British Journal of Mathematics & Computer Science
3(1): 73-85, 2013

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Asymptotic Expressions of Several Distributions on the Sphere

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Research Article

Received: 15 July 2012
Accepted: 15 October 2012
Published: 09 March 2013

Abstract

How to define the products of distributions is a difficult and not completely understood problem, and has been studied from several points of views since Schwartz established the theory of distributions by treating singular functions as linear and continuous functions on the testing function space. Many fields, such as differential equations or quantum mechanics, require such multiplications. In this paper, we use the Temple delta sequence and the convolution given on the regular manifolds to derive an invariant theorem, that powerfully changes the products of distributions of several dimensional spaces into the well-defined products of a single variable. With the help of the invariant theorem, we solve a couple of particular distributional products and hence we are able to obtain asymptotic expressions for $\delta^{(k)}(\frac{1}{a(r)}(r-t))$ as well as the distribution $\delta^{(k)}(\frac{1}{a(r)}(r^2-t^2))$ by the Fourier transform, where the distribution $\delta^{(k)}(r-t)$ focused on the sphere O_t is defined by

$$(\delta^{(k)}(r-t),\, \phi) = \frac{(-1)^k}{t^{n-1}} \int_{O_t} \frac{\partial^k}{\partial r^k}(\phi r^{n-1}) dO_t.$$

Keywords: Distribution, product, asymptotic expansion, Fourier transform, invariant theorem and Pizzetti's formula

2010 Mathematics Subject Classification: 46F10

1 Introduction

We let $\mathcal{D}(R^n)$ be the Schwartz space of the testing functions with bounded support in R^n and let $r^2 = \sum_{i=1}^n x_i^2$. The distribution $\delta(r-t)$ concentrated on the sphere O_t of $r - t = 0$ is defined as

$$(\delta(r-t),\, \phi) = \int_{O_t} \phi dO_t$$

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where dO_t is the Euclidean element on the sphere $r - t = 0$. Let $S_\phi(r)$ be the mean value of $\phi(x) \in \mathcal{D}(R^n)$ on the sphere of radius r given by

$$S_\phi(r) = \frac{1}{\Omega_n} \int_{r=1} \phi(r\sigma) dO_1 \tag{1.1}$$

where $\Omega_n = 2\pi^{\frac{n}{2}}/\Gamma(\frac{n}{2})$ is the area of the unit sphere Ω $(= O_1)$. We can write out an asymptotic expression for $S_\phi(r)$ (1), namely

$$\begin{aligned} S_\phi(r) &\sim \phi(0) + \frac{1}{2!} S''_\phi(0) r^2 + \cdots + \frac{1}{(2k)!} S_\phi^{(2k)}(0) r^{2k} + \cdots \\ &= \sum_{k=0}^{\infty} \frac{\triangle^k \phi(0) r^{2k}}{2^k\, k!\, n(n+2)\cdots(n+2k-2)} \quad (\triangle \text{ is the Laplacian}) \end{aligned}$$

which is the well-known Pizzetti's formula and it plays an important role in the work of Li, Aguirre and Fisher [(2), (3), (4) and (5)]. Recently, it served as a foundation for building the gravity formula on the algebra (6).

Remark 1: Pizzetti's formula is not a convergent series for $\phi \in \mathcal{D}(R^n)$ from the counterexample below.

$$\phi(x) = \begin{cases} \exp\{-\frac{1}{r^2(1-r^2)}\} & \text{if } 0 < r < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\phi(x) \in \mathcal{D}(R^n)$ and $S_\phi(r) \neq 0$ for $0 < r < 1$, but the series in the formula is identically equal to zero. Obviously, $S_\phi(r) \to 0$ as $r \to 0$. However, it converges in the space of analytic functions from the reference (7).

By equation (1.1) and Pizzetti's formula,

$$\begin{aligned} S_\phi(r) &= \frac{1}{t^{n-1}\,\Omega_n} (\delta(r-t),\, \phi) \\ &\sim \sum_{k=0}^{\infty} \frac{S_\phi^{(2k)}(0)}{(2k)!}\, r^{2k} = \sum_{k=0}^{\infty} \frac{(\delta^{(2k)}(r),\, \phi)}{(2k)!}\, r^{2k} \\ &= \frac{1}{t^{n-1}\,\Omega_n} \left(\frac{2\pi^{\frac{n}{2}}\, t^{n-1}}{\Gamma(\frac{n}{2})} \sum_{k=0}^{\infty} \frac{\delta^{(2k)}(r)}{(2k)!} r^{2k},\, \phi \right) \end{aligned}$$

where $S_\phi^{(2k)}(0) = (\delta^{(2k)}(r),\, \phi)$. Hence

$$\delta(r-t) \sim \frac{2\pi^{\frac{n}{2}}\, t^{n-1}}{\Gamma(\frac{n}{2})} \sum_{k=0}^{\infty} \frac{\delta^{(2k)}(r)}{(2k)!} t^{2k},$$

which is equivalent to Pizzetti's formula. It follows from reference (8) that

$$\frac{\Omega_n\, \delta^{(2k)}(r)}{(2k)!} = \text{res}_{\lambda=-n-2k}\, r^\lambda = \frac{\Omega_n \triangle^k \delta(x)\, \Gamma(\frac{n}{2})}{2^k\, k!\, 2^k\, \Gamma(\frac{n}{2}+k)},$$

which implies

$$\triangle^k \delta(x) = \frac{2^{2k}\, k!\, \Gamma(\frac{n}{2}+k)}{(2k)!\, \Gamma(\frac{n}{2})} \delta^{(2k)}(r). \tag{1.2}$$

Therefore,

$$\begin{aligned}
\delta(r-t) &\sim 2\pi^{\frac{n}{2}}t^{n-1}\sum_{k=0}^{\infty}\frac{t^{2k}\triangle^k\delta(x)}{2^{2k}\,k!\,\Gamma(\frac{n}{2}+k)} \\
&= \Omega_n t^{n-1}\delta(x)+\frac{\Omega_n\,t^{n+1}}{2n}\triangle\delta(x)+\frac{\Omega_n t^{n+3}}{4n(2n+4)}\triangle^2\delta(x)+\cdots \\
&+ \frac{2\pi^{\frac{n}{2}}t^{n-1+2k}}{2^{2k}\,k!\,\Gamma(\frac{n}{2}+k)}\triangle^k\delta(x)+\cdots \\
&= \Omega_n t^{n-1}\sum_{k=0}^{\infty}\frac{t^{2k}}{(2k)!}\delta^{(2k)}(r) \\
&= \Omega_n t^{n-1}\delta(r)+\frac{\Omega_n t^{n+1}}{2!}\delta^{(2)}(r)+\frac{\Omega_n t^{n+3}}{4!}\delta^{(4)}(r)+\cdots \qquad (1.3)
\end{aligned}$$

in $\mathcal{D}'(R^n)$.

Note that $\delta(r)=\delta(x)$, since

$$(\delta(r),\,\phi(x))=S_\phi(0)=\phi(0)=(\delta(x),\,\phi(x)).$$

Clearly, this also can be seen by setting $k=0$ in equation (1.2).

For any Schwartz testing function ϕ, the distribution $\delta^{(k)}(r^2-t^2)$ focused on the sphere O_t of $r=t$ in R^n is defined by

$$(\delta^{(k)}(r^2-t^2),\,\phi)=\frac{(-1)^k}{2t^{n-1}}\int_{O_t}\left(\frac{\partial}{2r\partial r}\right)^k(\phi r^{n-2})dO_t,$$

which is the solution of the wave equation with the initial conditions described below in a space of odd dimension (7) for $k=(n-3)/2$:

$$\begin{aligned}
&(\triangle-\frac{\partial^2}{\partial t^2})u=0 \\
&u(x,0)=0,\quad \frac{\partial u(x,0)}{\partial t}=(-1)^k 2\pi^{k+1}\delta(x),
\end{aligned}$$

where $n\geq 3$.

Li recently investigated an asymptotic expression of $\delta^{(k)}(r^2-t^2)$ in a space of even dimension and obtained the following result (9).

Theorem 1.1. *The following asymptotic expansions hold in a space of even dimension and for* $k\leq(n-2)/2$*,*

$$\begin{aligned}
&\delta^{(k)}(r^2-t^2) \\
&\sim \frac{(-1)^k\Omega_n t^{n-2-2k}}{2^{k+1}}\sum_{j=0}^{\infty}\frac{(n-2+2j)\cdots(n+2j-2k)\,t^{2j}}{2^j\,j!\,n(n+2)\cdots(n+2j-2)}\triangle^j\delta(x),
\end{aligned}$$

and for $k>(n-2)/2$

$$\begin{aligned}
&\delta^{(k)}(r^2-t^2) \\
&\sim \frac{(-1)^k\Omega_n t^{n-2-2k}}{2^{k+1}}\sum_{j=\frac{2k-n+2}{2}}^{\infty}\frac{(n-2+2j)\cdots(n+2j-2k)\,t^{2j}}{2^j\,j!\,n(n+2)\cdots(n+2j-2)}\triangle^j\delta(x).
\end{aligned}$$

We begin in this paper to study an invariant theorem by a delta sequence and the convolution given on the regular manifolds. By the invariant theorem, we obtain an asymptotic expression of $\delta(\frac{1}{a(r)}(r-t))$, where $a(r)$ is a non-zero and smooth function. Furthermore, we get an asymptotic expression for $\delta^{(k)}(r-t)$ by the Fourier transform and the Bessel function given by

$$J_\nu(x) = \frac{1}{2^\nu\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\int_0^\pi e^{ix\cos\theta}x^\nu \sin^{2\nu}\theta d\theta,$$

which is used to derive an asymptotic expression for $\delta^{(k)}(\frac{1}{a(r)}(r-t))$. Finally, we can write out an expression for the distribution $\delta^{(k)}(\frac{1}{a(r)}(r^2-t^2))$.

2 The invariant theorem

The problem of defining products of distributions in R^n has been very difficult since there is a serious lack of definitions for multiplications overall. In this section, we are going to provide an invariant theorem that first appeared in (10) with an error. This theorem can convert the distributional products of several variables into the multiplications of a single variable which we are able to deal with by existing methods such as sequential approaches (3). We should note that Aguirre initially investigated an invariant theorem by a definition of distributional products (11), which is different from our Definition (2.3) below. This invariant theorem will be used in section (3) to compute the products occurring in the asymptotic expressions.

Let $\rho(x)$ be a fixed infinitely differentiable function on R with four properties

(i) $\rho(x) \geq 0,$

(ii) $\rho(x) = 0$ for $|x| \geq 1,$

(iii) $\rho(x) = \rho(-x),$

(iv) $\int_{-1}^{1} \rho(x)dx = 1$

Obviously, the Temple sequence $\delta_m(x) = m\rho(mx)$ is an infinitely differentiable sequence converging to δ in $\mathcal{D}'(R)$. Let f be an arbitrary distribution in $\mathcal{D}'(R)$. We define

$$f_m(x) = (f * \delta_m)(x) = (f(t),\ \delta_m(x-t))$$

for $m = 1, 2, \cdots$. It follows that $\{f_m(x)\}$ is a regular sequence converging to the distribution f in $\mathcal{D}'(R)$. The definition of the product of a distribution and an infinitely differentiable function is as follows (7).

Definition 2.1. Let f be a distribution in $\mathcal{D}'(R)$ and let g be an infinitely differentiable function. Then the product fg is defined by

$$(fg,\ \phi) = (f,\ g\phi)$$

for all functions ϕ in $\mathcal{D}(R)$.

It follows from definition (2.1) that

Lemma 2.1.

$$x^k\delta^{(m)}(x) = \begin{cases} (-1)^k k!\begin{pmatrix} m \\ k \end{pmatrix}\delta^{(m-k)}(x) & \textit{if } k \leq m, \\ 0 & \textit{otherwise} \end{cases}$$

for $k,\ m = 0, 1, 2, \cdots$.

Indeed,

$$\begin{aligned}(x^k\delta^{(m)}(x),\, \phi(x)) &= (-1)^m (x^k\phi(x))^{(m)}\Big|_{x=0} \\ &= (-1)^m k! \binom{m}{k} \phi^{(m-k)}(0) \\ &= (-1)^k k! \binom{m}{k} (\delta^{(m-k)}(x),\, \phi(x)\end{aligned}$$

for $k \le m$.

On the other hand, we have

$$x^k\delta^{(m)}(x) = 0$$

if $k > m$.

We use the following definition (3) for the commutative neutrix products of distributions in a single variable.

Definition 2.2. Let f and g be distributions in $\mathcal{D}'(R)$ and let $f_m(x) = (f * \delta_m)(x)$ and $g_m(x) = (g * \delta_m)(x)$. Then the commutative neutrix product fg of f and g exists and is equal to h if

$$N - \lim_{m\to\infty} \frac{1}{2}\{(f_m g,\ \phi) + (f g_m,\ \phi)\} = (h,\ \phi)$$

for all function $\phi \in \mathcal{D}(R)$, where N is the neutrix (12) having domain $N' = \{1, 2, \cdots\}$ and range the real numbers, with negligible functions that are finite linear sums of functions

$$m^\lambda \ln^{r-1} m, \quad \ln^r m \quad (\lambda > 0,\ r = 1, 2, \cdots)$$

and all functions of m that converge to zero in the normal sense as m tends to infinity. If the normal limit exists, then it is simply called the commutative product.

To see Definition (2.2) extends Definition (2.1), we let g be a C^∞ function. Clearly, $g_m\phi$ has an uniform support and converges to $g\phi$ in $\mathcal{D}(R)$. For any f in $\mathcal{D}'(R)$, we imply that

$$\begin{aligned}(fg,\ \phi) &= N - \lim_{m\to\infty} \frac{1}{2}\{(f_m g,\ \phi) + (f g_m,\ \phi)\} \\ &= N - \lim_{m\to\infty} \frac{1}{2}\{(f_m,\ g\phi) + (f,\ g_m\phi)\} = (f,\ g\phi) = (fg,\ \phi).\end{aligned}$$

Let $f(t)$ be a distribution of one variable and P be a regular manifold given by Gelfand (7). We define for $\phi \in \mathcal{D}(R^n)$ (13) that

$$(f(P),\ \phi(x)) = (f(t),\ \psi(t))$$

where

$$\psi(t) = \int_{P(x)=t} \phi(x)\,\omega \quad \text{and} \quad dP\cdot\omega = dv,$$

and $dv = dx_1 \cdots dx_n$ and dP is the differential form of P.

Clearly $\psi(t) \in \mathcal{D}(R)$, since there is at least one unbounded x_j when t is large, which implies that ϕ vanishes.

As an example, we consider the functional r^λ defined by

$$(r^\lambda,\ \phi) = \int_{R^n} r^\lambda \phi(x) dx$$

for Re$\lambda > -n$. Using the spherical coordinates below

$$\begin{aligned}
&x_1 = r\cos\theta_1,\\
&x_2 = r\sin\theta_1\cos\theta_2,\\
&x_3 = r\sin\theta_1\sin\theta_2\cos\theta_3,\\
&\cdots\ \cdots\\
&x_{n-1} = r\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}\cos\theta_{n-1}\\
&x_n = r\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}\sin\theta_{n-1}
\end{aligned}$$

we write

$$\begin{aligned}
(r^\lambda,\ \phi) &= \int_0^\infty r^\lambda\{\int_{r=1}\phi(r\omega)r^{n-1}dO_1\}dr\\
&= \int_0^\infty r^\lambda\{\int_{r=1}\phi(r\omega)\omega\}dr = (r^\lambda,\ \int_{r=1}\phi(r\omega)\omega).
\end{aligned}$$

As we will see, the sequence $\delta_m(P)$ plays an important role in obtaining the invariant theorem. First, we claim that $\lim_{m\to\infty}\delta_m(P(x)) = \delta(P(x))$. Indeed,

$$\lim_{m\to\infty}(\delta_m(P(x)),\ \phi(x)) = \lim_{m\to\infty}(\delta_m(t),\ \psi(t)) = (\delta(t),\ \psi(t)) = (\delta(P(x)),\ \phi(x)).$$

Let f be a distribution of one variable. The convolution $f(P(x)) * \phi$ is defined by

$$f(P(x)) * \phi = \int_{-\infty}^\infty f(t)dt\int_{P(z)=t}\phi(z-x)\omega,$$

where $\phi \in \mathcal{D}(R^n)$.

Next, we shall prove that $\lim_{m\to\infty}\delta_m(P(x)) * f(P(x)) = f(P(x))$ if f is a distribution of a single variable and P is regular. Consider

$$\begin{aligned}
(\delta_m(P(x)) * f(P(x)),\ \phi(x)) &= \int_{R^n}\int_{-\infty}^\infty f(t)dt\int_{P(z)=t}\delta_m(z-x)\,\omega\,\phi(x)dx\\
&= \int_{-\infty}^\infty f(t)dt\int_{P(z)=t}\int_{R^n}\delta_m(z-x)\phi(x)dx\,\omega.
\end{aligned}$$

Since the sequence

$$\int_{R^n}\delta_m(z-x)\phi(x)dx$$

converges to $\phi(z)$ in $\mathcal{D}(R^n)$ as $m\to\infty$ by the four properties of $\rho(x)$, it follows that

$$\begin{aligned}
\lim_{m\to\infty}(\delta_m(P(x)) * f(P(x)),\ \phi(x)) &= \int_{-\infty}^\infty f(t)dt\int_{P(z)=t}\phi(z)\omega\\
&= (f(P(x)),\ \phi(x)),
\end{aligned}$$

which completes the proof.

The author would like to point out that Aguirre initially showed the following identities

$$\begin{aligned}
&\lim_{m\to\infty}\delta_m(P(x)) = \delta(P(x)),\\
&\lim_{m\to\infty}\delta_m(P(x)) * f(P(x)) = f(P(x))
\end{aligned}$$

with a different delta sequence (11) while he studied a definition of distributional products by variable changes in several dimensional spaces.

Definition 2.3. Let $f(t)$ and $g(t)$ be distributions of one variable and let $P(x)$ be a regular $(n-1)$ dimensional manifold. Then the commutative neutrix product $f(P(x))g(P(x))$ of $f(P(x))$ and $g(P(x))$ is defined as

$$\begin{aligned}&(f(P(x))g(P(x)),\ \phi)\\&= N-\lim_{m\to\infty}\frac{1}{2}\{((f(P(x))*\delta_m(P(x)))g(P(x)),\ \phi)\\&+(f(P(x))(g(P(x))*\delta_m(P(x))),\ \phi)\}\end{aligned}$$

if the left-hand side limit exists for all function $\phi \in \mathcal{D}(R^n)$. If the normal limit exists, then it is simply called the commutative product.

Theorem 2.2. *(Invariant Theorem) Assume $P(x)$ is a regular $(n-1)$ (with $n > 1$) dimensional manifold and the commutative neutrix product $h(t) = f(t)g(t)$ exists. Then the commutative neutrix product $f(P(x))g(P(x))$ also exists and*

$$f(P(x))g(P(x)) = h(P(x)).$$

Proof. It follows from Definition (2.3) that

$$\begin{aligned}&(f(P(x))g(P(x)),\ \phi)\\&= N-\lim_{m\to\infty}\frac{1}{2}\{((f(P(x))*\delta_m(P(x)))g(P(x)),\ \phi)\\&+(f(P(x))(g(P(x))*\delta_m(P(x))),\ \phi)\}\\&= N-\lim_{m\to\infty}\frac{1}{2}\{(f(t)(g(t)*\delta_m(t)),\ \psi(t))\\&+(g(t)(f(t)*\delta_m(t)),\ \psi(t))\}\\&= (h(t),\psi(t)) = (h(P(x)),\ \phi(x)),\end{aligned}$$

where

$$\psi(t) = \int_{P(x)=t}\phi(x)\,\omega \quad \text{for} \quad \phi(x)\in\mathcal{D}(R^n),$$

which completes the proof. □

As a simple example of the use of the invariant theorem, we let $P(x) = x_1 + x_2 + 1$, which is obviously regular. The functional $\theta(P)$ is given by

$$(\theta(P),\ \phi(x)) = \int_{x_1+x_2+1\geq 0}\phi(x_1,x_2)dx = \int_{x_1+x_2\geq -1}\phi(x_1,x_2)dx$$

and the functional $\delta(P)$ is defined as

$$(\delta(P),\ \phi(x)) = \int_{x_1+x_2+1=0}\phi(x_1,x_2)\omega = \int\phi(-1-x_2,x_2)dx_2.$$

It was proved in (3) that

$$\theta(t)\delta(t) = \frac{1}{2}\delta(t),$$

which implies

$$\theta(P)\delta(P) = \frac{1}{2}\delta(P)$$

by the invariant theorem. Note that it seems infeasible or hard to compute this product using existing methods like the differential form approach discussed in (7).

3 The distribution $\delta^{(k)}(\frac{1}{a(r)}(r-t))$

Asymptotic analysis is a subject that has found applications for many years in various fields of pure and applied mathematics, both classical and modern. Estrada and Kanwal presented a simplified approach, in a distributional sense, to asymptotic techniques for solving problems in different areas (14) and proved the following fact.

If $f \in \mathcal{D}(R)$, then

$$f(\lambda x) \sim \sum_{n=0}^{\infty} \frac{(-1)^n \mu_n \delta^{(n)}(x)}{n!\lambda^{n+1}}, \quad \text{as} \lambda \to \infty$$

where μ_n, for $n = 0, 1, 2, \cdots$, are the moments of f.

In this section, we plan to provide several asymptotic expressions of the distributions on the sphere by the invariant theorem and the Fourier transform.

Theorem 3.1. *Assume $a(r)$ is a non-zero smooth function on $R^+ = [0, \infty)$ with respect to all x_i and r for $i = 1, 2, \cdots, n$. Then*

$$\begin{aligned} \delta(\frac{1}{a(r)}(r-t)) &= a(r)\delta(r-t) \\ &\sim \frac{2\pi^{\frac{n}{2}} t^{n-1}}{\Gamma(\frac{n}{2})} \sum_{k=0}^{\infty} \sum_{j=0}^{2k} \frac{\binom{2k}{2j} a^{(2j)}(0)\delta^{(2k-2j)}(r)}{(2k)!} t^{2k}. \end{aligned} \tag{3.1}$$

Proof. By the invariant theorem and Lemma (2.1),

$$a(r)\delta^{(2r)}(r) = \sum_{j=0}^{2k} (-1)^j \binom{2k}{j} a^{(j)}(0)\delta^{(2k-j)}(r).$$

Since

$$(\delta^{(2k+1)}(r), \phi(x)) = S_\phi^{(2k+1)}(0) = 0$$

for $k = 0, 1, 2, \cdots$, this implies

$$a(r)\delta^{(2r)}(r) = \sum_{j=0}^{2k} \binom{2k}{2j} a^{(2j)}(0)\delta^{(2k-2j)}(r).$$

On the other hand, we are able to derive the product $a(r)\delta^{(2r)}(r)$ directly without the invariant theorem. Indeed,

$$\begin{aligned} (a(r)\delta^{(2r)}(r), \phi(x)) &= (\delta(r), (a(r)S_\phi(r))^{(2k)}) \\ &= \sum_{j=0}^{2k} \binom{2k}{j} a^{(j)}(r) S_\phi^{(2k-j)}(r) \Big|_{r=0} \\ &= \sum_{j=0}^{2k} \binom{2k}{2j} a^{(2j)}(0) S_\phi^{(2k-2j)}(0) \\ &= \sum_{j=0}^{2k} \binom{2k}{2j} a^{(2j)}(0)(\delta^{(2k-2j)}(r), \phi(x)). \end{aligned} \tag{3.2}$$

Applying the identity (10)

$$\delta(aP) = a^{-1}\delta(P),$$

we get

$$\begin{aligned}
\delta(\frac{1}{a(r)}(r-t)) &= a(r)\delta(r-t) \\
&\sim a(r)\frac{2\pi^{\frac{n}{2}}t^{n-1}}{\Gamma(\frac{n}{2})}\sum_{k=0}^{\infty}\frac{\delta^{(2k)}(r)}{(2k)!}t^{2k} \\
&= \frac{2\pi^{\frac{n}{2}}t^{n-1}}{\Gamma(\frac{n}{2})}\sum_{k=0}^{\infty}\sum_{j=0}^{2k}\frac{\binom{2k}{2j}a^{(2j)}(0)\delta^{(2k-2j)}(r)}{(2k)!}t^{2k}.
\end{aligned}$$

This completes the proof of Theorem 3.1. □

In particular, we come to

$$\begin{aligned}
\delta(r^2-t^2) &= \delta((r+t)(r-t)) = \frac{1}{2t}\delta(r-t) \\
&\sim \frac{\pi^{\frac{n}{2}}t^{n-2}}{\Gamma(\frac{n}{2})}\sum_{k=0}^{\infty}\frac{\delta^{(2k)}(r)}{(2k)!}t^{2k} \\
&= \pi^{\frac{n}{2}}t^{n-2}\sum_{k=0}^{\infty}\frac{t^{2k}\triangle^k\delta(x)}{2^{2k}\,k!\,\Gamma(\frac{n}{2}+k)}
\end{aligned}$$

from equation (1.2).

In order to obtain an asymptotic expression of $\delta^{(k)}(\frac{1}{a(r)}(r-t))$, we are going to apply the Fourier transform and the following formula

$$(\delta^{(k)}(r-t),\,\phi) = \frac{(-1)^k}{t^{n-1}}\int_{O_t}\frac{\partial^k}{\partial r^k}(\phi r^{n-1})dO_t$$

to derive an asymptotic expansion of $\delta^{(k)}(r-t)$.

The Fourier transform of $\delta^{(k)}(r-t)$ in the space of analytic functions is defined as

$$F(\delta^{(k)}(r-t)) = (\delta^{(k)}(r-t), e^{i(x,\,\sigma)}) = \int_{O_t}\delta^{(k)}(r-t)e^{i(x,\,\sigma)}dx.$$

Employing the spherical coordinates provided in Section (2), we come to

$$F(\delta^{(k)}(r-t)) = (-1)^k\Omega_{n-1}\int_0^{\pi}\frac{\partial^k}{\partial r^k}(e^{ir\rho\cos\theta}\,r^{n-1})\Big|_{r=t}\sin^{n-2}\theta d\theta.$$

It follows from the following equation (15)

$$J_\nu(x) = \frac{1}{2^\nu\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\int_0^{\pi}e^{ix\cos\theta}x^\nu\sin^{2\nu}\theta d\theta$$

that

$$\begin{aligned}
&2^{\frac{n-2}{2}}\sqrt{\pi}\Gamma(\frac{n-1}{2})\rho^{1-\frac{n}{2}}\ \frac{\partial^k}{\partial r^k}(r^{\frac{n}{2}}J_{\frac{n-2}{2}}(r\rho))\Big|_{r=t} \\
&= \int_0^{\pi}\frac{\partial^k}{\partial r^k}(e^{ir\rho\cos\theta}\,r^{n-1})\Big|_{r=t}\sin^{n-2}\theta d\theta.
\end{aligned}$$

Therefore,

$$F(\delta^{(k)}(r-t)) = (-1)^k 2^{\frac{n}{2}} \pi^{\frac{n}{2}} \rho^{1-\frac{n}{2}} \ \frac{\partial^k}{\partial r^k}(r^{\frac{n}{2}} J_{\frac{n-2}{2}}(r\rho))\bigg|_{r=t}.$$

Since

$$z(z-1)\cdots(z-k+1) = \frac{\Gamma(z+1)}{\Gamma(z-k+1)},$$

we can directly compute the factor $\frac{\partial^k}{\partial r^k}(r^{\frac{n}{2}} J_{\frac{n-2}{2}}(r\rho))\Big|_{r=t}$ to obtain

$$\begin{aligned}
&\frac{\partial^k}{\partial r^k}(r^{\frac{n}{2}} J_{\frac{n-2}{2}}(r\rho))\bigg|_{r=t} \\
&= \begin{cases} \frac{2t^{n-1-k}\rho^{\frac{n}{2}-1}}{2^{\frac{n}{2}}} \displaystyle\sum_{j=0}^{\infty} \frac{(-1)^j \rho^{2j} \Gamma(n+2j) t^{2j}}{2^{2j}\, j!\, \Gamma(\frac{n}{2}+j)\, \Gamma(n+2j-k)} & \text{if } k \le n-1, \\ \frac{2t^{n-1-k}\rho^{\frac{n}{2}-1}}{2^{\frac{n}{2}}} \displaystyle\sum_{j=\lceil \frac{k-n+1}{2} \rceil}^{\infty} \frac{(-1)^j \rho^{2j} \Gamma(n+2j) t^{2j}}{2^{2j}\, j!\, \Gamma(\frac{n}{2}+j)\, \Gamma(n+2j-k)} & \text{if } k > n-1, \end{cases}
\end{aligned}$$

where $\lceil x \rceil$ represents the ceiling number of x, for example $\lceil 3.5 \rceil = 4$. This implies

$$\begin{aligned}
&F(\delta^{(k)}(r-t)) \\
&\sim \begin{cases} (-1)^k 2\pi^{\frac{n}{2}} t^{n-1-k} \displaystyle\sum_{j=0}^{\infty} \frac{(-1)^j \rho^{2j} \Gamma(n+2j) t^{2j}}{2^{2j}\, j!\, \Gamma(\frac{n}{2}+j)\, \Gamma(n+2j-k)} & \text{if } k \le n-1, \\ (-1)^k 2\pi^{\frac{n}{2}} t^{n-1-k} \displaystyle\sum_{j=\lceil \frac{k-n+1}{2} \rceil}^{\infty} \frac{(-1)^j \rho^{2j} \Gamma(n+2j) t^{2j}}{2^{2j}\, j!\, \Gamma(\frac{n}{2}+j)\, \Gamma(n+2j-k)} & \text{if } k > n-1 \end{cases}
\end{aligned}$$

in $\mathcal{Z}'(R^n)$.

Again using the identity (7)

$$F(\triangle^j \delta(x)) = (-1)^j \rho^{2j}$$

we come to

$$\delta^{(k)}(r-t) \sim \begin{cases} \displaystyle\sum_{j=0}^{\infty} \frac{C_{n,t,k} \triangle^j \delta(x)\, \Gamma(n+2j) t^{2j}}{2^{2j}\, j!\, \Gamma(\frac{n}{2}+j)\, \Gamma(n+2j-k)} & \text{if } k \le n-1, \\ \displaystyle\sum_{j=\lceil \frac{k-n+1}{2} \rceil}^{\infty} \frac{C_{n,t,k} \triangle^j \delta(x)\, \Gamma(n+2j) t^{2j}}{2^{2j}\, j!\, \Gamma(\frac{n}{2}+j)\, \Gamma(n+2j-k)} & \text{if } k > n-1 \end{cases} \tag{3.3}$$

where $C_{n,t,k} = (-1)^k 2\pi^{\frac{n}{2}} t^{n-1-k}$. In particular, we have for $k = 0$

$$\delta(r-t) \sim 2\pi^{\frac{n}{2}} t^{n-1} \sum_{j=0}^{\infty} \frac{t^{2j}}{2^{2j}\, j!\, \Gamma(\frac{n}{2}+j)} \triangle^j \delta(x)$$

which coincides with equation (1.3) in Section (2).

It follows from equation (1.2) that

$$\delta^{(k)}(r-t) \sim \begin{cases} \frac{C_{n,t,k}}{\Gamma(\frac{n}{2})} \displaystyle\sum_{j=0}^{\infty} \frac{\delta^{(2j)}(r)\, \Gamma(n+2j) t^{2j}}{(2j)!\, \Gamma(n+2j-k)} & \text{if } k \le n-1, \\ \frac{C_{n,t,k}}{\Gamma(\frac{n}{2})} \displaystyle\sum_{j=\lceil \frac{k-n+1}{2} \rceil}^{\infty} \frac{\delta^{(2j)}(r)\, \Gamma(n+2j) t^{2j}}{(2j)!\, \Gamma(n+2j-k)} & \text{if } k > n-1. \end{cases} \tag{3.4}$$

In particular for $k = 0$, we get

$$\delta(r-t) \sim \Omega_n t^{n-1} \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} \delta^{(2k)}(r).$$

Remark 2: The above expression of $\delta^{(k)}(r-t)$ was also investigated by Li and Aguirre in (16), where they made a minor error and derived an expression which is not asymptotic.

Theorem 3.2. *Assume $a(r)$ is a non-zero smooth function on $R^+ = [0, \infty)$ with respect to all x_i and r for $i = 1, 2, \cdots, n$. Then the following asymptotic expansions hold respectively for $k \leq n-1$,*

$$\delta^{(k)}(\frac{1}{a(r)}(r-t)) = a^{k+1}(r)\delta^{(k)}(r-t)$$
$$\sim \frac{C_{n,t,k}}{\Gamma(\frac{n}{2})} \sum_{j=0}^{\infty} \sum_{m=0}^{2j} \frac{\binom{2j}{2m} \left(a^{k+1}(r)\right)^{(2m)}\Big|_{r=0} \Gamma(n+2j)\, \delta^{(2j-2m)}(r)\, t^{2j}}{(2j)!\, \Gamma(n+2j-k)},$$

and for $k > n-1$

$$\delta^{(k)}(\frac{1}{a(r)}(r-t)) = a^{k+1}(r)\delta^{(k)}(r-t)$$
$$\sim \frac{C_{n,t,k}}{\Gamma(\frac{n}{2})} \sum_{j=\lceil \frac{k-n+1}{2} \rceil}^{\infty} \sum_{m=0}^{2j} \frac{\binom{2j}{2m} \left(a^{k+1}(r)\right)^{(2m)}\Big|_{r=0} \Gamma(n+2j)\, \delta^{(2j-2m)}(r)\, t^{2j}}{(2j)!\, \Gamma(n+2j-k)}.$$

Proof. It follows from the following identity (10) and equations (3.2) and (3.3)

$$\delta^{(k)}(aP) = a^{-(k+1)}\delta^{(k)}(P).$$

□

Similarly, we come to

Theorem 3.3. *Assume $a(r)$ is a non-zero smooth function on $R^+ = [0, \infty)$ with respect to all x_i and r for $i = 1, 2, \cdots, n$. Then the following asymptotic expansions hold respectively for $k \leq n-1$,*

$$\delta^{(k)}(\frac{1}{a(r)}(r^2-t^2)) = \left(\frac{a(r)}{r+t}\right)^{k+1} \delta^{(k)}(r-t)$$
$$\sim \frac{C_{n,t,k}}{\Gamma(\frac{n}{2})} \sum_{j=0}^{\infty} \sum_{m=0}^{2j} \frac{\binom{2j}{2m} \left(\left(\frac{a(r)}{r+t}\right)^{k+1}\right)^{(2m)}\Bigg|_{r=0} \Gamma(n+2j)\, \delta^{(2j-2m)}(r)\, t^{2j}}{(2j)!\, \Gamma(n+2j-k)},$$

and for $k > n-1$

$$\delta^{(k)}(\frac{1}{a(r)}(r^2-t^2)) = \left(\frac{a(r)}{r+t}\right)^{k+1} \delta^{(k)}(r-t)$$
$$\sim \frac{C_{n,t,k}}{\Gamma(\frac{n}{2})} \sum_{j=\lceil \frac{k-n+1}{2} \rceil}^{\infty} \sum_{m=0}^{2j} \frac{\binom{2j}{2m} \left(\left(\frac{a(r)}{r+t}\right)^{k+1}\right)^{(2m)}\Bigg|_{r=0} \Gamma(n+2j)\, \delta^{(2j-2m)}(r)\, t^{2j}}{(2j)!\, \Gamma(n+2j-k)}.$$

4 Conclusions

This paper contains three sections providing several new ideas in the theory of distributions.

a In the section (1), we have proven that the well-known Pizzetti's formula is equivalent to the following

$$\delta(r-t) \sim \frac{2\pi^{\frac{n}{2}} t^{n-1}}{\Gamma(\frac{n}{2})} \sum_{k=0}^{\infty} \frac{\delta^{(2k)}(r)}{(2k)!} t^{2k},$$

which can be further described in equation (1.3) by equation (1.2).

b In the section (2), we have studied an invariant theorem by the delta sequence and neutrix calculus due to van der Corput. This is very useful and powerful in computing the distributional products of R^n based on the multiplications of a single variable, which is much easier to carry out.

c In the section (3), two asymptotic expressions for $\delta^{(k)}(\frac{1}{a(r)}(r-t))$ as well as the distribution $\delta^{(k)}(\frac{1}{a(r)}(r^2-t^2))$ have been obtained by the invariant theorem and the Fourier transform. We should note that the asymptotic expression of $\delta^{(k)}(\frac{1}{a(r)}(r-t))$ is a generalization of equation (1.3).

d A challenge problem is how to deduce an asymptotic expression for $\delta(P)$, where P is not a sphere in R^n. The author welcomes and appreciates any discussion from interested readers.

Acknowledgment

The author is grateful to Dr. J. Williams who carefully read the paper with several productive suggestions, which improved the quality of the work.

Competing interests

The author declares that no competing interests exist.

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