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Several Asymptotic Products of Particular Distributions

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Research Article

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Abstract

The problem of defining products of distributions is a difficult and not completely understood problem, studied from several points of views since Schwartz established the theory of distributions around 1950. Many fields, such as wave propagation or quantum mechanics, require such multiplications. The product of an infinitely differentiable function $\phi(x)$ and distribution $\triangle^k\delta(x)$ in R^n is well defined by

$$(\phi(x)\triangle^k\delta(x),\ \psi) = (\delta(x),\ \triangle^k(\phi\psi)),$$

since $\triangle^k(\phi\psi) \in \mathcal{D}(R^n)$. Using an induction, we derive an interesting formula for $\triangle^k(\phi(x)\psi(x))$ and hence we are able to write out an explicit expression of the product $\phi(x)\triangle^k\delta(x)$. In particular, we imply the product $X^s\triangle^k\delta(x)$ with a few applications in further simplifying existing distributional products. Furthermore, we obtain an asymptotic expression for $\delta(r-a)$ in terms of $\triangle^k\delta(x)$, which is equivalent to the well-known Pizzetti's formula. Several asymptotic products including $\phi(x)\,\delta(r-1)$, $X^s\,\delta(r-1)$ as well as the more generalized $\phi(x)\,\delta^{(k)}(r-1)$ are calculated and presented as infinitely series.

Keywords: Distribution, product, asymptotic expansion, asymptotic product, neutrix limit and Pizzetti's formula

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1 Introduction

Physicists have long been using the singular function $\delta(x)$, although it cannot be properly defined within the structure of classical function theory. In elementary physics, one finds the need to evaluate δ^2 when calculating the transition rates of certain particle interactions (12). Schwartz established theory of distributions by treating singular functions as linearly continuous functionals on the testing function space whose elements have compact support. Although they are of great importance to quantum field theory, it is difficult to define products, convolutions and compositions of distributions in general. The sequential method [(7); (8); (9); (23)] and complex analysis approach [(4); (1)], including

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nonstandard analysis (15), have been the main tools in dealing with those non-linear operations of distributions in the Schwartz space $\mathcal{D}'(R)$. On the other hand, Oberguggenberger (25) wrote a review book on the theories of (new) generalized functions, initiated by J.F. Colombeau and others in 1985, that deal with differential algebras larger than the space of distributions and are based on the "sequential approach", i.e. the generalized functions are approximately by (not necessarily discrete) "sequences" of smooth functions. They can accommodate most of the particular or intrinsic definitions of distributional multiplications. Franssens (11) recently investigated the convolution and multiplication of one-dimensional associated homogeneous distributions by the multi-valued methods used in quantum field theory. The derived products may involve at most one arbitrary constant. However, little progress has been made so far towards obtaining the products on manifolds in R^n, such as the product $\phi(x)\,\delta(r-1)$ on the unit sphere, since Gel'fand introduced special types of generalized functions. As outlined in the abstract, we start to evaluate the product $\phi(x)\triangle^k\delta(x)$ based on the formula of $\triangle^k(\phi(x)\psi(x))$, and hence we are able to represent the more generalized $\phi(x)\,\delta^{(k)}(r-1)$ as an asymptotic expression by the Fourier transform.

2 The product $X^s\triangle^k\delta(x)$ in R^n

Lemma 2.1. *Let $\phi(x)$ and $\psi(x)$ be infinitely differentiable functions. Then for $k = 0, 1, 2, \cdots$*

$$\triangle^k(\phi\psi) = \sum_{m+i+l=k} 2^i \binom{m+l}{m}\binom{k}{m+l} \nabla^i\triangle^m\phi\cdot\nabla^i\triangle^l\psi \tag{2.1}$$

where

$$\nabla^i\phi\cdot\nabla^i\psi = \begin{cases} \phi\psi & \text{if } i=0, \\ \displaystyle\sum_{j=1}^n \frac{\partial^i}{\partial x_j^i}\phi\frac{\partial^i}{\partial x_j^i}\psi & \text{if } i>0. \end{cases}$$

Proof. We use an induction to prove the formula. Assume that $k=0$, it is clearly true since both sides are equal to $\phi\psi$. Suppose it holds for some integer $k>0$ and we need to consider $k+1$ case. Obviously,

$$\triangle^{k+1}(\phi\psi) = \sum_{m+i+l=k} 2^i \binom{m+l}{m}\binom{k}{m+l} \triangle(\nabla^i\triangle^m\phi\cdot\nabla^i\triangle^l\psi)$$

and

$$\begin{aligned} &\triangle(\nabla^i\triangle^m\phi\cdot\nabla^i\triangle^l\psi) \\ &= \nabla^i\triangle^{m+1}\phi\cdot\nabla^i\triangle^l\psi + \nabla^i\triangle^m\phi\cdot\nabla^i\triangle^{l+1}\psi + 2\nabla^{i+1}\triangle^m\phi\cdot\nabla^{i+1}\triangle^l\psi \\ &\triangleq I_1 + I_2 + I_3 \end{aligned}$$

by a simple calculation.

Replacing $m+1$ by m, we calculate

$$\begin{aligned} &\sum_{m+i+l=k} 2^i \binom{m+l}{m}\binom{k}{m+l} I_1 \\ &= \sum_{m+i+l=k} 2^i \binom{m+l}{m}\binom{k}{m+l} \nabla^i\triangle^{m+1}\phi\cdot\nabla^i\triangle^l\psi \\ &= \sum_{m+i+l=k+1} 2^i \binom{m-1+l}{m-1}\binom{k}{m-1+l} \nabla^i\triangle^m\phi\cdot\nabla^i\triangle^l\psi. \end{aligned}$$

Similarly,

$$\begin{aligned}
&\sum_{m+i+l=k} 2^i \binom{m+l}{m}\binom{k}{m+l} I_2 \\
&= \sum_{m+i+l=k} 2^i \binom{m+l}{m}\binom{k}{m+l} \nabla^i \triangle^m \phi \cdot \nabla^i \triangle^{l+1} \psi \\
&= \sum_{m+i+l=k+1} 2^i \binom{m+l-1}{m}\binom{k}{m+l-1} \nabla^i \triangle^m \phi \cdot \nabla^i \triangle^l \psi.
\end{aligned}$$

As for I_3,

$$\begin{aligned}
&\sum_{m+i+l=k} 2^i \binom{m+l}{m}\binom{k}{m+l} I_3 \\
&= \sum_{m+i+l=k} 2^{i+1} \binom{m+l}{m}\binom{k}{m+l} \nabla^{i+1} \triangle^m \phi \cdot \nabla^{i+1} \triangle^l \psi \\
&= \sum_{m+i+l=k+1} 2^i \binom{m+l}{m}\binom{k}{m+l} \nabla^i \triangle^m \phi \cdot \nabla^i \triangle^l \psi.
\end{aligned}$$

By direct calculation,

$$\begin{aligned}
&\binom{m-1+l}{m-1}\binom{k}{m-1+l} + \binom{m+l-1}{m}\binom{k}{m+l-1} + \binom{m+l}{m}\binom{k}{m+l} \\
&= \binom{m+l}{m}\binom{k+1}{m+l}.
\end{aligned}$$

This completes the proof of the Lemma. $\square$

Remark 1: Lemma 2.1 was first presented in (22) with ambiguity and confusion between ordinary function multiplication and the $\cdot$ operation given in Lemma 2.1, which cause errors in computing several distributional products in the paper later on.

Theorem 2.2. *Let $\phi(x) \in C^\infty(R^n)$. Then the distributional product $\phi(x)$ and $\triangle^k \delta(x)$ exists and*

$$\phi(x) \triangle^k \delta(x) = \sum_{m+i+l=k} 2^i (-1)^i \binom{m+l}{m}\binom{k}{m+l} \nabla^i \triangle^m \phi(0) \cdot \nabla^i \triangle^l \delta(x), \tag{2.2}$$

for $k = 0, 1, 2, \cdots$.

Proof. Clearly, $\phi(x)\psi(x) \in \mathcal{D}(R^n)$ if $\psi(x) \in \mathcal{D}(R^n)$ and $\phi(x) \in C^\infty(R^n)$. Hence

$$\begin{aligned}
&(\phi(x)\triangle^k \delta(x),\, \psi(x)) = (\triangle^k \delta(x),\, \phi(x)\psi(x)) \\
&= \sum_{m+i+l=k} 2^i \binom{m+l}{m}\binom{k}{m+l} \nabla^i \triangle^m \phi(0) \cdot \nabla^i \triangle^l \psi(0),
\end{aligned}$$

by Lemma 2.1. The result follows from

$$\nabla^i \triangle^l \psi(0) = (-1)^i (\nabla^i \triangle^l \delta(x),\, \psi(x)).$$

This completes the proof of the theorem. $\square$

It follows from Theorem 2.2 that

$$\begin{aligned}
&X\delta(x) = 0,\\
&X\triangle\delta(x) = -2\nabla\delta(x),\\
&X\triangle^2\delta(x) = -4\nabla\triangle\delta(x),\\
&X\triangle^k\delta(x) = -2k\nabla\triangle^{k-1}\delta(x),\\
&X^2\triangle\delta(x) = 2n\delta(x),\\
&X^2\triangle^k\delta(x) = 2nk\triangle^{k-1}\delta(x) + 2^2k(k-1)\nabla^2\triangle^{k-2}\delta(x),\\
&X^3\triangle\delta(x) = 0,\\
&X^3\triangle^k\delta(x) = -12nk(k-1)\nabla\triangle^{k-2}\delta(x) - 2^3k(k-1)(k-2)\nabla^3\triangle^{k-3}\delta(x),
\end{aligned}$$

where $X = \sum_{i=1}^n x_i$ and $k = 0, 1, 2, \cdots$.

On the other hand, we can directly use an induction to show that

$$\triangle^k(X\phi) = 2k\nabla\triangle^{k-1}\phi + X\triangle^k\phi$$

which also claims that $X\triangle^k\delta(x) = -2k\nabla\triangle^{k-1}\delta(x)$ in the above. It is obviously true for $k = 0$. Assume it holds for the case of $k > 0$, that is

$$\triangle^k(X\phi) = 2k\nabla\triangle^{k-1}\phi + X\triangle^k\phi.$$

Therefore,

$$\begin{aligned}
&\triangle^{k+1}(X\phi) = \triangle\triangle^k(X\phi) = \triangle(2k\nabla\triangle^{k-1}\phi + X\triangle^k\phi)\\
&= 2k\nabla\triangle^k\phi + \triangle(X\triangle^k\phi)\\
&= 2(k+1)\nabla\triangle^k\phi + X\triangle^{k+1}\phi.
\end{aligned}$$

Similarly, we can get

$$\triangle^k(X^2\phi(x))\Big|_{x=0} = 2nk\triangle^{k-1}\phi(0) + 2^2k(k-1)\nabla^2\triangle^{k-2}\phi(0),$$

which claims

$$X^2\triangle^k\delta(x) = 2nk\triangle^{k-1}\delta(x) + 2^2k(k-1)\nabla^2\triangle^{k-2}\delta(x).$$

However, it seems infeasible to write out an explicit formula for the important product $X^s\triangle^k\delta(x)$, for any positive integer s, by a direct computation without employing Theorem 2.2 (21). We shall provide a few interesting applications of the product in simplifying other existing distributional multiplications in R^n, as well as in obtaining some asymptotic products related to the delta functions on unit spheres in the following section after completing Theorem 2.3 below.

Theorem 2.3. *The distributional product $X^s\triangle^k\delta(x)$ exists and*

$$X^s\triangle^k\delta(x) = \begin{cases} 2^s k!s!\displaystyle\sum_{j=0}^{s/2}\frac{n^j\nabla^{s-2j}\triangle^{k-s+j}\delta(x)}{2^{2j}j!(k-s+j)!(s-2j)!} & \textit{if } s \textit{ is even,}\\ -2^s k!s!\displaystyle\sum_{j=0}^{\lfloor s/2\rfloor}\frac{n^j\nabla^{s-2j}\triangle^{k-s+j}\delta(x)}{2^{2j}j!(k-s+j)!(s-2j)!} & \textit{if } s \textit{ is odd,}\end{cases}$$

where $\triangle^{-p} = 0$ for any positive integer p and $k, s = 0, 1, 2, \cdots$.

Proof. Assume $\phi(x) = X^s$ and s is even. By Theorem 2.2,

$$\phi(x)\triangle^k\delta(x) = \sum_{m+i+l=k} 2^i(-1)^i\binom{m+l}{m}\binom{k}{m+l}\nabla^i\triangle^m\phi(0)\cdot\nabla^i\triangle^l\delta(x).$$

Note that all non-zero terms in the above sum require $2m + i = s$. So,

$$
\begin{aligned}
&\phi(x)\triangle^k\delta(x) \\
&= 2^s(-1)^s\binom{k-s}{0}\binom{k}{k-s}\nabla^s\phi(0)\cdot\nabla^s\triangle^{k-s}\delta(x)+ \\
&\quad 2^{s-2}(-1)^{s-2}\binom{k-s+2}{1}\binom{k}{k-s+2}\nabla^{s-2}\triangle\phi(0)\cdot\nabla^{s-2}\triangle^{k-s+1}\delta(x)+ \\
&\quad 2^{s-4}(-1)^{s-4}\binom{k-s+4}{2}\binom{k}{k-s+4}\nabla^{s-4}\triangle^2\phi(0)\cdot\nabla^{s-4}\triangle^{k-s+2}\delta(x)+ \\
&\quad \cdots+ \\
&\quad 2^0(-1)^0\binom{k-1}{s/2}\binom{k}{k-1}\nabla^0\triangle^{s/2}\phi(0)\cdot\nabla^0\triangle^{k-s/2}\delta(x).
\end{aligned}
$$

Clearly, we have

$$
\begin{aligned}
&\nabla^s\phi(0)\cdot\nabla^s\triangle^{k-s}\delta(x) = s!\nabla^s\triangle^{k-s}\delta(x), \\
&\nabla^{s-2}\triangle\phi(0)\cdot\nabla^{s-2}\triangle^{k-s+1}\delta(x) = ns!\nabla^{s-2}\triangle^{k-s+1}\delta(x), \\
&\cdots \\
&\nabla^0\triangle^{s/2}\phi(0)\cdot\nabla^0\triangle^{k-s/2}\delta(x) = n^{s/2}s!\nabla^0\triangle^{k-s/2}\delta(x).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\phi(x)\triangle^k\delta(x) \\
&= 2^s\frac{k!s!}{0!(k-s)!s!}\nabla^s\triangle^{k-s}\delta(x)+ \\
&\quad 2^{s-2}\frac{nk!s!}{1!(k-s+1)!(s-2)!}\nabla^{s-2}\triangle^{k-s+1}\delta(x)+ \\
&\quad 2^{s-4}\frac{n^2k!s!}{2!(k-s+2)!(s-4)!}\nabla^{s-4}\triangle^{k-s+2}\delta(x)+ \\
&\quad \cdots+ \\
&\quad 2^0\frac{n^{s/2}k!s!}{(s/2)!(k-s/2)!0!}\nabla^0\triangle^{k-s/2}\delta(x) \\
&= 2^sk!s!\sum_{j=0}^{s/2}\frac{n^j\nabla^{s-2j}\triangle^{k-s+j}\delta(x)}{2^{2j}j!(k-s+j)!(s-2j)!}.
\end{aligned}
$$

The case that s is odd follows similarly. This completes the proof of the theorem. □

Theorem 2.4. *Let $f(x) \in C^\infty(R)$. Then the distributional product $f(X)\triangle^k\delta(x)$ exists and*

$$
f(X)\triangle^k\delta(x) = \sum_{m+i+l=k} 2^i(-1)^i\binom{m+l}{m}\binom{k}{m+l}n^m f^{(2m+i)}(0)\nabla^i\triangle^l\delta(x),
$$

for $k = 0, 1, 2, \cdots$

Proof. It easily follows from Theorem 2.2. □

Theorem 2.5. *Let $\phi(x) \in C^{\infty}(R^n)$. Then the distributional product $\phi(x)\nabla^k\delta(x)$ exists and*

$$\phi(x)\nabla^k\delta(x) = \sum_{j=0}^{k}(-1)^j\binom{k}{j}\nabla^j\phi(0)\nabla^{k-j}\delta(x),$$

for $k = 0, 1, 2, \cdots$.

Proof. It follows from the identity below claimed by induction

$$\nabla^k(\phi\psi) = \sum_{j=0}^{k}\binom{k}{j}\nabla^j\phi\nabla^{k-j}\psi.$$

□

In particular, we come to

$$f(X)\nabla^k\delta(x) = \sum_{j=0}^{k}(-1)^j\binom{k}{j}n^j f^{(j)}(0)\nabla^{k-j}\delta(x)$$

for $f(x) \in C^{\infty}(R)$ and $k = 0, 1, 2, \cdots$ and

$$X^s\nabla^k\delta(x) = \begin{cases} (-1)^s\binom{k}{s}n^s s!\nabla^{k-s}\delta(x) & \text{if } s \le k, \\ 0 & \text{if } s > k. \end{cases}$$

At the end of this section, we would like to supply a couple of appealing applications of the product $X^s\triangle^k\delta(x)$ in simplifying several multiplications obtained in [(24); (1)].

Let $r = (x_1^2 + \cdots + x_n^2)^{1/2}$ and let $\rho(s)$ be a fixed infinitely differentiable function defined on $R^+ = [0,\ \infty)$ having the properties:

(i) $\rho(s) \ge 0$;

(ii) $\rho(s) = 0$ for $s \ge 1$;

(iii) $\displaystyle\int_{R^n} \delta_m(x)dx = 1$;

where $\delta_m(x) = c_n m^n \rho(m^2r^2)$ and c_n is the constant satisfying (iii).

It follows that $\{\delta_m(x)\}$ is a regular δ - sequence of infinitely differentiable functions converging to $\delta(x)$ in $\mathcal{D}'(R^n)$ (an n-dimensional space of distributions).

Definition 2.1. Let f and g be distributions in $\mathcal{D}'(R^n)$ and let

$$g_m(x) = (g * \delta_m)(x) = (g(x-t),\ \delta_m(t))$$

where $t = (t_1, t_2, \cdots, t_n)$. The noncommutative neutrix product $f.g$ of f and g exists and is equal to h if

$$N - \lim_{m\to\infty}(fg_m,\ \phi) = (h,\ \phi),$$

where $\phi \in \mathcal{D}(R^n)$ and the N-limit is the neutrix limit defined in (26).

Note that Fisher [(7); (8); (9); (10)], for example) has actively used Jones' δ-sequence $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \cdots$, and the concept of neutrix limit to deduce numerous products, powers, convolutions, and compositions of distributions on R since 1969.

With the above definition, Li and Zou (24) showed that the noncommutative neutrix product $r^{-k}.\nabla^{2l}\delta(x)$ exists for $k = 1, 2, \cdots$ and $l = 0, 1, 2, \cdots$, and

$$r^{-2k}.\nabla^{2l}\delta(x) = \sum_{j=0}^{l} \frac{b_j^l n^{l-j}(-1)^{l+j} X^{2j} \triangle^{k+l+j}\delta(x)}{(k+l+j)!\ 2^{k+l+j}\ (n+2l+2j)\cdots(n+2k+2l+2j-2)}$$

where b_j^l is a constant satisfying a certain recursion.

It is clear to see that the above result can be further calculated and simplified with Theorem 2.3, since

$$X^{2j}\triangle^{k+l+j}\delta(x) = 2^{2j}(k+l+j)!(2j)!\sum_{i=0}^{j}\frac{n^i\nabla^{2j-2i}\triangle^{k+l-j+i}\delta(x)}{2^{2i}i!(k+l-j+i)!(2j-2i)!}.$$

On the other hand, Aguirre (1) used the Laurent series of r^λ in a neighborhood of $\lambda = -n-2j$ for $j = 0, 1, 2, \cdots$ and the following identity (where P is a regular manifold defined in (13)

$$\frac{\partial}{\partial x_j}\delta^{(k)}(P) = \frac{\partial P}{\partial x_j}\delta^{(k+1)}(P),$$

to show that the product r^{-2k} and $\nabla\triangle^j\delta(x)$ exists and

$$r^{-2k}\nabla\triangle^j\delta(x) = \frac{-(n+2j)(2j)!X\triangle^{k+j+1}\delta(x)}{(k+j+1)!2^{k+j+1}n(n+2)\cdots(n+2k+2j)}.$$

This, again, can be further simplified by applying the identity below

$$X\triangle^{k+j+1}\delta(x) = -2(k+j+1)\nabla\triangle^{k+j}\delta(x).$$

3 The asymptotic products on unit spheres

The distribution $\delta^{(k)}(r-1)$ concentrated on unit sphere $r-1=0$ is defined as

$$(\delta^{(k)}(r-1),\ \phi) = (-1)^k\int_{r=1}\frac{\partial^k}{\partial r^k}(\phi r^{n-1})d\omega$$

where $d\omega$ is the Euclidean element on $r=1$ and $\phi \in \mathcal{D}(R^n)$. With the expansion formula

$$\int_{r=1}\frac{\partial^k}{\partial r^k}\phi(r\omega)d\omega = (-1)^k\left(\sum_{i=0}^{k}\binom{k}{i}C(m,i)\delta^{(k-i)}(r-1),\ \phi(x)\right),$$

Li evaluated the product of $f(r)$ and $\delta^{(k)}(r-1)$ (17), for any infinitely differentiable $f(x)$ at $x=1$, and obtained that

$$f(r)\delta^{(k)}(r-1) = \sum_{j=0}^{k}\sum_{i=0}^{j}\sum_{s=0}^{k-j}\frac{(-1)^j k! f^{(i)}(1)}{i!s!(j-i)!(k-j-s)!} \cdot\chi(m,i,j)C(m,s)\delta^{(k-j-s)}(r-1)$$

where $\chi(m,i,j)$ and $C(m,s)$ are constants depending on the indices and $k \le n-1$.

Later on, Aguirre and Li (2) simplified the product of $f(r)$ and $\delta^{(k)}(r-1)$, and proved that for any $k = 0, 1, \cdots$,

$$f(r)\delta^{(k)}(r-1) = \sum_{j=0}^{k}(-1)^j \binom{k}{j} f^{(j)}(1)\delta^{(k-j)}(r-1).$$

In particular, we have

$$\begin{aligned} &f(r)\,\delta(r-1) = f(1)\delta(r-1); \\ &f(r)\,\delta'(r-1) = f(1)\delta'(r-1) - f'(1)\delta(r-1); \\ &\frac{1}{r}\,\delta^{(k)}(r-1) = \sum_{j=0}^{k}\binom{k}{j} j!\delta^{(k-j)}(r-1). \end{aligned}$$

Furthermore, we assume that $H(x_1, x_2, \cdots, x_m)$ is any sufficiently smooth function such that on $H = 0$ we have

$$\text{grad}H \neq 0$$

which means that there are no singular points on $H = 0$. Then the generalized function $\delta(H)$ can be defined in the following way.

$$(\delta(H),\,\phi) = \int_{H=0} \psi(0, u_2, \cdots, u_m)du_2 \cdots du_m$$

where $\phi_1(u_1, \cdots, u_m) = \phi(x_1, \cdots x_m)$ and $\psi = \phi_1(u)D\binom{x}{u}$.

Similarly, we shall define

$$(\delta^{(k)}(H),\,\phi) = (-1)^k \int_{H=0} \psi^{(k)}_{u_1}(0, u_2, \cdots, u_m)du_2 \cdots du_m.$$

As an example, we consider the generalized function $\delta(\alpha_1 x_1 + \cdots + \alpha_m x_m)$, where $\sum_{i=1}^m \alpha_i^2 = 1$. The equation

$$\alpha_1 x_1 + \cdots + \alpha_m x_m = 0$$

determines a hypersurface which passes through the origin and is orthogonal to the unit vector α. Making the substitution

$$u_1 = \alpha_1 x_1 + \cdots + \alpha_m x_m, \quad u_2 = x_2, \cdots, u_m = x_m,$$

we thus arrive at

$$(\delta(\alpha_1 x_1 + \cdots + \alpha_m x_m), \phi) = \int_{\sum \alpha_i x_i = 0} \phi\, du_2 \cdots du_m.$$

Let $f(x)$ be a $C^\infty(R)$ function and let H be defined as above. Then the product $f(H)\,\delta^{(k)}(H)$ exists (2) for $k = 0, 1, 2, \cdots$, and

$$f(H)\,\delta^{(k)}(H) = \sum_{i=0}^{k}\binom{k}{i}(-1)^i f^{(i)}(0)\,\delta^{(k-i)}(H).$$

We define $S_\phi(r)$ as the mean value of $\phi(x) \in \mathcal{D}(R^n)$ on the sphere of radius r by

$$S_\phi(r) = \frac{1}{\Omega_n}\int_{r=1} \phi(r\sigma)d\sigma$$

where $\Omega_n = 2\pi^{\frac{n}{2}}/\Gamma(\frac{n}{2})$ is the surface area of unit sphere $r = 1$. We can write out an asymptotic expression for $S_\phi(r)$ (see (6)), namely

$$\begin{aligned} S_\phi(r) \quad &\sim \quad \phi(0) + \frac{1}{2!}S''_\phi(0)r^2 + \cdots + \frac{1}{(2k)!}S^{(2k)}_\phi(0)r^{2k} + \cdots \\ &= \quad \sum_{k=0}^{\infty}\frac{\triangle^k\phi(0)r^{2k}}{2^k\,k!\,n(n+2)\cdots(n+2k-2)} \end{aligned}$$

which is the well-known Pizzetti's formula and it plays an important role in the work of Li, Aguirre and Fisher [(23); (16); (19); (5); (3); (18)].

Remark 2: Pizzetti's formula is not a convergent series for $\phi \in \mathcal{D}(R^n)$ from the following counterexample.

$$\phi(x) = \begin{cases} \exp\{-\frac{1}{r^2(1-r^2)}\} & \text{if } 0 < r < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\phi(x) \in \mathcal{D}(R^n)$ and $S_\phi(r) \neq 0$ for $0 < r < 1$, but the series in the formula is identically equal to zero. Obviously, $S_\phi(r) \to 0$ as $r \to 0$. However, it converges in the space of analytic functions from the reference (13).

Now, we turn our attentions to studying the product $\phi(x)\,\delta(r-1)$, where $\phi(x) \in C^\infty(R^n)$ and the more generalized multiplication $\phi(x)\,\delta^{(k)}(r-1)$. It seems impossible to get them by either following the computational patterns of the products $f(r)\,\delta^{(k)}(r-1)$ and $f(H)\,\delta^{(k)}(H)$, or any existing methods including invariant theorem (20). However, we shall be able to derive the asymptotic products below to approximate these products. This idea will have many applications in dealing with complex products in R^n.

Theorem 3.1. *The asymptotic expression*

$$\delta(r-1) \sim 2\pi^{\frac{n}{2}} \sum_{k=0}^{\infty} \frac{\triangle^k \delta(x)}{2^{2k}\, k!\, \Gamma(\frac{n}{2}+k)} \tag{3.1}$$

holds in $\mathcal{D}'(R^n)$.

Proof. Since

$$(\delta(r-a),\, \phi) = \int_{r=a} \phi d\sigma = a^{n-1} \int_{r=1} \phi(r\sigma) d\sigma,$$

for any $\phi \in \mathcal{D}(R^n)$. We come to

$$\begin{aligned} S_\phi(r) &= \frac{1}{a^{n-1}\,\Omega_n}(\delta(r-a),\, \phi) \\ &\sim \sum_{k=0}^{\infty} \frac{S_\phi^{(2k)}(0)}{(2k)!}\, r^{2k} = \sum_{k=0}^{\infty} \frac{(\delta^{(2k)}(r),\, \phi)}{(2k)!}\, r^{2k} \\ &= \frac{1}{a^{n-1}\,\Omega_n} \left(\frac{2\pi^{\frac{n}{2}} a^{n-1}}{\Gamma(\frac{n}{2})} \sum_{k=0}^{\infty} \frac{\delta^{(2k)}(r)}{(2k)!} r^{2k},\, \phi \right) \end{aligned}$$

where $S_\phi^{(2k)}(0) = (\delta^{(2k)}(r),\, \phi)$. Hence

$$\delta(r-a) \sim \frac{2\pi^{\frac{n}{2}} a^{n-1}}{\Gamma(\frac{n}{2})} \sum_{k=0}^{\infty} \frac{\delta^{(2k)}(r)}{(2k)!} a^{2k}.$$

It follows from reference (1) that

$$\frac{\Omega_n\, \delta^{(2k)}(r)}{(2k)!} = \text{res}_{\lambda=-n-2k}\, r^\lambda = \frac{\Omega_n\, \triangle^k \delta(x)\, \Gamma(\frac{n}{2})}{2^k\, k!\, 2^k\, \Gamma(\frac{n}{2}+k)},$$

which implies

$$\triangle^k \delta(x) = \frac{2^{2k}\, k!\, \Gamma(\frac{n}{2}+k)}{(2k)!\, \Gamma(\frac{n}{2})} \delta^{(2k)}(r). \tag{3.2}$$

Therefore,

$$\delta(r-a) \sim 2\pi^{\frac{n}{2}}\, a^{n-1} \sum_{k=0}^{\infty} \frac{a^{2k} \triangle^k \delta(x)}{2^{2k}\, k!\, \Gamma(\frac{n}{2}+k)} \tag{3.3}$$

in $\mathcal{D}'(R^n)$. This completes the proof of the theorem by setting $a = 1$. □

Remark 3: Clearly, Pizzetti's formula is equivalent to equation (3.3), which can be directly obtained (22) by the Fourier transform and the following formula in (14)

$$J_\nu(x) = \frac{1}{2^\nu \sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} \int_0^\pi e^{ix\cos\theta} x^\nu \sin^{2\nu}\theta d\theta. \tag{3.4}$$

Furthermore, the equality of asymptotic expression (3.3) holds in the space of analytic functions (13).

It follows from Theorem 2.2 and equation (3.2) that

$$\begin{aligned} &\phi(x)\,\delta^{(2k)}(r) \\ &= \frac{(2k)!\,\Gamma(\frac{n}{2})}{2^{2k}\,k!\,\Gamma(\frac{n}{2}+k)} \sum_{m+i+l=k} 2^i(-1)^i \binom{m+l}{m}\binom{k}{m+l} \nabla^i \triangle^m \phi(0)\cdot\nabla^i\triangle^l\delta(x), \end{aligned}$$

for any $\phi(x) \in C^\infty(R^n)$ and $k = 0, 1, 2, \cdots$.

In particular, we have for $k = 0$ that

$$\phi(x)\,\delta(r) = \phi(0)\delta(x).$$

Theorem 3.2. *The asymptotic product of $\phi(x)$ and $\delta(r-1)$ exists and*

$$\begin{aligned} &\phi(x)\,\delta(r-1) \\ &\sim 2\pi^{\frac{n}{2}} \sum_{k=0}^\infty \sum_{m+i+l=k} (-1)^i \frac{\binom{m+l}{m}\binom{k}{m+l}\nabla^i\triangle^m\phi(0)\cdot\nabla^i\triangle^l\delta(x)}{2^{2k-i}\,k!\,\Gamma(\frac{n}{2}+k)} \end{aligned}$$

for any $\phi(x) \in C^\infty(R^n)$.

Proof. It immediately follows from Theorems 3.1 and 2.2. □

Theorem 3.3. *The asymptotic product of X^s and $\delta(r-1)$ exists and*

$$\begin{aligned} &X^s\,\delta(r-1) \\ &\sim \begin{cases} 2^{s+1}\pi^{\frac{n}{2}}\, s! \displaystyle\sum_{k=0}^\infty \sum_{j=0}^{s/2} \frac{n^j\,\nabla^{s-2j}\triangle^{k-s+j}\delta(x)}{2^{2j+2k} j!(k-s+j)!(s-2j)!\,\Gamma(\frac{n}{2}+k)} & \text{if } s \text{ is even,} \\ -2^{s+1}\pi^{\frac{n}{2}}\, s! \displaystyle\sum_{k=0}^\infty \sum_{j=0}^{\lfloor s/2\rfloor} \frac{n^j\,\nabla^{s-2j}\triangle^{k-s+j}\delta(x)}{2^{2j+2k} j!(k-s+j)!(s-2j)!\,\Gamma(\frac{n}{2}+k)} & \text{if } s \text{ is odd.} \end{cases} \end{aligned}$$

Proof. It directly follows from Theorems 3.1 and 2.3. □

Similarly, we can follow the ideas presented in (22) and the Fourier transform to obtain

$$\begin{aligned} &\delta^{(k)}(r-a) \\ &\sim \begin{cases} (-1)^k 2\pi^{\frac{n}{2}} a^{n-1-k} \displaystyle\sum_{j=0}^\infty \frac{\Gamma(n+2j)a^{2j}\,\triangle^j\delta(x)}{2^{2j}\,j!\,\Gamma(\frac{n}{2}+j)\,\Gamma(n+2j-k)} & \text{if } k \le n-1, \\ (-1)^k 2\pi^{\frac{n}{2}} a^{n-1-k} \displaystyle\sum_{j=\lceil\frac{k-n+1}{2}\rceil}^\infty \frac{\Gamma(n+2j)a^{2j}\,\triangle^j\delta(x)}{2^{2j}\,j!\,\Gamma(\frac{n}{2}+j)\,\Gamma(n+2j-k)} & \text{if } k > n-1. \end{cases} \end{aligned}$$

In particular, we have for $k = 0$

$$\delta(r-a) \sim 2\pi^{\frac{n}{2}} a^{n-1} \sum_{j=0}^\infty \frac{a^{2j}}{2^{2j}\,j!\,\Gamma(\frac{n}{2}+j)} \triangle^j\delta(x)$$

which coincides with equation (3.3).

It follows from equation (3.2) that

$$\delta^{(k)}(r-a) \sim \begin{cases} \dfrac{(-1)^k 2\pi^{\frac{n}{2}} a^{n-1-k}}{\Gamma(\frac{n}{2})} \displaystyle\sum_{j=0}^{\infty} \frac{\Gamma(n+2j) a^{2j}\, \delta^{(2j)}(r)}{(2j)!\,\Gamma(n+2j-k)} & \text{if } k \le n-1, \\ \dfrac{(-1)^k 2\pi^{\frac{n}{2}} a^{n-1-k}}{\Gamma(\frac{n}{2})} \displaystyle\sum_{j=\lceil \frac{k-n+1}{2} \rceil}^{\infty} \frac{\Gamma(n+2j) a^{2j}\, \delta^{(2j)}(r)}{(2j)!\,\Gamma(n+2j-k)} & \text{if } k > n-1. \end{cases}$$

Clearly, we get from $k = 0$

$$\delta(r-a) \sim \frac{2\pi^{\frac{n}{2}} a^{n-1}}{\Gamma(\frac{n}{2})} \sum_{j=0}^{\infty} \frac{\delta^{(2j)}(r)}{(2j)!} a^{2j}$$

which is the same result obtained previously.

Setting $a = 1$, we have the following generalized products from Theorem 2.2.

$$\phi(x)\,\delta^{(k)}(r-1) \sim \begin{cases} 2\pi^{\frac{n}{2}}(-1)^k \displaystyle\sum_{j=0}^{\infty} \frac{\phi(x)\,\triangle^j \delta(x)\,\Gamma(n+2j)}{2^{2j}\,j!\,\Gamma(\frac{n}{2}+j)\,\Gamma(n+2j-k)} \\ = 2\pi^{\frac{n}{2}}(-1)^k \displaystyle\sum_{j=0}^{\infty} \frac{\Gamma(n+2j)}{2^{2j}\,j!\,\Gamma(\frac{n}{2}+j)\,\Gamma(n+2j-k)} \cdot \\ \displaystyle\sum_{m+i+l=j} 2^i(-1)^i \binom{m+l}{m}\binom{j}{m+l} \nabla^i \triangle^m \phi(0) \cdot \nabla^i \triangle^l \delta(x) & \text{if } k \le n-1, \\ 2\pi^{\frac{n}{2}}(-1)^k \displaystyle\sum_{j=\lceil \frac{k-n+1}{2} \rceil}^{\infty} \frac{\phi(x)\,\triangle^j \delta(x)\,\Gamma(n+2j)}{2^{2j}\,j!\,\Gamma(\frac{n}{2}+j)\,\Gamma(n+2j-k)} \\ = 2\pi^{\frac{n}{2}}(-1)^k \displaystyle\sum_{j=\lceil \frac{k-n+1}{2} \rceil}^{\infty} \frac{\Gamma(n+2j)}{2^{2j}\,j!\,\Gamma(\frac{n}{2}+j)\,\Gamma(n+2j-k)} \cdot \\ \displaystyle\sum_{m+i+l=j} 2^i(-1)^i \binom{m+l}{m}\binom{j}{m+l} \nabla^i \triangle^m \phi(0) \cdot \nabla^i \triangle^l \delta(x) & \text{if } k > n-1. \end{cases}$$

The product $X^s\,\delta^{(k)}(r-1)$ can be derived easily using Theorem 2.3 and we leave it to interested readers.

Competing Interests

The author declares that no competing interests exist.

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