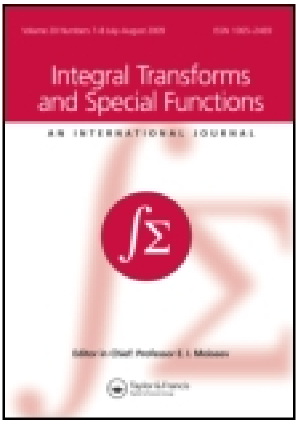

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Several results on the commutative neutrix product of distributions

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Several results on the commutative neutrix product of distributions

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Let $H(x)$ denote Heaviside's function. The goal of this paper is to evaluate the commutative neutrix products $f_+(x) \diamond \delta^{(r)}(x)$ and $f_-(x) \diamond \delta^{(r)}(x)$ for $r = 0, 1, 2, \ldots$, where $f(x)$ is only the r-th differentiable on an open interval containing the origin and $f_+(x) = H(x)f(x)$ and $f_-(x) = H(-x)f(x)$. We also obtain the products $(\psi(x)/x) \diamond \delta(x)$, including a few examples as well as $x_+^{-r-(1/2)} \diamond x_-^{-r-(1/2)}$.

Keywords: Product; δ-sequence; Neutrix and distribution

2000 Mathematics Subject Classification: Primary: 46F10

1. Introduction

One of the problems in distribution theory is the lack of definitions for products and powers, in general. In physics (see *e.g.* [1], p. 141), one finds the need to evaluate δ^2 when calculating the transition rates of certain particle interactions. Embacher, Grübl and Oberguggenberger [2] studied the parameter products of distributions in several variables arising from the computations of the one-loop vacuum polarization of zero-mass QED_2 to avoid the occurrence of renormalization ambiguities. In [3], a definition for product of distributions was given by delta sequences. However, δ^2 as a product of δ with itself was shown not to exist. Bremermann [4] used the Cauchy representations of distributions with compact supports to define $\sqrt{\delta_+}$ and $\log \delta_+$. Unfortunately, his definition did not carry over to $\sqrt{\delta}$ and $\log \delta$.

Now let $\mathcal{D}$ be the testing function space of infinitely differentiable functions of a single variable with compact support, and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. The definition of the product of a distribution and an infinitely differentiable function is the following (see for example [5]).

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Definition 1.1 *Let f be a distribution in $\mathcal{D}'$ and let g be an infinitely differentiable function. Then the product fg is defined by*

$$(fg, \phi) = (f, g\phi)$$

for all functions ϕ in $\mathcal{D}$.

An extension of the product of a distribution and an infinitely differentiable function was given by Fisher in [6].

Definition 1.2 *Let f and g be distributions in $\mathcal{D}'$ for which, on the interval (a, b), f is the k-th derivative of a locally summable function F in $L^p(a, b)$ and $g^{(k)}$ is a locally summable function in $L^q(a, b)$ with $(1/p) + (1/q) = 1$. Then the product $fg = gf$ of f and g is defined on the interval (a, b) by*

$$fg = \sum_{i=0}^{k} \binom{k}{i} (-1)^i [Fg^{(i)}]^{(k-i)}.$$

To further extend Definition 1.2, Fisher, with his collaborators [6–12], has actively used Temples' δ-sequence $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots,$ where $\rho(x)$ is a fixed infinitely differentiable function on R with four properties:

(i) $\rho(x) \geq 0$,
(ii) $\rho(x) = 0$ for $|x| \geq 1$,
(iii) $\rho(x) = \rho(-x)$,
(iv) $\int_{-1}^{1} \rho(x) \mathrm{d}x = 1$

and the concept of neutrix limit of van der Corput [13] (in order to abandon unwanted infinite quantities from asymptotic expressions) to deduce numerous products, powers, convolutions and compositions of distributions on R. One of Fisher's definitions for the commutative product of distributions was given as follows.

Let f be an arbitrary distribution in $\mathcal{D}'$, we define

$$f_n(x) = (f * \delta_n)(x) = (f(t), \delta_n(x - t))$$

for $n = 1, 2, \ldots$. It follows that $\{f_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $f(x)$ in $\mathcal{D}'$.

Definition 1.3 *Let f and g be distributions in $\mathcal{D}'$ and let $f_n = f * \delta_n$ and $g_n = g * \delta_n$. We say that the commutative neutrix product $f \cdot g$ of f and g exists and is equal to h if*

$$N - \lim_{n \to \infty} (f_n g_n, \phi) = (h, \phi)$$

for all functions ϕ in $\mathcal{D}$, where N is the neutrix (see [13]) having domain $N' = \{1, 2, \ldots\}$ and range the real numbers, with negligible functions that are finite linear sums of the functions

$$n^{\lambda} \ln^{r-1} n, \quad \ln^r n \ \ (\lambda > 0, r = 1, 2, \ldots)$$

and all functions of n that converge to zero in the normal sense as n tends to infinity.

Recently, Fisher, Özçağ and Gülen [14] employed the above definition to imply the following theorem.

THEOREM 1.1 *Let f be a function, which is infinitely differentiable on an open interval containing the origin and let $f_+(x) = H(x)f(x)$ and $f_-(x) = H(-x)f(x)$. Then, the commutative neutrix products $f_+(x) \cdot \delta^{(r)}(x)$ and $f_-(x) \cdot \delta^{(r)}(x)$ exist and*

$$f_+(x) \cdot \delta^{(r)}(x) = f_-(x) \cdot \delta^{(r)}(x) = \sum_{k=0}^{r} \frac{(-1)^{r-k}}{2} \binom{r}{k} f^{(r-k)}(0)\delta^{(k)}(x),$$

where $r = 0, 1, 2, \ldots$.

We begin to compute the commutative neutrix products $f_+(x) \diamond \delta^r(x)$ and $f_-(x) \diamond \delta^r(x)$ under the weaker condition that f is only the r-th differentiable with a new definition. Then we obtain several interesting corollaries. Furthermore, we derive the commutative product $(\psi(x)/x) \diamond \delta(x)$, as well as the product $x_+^{-r-(1/2)}$ and $x_-^{-r-(1/2)}$.

2. New results

Let $\delta_n(x) = n\rho(nx)$ be defined as in the introduction. We use the following definition for the commutative neutrix product of a single variable (see [15]).

DEFINITION 2.1 *Let f and g be distributions in $\mathcal{D}'$ and let $f_n = f * \delta_n$ and $g_n = g * \delta_n$. We say that the commutative neutrix product $f \diamond g$ of f and g exists and is equal to h if*

$$N - \lim_{n\to\infty} \frac{1}{2}\{(f_n g, \phi) + (f g_n, \phi)\} = (h, \phi)$$

for all functions ϕ in $\mathcal{D}$. If the normal limit exists, then it is simply called the commutative product.

THEOREM 2.1 *Definition* 2.1 *extends Definitions* 1.1 *and* 1.2.

Proof Let g be a C^∞ function. Obviously, $g_n\phi$ has an uniform support and converges to $g\phi$ in $\mathcal{D}$. For any f in $\mathcal{D}'$, we imply that

$$\begin{aligned}(f \diamond g, \phi) &= N - \lim_{n\to\infty} \frac{1}{2}\{(f_n g, \phi) + (f g_n, \phi)\} \\ &= N - \lim_{n\to\infty} \frac{1}{2}\{(f_n, g\phi) + (f, g_n\phi)\} = (f, g\phi).\end{aligned}$$

■

To see Definition 2.1 is an extension of Definition 1.2, we let f and g be functions given in Definition 1.2. Then we have

$$\lim_{n\to\infty} (f g_n, \phi) = \lim_{n\to\infty} (f_n g, \phi) = \left(\sum_{i=0}^{k} \binom{k}{i} (-1)^i \left[F g^{(i)}\right]^{(k-i)}, \phi \right)$$

by Theorem 2 in [8]. This completes the proof.

At this moment, it is not quite clear if Definition 2.1 is equivalent to Definition 1.3, although both define the commutative neutrix products. In fact,

$$\begin{aligned}&N - \lim_{n\to\infty} \frac{1}{2}\{(f_n g, \phi) + (f g_n, \phi)\} - N - \lim_{n\to\infty} (f_n g_n, \phi) \\ &\quad - N - \lim_{n\to\infty} \frac{1}{2}\{(f_n(g - g_n) + g_n(f - f_n), \phi)\}.\end{aligned}$$

One can neither prove the above being zero nor non-zero by a counter example. The author welcomes and appreciates any discussion from interested readers.

However, we are able to claim that the two definitions are different if we remove the neutrix limits from both. Indeed, we have $\delta'(x) \diamond \delta'(x) = 0$ on subspace $\mathcal{D}_0(R) = \{\phi(x) \in \mathcal{D}(R) \mid \phi(0) = 0\}$ because

$$\begin{aligned}(\delta_n'(x)\delta'(x), \phi(x)) &= -(\delta_n'(x)\phi(x))'|_{x=0} \\ &= -(\delta_n''(0)\phi(0) + \delta_n'(0)\phi'(0)) = 0\end{aligned}$$

for all n. But it is impractical to compute the product $\delta'(x) \cdot \delta'(x)$ without recourse to neutrix limit, since $(\delta_n'(x)\delta_n'(x), \phi(x))$ diverges.

THEOREM 2.2 *Let f and g be distributions in $\mathcal{D}'$ and suppose that the commutative neutrix products $f \diamond g$ and $f' \diamond g$ exist. Then the commutative neutrix product $f \diamond g'$ exists and*

$$(f \diamond g)' = f' \diamond g + f \diamond g'.$$

Proof The result follows immediately from Definition 2.1. ■

THEOREM 2.3 *Let f be a function which is the r-th differentiable on an open interval containing the origin. Then the commutative neutrix products $f_+(x) \diamond \delta^{(r)}(x)$ and $f_-(x) \diamond \delta^{(r)}(x)$ exist and*

$$f_+(x) \diamond \delta^{(r)}(x) = f_-(x) \diamond \delta^{(r)}(x) = \sum_{k=0}^{r} \frac{(-1)^{r-k}}{2} \binom{r}{k} f^{(r-k)}(0)\delta^{(k)}(x),$$

where $r = 0, 1, 2, \ldots$.

Proof For any $\phi \in \mathcal{D}$, we use integration by parts repeatedly

$$\begin{aligned}&(f_+(x)(\delta^{(r)}(x) * \delta_n(x)), \phi(x)) \\ &= \int_0^\infty \delta_n^{(r)}(x) f(x)\phi(x)\mathrm{d}x \\ &= f(x)\phi(x)\delta_n^{(r-1)}(x)\Big|_0^\infty - \int_0^\infty (f(x)\phi(x))'\delta_n^{(r-1)}(x)\mathrm{d}x \\ &= -f(0)\phi(0)\delta_n^{(r-1)}(0) - (f(x)\phi(x))'\delta_n^{(r-2)}(x)\Big|_0^\infty \\ &\quad + \int_0^\infty (f(x)\phi(x))''\delta_n^{(r-2)}(x)\mathrm{d}x \\ &= -f(0)\phi(0)\delta_n^{(r-1)}(0) - (f(0)\phi(0))'\delta_n^{(r-2)}(0) \\ &\quad + \int_0^\infty (f(x)\phi(x))''\delta_n^{(r-2)}(x)\mathrm{d}x \\ &= \cdots \quad \cdots \\ &= -f(0)\phi(0)\delta_n^{(r-1)}(0) + (f(0)\phi(0))'\delta_n^{(r-2)}(0) + \cdots \\ &\quad + (-1)^r \int_0^\infty (f(x)\phi(x))^r \delta_n(x)\mathrm{d}x.\end{aligned}$$

It follows that

$$N-\lim_{n\to\infty}(f_+(x)(\delta^{(r)}(x)*\delta_n(x)),\phi(x))$$
$$=\frac{1}{2}(-1)^r(f(0)\phi(0))^{(r)}$$
$$=\frac{1}{2}(-1)^r\sum_{k=0}^{r}\binom{r}{k}f^{(r-k)}(0)\phi^{(k)}(0)$$

by noting that

$$\int_0^\infty \delta_n(x)\mathrm{d}x=\frac{1}{2}.$$

On the other hand, we have

$$((f_+(x)*\delta_n(x))\delta^{(r)}(x),\phi(x))$$
$$=\left(\delta^{(r)}(x),\int_{-\infty}^{\infty}H(t)f(t)\delta_n(x-t)\mathrm{d}t\,\phi(x)\right)$$
$$=(-1)^r\left(\int_0^\infty f(t)\delta_n(x-t)\mathrm{d}t\,\phi(x)\right)^{(r)}\Bigg|_{x=0}$$
$$=(-1)^r\sum_{k=0}^{r}\binom{r}{k}\int_0^\infty f(t)\delta_n^{(r-k)}(-t)\mathrm{d}t\,\phi^{(k)}(0).$$

Making the substitution $u=-t$, we arrive at

$$\lim_{n\to\infty}\int_0^\infty f(t)\delta_n^{(r-k)}(-t)\mathrm{d}t=\lim_{n\to\infty}\int_{-\infty}^{0}f(-u)\delta_n^{(r-k)}(u)\mathrm{d}u=\frac{f^{(r-k)}(0)}{2}$$

due to

$$\int_{-\infty}^0 \delta_n(x)\mathrm{d}x=\frac{1}{2}.$$

It follows that

$$\lim_{n\to\infty}((f_+(x)*\delta_n(x))\delta^{(r)}(x),\phi(x))=\frac{1}{2}(-1)^r\sum_{k=0}^{r}\binom{r}{k}f^{(r-k)}(0)\phi^{(k)}(0).$$

Adding the above two terms clearly gives us

$$f_+(x)\diamond\delta^{(r)}(x)=\sum_{k=0}^{r}\frac{(-1)^{r-k}}{2}\binom{r}{k}f^{(r-k)}(0)\delta^{(k)}(x).$$

To obtain the product $f_-(x)\diamond\delta^r(x)$, note that

$$(H(x)f(-x))\diamond\delta^{(r)}(x)=\sum_{k=0}^{r}\frac{1}{2}\binom{r}{k}f^{r-k}(0)\delta^k(x)$$

and replace x by $-x$ to get

$$f_-(x)\diamond\delta^r(x)=\sum_{k=0}^{r}\frac{(-1)^{r-k}}{2}\binom{r}{k}f^{(r-k)}(0)\delta^{(k)}(x)$$

by using $\delta^{(k)}(x)=(-1)^k\delta^{(k)}(-x)$ for $k=0,1,\dots,r$. This completes the proof. ■

COROLLARY 2.1 *Let f be a function, which is the r-th differentiable on an open interval containing the origin. Then the commutative neutrix product* $f(x) \diamond \delta^{(r)}(x)$ *exists and*

$$f(x) \diamond \delta^{(r)}(x) = \sum_{k=0}^{r}(-1)^{r-k}\binom{r}{k} f^{(r-k)}(0)\delta^{(k)}(x),$$

where $r = 0, 1, 2, \ldots$.

COROLLARY 2.2 $H(x) \diamond \delta^{(r)}(x) = H(-x) \diamond \delta^{(r)}(x) = (1/2)\delta^{(r)}(x)$ *for any non-negative integer r.*

COROLLARY 2.3 $\delta^2(x) = \delta(x) \diamond \delta(x) = 0.$

Proof From Corollary 2.2, we have $H(x) \diamond \delta(x) = \delta(x)/2$ and differentiating both sides by Theorem 2.2 we get

$$\delta(x) \diamond \delta(x) + H(x) \diamond \delta'(x) = \frac{1}{2}\delta'(x),$$

which shows that $\delta^2(x) = \delta(x) \diamond \delta(x) = 0$.

Similarly, we can derive $\delta(x) \diamond \delta'(x) = \delta'(x) \diamond \delta(x) = 0$ by noting that

$$\delta'(x) \diamond \delta(x) + \delta(x) \diamond \delta'(x) = 0.$$

■

THEOREM 2.4 *The commutative product* $(1/x) \diamond \delta(x)$ *exists and*

$$\frac{1}{x} \diamond \delta(x) = -\frac{1}{2}\delta'(x).$$

Proof By the mean value theorem, we get

$$\begin{aligned}\left(\frac{1}{x}\delta_n(x), \phi(x)\right) &= \int_0^{\infty} \frac{\delta_n(x)\phi(x) - \delta_n(-x)\phi(-x)}{x}\mathrm{d}x \\ &= \int_0^{\infty} \delta_n(x)\frac{\phi(x) - \phi(-x)}{x}\mathrm{d}x = \frac{\phi(\zeta) - \phi(-\zeta)}{2\zeta},\end{aligned}$$

where $0 < \zeta < 1/n$.

It follows that

$$\begin{aligned}\lim_{n\to\infty}\left(\frac{1}{x}\delta_n(x), \phi(x)\right) &= \lim_{\zeta\to 0}\frac{\phi(\zeta) - \phi(-\zeta)}{2\zeta} \\ &= \phi'(0).\end{aligned}$$

On the other hand, we have

$$\frac{1}{x} * \delta_n(x) = \left(\frac{1}{t}, \delta_n(x - t)\right)$$

and

$$\begin{aligned}&\left(\left(\frac{1}{x} * \delta_n(x)\right)\delta(x), \phi(x)\right) \\ &= \left(\delta(x), \left(\frac{1}{x} * \delta_n(x)\right)\phi(x)\right) \\ &= \left(\frac{1}{t}, \delta_n(-t)\right)\phi(0) = \int_0^{\infty}\frac{\delta_n(t) - \delta_n(-t)}{t}\mathrm{d}t\,\phi(0) = 0.\end{aligned}$$

By Definition 2.1, we complete the proof. ■

COROLLARY 2.4 *The commutative product* $(\psi(x)/x) \diamond \delta(x)$ *exists and*

$$\frac{\psi(x)}{x} \diamond \delta(x) = -\frac{\psi(0)}{2}\delta'(x) + \psi'(0)\delta(x).$$

Proof Clearly we have

$$\frac{\psi(x)}{x} = \frac{\psi(0)}{x} + \psi'(\zeta x),$$

where $0 < \zeta < 1$ and $\psi'(\zeta x)$ is continuous. Hence,

$$\frac{\psi(x)}{x} \diamond \delta(x) = -\frac{\psi(0)}{2}\delta'(x) + \psi'(\zeta x) \diamond \delta(x) = -\frac{\psi(0)}{2}\delta'(x) + \psi'(0)\delta(x),$$

by Theorem 2.4 and Corollary 2.1. ■

In particular, we obtain the following products from Corollary 2.4.

$$\begin{aligned}
&\frac{1}{x} \diamond \delta(x) = -\frac{1}{2}\delta'(x)\\
&\frac{\sin x}{x} \diamond \delta(x) = \delta(x)\\
&\frac{\cos x}{x} \diamond \delta(x) = -\frac{1}{2}\delta'(x)\\
&\frac{e^x}{x} \diamond \delta(x) = -\frac{1}{2}\delta'(x) + \delta(x).
\end{aligned}$$

THEOREM 2.5 *The commutative product* $x_+^{-r-(1/2)}$ *and* $x_-^{-r-(1/2)}$ *exists and*

$$x_+^{-r-(1/2)} \diamond x_-^{-r-(1/2)} = \frac{(-1)^r\pi}{2(2r)!}\delta^{(2r)}(x),$$

where $r = 0, 1, 2, \ldots$.

Proof Using

$$\begin{aligned}
x_-^{-r-(1/2)} &= \frac{2^r}{1\cdot 3\cdot\cdots\cdot(2r-1)}\frac{d^r}{dx^r}x_-^{-(1/2)} = c_1\frac{d^r}{dx^r}x_-^{-(1/2)} \quad \text{and}\\
x_+^{-r-(1/2)} &= \frac{(-1)^r 2^r}{1\cdot 3\cdot\cdots\cdot(2r-1)}\frac{d^r}{dx^r}x_+^{-(1/2)} = (-1)^r c_1\frac{d^r}{dx^r}x_+^{-(1/2)},
\end{aligned}$$

we have

$$\begin{aligned}
x_-^{-r-(1/2)} * \delta_n &= c_1(x_-^{-(1/2)} * \delta_n^{(r)}) = c_1\int_x^{(1/n)}(t-x)^{-(1/2)}\delta_n^{(r)}(t)dt \quad \text{and}\\
x_+^{-r-(1/2)} * \delta_n &= (-1)^r c_1(x_+^{-(1/2)} * \delta_n^{(r)}) = (-1)^r c_1\int_{-(1/n)}^{x}(x-t)^{-(1/2)}\delta_n^{(r)}(t)\,dt.
\end{aligned}$$

■

By Definition 2.1, we need to compute

$$I = \frac{1}{2}\{(x_+^{-r-(1/2)}, (x_-^{-r-(1/2)} * \delta_n)\phi(x)) + (x_-^{-r-(1/2)}, (x_+^{-r-(1/2)} * \delta_n)\phi(x))\}$$
$$= \frac{1}{2}(I_1 + I_2).$$

Consider each term separately,

$$I_1 = (x_+^{-r-(1/2)}, (x_-^{-r-(1/2)} * \delta_n)\phi(x))$$
$$= c_1^2 \sum_{j=0}^{r} \binom{r}{j} \int_0^{(1/n)} x^{-(1/2)} \int_x^{(1/n)} (t-x)^{-(1/2)} \delta_n^{(r+j)}(t)\mathrm{d}t\, \phi^{(r-j)}(x)\mathrm{d}x$$
$$= c_1^2 \sum_{j=0}^{r} \binom{r}{j} \int_0^{(1/n)} \delta_n^{(r+j)}(t)\mathrm{d}t \int_0^t x^{-(1/2)}(t-x)^{-(1/2)}\phi^{(r-j)}(x)\mathrm{d}x \quad \text{and}$$
$$I_2 = (x_-^{-r-(1/2)}, (x_+^{-r-(1/2)} * \delta_n)\phi(x))$$
$$= c_1^2 \sum_{j=0}^{r} \binom{r}{j} \int_0^{(1/n)} \delta_n^{(r+j)}(t)\mathrm{d}t \int_0^t x^{-(1/2)}(t-x)^{-(1/2)}(-1)^{r+j}\phi^{(r-j)}(-x)\mathrm{d}x.$$

Adding I_1 and I_2 to get,

$$I = \frac{c_1^2}{2} \sum_{j=0}^{r} \binom{r}{j} \int_0^{(1/n)} \delta_n^{(r+j)}(t)\mathrm{d}t \int_0^t x^{-(1/2)}(t-x)^{-(1/2)}$$
$$\times (\phi^{(r-j)}(x) + (-1)^{(r+j)}\phi^{(r-j)}(-x))\mathrm{d}x.$$

By Taylor series,

$$\phi^{(r-j)}(x) + (-1)^{(r+j)}\phi^{(r-j)}(-x)$$
$$= \sum_{i=0}^{j+r-1} \frac{1}{i!}[\phi^{(r-j+i)}(0) + (-1)^{r+j+i}\phi^{(r-j+i)}(0)]x^i$$
$$+ \frac{2\phi^{(2r)}(0)}{(j+r)!}x^{j+r} + \frac{1}{(r+j+1)!}[\phi^{(2r+1)}(\eta x) - \phi^{(2r+1)}(-\eta x)]x^{r+j+1},$$

where $0 < \eta < 1$. Integrating by parts, we come to conclude

$$\int_0^{(1/n)} \delta_n^{(r+j)}(t)\mathrm{d}t \int_0^t x^{-(1/2)}(t-x)^{-(1/2)} \frac{2\phi^{(2r)}(0)}{(j+r)!}x^{j+r}\mathrm{d}x$$
$$= \frac{2\phi^{(2r)}(0)}{(j+r)!} \int_0^{(1/n)} \delta_n^{(r+j)}(t)t^{j+r}\mathrm{d}t\, \beta\left(r+j+\frac{1}{2}, \frac{1}{2}\right)$$
$$= (-1)^{(r+j)}\beta\left(r+j+\frac{1}{2}, \frac{1}{2}\right)\phi^{(2r)}(0)$$

by using

$$\int_0^t x^{r+j-(1/2)}(t-x)^{-(1/2)}\,\mathrm{d}x = t^{(r+j)}\beta\left(r+j+\frac{1}{2}, \frac{1}{2}\right),$$

where β is the Beta function. Similarly, we can derive that

$$\lim_{n\to\infty}\frac{c_1^2}{2}\sum_{j=0}^{r}\binom{r}{j}\int_0^{(1/n)}\delta_n^{(r+j)}(t)\mathrm{d}t\int_0^t x^{-(1/2)}(t-x)^{-(1/2)}\cdot$$

$$\frac{1}{(r+j+1)!}[\phi^{(2r+1)}(\eta x)-\phi^{(2r+1)}(-\eta x)]x^{r+j+1}\,\mathrm{d}x=0.$$

On the other hand, we see that

$$\int_0^{(1/n)}\delta_n^{(r+j)}(t)t^i\,\mathrm{d}t=0,$$

if $(r+j+i)$ is even and positive. This implies that

$$\frac{c_1^2}{2}\sum_{j=0}^{r}\binom{r}{j}\int_0^{(1/n)}\delta_n^{(r+j)}(t)\mathrm{d}t\int_0^t x^{-(1/2)}(t-x)^{-(1/2)}\cdot$$

$$\sum_{i=0}^{j+r-1}\frac{1}{i!}[\phi^{(r-j+i)}(0)+(-1)^{r+j+i}\phi^{(r-j+i)}(0)]x^i\,\mathrm{d}x=0.$$

By the following identity in [16]

$$\sum_{j=0}^{r}\binom{r}{j}(-1)^j\beta\left(r+j+\frac{1}{2},\frac{1}{2}\right)=\beta\left(r+\frac{1}{2},r+\frac{1}{2}\right),$$

we obtain

$$\begin{aligned}x_+^{-r-(1/2)}\diamond x_-^{-r-(1/2)}&=\frac{c_1^2}{2}\sum_{j=0}^{r}\binom{r}{j}(-1)^{(r+j)}\beta\left(r+j+\frac{1}{2},\frac{1}{2}\right)\delta^{(2r)}(x)\\&=\frac{c_1^2}{2}(-1)^r\beta\left(r+\frac{1}{2},r+\frac{1}{2}\right)\delta^{(2r)}(x)=\frac{(-1)^r\pi}{2(2r)!}\delta^{(2r)}(x)\end{aligned}$$

due to the fact that

$$\beta\left(r+\frac{1}{2},r+\frac{1}{2}\right)=\frac{\Gamma^2(r+1/2)}{(2r)!}=\frac{\pi}{(2r)!c_1^2}.$$

Remark 1 This result is obviously identical with one obtained by Fisher in [12]. It is interesting to point out that both $(x_+^{-r-(1/2)},(x_-^{-r-(1/2)}*\delta_n)\phi(x))$ and $(x_-^{-r-(1/2)},(x_+^{-r-(1/2)}*\delta_n)\phi(x))$ are divergent in the normal sense $n\to\infty$, but the addition is convergent.

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