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THE HANKEL TRANSFORMATION OF BANACH-SPACE-VALUED GENERALIZED FUNCTIONS

E. L. KOH AND C. K. LI

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ABSTRACT. The object of this paper is to study Banach-space-valued generalized functions belonging to $[H_\mu(A); B]$ for which the Hankel transformation may be defined. In *Realizability theory for continuous linear systems* (Academic Press, New York, 1972), Zemanian considered certain ρ -type testing function spaces for which the Laplace transformation is defined. Tiwari (*Banach space valued distributional Mellin transform and form invariant linear filtering*, Indian J. Pure Appl. Math. **20** (1989), 493–504) follows Zemanian in extending the Mellin transform. Their works are based on the denseness of the Schwartz space $D^m(A)$ in the testing function spaces of interest. This method is not possible here since the space $D^m(A)$ is not dense in $H_\mu(A)$, and the structure of $H_\mu(A)$ is quite different from that of $D^m(A)$, which has an inductive-limit topology. Thus, it is necessary to introduce a dense subspace ${}_\mu D_l(A)$ of $H_\mu(A)$ to derive some properties of $H_\mu(A)$. We then define the Hankel transformation on $[H_\mu(A); B]$. We end this paper with some operational formulas, which are analogous with those given by the first author in *SIAM J. Math. Anal.* **1** (1970), 322–327.

1. INTRODUCTION

Zemanian studied the theory of Banach-space-valued testing functions and distributions in [2], which is somewhat more general than that of scalar distributions. He constructed $D^m(A)$ as the inductive-limit space given by

$$D^m(A) = D_{R^n}^m(A) = \bigcup_{j=1}^{\infty} D_{K_j}^m(A)$$

where $D_{K_j}^m(A)$ denotes the linear space of all smooth functions ϕ from R^n into a Banach space A such that $\text{supp } \phi \subset K_j$. K_j are compact subsets of R^n and $K_j \subset K_{j+1}$, $\bigcup_{j=1}^{\infty} K_j = R^n$. We assign to $D_{K_j}^m(A)$ the topology generated by the collection $\{\gamma_k; 0 \leq k \leq m\}$ of seminorms, where $\gamma_k(\phi) \triangleq \sup_{t \in K_j} \|\phi^{(k)}(t)\|_A$.

Applying the interpolation theory, he describes the local structure property below.

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Theorem 1.1. Let $f \in [D^m(A); B]$ and K be a compact interval in R^n . Then there exists an integer $p \in R^n$ with $0 \leq p \leq m$ and a continuous $[A; B]$ -valued function h on K such that, for all $\phi \in D_K^{m+[2]}(A)$,

$$\langle f, \phi \rangle = \int_K h(t) D^{p+[2]} \phi(t) dt.$$

In general, p and h depend on f and K .

Finally, he mentions

Theorem 1.2. If \mathcal{T}^m and $\mathcal{T}^m(A)$ are normal spaces (i.e., D is dense in \mathcal{T}^m and $\mathcal{T}^m(A)$), then there exists a bijection from $[\mathcal{T}^m(A); B]$ onto $[\mathcal{T}^m; [A; B]]$ defined by

$$\langle g, \psi \rangle a \triangleq \langle f, \psi a \rangle, \quad \psi \in \mathcal{T}^m, a \in A,$$

where $g \in [\mathcal{T}^m; [A; B]]$, $f \in [\mathcal{T}^m(A); B]$.

Tiwari [3] mimics the method of Zemanian in defining Banach-space-valued distributions for which a Mellin transform can be given. Several properties including a Mellin-type convolution theorem are proved. These results are similar to those of Zemanian [1].

In this paper, we introduce a dense subspace ${}_\mu D_I(A)$ of $H_\mu(A)$. It does not have an inductive-limit topology. The local structure theorem is no longer discussed in $[H_\mu(A); B]$. However, with a different method, we show that there is still a bijection from $[H_\mu(A); B]$ onto $[H_\mu; [A; B]]$. Further, we are able to define the Hankel transformation of arbitrary order on $H_\mu(A)$, which is still an automorphism on $H_\mu(A)$. We give some operational formulas at the end of this paper.

Our notation is similar to that used in [1, 2, 4]. Given any two topological vector spaces A and B , $[A; B]$ denotes the linear space of all continuous linear mappings of A into B . The element of B assigned by $f \in [A; B]$ to $\phi \in A$ is denoted by $\langle f, \phi \rangle$. $[A; B]$ is supplied with the topology of uniform convergence on bounded sets in A . $\|\cdot\|_B$ denotes the norm in any Banach space B . R and C are the real and complex number fields. I is the open interval $(0, +\infty)$. Other notation will be introduced as the need arises.

2. THE CORRESPONDENCE BETWEEN $[H_\mu(A); B]$ AND $[H_\mu; [A; B]]$ FOR $\mu \geq -\frac{1}{2}$

Following Zemanian, $H_\mu(A)$ is defined as follows:

Definition 2.1. Let x be a real variable restricted to I . For each real number μ , $\phi(x) \in H_\mu(A)$ iff it is defined on I , takes its values in A , is smooth, and for each pair of nonnegative integers m and k

$$\gamma_{m,k}^\mu(\phi) \triangleq \sup_{x \in I} \|x^m (x^{-1} D)^k x^{-\mu-1/2} \phi(x)\|_A$$

is finite. $H_\mu(A)$ is a linear space. The topology of $H_\mu(A)$ is that generated by $\{\gamma_{m,k}^\mu\}_{m,k=0}^\infty$.

Definition 2.2. $\phi(x) \in D_I(A)$ iff ϕ is defined on I , takes its value in A , is smooth, and for every ϕ there exists $b \in I$ such that $\phi(x) = 0$ for $x \in [b, +\infty)$. Let ${}_\mu D_I(A) \triangleq D_I(A) \cap H_\mu(A)$.

Theorem 2.1. *The subspace ${}_{\mu}D_I(A)$ is dense in $H_{\mu}(A)$ for all $\mu \in R$.*

Proof. Let $\lambda(x) \in D_I(C)$ such that $\lambda(x) = 1$ for $0 < x \leq 1$ and $\lambda(x) = 0$ for $x \geq 2$.

For arbitrary $\phi(x) \in H_{\mu}(A)$ and each pair of nonnegative integers m and k we consider

$$\begin{aligned} & x^m(x^{-1}D)^k x^{-\mu-1/2}[\lambda(x/N)\phi(x) - \phi(x)] \\ &= x^{m+1} \sum_{v=0}^k \binom{k}{v} (x^{-1}D)^{k-v} x^{-\mu-1/2} \phi \frac{(x^{-1}D)^v [\lambda(x/N) - 1]}{x}. \end{aligned}$$

Therefore,

$$\begin{aligned} & \sup_{x \in I} \|x^m(x^{-1}D)^k x^{-\mu-1/2}(\lambda(x/N)\phi(x) - \phi(x))\|_A \\ & \leq \sum_{v=0}^k \binom{k}{v} \sup_{x \in I} \|x^{m+1}(x^{-1}D)^{k-v} x^{-\mu-1/2} \phi\|_A \cdot \sup_{x \geq N} \left| \frac{(x^{-1}D)^v [\lambda(x/N) - 1]}{x} \right|. \end{aligned}$$

It follows from $\phi \in H_{\mu}(A)$ that

$$\sup_{x \in I} \|(x^{-1}D)^{k-v} x^{-\mu-1/2} \phi\|_A$$

is finite.

Since $\lambda(x)$ and its derivatives are bounded, it follows that

$$\sup_{x \geq N} \left| \frac{(x^{-1}D)^v [\lambda(x/N) - 1]}{x} \right| \rightarrow 0 \quad \text{as } N \rightarrow \infty, \text{ for fixed } k \text{ and } 0 \leq v \leq k,$$

whence our assertion.

$H_{\mu}(A)$ is not a ρ -type testing function space in the sense of Zemanian [2]. To see this, we choose $\phi(x) = x^{\mu+1/2}e^{-x^2}a_0$, $a_0 \in A$ and $a_0 \neq 0$. Then for all ψ , which is smooth from I into A with compact support contained in I , $\gamma_{0,0}^{\mu}(\phi - \psi) \geq \|a_0\|/2 > 0$. This means the balloon

$$\{\theta; \theta \in H_{\mu}(A), \gamma_{0,0}^{\mu}(\phi - \theta) \leq \|a_0\|/3\}$$

does not contain any element of $D^m(A)$. Thus our result is true.

The following lemmas will be used subsequently (see [1, 2]).

Lemma 2.1. *Let V, W be locally convex spaces, and let Γ and P generate families of seminorms of the topologies of V and W , respectively. Let f be a linear mapping of V into W . The following four assertions are equivalent:*

- (i) f is continuous.
- (ii) f is continuous at the origin.
- (iii) For every continuous seminorm ρ on W , there exists a continuous seminorm γ on V such that $\rho(f(\theta)) \leq \gamma(\theta)$ for all θ .
- (iv) For every $\rho \in P$, there exists a constant $M > 0$ and a finite collection $\{\gamma_1, \gamma_2, \dots, \gamma_m\} \subset \Gamma$ such that

$$\rho(f(\theta)) \leq M \max_{0 \leq k \leq m} \gamma_k(\theta)$$

for all $\theta \in V$.

Lemma 2.2. *For $\mu \geq -\frac{1}{2}$, the conventional h_{μ} is an automorphism on $H_{\mu}(A)$.*

Proof. Very similar to Theorem 5.4.-1, p. 141 of [1].

Theorem 2.2. Every $f \in [H_\mu(A); B]$ uniquely defines a $g \in [H_\mu; [A; B]]$ through the equation

$$\langle g, \theta \rangle a \triangleq \langle f, \theta a \rangle, \quad \theta \in H_\mu, \quad a \in A,$$

for all $\mu \in \mathbb{R}$.

Proof. Fixing upon some $\theta \in H_\mu$ we define a mapping j_θ of A into B by $j_\theta a = \langle f, \theta a \rangle$ for all $a \in A$. It readily follows that j_θ is linear. By Lemma 2.1(iv) there exist positive integers m_0, k_0 and a constant $M > 0$ such that

$$\|j_\theta a\|_B = \|\langle f, \theta a \rangle\|_B \leq M \max_{\substack{0 \leq k \leq k_0 \\ 0 \leq m \leq m_0}} \gamma_{m,k}^\mu(\theta a)$$

where

$$\begin{aligned} \gamma_{m,k}^\mu(\theta a) &= \sup_{x \in I} \|x^m(x^{-1}D)^k x^{-\mu-1/2} \theta a\|_A \\ &= \|a\|_A \sup_{x \in I} |x^m(x^{-1}D)^k x^{-\mu-1/2} \theta|. \end{aligned}$$

Hence

$$\|j_\theta a\|_B \leq M \|a\|_A \max_{\substack{0 \leq k \leq k_0 \\ 0 \leq m \leq m_0}} \gamma_{m,k}^\mu(\theta)$$

and

$$(2.1) \quad \|j_\theta\|_{[A; B]} \leq M \max_{\substack{0 \leq k \leq k_0 \\ 0 \leq m \leq m_0}} \gamma_{m,k}^\mu(\theta).$$

Next, set $\langle g, \theta \rangle \triangleq j_\theta$. This uniquely defines g as a mapping from H_μ into $[A; B]$. g is linear because, for any $a \in A, \alpha, \beta \in \mathbb{C}$, and $\theta, \psi \in H_\mu$

$$\begin{aligned} \langle g, \alpha\theta + \beta\psi \rangle a &= \langle f, \alpha\theta a + \beta\psi a \rangle = \alpha \langle f, \theta a \rangle + \beta \langle f, \psi a \rangle \\ &= (\alpha \langle g, \theta \rangle + \beta \langle g, \psi \rangle) a. \end{aligned}$$

Moreover, (2.1) implies that g is continuous.

Let ${}_\mu D_I \odot A$ denote the linear space of all $\varphi \in {}_\mu D_I(A)$ having representations of the form $\varphi = \sum \theta_k a_k$ where $\theta_k \in {}_\mu D_I, a_k \in A$, and the summation is over a finite number of terms.

Theorem 2.3. The space ${}_\mu D_I \odot A$ is dense in $H_\mu(A)$ for $\mu \geq -\frac{1}{2}$.

Proof. Let $\lambda(x)$ be defined as in the proof of Theorem 2.1. For $\varphi \in {}_\mu D_I(A)$, we first show that

$$\lambda(x/N) h_\mu(\varphi) \rightarrow h_\mu(\varphi) \quad \text{in } H_\mu(A) \text{ as } N \rightarrow +\infty \text{ for all } \mu \in \mathbb{R}.$$

The following equation will be used (see [1]):

$$(2.2) \quad \begin{aligned} &(-1)^{m+k} y^m (y^{-1}D)^k y^{-\mu-1/2} h_\mu(\varphi)(y) \\ &= \int_0^{+\infty} x^{2\mu+2k+m+1} [(x^{-1}D)^m x^{-\mu-1/2} \varphi(x)] \frac{J_{\mu+k+m}(xy)}{(xy)^{\mu+k}} dx. \end{aligned}$$

Hence

$$\begin{aligned} &\sup_{x \in I} \left\| x^m (x^{-1}D)^k x^{-\mu-1/2} h_\mu(\varphi) \left[\lambda\left(\frac{x}{N}\right) - 1 \right] \right\|_A \\ &\leq \sum_{v=0}^k \binom{k}{v} \sup_{x \geq N} \left| \frac{(x^{-1}D)^v [\lambda(x/N) - 1]}{x} \right| \sup_{x \in I} \|x^{m+1} (x^{-1}D)^{k-v} x^{-\mu-1/2} h_\mu(\varphi)\|_A. \end{aligned}$$

By what we have proved in Theorem 2.1,

$$\sup_{x \geq N} \left| \frac{(x^{-1}D)^v [\lambda(x/N) - 1]}{x} \right| \rightarrow 0 \quad \text{as } N \rightarrow \infty \text{ for fixed } k \text{ and } 0 \leq v \leq k.$$

Using equation (2.2) and noting that $J_{\mu+k-v+m+1}(xy)/(xy)^{\mu+k-v}$ is bounded, say by $B_{k,v,m}$, we get

$$\begin{aligned} & \sup_{x \in I} \|x^{m+1}(x^{-1}D)^{k-v}x^{-\mu-1/2}h_\mu(\varphi)\|_A \\ &= \sup_{x \in I} \left\| \int_0^{+\infty} y^{2\mu+2(k-v)+m+2} [(y^{-1}D)^{m+1}y^{-\mu-1/2}\varphi(y)] \frac{J_{\mu+k-v+m+1}(xy)}{(xy)^{\mu+k-v}} dy \right\|_A. \end{aligned}$$

Choosing a positive integer n such that

$$y^{2\mu+2(k-v)+m+2} \leq (1+y^2)^n \quad \text{for all } y \in I$$

we have

$$\begin{aligned} & \sup_{y \in I} \|y^{2\mu+2(k-v)+m+2} [(y^{-1}D)^{m+1}y^{-\mu-1/2}\varphi(y)]\|_A \\ & \leq \sup_{y \in I} \|(1+y^2)^n [(y^{-1}D)^{m+1}y^{-\mu-1/2}\varphi(y)]\|_A. \end{aligned}$$

Since $\varphi \in {}_\mu D_I(A)$, there exists $b \in I$ such that $\varphi(x) = 0$ for $x \in [b, +\infty)$. It follows that

$$\begin{aligned} & \sup_{x \in I} \|x^{m+1}(x^{-1}D)^{k-v}x^{-\mu-1/2}h_\mu(\varphi)\|_A \\ & \leq B_{k,v,m} b \sup_{y \in I} \|(1+y^2)^n [(y^{-1}D)^{m+1}y^{-\mu-1/2}\varphi(y)]\|_A \end{aligned}$$

is finite. Therefore,

$$\lambda(x/N)h_\mu(\varphi) \rightarrow h_\mu(\varphi) \quad \text{in } H_\mu(A) \text{ as } N \rightarrow +\infty.$$

Second, we prove that ${}_\mu D_I \odot A$ is dense in $H_\mu(A)$ for $\mu \geq -\frac{1}{2}$. For a positive integer m_1 , we have

$$\sqrt{xy}J_\mu(xy) = \sum_{j=0}^{m_1} \frac{(xy)^{1/2}(-1)^j(xy/2)^{\mu+2j}}{j!\Gamma(\mu+j+1)} + \sum_{j=m_1+1}^{\infty} \frac{(xy)^{1/2}(-1)^j(xy/2)^{\mu+2j}}{j!\Gamma(\mu+j+1)}.$$

For every $\varphi \in {}_\mu D_I(A)$, the term

$$T_{N,m_1} = \lambda \left(\frac{x}{N} \right) \int_0^{+\infty} \varphi(t) \sum_{j=0}^{m_1} \frac{(xt)^{1/2}(-1)^j(xt/2)^{\mu+2j}}{j!\Gamma(\mu+j+1)} dt,$$

where $N, m_1 = 1, 2, \dots$, belongs to ${}_\mu D_I \odot A$ since $\mu \geq -\frac{1}{2}$. Now

$$\begin{aligned} & T_{N,m_1} - \int_0^{+\infty} \varphi(t)\sqrt{xt}J_\mu(xt) dt \\ &= T_{N,m_1} - \lambda \left(\frac{x}{N} \right) \int_0^{+\infty} \varphi(t)\sqrt{xt}J_\mu(xt) dt \\ & \quad + \lambda \left(\frac{x}{N} \right) \int_0^{+\infty} \varphi(t)\sqrt{xt}J_\mu(xt) dt - \int_0^{+\infty} \varphi(t)\sqrt{xt}J_\mu(xt) dt. \end{aligned}$$

By what we have just proved, for arbitrary $\varepsilon > 0$, there exists an N_1 such that for $N \geq N_1$, we have

$$\sup_{x \in I} \|x^m(x^{-1}D)^k x^{-\mu-1/2} [\lambda(x/N)h_\mu(\varphi) - h_\mu(\varphi)]\|_A < \varepsilon/2.$$

Fixing $N (\geq N_1)$, then

$$\lambda\left(\frac{x}{N}\right) \left[\sum_{j=0}^{m_1} \frac{(xt)^{1/2}(-1)^j(xt/2)^{\mu+2j}}{j!\Gamma(\mu+j+1)} - \sqrt{xt}J_\mu(xt) \right]$$

and its derivatives with respect to x converge to zero uniformly on every compact subset of I . It has a uniformly bounded support. Therefore it converges in the sense of Schwartz, whose topology is stronger than that of H_μ (see [1]). It follows that there exists an $L \in I$ such that as long as $m_1 \geq L$, then for all $t \leq b$,

$$\begin{aligned} \sup_{x \in I} \left\| x^m(x^{-1}D)^k x^{-\mu-1/2} \lambda\left(\frac{x}{N}\right) \left[\sum_{j=0}^{m_1} \frac{(xt)^{1/2}(-1)^j(xt/2)^{\mu+2j}}{j!\Gamma(\mu+j+1)} - \sqrt{xt}J_\mu(xt) \right] \right\| \\ \leq \frac{\varepsilon}{2M_1}, \end{aligned}$$

where $M_1 = b \sup_{t \in I} \|\varphi(t)\|_A$. If $M_1 = 0$, then there is nothing to be proved. Therefore,

$$\sup_{x \in I} \left\| x^m(x^{-1}D)^k x^{-\mu-1/2} \left[T_{N, m_1} - \int_0^{+\infty} \varphi(t)\sqrt{xt}J_\mu(xt) dt \right] \right\|_A < \varepsilon$$

provided $N \geq N_1, m_1 \geq L$.

Since h_μ is an automorphism on $H_\mu(A)$ for $\mu \geq -\frac{1}{2}$ by Lemma 2.2 and ${}_\mu D_I(A)$ is dense in $H_\mu(A)$, it follows that $h_\mu({}_\mu D_I(A))$ is dense in $H_\mu(A)$. Our assertion follows directly from the fact that ${}_\mu D_I \odot A$ is dense in $h_\mu({}_\mu D_I(A))$.

Theorem 2.4. *There is a bijection from $[H_\mu(A); B]$ onto $[H_\mu; [A; B]]$ defined by*

$$\langle g, \theta \rangle a = \langle f, \theta a \rangle$$

where $a \in A, g \in [H_\mu; [A; B]]$, and $f \in [H_\mu(A); B], \theta \in H_\mu$ for $\mu \geq -\frac{1}{2}$.

Proof. By Theorem 2.2, every $f \in [H_\mu(A); B]$ uniquely defines a $g \in [H_\mu; [A; B]]$ through the equation

$$\langle g, \theta \rangle a \triangleq \langle f, \theta a \rangle \quad \text{for all } \mu \in \mathbb{R}.$$

Let us consider the converse. For every $\varphi \in {}_\mu D_I \odot A$, we define

$$\langle f, \varphi \rangle = \sum \langle g, \theta_k \rangle a_k \quad \text{for } \varphi = \sum \theta_k a_k.$$

It follows from the definition that f is linear on ${}_\mu D_I \odot A$. We wish to show that f is continuous. Indeed, for arbitrary $\varepsilon > 0$, as long as θa ($\theta \in {}_\mu D_I, a \in A$) belongs to the balloon $\{\varphi; \gamma_{m,k}^\mu(\varphi) < \varepsilon/M, m = 0, 1, \dots, m_0, k = 0, 1, \dots, k_0\}$. M, m_0, k_0 are defined as follows. We infer that

$$\|\langle f, \theta a \rangle\|_B = \|\langle g, \theta \rangle a\|_B \leq \|a\|_A \cdot \|\langle g, \theta \rangle\|_{[A; B]}.$$

By Lemma 2.1(iv), there exist $M > 0$ and positive integers m_0, k_0 such that

$$\| \langle f, \theta a \rangle \|_B \leq \| a \|_A \cdot M \max_{\substack{0 \leq k \leq k_0 \\ 0 \leq m \leq m_0}} \gamma_{m,k}^\mu(\theta) < M \cdot \varepsilon / M = \varepsilon.$$

Therefore f is continuous at the origin. By Lemma 2.1(ii), f is continuous on ${}_\mu D_I \odot A$. According to Theorem 2.3, ${}_\mu D_I \odot A$ is dense in $H_\mu(A)$ for $\mu \geq -\frac{1}{2}$. Thus our assertion is true.

**3. THE HANKEL TRANSFORMATION h_μ ON $H_\mu(A)$
AND THE CORRESPONDENCE BETWEEN $[H_\mu(A); B]$ AND $[H_\mu; [A; B]]$
FOR ARBITRARY μ**

We shall use the following differential and integral operators due to Zemanian [1]:

$$\begin{aligned} N_\mu \varphi(x) &\triangleq x^{\mu+1/2} D x^{-\mu-1/2} \varphi(x), \\ M_\mu \varphi(x) &\triangleq x^{-\mu-1/2} D x^{\mu+1/2} \varphi(x), \\ N_\mu^{-1} \varphi(x) &\triangleq x^{\mu+1/2} \int_\infty^x t^{-\mu-1/2} \varphi(t) dt. \end{aligned}$$

Lemma 3.1. N_μ is a continuous linear mapping of $H_\mu(A)$ into $H_{\mu+1}(A)$.

Indeed, $\gamma_{m,k}^{\mu+1}(N_\mu \varphi) = \gamma_{m,k+1}^\mu(\varphi)$ for every $\varphi \in H_\mu(A)$ and every choice of m and k .

Lemma 3.2. N_μ^{-1} is a continuous linear mapping of $H_{\mu+1}(A)$ into $H_\mu(A)$.

Proof. Assume that $\varphi(x) \in H_{\mu+1}(A)$ and k is a fixed positive integer. Then

$$\begin{aligned} (x^{-1}D)^k x^{-\mu-1/2} N_\mu^{-1} \varphi(x) &= (x^{-1}D)^k x^{-\mu-1/2} x^{\mu+1/2} \int_\infty^x t^{-\mu-1/2} \varphi(t) dt \\ &= (x^{-1}D)^{k-1} x^{-\mu-3/2} \varphi(x). \end{aligned}$$

Hence

$$\gamma_{m,k}^\mu(N_\mu^{-1} \varphi) = \gamma_{m,k-1}^{\mu+1}(\varphi), \quad k = 1, 2, 3, \dots, m = 0, 1, 2, \dots$$

A similar result for the case $k = 0$ can be derived as follows:

$$\begin{aligned} \| x^m x^{-\mu-1/2} N_\mu^{-1} \varphi(x) \|_A &\leq x^m \int_x^\infty \| t^{-\mu-1/2} \varphi(t) \|_A dt \\ &\leq \int_x^\infty \| t^m t^{-\mu-1/2} \varphi(t) \|_A dt \\ &\leq \int_0^{+\infty} \left\| \frac{1}{1+t^2} (t^{m+1} + t^{m+2}) t^{-\mu-3/2} \varphi(t) \right\|_A dt \\ &\leq \int_0^\infty \frac{1 dt}{1+t^2} \cdot \sup_{t \in I} \| (t^{m+1} + t^{m+3}) t^{-\mu-3/2} \varphi(t) \|_A. \end{aligned}$$

Therefore

$$\gamma_{m,0}^\mu(N_\mu^{-1} \varphi) \leq \frac{\pi}{2} [\gamma_{m+1,0}^{\mu+1}(\varphi) + \gamma_{m+3,0}^{\mu+1}(\varphi)], \quad m = 0, 1, 2, \dots$$

It follows from the above that N_μ^{-1} is a continuous linear mapping of $H_{\mu+1}(A)$ into $H_\mu(A)$.

Let $\mu \in \mathbb{R}$ and let k be a positive integer such that $\mu + k \geq -\frac{1}{2}$. Assume that $\varphi \in H_\mu(A)$ and define $h_{\mu,k}$ on $H_\mu(A)$ by (see Koh [4])

$$\Phi(x) = h_{\mu,k}(\varphi(y)) \triangleq (-1)^k x^{-k} h_{\mu+k} N_{\mu+k-1} \cdots N_{\mu+1} N_\mu \varphi(y).$$

Let $\Phi(x) \in H_\mu(A)$ and define $h_{\mu,k}^{-1}$ on $H_\mu(A)$ by

$$\varphi(y) = h_{\mu,k}^{-1}(\Phi(x)) \triangleq (-1)^k N_\mu^{-1} N_{\mu+1}^{-1} \cdots N_{\mu+k-1}^{-1} h_{\mu+k} x^k \Phi(x).$$

Theorem 3.1. $h_{\mu,k}$ is an automorphism on $H_\mu(A)$. Its inverse is $h_{\mu,k}^{-1}$ and $h_{\mu,k} = h_\mu$ if $\mu \geq -\frac{1}{2}$.

Proof. By Lemmas 3.1 and 3.2, $\varphi \rightarrow N_\mu N_{\mu+1} \cdots N_{\mu+k-1} \varphi$ is an isomorphism from $H_\mu(A)$ onto $H_{\mu+k}(A)$.

By Lemma 2.2, $h_{\mu+k}$ is an automorphism on $H_{\mu+k}(A)$ for $\mu + k \geq -\frac{1}{2}$. It follows from $y_m^\mu(x^{-k}\varphi) = y_m^{\mu+k}(\varphi)$ that $\varphi \rightarrow x^{-k}\varphi$ is an isomorphism from $H_{\mu+k}(A)$ onto $H_\mu(A)$. Therefore $h_{\mu,k}$ is an automorphism on $H_\mu(A)$. Similarly $h_{\mu,k}^{-1}$ is an automorphism on $H_\mu(A)$ and is inverse to $h_{\mu,k}$ because $h_{\mu+k}^{-1} = h_{\mu+k}$ and the inverse of $N_{\mu+k-1} \cdots N_\mu$ is $N_\mu^{-1} \cdots N_{\mu+k-1}^{-1}$.

To prove the last statement, let $\varphi(y) \in H_\mu(A)$, $\mu \geq -\frac{1}{2}$, and consider $k = 1$;

$$\begin{aligned} h_{\mu,1}\varphi &= -x^{-1} h_{\mu+1} N_\mu \varphi = -x^{-1} \int_0^\infty y^{\mu+1/2} [D_y y^{-\mu-1/2} \varphi(y)] \sqrt{xy} J_{\mu+1}(xy) dy \\ &= -x^{-1} \sqrt{xy} J_{\mu+1}(xy) \varphi(y) \Big|_0^\infty + \int_0^\infty \varphi(y) \sqrt{xy} J_\mu(xy) dy. \end{aligned}$$

Since $\varphi(y)$ is of rapid descent and $\sqrt{xy} J_{\mu+1}(xy)$ is bounded as $y \rightarrow \infty$, while $\varphi(y) = O(y^{\mu+1/2})$ and $\sqrt{xy} J_{\mu+1}(xy) = O(y^{\mu+3/2})$ as $y \rightarrow 0^+$, the limit terms are zero for $\mu \geq -\frac{1}{2}$. Thus $h_{\mu,1}\varphi = h_\mu \varphi$. By induction, $h_{\mu,k} = h_\mu$ for $\mu \geq -\frac{1}{2}$.

Note that the definition of $h_{\mu,k}$ is independent of the choice of k so long as $k + \mu \geq -\frac{1}{2}$. Indeed, if $k > p \geq -\mu - \frac{1}{2}$, then $h_{\mu+p,k-p} = h_{\mu+p}$ by Theorem 3.1; hence

$$\begin{aligned} h_{\mu,k}\varphi &= (-1)^k x^{-k} h_{\mu+k} N_{\mu+k-1} \cdots N_\mu \varphi \\ &= (-1)^p x^{-p} (-1)^{k-p} x^{-(k-p)} h_{\mu+p+k-p} N_{\mu+p+k-p-1} \cdots N_{\mu+p} N_{\mu+p-1} \cdots N_\mu \varphi \\ &= (-1)^p x^{-p} h_{\mu+p,k-p} N_{\mu+p-1} \cdots N_\mu \varphi \\ &= (-1)^p x^{-p} h_{\mu+p} N_{\mu+p-1} \cdots N_\mu \varphi = h_{\mu,p}\varphi. \end{aligned}$$

Definition 3.1. Let $\mu \in \mathbb{R}$ and k be a positive integer such that $\mu + k \geq -\frac{1}{2}$. For any $f \in [H_\mu(A); B]$, the generalized Hankel transform $h'_\mu f$ is defined by

$$\langle h'_\mu f, \varphi \rangle = \langle f, h_{\mu,k}\varphi \rangle, \quad \varphi \in H_\mu(A).$$

By Theorem 3.1 and the fact that h'_μ is the adjoint operator of $h_{\mu,k}$ on $H_\mu(A)$, we have

Theorem 3.2. h'_μ is an automorphism on $[H_\mu(A); B]$ for all $\mu \in \mathbb{R}$.

Applying operator $T \triangleq N_{\mu+k-1} \cdots N_{\mu}$, we have

Theorem 3.3. *Let A and B be two Banach spaces. There is a bijection from $[H_{\mu}(A); B]$ onto $[H_{\mu}; [A; B]]$ defined by*

$$\langle g, \theta \rangle a = \langle f, \theta a \rangle$$

where $a \in A$, $\theta \in H_{\mu}$, $g \in [H_{\mu}; [A; B]]$, and $f \in [H_{\mu}(A); B]$, $\mu \in R$.

Proof. For arbitrary $\mu \in R$, we choose a positive integer k such that $\mu + k \geq -\frac{1}{2}$. The operator T is an isomorphism from ${}_{\mu}D_I \odot A$ onto ${}_{\mu+k}D_I \odot A$, which is dense in $H_{\mu+k}(A)$. Also T is an isomorphism from $H_{\mu}(A)$ onto $H_{\mu+k}(A)$. Therefore, ${}_{\mu}D_I \odot A$ is dense in $H_{\mu}(A)$. By Theorems 2.3 and 2.4, there is a bijection from $[H_{\mu}(A); B]$ onto $[H_{\mu}; [A; B]]$ satisfying the above equation.

4. SOME OPERATIONAL FORMULAS

We now establish certain transformation formulas relating to the Bessel-type differential operator $M_{\mu}N_{\mu}$, which are similar to those obtained in [4], but on $H_{\mu}(A)$.

Lemma 4.1. *Let μ be a fixed real number and k a positive integer $\geq -\mu - \frac{1}{2}$. Then for every $\varphi \in H_{\mu}(A)$, $h_{\mu+1,k}(N_{\mu}\varphi) = -xh_{\mu,k}(\varphi)$.*

Proof. By definition

$$\begin{aligned} h_{\mu+1,k}(N_{\mu}\varphi) &= (-1)^k x^{-k} h_{\mu+1+k} N_{\mu+1+k} \cdots N_{\mu+1} N_{\mu} \varphi \\ &= -x h_{\mu,k+1}(\varphi) = -x h_{\mu,k}(\varphi). \end{aligned}$$

Turning to the linear operator M_{μ} , we prove that $\varphi \rightarrow M_{\mu}\varphi$ is a continuous linear mapping of $H_{\mu+1}(A)$ onto $H_{\mu}(A)$. Indeed, for $\varphi \in H_{\mu+1}(A)$ and any choice m and k

$$\begin{aligned} \gamma_{m,k}^{\mu}(M_{\mu}\varphi) &= \sup_{x \in I} \|x^m (x^{-1}D)^k x^{-2\mu-1} D x^{2\mu+2} x^{-\mu-3/2} \varphi(x)\|_A \\ &= \sup_{x \in I} \|(2\mu+2)x^m (x^{-1}D)^k x^{-\mu-3/2} \varphi(x) \\ &\quad + x^m (x^{-1}D)^k x^2 (x^{-1}D) x^{-\mu-3/2} \varphi(x)\|_A \\ &= \cdots = \sup_{x \in I} \|2(\mu+k+1)x^m (x^{-1}D)^k x^{-\mu-3/2} \varphi(x) \\ &\quad + x^{m+2} (x^{-1}D)^{k+1} x^{-\mu-3/2} \varphi(x)\|_A \\ &\leq 2|\mu+k+1| \gamma_{m,k}^{\mu+1}(\varphi) + \gamma_{m+2,k+1}^{\mu+1}(\varphi). \end{aligned}$$

This implies our assertion.

Lemma 4.2. *Let μ and k be as in Lemma 4.1. Then for every $\varphi \in H_{\mu+1}(A)$*

$$h_{\mu,k}(M_{\mu}\varphi) = x h_{\mu+1,k}(\varphi).$$

Proof. Using the relation

$$N_{\mu+k-1} \cdots N_{\mu+1} N_{\mu} \varphi(x) = x^{\mu+k+1/2} (x^{-1}D)^k x^{-\mu-1/2} \varphi(x)$$

we have

$$\begin{aligned} (4.1) \quad h_{\mu,k}(M_{\mu}\varphi) &= (-1)^k x^{-k} \int_0^{\infty} \sqrt{xy} J_{\mu+k}(xy) y^{\mu+k+1/2} y^2 (y^{-1}D)^{k+1} y^{-\mu-1-1/2} \varphi dy \\ &\quad + (-1)^k x^{-k} (2\mu+2k+2) \\ &\quad \times \int_0^{\infty} \sqrt{xy} J_{\mu+k}(xy) \cdot y^{\mu+k+1/2} (y^{-1}D)^k y^{-\mu-1/2-1} \varphi dy. \end{aligned}$$

We now show that $xh_{\mu+1,k}(\varphi)$ reduces to (4.1). Indeed

$$xh_{\mu+1,k}(\varphi) = (-1)^k x^{-k+1} \int_0^\infty \sqrt{xy} J_{\mu+k+1}(xy) y^{\mu+k+1+1/2} (y^{-1}D)^k y^{-\mu-3/2} \varphi dy.$$

From the formula (see [5])

$$J_{\mu+k+1}(xy) = -x^{-1} y^{\mu+k} D y^{-\mu-k} J_{\mu+k}(xy)$$

and an integration by parts, we obtain

$$\begin{aligned} xh_{\mu+1,k}(\varphi) &= (-1)^{k+1} x^{-k+1/2} \\ &\quad \times \int_0^\infty y^{2\mu+2k+2} (y^{-1}D)^k y^{-\mu-3/2} \varphi \cdot D[y^{-\mu-k} J_{\mu+k}(xy)] dy \\ &= (-1)^{k+1} x^{-k+1/2} \left\{ y^{\mu+k+2} J_{\mu+k}(xy) (y^{-1}D)^k y^{-\mu-3/2} \varphi \Big|_0^\infty \right. \\ &\quad \left. - \int_0^\infty y^{-\mu-k} J_{\mu+k}(xy) \right. \\ &\quad \left. \times D[y^{2\mu+2k+2} (y^{-1}D)^k \cdot y^{-\mu-3/2} \varphi] dy \right\}. \end{aligned}$$

The limit terms vanish because $\varphi \in H_{\mu+1}(A)$. Since

$$\begin{aligned} D[y^{2\mu+2k+2} (y^{-1}D)^k y^{-\mu-3/2} \varphi] &= y^{2\mu+2k+3} (y^{-1}D)^{k+1} y^{-\mu-3/2} \\ &\quad + (2\mu + 2k + 2) \cdot y^{2\mu+2k+1} (y^{-1}D) y^{-\mu-3/2} \varphi, \end{aligned}$$

we see that $xh_{\mu+1,k}(\varphi)$ equals the right-hand side of (4.1). This completes the proof.

Lemma 4.3. *Let μ be any fixed real number and k a positive integer $\geq -\mu - \frac{1}{2}$. Then, for every $\varphi \in H_\mu(A)$,*

$$h_{\mu,k}(M_\mu N_\mu \varphi) = -x^2 h_{\mu,k}(\varphi).$$

Proof. From Lemmas 4.1 and 4.2.

Similarly, we can show

Lemma 4.4. *Let μ be any fixed real number and k a positive integer $\geq -\mu - \frac{1}{2}$. Then, for every $\varphi \in H_\mu(A)$,*

$$M_\mu N_\mu h_{\mu,k} \varphi = h_{\mu,k}(-x^2 \varphi).$$

Theorem 4.1. *For any real μ and $f \in [H_\mu(A); B]$,*

$$M_\mu N_\mu h'_\mu f = h'_\mu[-x^2 f].$$

Proof. It follows from Lemma 4.3 that

$$\begin{aligned} \langle h'_\mu[-x^2 f], \varphi \rangle &= \langle -x^2 f, h_{\mu,k} \varphi \rangle = \langle f, -x^2 h_{\mu,k} \varphi \rangle = \langle f, h_{\mu,k}(M_\mu N_\mu \varphi) \rangle \\ &= \langle h'_\mu f, M_\mu N_\mu \varphi \rangle = \langle M_\mu N_\mu h'_\mu f, \varphi \rangle. \end{aligned}$$

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