PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 123, Number 1, January 1995

A KERNEL THEOREM ON THE SPACE $[H_\mu \times A; B]$

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(Communicated by Palle E. T. Jorgensen)

ABSTRACT. Recently, we introduced a space $[H_\mu(A); B]$ which consists of Banach space-valued distributions for which the Hankel transformation is an automorphism (*The Hankel transformation of a Banach space-valued generalized function*, Proc. Amer. Math. Soc. **119** (1993), 153–163). One of the cornerstones in distribution theory is the kernel theorem of Schwartz which characterizes continuous bilinear functionals as kernel operators. The object of this paper is to prove a kernel theorem which states that for an arbitrary element of $[H_\mu \times A; B]$, it can be uniquely represented by an element of $[H_\mu(A); B]$ and hence of $[H_\mu; [A; B]]$. This is motivated by a generalization of Zemanian (*Realizability theory for continuous linear systems*, Academic Press, New York, 1972) for the product space $D_{R^n} \times V$ where V is a Fréchet space. His work is based on the facts that the space D_{R^n} is an inductive limit space and the convolution product is well defined in D_{K_j}. This is not possible here since the space $H_\mu(A)$ is not an inductive limit space. Furthermore, $D(A)$ is not dense in $H_\mu(A)$. To overcome this, it is necessary to apply some results from our aforementioned paper. We close this paper with some applications to integral transformations by a suitable choice of A.

1. INTRODUCTION

In 1957, L. Schwartz showed that every bilinear continuous functional $f(\varphi, \psi)$ on the space $D(\Omega_1) \times D(\Omega_2)$ may be represented by a linear continuous functional g on the space $D(\Omega_1 \times \Omega_2)$, i.e.,

$$f(\varphi, \psi) = g(\varphi \times \psi) \quad \text{for } \varphi \in D(\Omega_1), \ \psi \in D(\Omega_2)$$

where $(\varphi \times \psi)(x_1, x_2) = \varphi(x_1) \cdot \psi(x_2)$ for $x_i \in \Omega_i$, $i = 1, 2$.

Zemanian extended the theorem to a more general type of product space $D_{R^n} \times V$. Let V be the strict inductive limit of a sequence $\{v_j\}_{j=1}^\infty$ of Fréchet spaces, and let $\{K_j\}_{j=1}^\infty$ be a sequence of compact intervals in R^n such that $K_j \subset \operatorname{int}(K_{j+1})$ for every j and $\bigcup K_j = R^n$. We let $H \triangleq D_{R^n}(V)$ denote the linear space of all smooth V-valued functions on R^n having compact supports. We now let $H_j \triangleq D_{K_j}(v_j)$ be the linear space of all $h \in H$ such that $h(R^n) \subset v_j$ and $\operatorname{supp} h \subset K_j$. Thus $H_j \subset H_{j+1}$ for every j, and $H = \bigcup_{j=1}^\infty H_j$.

Received by the editors April 16, 1993; this paper was presented on January 15, 1994 at the 100th annual meeting of the American Mathematical Society in Cincinnati, Ohio.

1991 *Mathematics Subject Classification*. Primary 46F10.

Key words and phrases. Kernel Theorem, the Hankel transformation, inductive-limit topology, generalized function.

Zemanian proved the kernel theorem as follows.

Theorem 1.1. *Corresponding to every separately continuous bilinear mapping f of $D_{R^n} \times V$ into B there exists one and only one $g \in [H; B]$ such that*

$$f(\varphi, \psi) = g(\varphi\psi) \tag{1}$$

for all $\varphi \in D_{R^n}$ and $\psi \in V$. B is a Banach space and $[H; B]$ is the linear space of all continuous linear mappings of H into B.

In this paper, we consider a new product space $H_\mu \times A$, where H_μ is Zemanian's space for the Hankel transformation and A is a Banach space. H_μ does not have an inductive-limit topology. Moreover, $D_I \subset H_\mu$, yet D_I is not dense in H_μ. A is a special case of V. We will show that for each element f of the space $[H_\mu \times A; B]$, there is a unique element g of $[H_\mu(A); B]$ such that $f(\varphi, \psi) = g(\varphi\psi)$.

Our notation is similar to that used in [1, 2]. Given any two topological vector spaces A and B, $[A; B]$ denotes the linear space of all continuous linear mappings of A into B. The element of B assigned by $f \in [A; B]$ to $\varphi \in A$ is denoted by (f, φ). The norm in any Banach space B is denoted by $\|\cdot\|_B$. R and C are the real and complex number fields. I is the open interval $(0, \infty)$. Other notation will be introduced as the need arises.

2. Main result

Following Zemanian, $H_\mu(A)$ is defined as follows.

Definition 2.1. Let x be a real variable restricted to I. For each real number μ, $\varphi(x) \in H_\mu(A)$ iff it is defined on I, takes it value in A, is smooth, and for each pair of nonnegative integers m and k

$$\gamma^\mu_{m,k}(\varphi) = \sup_{x \in I} \|x^m (x^{-1}D)^k x^{-\mu-1/2}\varphi(x)\|_A$$

is finite. $H_\mu(A)$ is a linear space. The topology of $H_\mu(A)$ is that generated by $\{\gamma^\mu_{m,k}\}^\infty_{m,k=0}$.

Definition 2.2. $\varphi(x) \in D_I(A)$ iff φ is defined on I, takes its value in A, is smooth, and for every φ there exists $b \in I$ such that $\varphi(x) = 0$ for $x \in [b, \infty)$. Let ${}_\mu D_I(A) \triangleq D_I(A) \cap H_\mu(A)$.

Let ${}_\mu D_I \odot A$ denote the linear space of all $\varphi \in {}_\mu D_I(A)$ having representation of the form $\varphi = \sum \theta_k a_k$ where $\theta_k \in {}_\mu D_I$, $a_k \in A$, and the summation is over a finite number of terms.

The following result can be found in [3].

Theorem 2.1. *The space ${}_\mu D_I \odot A$ is dense in $H_\mu(A)$ for all $\mu \in R$.*

The following two lemmas can be found in [2].

Lemma 2.1. *Let V, W be locally convex spaces and Γ and P be generating families of seminorms for the topologies of V and W, respectively. Let f be a linear mapping of V into W. The following four assertions are equivalent.*

(i) *f is continuous.*
(ii) *f is continuous at the origin.*
(iii) *For every continuous seminorm ρ on W, there exists a continuous seminorm γ on V such that $\rho(f(\theta)) \le \gamma(\theta)$ for all $\theta \in V$.*

(iv) *For every* $\rho \in P$, *there exist a constant* $M > 0$ *and a finite collection* $\{\gamma_1, \gamma_2, \dots, \gamma_m\} \subset \Gamma$ *such that*

$$\rho(f(\theta)) \le M \max_{1\le k\le m} \gamma_k(\theta) \quad \text{for all } \theta \in V.$$

Lemma 2.2. *Let* W *be a locally convex space, and let* Γ *be a generating family of seminorms for the topology of* W. *Let* V_1 *and* V_2 *be Fréchet spaces. Let* μ_1 *and* μ_2 *be dense linear subspaces of* V_1 *and* V_2, *respectively. Supply* $V_1 \times V_2$ *with the product topology and* $\mu_1 \times \mu_2$ *with the induced topology. Assume that* f *is a continuous sesquilinear mapping of* $\mu_1 \times \mu_2$ *into* W. *The continuity property is equivalent to the condition that, given any* $\rho \in \Gamma$, *there is a constant* $M > 0$ *and two continuous seminorms* γ_1 *and* γ_2 *on* V_1 *and* V_2, *respectively, for which*

$$(2) \qquad \rho[f(\varphi_1, \varphi_2)] \le M\gamma_1(\varphi_1)\gamma_2(\varphi_2), \qquad \varphi_1 \in \mu_1, \ \varphi_2 \in \mu_2.$$

We can conclude that there exists a unique continuous sesquilinear mapping g *of* $V_1 \times V_2$ *into* W *such that* $g(\varphi_1, \varphi_2) = f(\varphi_1, \varphi_2)$ *for all* $\varphi_i \in \mu_i$. *Moreover,* (2) *holds again for* f *replaced by* g *and for all* $\varphi_1 \in V_1$ *and* $\varphi_2 \in V_2$.

In particular, Lemma 2.2 still works for bilinear f.

Our main result is stated as follows.

Theorem 2.2. *Corresponding to every continuous bilinear mapping* f *of* $H_\mu \times A$ *into* B, *i.e.,* $f \in [H_\mu \times A; B]$, *there exists one and only one* $g \in [H_\mu(A); B]$ *such that*

$$(3) \qquad f(\varphi, \psi) = g(\varphi\psi)$$

for all $\varphi \in H_\mu$ *and* $\psi \in A$.

Proof. First of all, let us consider the converse. Since g is linear, by (3), f is bilinear. Let $\varphi_n \to \varphi$ in H_μ and $\psi_n \to \psi$ in A. Then

$$\begin{aligned}\gamma^\mu_{m,k}(\varphi_n\psi_n - \varphi\psi) &\triangleq \sup_{x\in I} \|x^m(x^{-1}D)^k x^{-\mu-1/2}(\varphi_n\psi_n - \varphi\psi)\|_A \\ &\le \sup_{x\in I} |x^m(x^{-1}D)^k x^{-\mu-1/2}\varphi_n| \cdot \|\psi_n - \psi\| \\ &\quad + \sup_{x\in I} |x^m(x^{-1}D)^k x^{-\mu-1/2}(\varphi_n - \varphi)| \cdot \|\psi\|_A \to 0 \quad \text{as } n\to\infty\end{aligned}$$

for $\sup_{x\in I} |x^m(x^{-1}D)^k x^{-\mu-1/2}\varphi_n|$ is bounded by a constant which does not depend on n.

Since g is continuous on $H_\mu(A)$, it follows that f is continuous on $H_\mu \times A$.

Let f be given as in Theorem 2.2. For $\varphi \in {}_\mu D_I \odot A$, we define

$$g(\varphi) \triangleq \sum_{k=1}^r f(\theta_k, a_k) \quad \text{for } \varphi = \sum_{k=1}^r \theta_k a_k.$$

To justify this definition, we have to show that the right-hand side does not depend on the choice of the representation for φ. Let $\varphi = \sum_{i=1}^s h_i b_i$ where $h_i \in {}_\mu D_I$, $b_i \in A$, be another representation. Now, we find l linearly independent elements $e_1, e_2, \dots, e_l \in A$ such that, for each k and i,

$$a_k = \sum_{j=1}^l \alpha_{k_j} e_j, \qquad b_i = \sum_{j=1}^l \beta_{i_j} e_j$$

where α_{k_j}, $\beta_{i_j} \in C$. Upon substituting these sums into the two representations of φ and invoking the linear independence of e_j, we obtain

$$\sum_{k=1}^{r} \theta_k \alpha_{k_j} = \sum_{i=1}^{s} h_i \beta_{i_j}.$$

Hence,

$$\begin{aligned}
\sum_{k=1}^{r} f(\theta_k, a_k) &= \sum_{k=1}^{r} f\left(\theta_k, \sum_{j=1}^{l} \alpha_{k_j} e_j\right) = \sum_{k=1}^{r} \sum_{j=1}^{l} \alpha_{k_j} f(\theta_k, e_j) \\
&= \sum_{j=1}^{l} f\left(\sum_{k=1}^{r} \alpha_{k_j} \theta_k, e_j\right) = \sum_{i=1}^{s} f\left(h_i, \sum_{j=1}^{l} \beta_{i_j} e_j\right) \\
&= \sum_{i=1}^{s} f(h_i, b_i)
\end{aligned}$$

Furthermore, g is linear. Indeed, let $\varphi_1, \varphi_2 \in {}_\mu D_I \odot A$ such that $\varphi_1 = \sum_{k=1}^{r} \theta_k a_k$, $\varphi_2 = \sum_{i=1}^{s} h_i b_i$. Then $\varphi_1 + \varphi_2 = \sum_{k=1}^{r+s} \theta'_k a'_k$, where $\theta'_k = \theta_k$, $a'_k = a_k$ for $1 \le k \le r$ and $\theta'_{r+i} = h_i$, $a'_{r+i} = b_i$ for $1 \le i \le s$. Hence,

$$\begin{aligned}
g(\varphi_1 + \varphi_2) &\triangleq \sum_{k=1}^{r+s} f(\theta'_k, a'_k) = \sum_{k=1}^{r} f(\theta'_k, a'_k) + \sum_{k=r+1}^{r+s} f(\theta'_k, a'_k) \\
&= g(\varphi_1) + g(\varphi_2).
\end{aligned}$$

Obviously $g(\alpha\varphi) = \alpha g(\varphi)$ for $\alpha \in C$.

Now we wish to show that g is uniformly continuous on ${}_\mu D_I \odot A$. Indeed, for arbitrary $\varepsilon > 0$, as long as $\varphi\psi$ $(\varphi \in {}_\mu D_I,\ \psi \in A)$ belongs to the balloon $\{\varphi\colon \gamma^\mu_{m,k}(\varphi) < \frac{\varepsilon}{M},\ m = 0, 1, \dots, m_0,\ k = 0, 1, \dots, k_0\}$, then there exist $M > 0$ and positive integers m_0, k_0 such that

$$\|g(\varphi\psi)\|_B \le \|f(\varphi, \psi)\|_B \le M\gamma^\mu_{m_0, k_0}(\varphi)\|\psi\|_A < \varepsilon.$$

This follows from Lemma 2.2. Thus g is uniformly continuous at the origin. By Lemma 2.1(iii), g is uniformly continuous on ${}_\mu D_I \odot A$. Since ${}_\mu D_I \odot A$ is dense in $H_\mu(A)$, we can extend g to $H_\mu(A)$ uniquely.

For arbitrary $\varphi \in H_\mu$, Theorem 2.1 enables us to construct $\varphi_n \in {}_\mu D_I$, such that $\varphi_n \to \varphi$ in H_μ. Therefore, from $g(\varphi_n\psi) = f(\varphi_n, \psi)$, $\psi \in A$, and letting $n \to \infty$ we get $g(\varphi\psi) = f(\varphi, \psi)$. Such a g is unique. This completes the proof. □

We invoke the following theorem (see [3]) to establish the kernel theorem.

Theorem 2.3. *There is a bijection from* $[H_\mu(A); B]$ *onto* $[H_\mu; [A; B]]$ *defined by* $(g, \theta)a = (f, \theta a)$ *where* $a \in A$, $\theta \in H_\mu$, $g \in [H_\mu; [A; B]]$, *and* $f \in [H_\mu(A); B]$.

Theorem 2.4 (Kernel Theorem). *Corresponding to every continuous bilinear mapping* f *of* $H_\mu \times A$ *into* B, *i.e.,* $f \in [H_\mu \times A; B]$, *there exists one and only one* $g \in [H_\mu; [A; B]]$ *such that* $f(\varphi, \psi) = (g, \varphi)\psi$ *where* $\varphi \in H_\mu$ *and* $\psi \in A$.

3. Some applications of the kernel theorem

We always take $B = C$ in the following examples.

Example 1 (Laplace transformation). We choose $A = L^p(0, \infty)$ in Theorem 2.4. Since $[L^p(0, \infty); C] = L^q(0, \infty)$ (p, q are conjugate numbers), by applying the kernel theorem, we know that for arbitrary $f \in [H_\mu \times L^p; C]$, there exists a unique $g \in [H_\mu; L^q]$ such that $f(\varphi, \psi) = (g, \varphi)\psi$ where $\varphi \in H_\mu$, $\psi \in L^p$.

Define a family of functions g_s ($s \in I$) on H_μ by $(g_s, \varphi) = \varphi(\sqrt{sx})$, $x \in I$; then $g_s \in [H_\mu; L^q]$. In fact,

$$\int_0^\infty |\varphi(\sqrt{sx})|^q \, dx = \int_0^\infty |\varphi(u)|^q \frac{2u}{s} \, du < \infty$$

since $\varphi \in H_\mu$. The topology of H_μ is stronger than that of L^q. Hence the assertion follows.

Therefore,

$$f(\varphi, \psi) = (g, \varphi)\psi = \int_0^\infty \varphi(\sqrt{sx})\psi(x)\, dx.$$

Set $\mu = -\frac{1}{2}$; then $\varphi = e^{-t^2} \in H_{-1/2}$, and

$$f(e^{-t^2}, \psi) = \int_0^\infty e^{-sx}\psi(x)\, dx$$

which is the Laplace transformation on L^p.

Example 2. We take $A = l^p$ in Theorem 2.4. By using the fact $[l^p; C] = l^q$, it follows that for $f \in [H_\mu \times l^p; C]$, there exists a unique $g \in [H_\mu; l^q]$ such that

$$f(\varphi, \psi) = (g, \varphi)\psi$$

where $\varphi \in H_\mu$ and $\psi \in l^p$.

We define

$$(g_s, \varphi) = \{i^s\varphi(i)\}_{i=1}^{+\infty} \quad \text{for } s \in R.$$

Then $g_s \in [H_\mu; l^q]$ since $\varphi(x)$ is a rapid decent function. And

$$f(\varphi, \psi) = \sum_{i=1}^{\infty} i^s \varphi(i) y_i$$

where $\psi = \{y_i\}_{i=1}^{\infty} \in l^p$.

Example 3 (Mellin transformation). Set

$$A = \{\psi \in C_I^\infty;\ \exists \text{ polynomial } P_\psi \text{ such that } |x\psi| \le P_\psi\}.$$

The norm is defined as

$$\|\psi\| = \sup_{x\in I} |e^{-x} x\psi(x)|.$$

It is easily verified that A is a Banach space. We define

$$(g, \varphi)\psi = \int_0^\infty \varphi(x)\psi(x)\, dx$$

where $\psi \in A$.

In particular, $\psi_s = x^{s-1} \in A$ for $s > 0$. We get the following Mellin transformation on H_μ $(\mu \geq -\frac{1}{2})$

$$f(\varphi, \psi_s) = \int_0^\infty \varphi(x) x^{s-1} dx$$

where $s > 0$.

Example 4 (Hankel transformation). Set

$$A = \{\psi(x) \in C_I^\infty; \ \psi \text{ is bounded}\}.$$

The norm is defined as $\|\psi\| = \sup_{x\in I} |\psi(x)|$.

It follows that A is a Banach space. We define

$$(g, \varphi)\psi = \int_0^\infty \varphi(x)\psi(x)\, dx$$

where $\psi(x) \in A$.

In particular, $\psi_y(x) = \sqrt{xy} J_\mu(xy) \in A$ for $y > 0$. We have the Hankel transformation

$$f(\varphi, \sqrt{xy} J_\mu(xy)) = \int_0^\infty \varphi(x) \sqrt{xy} J_\mu(xy)\, dx.$$

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