A KERNEL THEOREM ON THE SPACE $[H_\mu \times A; B]$  

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ABSTRACT. Recently, we introduced a space $[H_\mu(A); B]$ which consists of Banach space-valued distributions for which the Hankel transformation is an automorphism (The Hankel transformation of a Banach space-valued generalized function, Proc. Amer. Math. Soc. 119 (1993), 153–163). One of the cornerstones in distribution theory is the kernel theorem of Schwartz which characterizes continuous bilinear functionals as kernel operators. The object of this paper is to prove a kernel theorem which states that for an arbitrary element of $[H_\mu \times A; B]$, it can be uniquely represented by an element of $[H_\mu(A); B]$ and hence of $[H_\mu(A; B)]$. This is motivated by a generalization of Zemanian (Realizability theory for continuous linear systems, Academic Press, New York, 1972) for the product space $D_{RE} \times V$ where $V$ is a Fréchet space. His work is based on the facts that the space $D_{RE}$ is an inductive limit space and the convolution product is well defined in $D_K$. This is not possible here since the space $H_\mu(A)$ is not an inductive limit space. Furthermore, $D(A)$ is not dense in $H_\mu(A)$. To overcome this, it is necessary to apply some results from our aforementioned paper. We close this paper with some applications to integral transformations by a suitable choice of $A$.  

1. INTRODUCTION  

In 1957, L. Schwartz showed that every bilinear continuous functional $f(\varphi, \psi)$ on the space $D(\Omega_1) \times D(\Omega_2)$ may be represented by a linear continuous functional $g$ on the space $D(\Omega_1 \times \Omega_2)$, i.e.,  

$$\quad f(\varphi, \psi) = g(\varphi \times \psi) \quad \text{for } \varphi \in D(\Omega_1), \ \psi \in D(\Omega_2)$$  

where $(\varphi \times \psi)(x_1, x_2) = \varphi(x_1) \cdot \psi(x_2)$ for $x_i \in \Omega_i$, $\ i = 1, 2$.  

Zemanian extended the theorem to a more general type of product space $D_{RE} \times V$. Let $V$ be the strict inductive limit of a sequence $\{v_j\}_{j=1}^\infty$ of Fréchet spaces, and let $\{K_j\}_{j=1}^\infty$ be a sequence of compact intervals in $R^n$ such that $K_j \subset \text{int}(K_{j+1})$ for every $j$ and $\bigcup K_j = R^n$. We let $H \triangleq D_{RE}(V)$ denote the linear space of all smooth $V$-valued functions on $R^n$ having compact supports. We now let $H_j \triangleq D_{K_j}(v_j)$ be the linear space of all $h \in H$ such that $h(R^n) \subset v_j$ and $\text{supp } h \subset K_j$. Thus $H_j \subset H_{j+1}$ for every $j$, and $H = \bigcup_{j=1}^\infty H_j$.  

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Zemanian proved the kernel theorem as follows.

**Theorem 1.1.** Corresponding to every separately continuous bilinear mapping \( f \) of \( D_{R^n} \times V \) into \( B \) there exists one and only one \( g \in [H; B] \) such that

\[
(1) \quad f(\varphi, \psi) = g(\varphi \psi)
\]

for all \( \varphi \in D_{R^n} \) and \( \psi \in V \). \( B \) is a Banach space and \([H; B]\) is the linear space of all continuous linear mappings of \( H \) into \( B \).

In this paper, we consider a new product space \( H_\mu \times A \), where \( H_\mu \) is Zemanian's space for the Hankel transformation and \( A \) is a Banach space. \( H_\mu \) does not have an inductive-limit topology. Moreover, \( D_I \subset H_\mu \), yet \( D_I \) is not dense in \( H_\mu \). \( A \) is a special case of \( V \). We will show that for each element \( f \) of the space \([H_\mu \times A; B]\), there is a unique element \( g \) of \([H_\mu(A); B]\) such that \( f(\varphi, \psi) = g(\varphi \psi) \).

Our notation is similar to that used in [1, 2]. Given any two topological vector spaces \( A \) and \( B \), \([A; B]\) denotes the linear space of all continuous linear mappings of \( A \) into \( B \). The element of \( B \) assigned by \( f \in [A; B] \) to \( \varphi \in A \) is denoted by \((f, \varphi)\). The norm in any Banach space \( B \) is denoted by \( \|\cdot\|_B \). \( R \) and \( C \) are the real and complex number fields. \( I \) is the open interval \((0, \infty)\). Other notation will be introduced as the need arises.

2. MAIN RESULT

Following Zemanian, \( H_\mu(A) \) is defined as follows.

**Definition 2.1.** Let \( x \) be a real variable restricted to \( I \). For each real number \( \mu \), \( \varphi(x) \in H_\mu(A) \) iff it is defined on \( I \), takes its value in \( A \), is smooth, and for each pair of nonnegative integers \( m \) and \( k \)

\[
\gamma^\mu_{m,k}(\varphi) = \sup_{x \in I} \|x^m(x^{-1}D)^k x^{-\mu - 1/2}\varphi(x)\|_A
\]

is finite. \( H_\mu(A) \) is a linear space. The topology of \( H_\mu(A) \) is that generated by \( \{\gamma^\mu_{m,k}\}_{m,k=0}^\infty \).

**Definition 2.2.** \( \varphi(x) \in D_I(A) \) iff \( \varphi \) is defined on \( I \), takes its value in \( A \), is smooth, and for every \( \varphi \) there exists \( b \in I \) such that \( \varphi(x) = 0 \) for \( x \in [b, \infty) \). Let \( \mu D_I(A) = D_I(A) \cap H_\mu(A) \).

Let \( \mu D_I \circ A \) denote the linear space of all \( \varphi \in \mu D_I(A) \) having representation of the form \( \varphi = \sum \theta_k a_k \) where \( \theta_k \in \mu D_I \), \( a_k \in A \), and the summation is over a finite number of terms.

The following result can be found in [3].

**Theorem 2.1.** The space \( \mu D_I \circ A \) is dense in \( H_\mu(A) \) for all \( \mu \in R \).

The following two lemmas can be found in [2].

**Lemma 2.1.** Let \( V, W \) be locally convex spaces and \( \Gamma \) and \( P \) be generating families of seminorms for the topologies of \( V \) and \( W \), respectively. Let \( f \) be a linear mapping of \( V \) into \( W \). The following four assertions are equivalent.

(i) \( f \) is continuous.

(ii) \( f \) is continuous at the origin.

(iii) For every continuous seminorm \( \rho \) on \( W \), there exists a continuous seminorm \( \gamma \) on \( V \) such that \( \rho(f(\theta)) \leq \gamma(\theta) \) for all \( \theta \in V \).
(iv) For every \( \rho \in \mathcal{P} \), there exist a constant \( M > 0 \) and a finite collection \( \{\gamma_1, \gamma_2, \ldots, \gamma_m\} \subset \Gamma \) such that
\[
\rho(f(\theta)) \leq M \max_{1 \leq k \leq m} \gamma_k(\theta) \quad \text{for all } \theta \in V.
\]

**Lemma 2.2.** Let \( W \) be a locally convex space, and let \( \Gamma \) be a generating family of seminorms for the topology of \( W \). Let \( V_1 \) and \( V_2 \) be Fréchet spaces. Let \( \mu_1 \) and \( \mu_2 \) be dense linear subspaces of \( V_1 \) and \( V_2 \), respectively. Supply \( V_1 \times V_2 \) with the product topology and \( Y_1 \times Y_2 \) with the induced topology. Assume that \( f \) is a continuous sesquilinear mapping of \( \mu_1 \times \mu_2 \) into \( W \). The continuity property is equivalent to the condition that, given any \( \rho \in \Gamma \), there is a constant \( M > 0 \) and two continuous seminorms \( \gamma_1 \) and \( \gamma_2 \) on \( V_1 \) and \( V_2 \), respectively, for which
\[
(2) \quad \rho(f(\varphi_1, \varphi_2)) \leq M \gamma_1(\varphi_1) \gamma_2(\varphi_2), \quad \varphi_1 \in \mu_1, \ \varphi_2 \in \mu_2.
\]
We can conclude that there exists a unique continuous sesquilinear mapping \( g \) of \( V_1 \times V_2 \) into \( W \) such that \( g(\varphi_1, \varphi_2) = f(\varphi_1, \varphi_2) \) for all \( \varphi_1 \in \mu_1 \). Moreover, (2) holds again for \( f \) replaced by \( g \) and for all \( \varphi_1 \in V_1 \) and \( \varphi_2 \in V_2 \).

In particular, Lemma 2.2 still works for bilinear \( f \).

Our main result is stated as follows.

**Theorem 2.2.** Corresponding to every continuous bilinear mapping \( f \) of \( H_\mu \times A \) into \( B \), i.e., \( f \in [H_\mu \times A; B] \), there exists one and only one \( g \in [H_\mu(A); B] \) such that
\[
(3) \quad f(\varphi, \psi) = g(\varphi \psi)
\]
for all \( \varphi \in H_\mu \) and \( \psi \in A \).

**Proof.** First of all, let us consider the converse. Since \( g \) is linear, by (3), \( f \) is bilinear. Let \( \varphi_n \to \varphi \) in \( H_\mu \) and \( \psi_n \to \psi \) in \( A \). Then
\[
\gamma_{m,k}(\varphi_n \psi_n - \varphi \psi) \lesssim \sup_{\xi \in I} \|x^m(x^{-1}D)^k x^{-\mu-1/2}(\varphi_n \psi_n - \varphi \psi)\|_A
\leq \sup_{\xi \in I} |x^m(x^{-1}D)^k x^{-\mu-1/2} \varphi_n| \cdot \|\psi_n - \psi\| + \sup_{\xi \in I} |x^m(x^{-1}D)^k x^{-\mu-1/2}(\varphi_n - \varphi)| \cdot \|\psi\|_A \to 0 \quad \text{as } n \to \infty
\]
for \( \sup_{\xi \in I} |x^m(x^{-1}D)^k x^{-\mu-1/2} \varphi_n| \) is bounded by a constant which does not depend on \( n \).

Since \( g \) is continuous on \( H_\mu(A) \), it follows that \( f \) is continuous on \( H_\mu \times A \).

Let \( f \) be given as in Theorem 2.2. For \( \varphi \in \mu D_I \cap A \), we define
\[
g(\varphi) \triangleq \sum_{k=1}^{r} f(\theta_k, a_k) \quad \text{for } \varphi = \sum_{k=1}^{r} \theta_k a_k.
\]

To justify this definition, we have to show that the right-hand side does not depend on the choice of the representation for \( \varphi \). Let \( \varphi = \sum_{i=1}^{s} h_i b_i \) where \( h_i \in \mu D_I \), \( b_i \in A \), be another representation. Now, we find \( l \) linearly independent elements \( e_1, e_2, \ldots, e_l \in A \) such that, for each \( k \) and \( i \),
\[
a_k = \sum_{j=1}^{l} \alpha_k e_j, \quad b_i = \sum_{j=1}^{l} \beta_i e_j
\]
where $\alpha_k, \beta_i \in C$. Upon substituting these sums into the two representations of $\varphi$ and invoking the linear independence of $e_j$, we obtain

$$\sum_{k=1}^{r} \theta_k \alpha_k = \sum_{i=1}^{s} h_i \beta_i.$$ 

Hence,

$$\sum_{k=1}^{r} f(\theta_k, a_k) = \sum_{k=1}^{r} \left( \sum_{j=1}^{l} \alpha_k \ e_j \right) = \sum_{k=1}^{r} \sum_{j=1}^{l} \alpha_k \ f(\theta_k, e_j) = \sum_{j=1}^{l} \left( \sum_{k=1}^{r} \alpha_k \theta_k \right) \ e_j = \sum_{i=1}^{s} h_i \ \sum_{j=1}^{l} \beta_i \ e_j = \sum_{i=1}^{s} f(h_i, b_i)$$

Furthermore, $g$ is linear. Indeed, let $\varphi_1, \varphi_2 \in \mu D_I \odot A$ such that $\varphi_1 = \sum_{k=1}^{r} \theta_k a_k$, $\varphi_2 = \sum_{i=1}^{s} h_i b_i$. Then $\varphi_1 + \varphi_2 = \sum_{k=1}^{r+s} \theta'_k a'_k$, where $\theta'_k = \theta_k$, $a'_k = a_k$ for $1 \leq k \leq r$ and $\theta'_{r+i} = h_i, a'_{r+i} = b_i$ for $1 \leq i \leq s$. Hence,

$$g(\varphi_1 + \varphi_2) = \sum_{k=1}^{r+s} f(\theta'_k, a'_k) = \sum_{k=1}^{r} f(\theta'_k, a'_k) + \sum_{k=r+1}^{r+s} f(\theta'_k, a'_k) = g(\varphi_1) + g(\varphi_2).$$

Obviously $g(\alpha \varphi) = \alpha g(\varphi)$ for $\alpha \in C$.

Now we wish to show that $g$ is uniformly continuous on $\mu D_I \odot A$. Indeed, for arbitrary $\varepsilon > 0$, as long as $\varphi \psi \ (\varphi \in \mu D_I, \ \psi \in A)$ belongs to the balloon $\{\varphi: r_{m,k}^\mu(\varphi) < \frac{1}{M}, \ m = 0, 1, \ldots, m_0, \ k = 0, 1, \ldots, k_0\}$, then there exist $M > 0$ and positive integers $m_0, k_0$ such that

$$\|g(\varphi \psi)\|_B \leq \|f(\varphi, \psi)\|_B \leq M j_{m_0, k_0}(\varphi) \|\psi\|_A < \varepsilon.$$

This follows from Lemma 2.2. Thus $g$ is uniformly continuous at the origin. By Lemma 2.1(iii), $g$ is uniformly continuous on $\mu D_I \odot A$. Since $\mu D_I \odot A$ is dense in $H_\mu(A)$, we can extend $g$ to $H_\mu(A)$ uniquely.

For arbitrary $\varphi \in H_\mu$, Theorem 2.1 enables us to construct $\varphi_n \in \mu D_I$, such that $\varphi_n \to \varphi$ in $H_\mu$. Therefore, from $g(\varphi_n \psi) = f(\varphi_n, \psi), \ \psi \in A$, and letting $n \to \infty$ we get $g(\varphi \psi) = f(\varphi, \psi)$. Such a $g$ is unique. This completes the proof.

We invoke the following theorem (see [3]) to establish the kernel theorem.

**Theorem 2.3.** There is a bijection from $[H_\mu(A); B]$ onto $[H_\mu; [A; B]]$ defined by $(g, \theta) a = (f, \theta a)$ where $a \in A, \ \theta \in H_\mu$, $g \in [H_\mu; [A; B]],$ and $f \in [H_\mu(A); B]$.

**Theorem 2.4 (Kernel Theorem).** Corresponding to every continuous bilinear mapping $f$ of $H_\mu \times A$ into $B$, i.e., $f \in [H_\mu \times A; B], \ \psi \in A$, there exists one and only one $g \in [H_\mu; [A; B]]$ such that $f(\varphi, \psi) = (g, \varphi) \psi$ where $\varphi \in H_\mu$ and $\psi \in A$. 

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3. SOME APPLICATIONS OF THE KERNEL THEOREM

We always take $B = C$ in the following examples.

**Example 1** (Laplace transformation). We choose $A = L^p(0, \infty)$ in Theorem 2.4. Since $[L^p(0, \infty); C] = L^q(0, \infty)$ ($p$, $q$ are conjugate numbers), by applying the kernel theorem, we know that for arbitrary $f \in [H_\mu \times L^p; C]$, there exists a unique $g \in [H_\mu; L^q]$ such that $f(\varphi, \psi) = (g, \varphi)\psi$ where $\varphi \in H_\mu$, $\psi \in L^p$.

Define a family of functions $g_s (s \in I)$ on $H_\mu$ by $(g_s, \varphi) = \varphi(\sqrt{s}x)$, $x \in I$; then $g_s \in [H_\mu; L^q]$. In fact,

$$\int_0^\infty |\varphi(\sqrt{s}x)|^q \, dx = \int_0^\infty |\varphi(u)|^q \frac{2u}{s} \, du < \infty$$

since $\varphi \in H_\mu$. The topology of $H_\mu$ is stronger than that of $L^q$. Hence the assertion follows.

Therefore,

$$f(\varphi, \psi) = (g, \varphi)\psi = \int_0^\infty \varphi(\sqrt{s}x)\psi(x) \, dx.$$

Set $\mu = -\frac{1}{2}$; then $\varphi = e^{-t^2} \in H_{-1/2}$, and

$$f(e^{-t^2}, \psi) = \int_0^\infty e^{-sx}\psi(x) \, dx$$

which is the Laplace transformation on $L^p$.

**Example 2.** We take $A = l^p$ in Theorem 2.4. By using the fact $[l^p; C] = l^q$, it follows that for $f \in [H_\mu \times l^p; C]$, there exists a unique $g \in [H_\mu; l^q]$ such that

$$f(\varphi, \psi) = (g, \varphi)\psi$$

where $\varphi \in H_\mu$ and $\psi \in l^p$.

We define

$$(g_s, \varphi) = \{i^s \varphi(i)\}_{i=1}^{+\infty} \text{ for } s \in R.$$  

Then $g_s \in [H_\mu; l^q]$ since $\varphi(x)$ is a rapid decent function. And

$$f(\varphi, \psi) = \sum_{i=1}^{+\infty} i^s \varphi(i) y_i,$$

where $\psi = \{y_i\}_{i=1}^{+\infty} \in l^p$.

**Example 3** (Mellin transformation). Set

$$A = \{\psi \in C^\infty_f; \exists \text{ polynomial } P_\psi \text{ such that } |x\psi| \leq P_\psi\}.$$  

The norm is defined as

$$||\psi|| = \sup_{x \in l} |e^{-x}x\psi(x)|.$$

It is easily verified that $A$ is a Banach space. We define

$$(g, \varphi)\psi = \int_0^\infty \varphi(x)\psi(x) \, dx$$

where $\psi \in A$. 

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In particular, \( \psi_s = x^{s-1} \in A \) for \( s > 0 \). We get the following Mellin transformation on \( H_\mu (\mu \geq -\frac{1}{2}) \)
\[
f(\varphi, \psi_s) = \int_0^\infty \varphi(x)x^{s-1}dx
\]
where \( s > 0 \).

**Example 4 (Hankel transformation).** Set

\[
A = \{ \psi(x) \in C_1^\infty; \psi \text{ is bounded} \}.
\]

The norm is defined as \( \| \psi \| = \sup_{x \in I} |\psi(x)| \).

It follows that \( A \) is a Banach space. We define

\[
(g, \varphi)\psi = \int_0^\infty \varphi(x)\psi(x)\,dx
\]
where \( \psi(x) \in A \).

In particular, \( \psi_y(x) = \sqrt{xy}J_\mu(xy) \in A \) for \( y > 0 \). We have the Hankel transformation

\[
f(\varphi, \sqrt{xy}J_\mu(xy)) = \int_0^\infty \varphi(x)\sqrt{xy}J_\mu(xy)\,dx.
\]

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