac{t^{q+1} + 2l + m}{n^{q+2l+m}(q + 2l + m)!} S_{\phi_i}^{(q+2l+m)}(0) + \frac{t^{q+1} + 2l + m}{n^{q+2l+m+1}(q + 2l + m + 1)!} \] \times S_{\phi_i}^{(q+2l+m+1)} \left( \frac{\theta_i t}{n} \right) dt,
\]

where \( 0 < \theta_i < 1 \).

Similarly, we can prove

\[
N - \lim_{n \to \infty} I_2 = \sum_{i=1}^{m} f(m, l) C_m \Omega_m \left[ \frac{S_{\phi_i}^{(q+2l+m)}(0)}{(q + 2l + m)!} \right] \int_{0}^{1} \rho^{(2l)}(t) t^{2l+m-1} dt
\]

\[
= \sum_{i=1}^{m} \frac{f(m, l) S_{\phi_i}^{(2l+k)}(0)}{(2l+k)!} C_m \Omega_m \int_{0}^{1} \rho^{(2l)}(t) t^{2l+m-1} dt
\]

\[
= \sum_{i=1}^{m} \frac{f(m, l) S_{\phi_i}^{(2l+k)}(0)}{(2l+k)!} (m + 2l - 1)(m + 2l - 2) \cdots m.
\]

This implies our assertion for \( k = m + 1, m + 2, \ldots \).

Obviously, we have the following

\[
\left( \sum_{i=1}^{m} \frac{x_i}{r^{2k}} \right) \cdot \Delta^l = - \frac{\nabla(\Delta^l \delta)}{2^{k-1}k!(m + 2)(m + 4) \cdots (m + 2k)}
\]

\[
\left( \sum_{i=1}^{m} \frac{x_i}{r^{2k-1}} \right) \cdot \Delta^l \delta = 0
\]

by setting \( l = 1 \) in the theorem.

From our result, we can easily derive the product \( \nabla r^{-k} \cdot \Delta^l \delta \) where \( \nabla = \sum_{i=1}^{m} \partial/\partial x_i \). The author leaves this for interested readers.
Acknowledgements

I am deeply indebted to Dr. Brian Fisher and my visiting professor, Vincent Zou, who made numerous minor changes which improved the readability of this paper. This research is supported by NSERC.

References