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THE PRODUCT OF r^{-k} AND $\nabla\delta$ ON $\mathbb{R}^m$

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ABSTRACT. In the theory of distributions, there is a general lack of definitions for products and powers of distributions. In physics (Gasiorowicz (1967), page 141), one finds the need to evaluate δ^2 when calculating the transition rates of certain particle interactions and using some products such as $(1/x)\cdot\delta$. In 1990, Li and Fisher introduced a "computable" delta sequence in an m-dimensional space to obtain a noncommutative neutrix product of r^{-k} and $\triangle\delta$ ($\triangle$ denotes the Laplacian) for any positive integer k between 1 and $m-1$ inclusive. Cheng and Li (1991) utilized a net $\delta_\epsilon(x)$ (similar to the $\delta_n(x)$) and the normalization procedure of $\mu(x)x_+^\lambda$ to deduce a commutative neutrix product of r^{-k} and δ for any positive real number k. The object of this paper is to apply Pizetti's formula and the normalization procedure to derive the product of r^{-k} and $\nabla\delta$ (∇ is the gradient operator) on $\mathbb{R}^m$. The nice properties of the δ-sequence are fully shown and used in the proof of our theorem.

Keywords and phrases. Pizetti's formula, delta sequence, neutrix limit and distribution.

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1. Introduction. Let $\rho(x)$ be a fixed infinitely differentiable function with the following properties:

(i) $\rho(x)\geq 0$,
(ii) $\rho(x)=0$ for $|x|\geq 1$,
(iii) $\rho(x)=\rho(-x)$,
(iv) $\int_{-1}^{1}\rho(x)dx=1$.

The function $\delta_n(x)$ is defined by $\delta_n(x)=n\rho(nx)$ for $n=1,2,\ldots$. It follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$.

Now let $\mathcal{D}$ be the space of infinitely differentiable functions of a single variable with compact support and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. Then if f is an arbitrary distribution in $\mathcal{D}'$, we define

$$f_n(x)=(f*\delta_n)(x)=\big(f(t),\delta_n(x-t)\big) \tag{1.1}$$

for $n=1,2,\ldots$. It follows that $\{f_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $f(x)$ in $\mathcal{D}'$.

The following definition for the noncommutative neutrix product $f\cdot g$ of two distributions f and g in $\mathcal{D}'$ was given by Fisher in [2].

DEFINITION 1.1. Let f and g be distributions in $\mathcal{D}'$ and let $g_n=g*\delta_n$. We say that the neutrix product $f\cdot g$ of f and g exists and is equal to h if

$$N-\lim_{n\to\infty}(fg_n,\phi)=(h,\phi) \tag{1.2}$$

for all functions ϕ in $\mathcal{D}$, where N is the neutrix (see [6]) having domain $N'=\{1,2,\dots\}$ and range N'' the real numbers, with negligible functions that are finite linear sums of the functions

$$n^\lambda \ln^{r-1} n,\ \ln^r n \quad (\lambda>0,\ r=1,2,\dots) \tag{1.3}$$

and all functions of n which converge to zero in the normal sense as n tends to infinity.

The product of Definition 1.1 is not symmetric and hence $f\cdot g\neq g\cdot f$ in general.

Extending definitions of products from one-dimensional space $\mathbb{R}$ to m-dimensional space $\mathbb{R}^m$ by using appropriate delta-sequences has recently been an interesting topic in distribution theory. In order to define a neutrix product of two separable forms of distributions in $\mathcal{D}'_m$ (an m-dimensional space of distributions), Fisher and Li provided the following definition in [3].

Definition 1.2. Let $f(x)$ and $g(x)$ be distributions in $\mathcal{D}'_m$, where $x=(x_1,x_2,\dots,x_m)$. The function $g_n(x)$ is defined by

$$g_n(x)=(g*\delta_n)(x), \tag{1.4}$$

where $\delta_n(x)=\delta_{n_1}(x_1)\cdots\delta_{n_m}(x_m)=n_1\rho(n_1x_1)\cdots n_m\rho(n_mx_m)$. Hence $\{\delta_n(x)\}$ is a regular sequence converging to the Dirac delta-function $\delta(x)$. The neutrix product $f\cdot g$ is defined to be equal to h if

$$N-\lim_{n_1\to\infty}\cdots N-\lim_{n_m\to\infty}(fg_n,\phi)=(h,\phi) \tag{1.5}$$

for all ϕ in $\mathcal{D}_m$ (an m-dimensional Schwartz space).

With Definition 1.2, Fisher and Li (also in [3]) show the following results.

Let

$$x^r=x_1^{-r_1}\cdots x_m^{-r_m} \qquad \text{and} \qquad \delta^{(p)}(x)=\delta^{(p_1)}(x_1)\cdots\delta^{(p_m)}(x_m). \tag{1.6}$$

Then the noncommutative neutrix product $x^{-r}\cdot\delta^{(p)}(x)$ exists and

$$x^{-r}\cdot\delta^{(p)}(x)=\frac{(-1)^r p!}{(p+r)!}\delta^{(p+r)}(x) \tag{1.7}$$

for $r_1,\dots,r_m=1,2,\dots$ and $p_1,\dots,p_m=0,1,2,\dots$.

The following work on the commutative neutrix product of distributions on $\mathbb{R}^m$ is due to Cheng and Li (see [1]).

Let $\mathbb{R}^m$ be an Euclidean space with dimension m, and let $\rho(s)$, for $s\in\mathbb{R}$, be a fixed infinitely differentiable function having the properties:

(i) $\rho(s)\geq 0$,

(ii) $\rho(s)=0$ for $|s|\geq 1$,

(iii) $\rho(s)=\rho(-s)$,

(iv) $\int_{|x|\le 1}\rho(|x|^2)dx = 1,\ x\in\mathbb{R}^m$.

The property (iv) in the spherical coordinates is represented as

(v) $\Omega_m\int_0^1\rho(s^2)s^{m-1}ds = 1$,

where Ω_m is the surface area of the unit sphere in $\mathbb{R}^m$. Putting $\delta_\epsilon(x) = \epsilon^{-m}\rho(|\epsilon^{-1}x|^2)$, where $\epsilon > 0$, it follows that ϵ-net $\{\delta_\epsilon(x)\}$ converges to the Dirac delta-function $\delta(x)$.

Definition 1.3. Let f and g be arbitrary distributions in $\mathcal{D}'_m$ and let

$$f_\epsilon = f*\delta_\epsilon, \qquad g_\epsilon = g*\delta_\epsilon. \tag{1.8}$$

We say that the neutrix product $f\cdot g$ of f and g exists and is equal to h on the open domain $\Omega\subseteq\mathbb{R}^m$ if the neutrix limit

$$N-\lim_{\epsilon\to 0^+}\frac{1}{2}\{(f\cdot g_\epsilon,\phi)+(g\cdot f_\epsilon,\phi)\} = (h,\phi) \tag{1.9}$$

for all test functions ϕ with compact support contained in the domain Ω, where N is the neutrix having domain $N' = \mathbb{R}^+$, the positive numbers, and range $N'' = \mathbb{R}$, the real numbers, with negligible functions that are linear sums of the functions

$$\epsilon^{-\lambda}\ln^{r-1}\epsilon, \qquad \ln^r\epsilon \tag{1.10}$$

for $\lambda > 0$ and $r = 1,2,\ldots$, and all functions of ϵ which converge to zero as ϵ tends to zero.

In this paper, we would like to give a definition for the noncommutative neutrix product $f\cdot g$ of two distributions f and g in $\mathcal{D}'_m$ by applying the below δ-sequence. This definition is particularly useful in computing products of distributions of the variable r (radius).

From now on we let $\rho(s)$ be a fixed infinitely differentiable function defined on $\mathbb{R}^+ = [0,\infty)$ having the properties:

(i) $\rho(s)\ge 0$,

(ii) $\rho(s) = 0$ for $s\ge 1$,

(iii) $\int_{\mathbb{R}^m}\delta_n(x)dx = 1$,

where $\delta_n(x) = C_m n^m\rho(n^2r^2)$ and C_m is the constant satisfying (iii).

It follows that $\{\delta_n(x)\}$ is a regular δ-sequence of infinitely differentiable functions converging to $\delta(x)$ because of the above three conditions. The following definition will be applied in Section 3 to evaluate our product mentioned in the abstract.

Definition 1.4. Let f and g be distributions in $\mathcal{D}'(m)$ and let

$$g_n(x) = (g*\delta_n)(x) = (g(x-t),\delta_n(t)), \tag{1.11}$$

where $t = (t_1,t_2,\ldots,t_m)$. We say that the neutrix product $f\cdot g$ of f and g exists and is equal to h if

$$N-\lim_{n\to\infty}(fg_n,\phi) = (h,\phi), \tag{1.12}$$

where $\phi\in\mathcal{D}_m$.

2. The distribution r^{λ}. Let $r = (x_1^2 + \cdots + x_m^2)^{1/2}$ and consider the functional r^{λ} (see [5]) defined by

$$(r^{\lambda}, \phi) = \int_{\mathbb{R}^m} r^{\lambda} \phi(x)\, dx, \tag{2.1}$$

where $\operatorname{Re}\lambda > -m$ and $\phi(x) \in \mathcal{D}_m$. Because the derivative

$$\frac{\partial}{\partial\lambda}(r^{\lambda}, \phi) = \int r^{\lambda} \ln r\, \phi(x)\, dx \tag{2.2}$$

exists, the functional r^{λ} is an analytic function of λ for $\operatorname{Re}\lambda > -m$.

For $\operatorname{Re}\lambda \le -m$, we should use the following identity (2.4) to define its analytic continuation. For $\operatorname{Re}\lambda > 0$, we could deduce

$$\triangle(r^{\lambda+2}) = (\lambda+2)(\lambda+m) r^{\lambda} \tag{2.3}$$

simply by calculating the left-hand side, where $\triangle$ is the Laplacian operator. By iteration we find, for any integer k, that

$$r^{\lambda} = \frac{\triangle^k r^{\lambda+2k}}{(\lambda+2)\cdots(\lambda+2k)(\lambda+m)\cdots(\lambda+m+2k-2)}. \tag{2.4}$$

On making substitution of spherical coordinates in (2.1), we come to

$$(r^{\lambda}, \phi) = \int_0^{\infty} r^{\lambda} \left\{ \int_{r=1} \phi(r\omega)\, d\omega \right\} r^{m-1}\, dr, \tag{2.5}$$

where $d\omega$ is the hypersurface element on the unit sphere. The integral appearing in the above integrand can be written in the form

$$\int_{r=1} \phi(r\omega)\, d\omega = \Omega_m S_{\phi}(r), \tag{2.6}$$

where Ω_m is the hypersurface area of the unit sphere imbedded in Euclidean space of m dimensions, and $S_{\phi}(r)$ is the mean value of ϕ on the sphere of radius r.

It was proved in [5] that $S_{\phi}(r)$ is infinitely differentiable for $r \ge 0$, has bounded support, and that

$$S_{\phi}(r) = \phi(0) + a_1 r^2 + a_2 r^4 + \cdots + a_k r^{2k} + o(r^{2k}) \tag{2.7}$$

for any positive integer k. From (2.5) and (2.6), we obtain

$$(r^{\lambda}, \phi) = \Omega_m \int_0^{\infty} r^{\lambda+m-1} S_{\phi}(r)\, dr \tag{2.8}$$

which indicates the application of $\Omega_m x_+^{\mu}$ with $\mu = \lambda + m - 1$ to the testing function $S_{\phi}(r)$. Using the following Laurent series for x_+^{λ} about $\lambda = -k$,

$$x_+^{\lambda} = \frac{(-1)^{k-1}\delta^{(k-1)}(x)}{(k-1)!(\lambda+k)} + x_+^{-k} + (\lambda+k) x_+^{-k} \ln x + \cdots \tag{2.9}$$

we could show that the residue of $(r^\lambda, \phi(x))$ at $\lambda = -m-2k$ for nonnegative integer k is given by

$$\Omega_m \frac{(\delta^{(2k)}, \phi(x))}{(2k)!} = \Omega_m \frac{S_\phi^{(2k)}(0)}{(2k)!}. \tag{2.10}$$

On the other hand, the residue of the function r^λ of (2.4) for the same value of λ is

$$\frac{\Omega_m \triangle^k \delta(x)}{2^k k! m(m+2)\cdots(m+2k-2)}. \tag{2.11}$$

(See [5].) Therefore we get

$$S_\phi^{(2k)}(0) = \frac{(2k)! \triangle^k \phi(0)}{2^k k! m(m+2)\cdots(m+2k-2)}. \tag{2.12}$$

This result can be used to write out the Taylor's series for $S_\phi(r)$, namely

$$\begin{aligned} S_\phi(r) &= \phi(0) + \frac{1}{2!} S_\phi''(0) r^2 + \cdots + \frac{1}{(2k)!} S_\phi^{(2k)}(0) r^{2k} + \cdots \\ &= \sum_{k=0}^{\infty} \frac{\triangle^k \phi(0) r^{2k}}{2^k k! m(m+2)\cdots(m+2k-2)} \end{aligned} \tag{2.13}$$

which is the well-known Pizetti's formula.

3. The product r^{-k} and $\nabla\delta$.

The following normalization procedure is needed in the proof of our theorem regarding the product of r^{-k} and $\nabla\delta$.

The distribution $\mu(x)x_+^\lambda$. Let $\mu(x)$ be an infinitely differentiable function on $\mathbb{R}^+$ having properties:

(i) $\mu(x) \geq 0$,
(ii) $\mu(0) \neq 0$,
(iii) $\mu(x) = 0$ for $x \geq 1$.

Let $\phi(x)$ be a testing function. Then the functional

$$(\mu(x)x_+^\lambda, \phi) = \int_0^1 \mu(x) x^\lambda \phi(x)\, dx \tag{3.1}$$

is regular for $\operatorname{Re}\lambda > -1$. It can be extended to the domain $\operatorname{Re}\lambda > -n-1$ ($\lambda \neq -1, -2, \ldots$) by analytic continuation along Gelfand and Shilov (see [5]):

$$\begin{aligned} (\mu(x)x_+^\lambda, \phi) &= \int_0^1 \mu(x) x^\lambda \phi(x)\, dx \\ &= \sum_{k=1}^{n} \frac{\phi^{(k-1)}(0)\mu(\theta_{k-1})}{(k-1)!(\lambda+k)} \\ &\quad + \int_0^1 \mu(x) x^\lambda \left[\phi(x) - \phi(0) - x\phi'(0) - \cdots - \frac{x^{n-1}}{(n-1)!}\phi^{(n-1)}(0) \right] dx \end{aligned} \tag{3.2}$$

on applying the mean value theorem with $0 < \theta_{k-1} < 1$ for $1 \leq k \leq n$. This means that the generalized function $\mu(x)x_+^\lambda$ is well defined for $\lambda \neq -1, -2, \ldots$.

We thus normalize the value of the functional $(\mu(x)x_+^\lambda,\phi)$ at $-n$ by

$$(\mu(x)x_+^{-n},\phi)=\sum_{k=1}^{n-1}\frac{\phi^{(k-1)}(0)\mu(\theta_{k-1})}{(k-1)!(-n+k)} + \int_0^1 \mu(x)x^{-n}\left[\phi(x)-\phi(0)-x\phi'(0)-\cdots-\frac{x^{n-1}}{(n-1)!}\phi^{(n-1)}(0)\right]dx. \tag{3.3}$$

Theorem 3.1. *The noncommutative neutrix product $r^{-k}\cdot\nabla\delta$ exists. Furthermore*

$$r^{-2k}\nabla\delta=-\frac{1}{2^{k+1}(k+1)!(m+2)\cdots(m+2k)}\sum_{i=1}^{m}(x_i\triangle^{k+1}\delta), \qquad r^{1-2k}\cdot\nabla\delta=0, \tag{3.4}$$

where k is a positive integer and ∇ is the gradient operator.

Proof. Since $\nabla=\partial/\partial x_1+\cdots+\partial/\partial x_m=\sum_{i=1}^m\partial/\partial x_i$, we have

$$\nabla\delta_n(x)=2C_m n^{m+2}\sum_{i=1}^m\rho'(n^2r^2)x_i=2C_m n^{m+2}\rho'(n^2r^2)\sum_{i=1}^m x_i. \tag{3.5}$$

We note that r^{-k} is a locally summable function on $\mathbb{R}^m$ for $k=1,2,\dots,m-1$. Therefore

$$\begin{aligned} I=(r^{-k}\cdot\nabla\delta_n,\phi)&=\int_{\mathbb{R}^m}r^{-k}\nabla\delta_n(x)\phi(x)dx\\ &=2C_m n^{m+2}\sum_{i=1}^m\int_{\mathbb{R}^m}r^{-k}\rho'(n^2r^2)x_i\phi(x)dx. \end{aligned}\tag{3.6}$$

On changing to spherical polar coordinates and then making the substitution $t=nr$, we arrive at

$$\begin{aligned} I&=2C_m\Omega_m n^{m+2}\sum_{i=1}^m\int_0^{1/n}r^{m-k-1}\rho'(n^2r^2)S_{\psi_i}(r)dr\\ &=2C_m\Omega_m n^{k+2}\sum_{i=1}^m\int_0^1 t^{m-k-1}\rho'(t^2)S_{\psi_i}\left(\frac{t}{n}\right)dt, \end{aligned}\tag{3.7}$$

where $\psi_i(x)=x_i\phi(x)$. By Taylor's formula, we obtain

$$S_{\psi_i}(r)=\sum_{j=0}^{k+1}\frac{S_{\psi_i}^{(j)}(0)}{j!}r^j+\frac{S_{\psi_i}^{(k+2)}(0)}{(k+2)!}r^{k+2}+\frac{S_{\psi_i}^{(k+3)}(\zeta r)}{(k+3)!}r^{k+3}, \tag{3.8}$$

where $0 < \zeta < 1$. Hence

$$\begin{aligned}
I &= 2C_m\Omega_m n^{m+2} \sum_{i=1}^{m}\sum_{j=0}^{k+1} \frac{S_{\psi_i}^{(j)}(0)}{j!}\int_0^{1/n} r^{m-k-1}\rho'(n^2r^2)r^j\,dr \\
&\quad + 2C_m\Omega_m n^{m+2}\sum_{i=1}^{m}\int_0^{1/n} r^{m-k-1}\rho'(n^2r^2)\frac{S_{\psi_i}^{(k+2)}(0)}{(k+2)!}r^{k+2}\,dr \\
&\quad + 2C_m\Omega_m n^{m+2}\sum_{i=1}^{m}\int_0^{1/n} r^{m-k-1}\rho'(n^2r^2)\frac{S_{\psi_i}^{(k+3)}(\zeta r)}{(k+3)!}r^{k+3}\,dr \\
&= I_1 + I_2 + I_3,
\end{aligned} \tag{3.9}$$

respectively. Employing $t = nr$ again, we get

$$I_1 = 2C_m\Omega_m\sum_{i=1}^{m}\sum_{j=0}^{k+1} n^{k+2-j}\frac{S_{\psi_i}^{(j)}(0)}{j!}\int_0^1 t^{m+j-k-1}\rho'(t^2)\,dt \tag{3.10}$$

whence

$$N-\lim_{n\to\infty} I_1 = 0 \tag{3.11}$$

as for

$$I_2 = 2C_m\Omega_m\sum_{i=1}^{m}\frac{S_{\psi_i}^{(k+2)}(0)}{(k+2)!}\int_0^1 t^{m+1}\rho'(t^2)\,dt \tag{3.12}$$

integrating by parts, we have

$$\begin{aligned}
2C_m\Omega_m\int_0^1 t^{m+1}\rho'(t^2)\,dt &= C_m\Omega_m\int_0^1 t^m\,d\rho(t^2) \\
&= -C_m\Omega_m\cdot m\int_0^1 t^{m-1}\rho(t^2)\,dt \\
&= -m\int_{\mathbb{R}^m}\delta_n(x)\,dx = -m.
\end{aligned} \tag{3.13}$$

Hence

$$I_2 = -m\sum_{i=1}^{m}\frac{S_{\psi_i}^{(k+2)}(0)}{(k+2)!} = -\frac{m}{(k+2)!}\sum_{i=1}^{m}S_{\psi_i}^{(k+2)}(0). \tag{3.14}$$

Putting

$$M = \sup\left\{\left|S_{\psi_i}^{(k+3)}(r)\right| : r\in\mathbb{R}^+ \text{ and } 1\le i\le m\right\}, \tag{3.15}$$

we obtain that

$$|I_3| \le 2C_m\Omega_m\frac{mM}{n(k+3)!}\int_0^1 t^{m+2}\left|\rho'(t^2)\right|dt \longrightarrow 0 \quad \text{as } n\longrightarrow\infty. \tag{3.16}$$

Hence it follows from above that

$$N-\lim_{n\to\infty} I = I_2 = -\frac{m}{(k+2)!}\sum_{i=1}^{m}S_{\psi_i}^{(k+2)}(0). \tag{3.17}$$

We now turn our attention to the product $r^{-k}\cdot\nabla\delta$ for $k\geq m$. Note that, in this case, the functional r^{-k} is not locally summable. We assume $k=m+q$ for $q=0,1,2,\ldots$, then $-k+m-1\leq -1$. We apply the regularization in (3.3) to I of (3.7) to deduce

$$\begin{aligned}
I = 2C_m\Omega_m n^{k+2}\sum_{i=1}^{m}\Bigg\{&\sum_{j=1}^{q=k-m}\frac{S_{\psi_i}^{(j-1)}(0)\rho'(\theta_{j-1}^2)}{(j-1)!(m-k-1+j)} \quad (=I_1)\\
&+\int_0^1\rho'(t^2)t^{m-k-1}\\
&\times\left[S_{\psi_i}\left(\frac{t}{n}\right)-S_{\psi_i}(0)-\cdots-\frac{t^q}{n^q q!}S_{\psi_i}^{(q)}(0)\right]dt\Bigg\} \quad (=I_2)\\
= I_1+I_2,&
\end{aligned}\tag{3.18}$$

respectively.

Clearly,

$$N-\lim_{n\to\infty} I_1 = 0. \tag{3.19}$$

Applying Taylor's theorem, we obtain

$$\begin{aligned}
I_2 = 2C_m\Omega_m n^{k+2}\sum_{i=1}^{m}\int_0^1\rho'(t^2)t^{m-k-1}\Bigg[&\frac{t^{q+1}}{n^{q+1}(q+1)!}S_{\psi_i}^{(q+1)}(0)+\cdots\\
&+\frac{t^{q+m+2}}{n^{q+m+2}(q+m+2)!}S_{\psi_i}^{(q+m+2)}(0)\\
&+\frac{t^{q+m+3}}{n^{q+m+3}(q+m+3)!}S_{\psi_i}^{(q+m+3)}\left(\frac{\theta t}{n}\right)\Bigg]dt,
\end{aligned}\tag{3.20}$$

where $0<\theta<1$. Similarly, we could prove

$$\begin{aligned}
N-\lim_{n\to\infty} I_2 &= 2C_m\Omega_m\int_0^1\rho'(t^2)t^{m+1}dt\sum_{i=1}^{m}\frac{S_{\psi_i}^{(q+m+2)}(0)}{(q+m+2)!}\\
&= -\frac{m}{(q+m+2)!}\sum_{i=1}^{m}S_{\psi_i}^{(q+m+2)}(0)\\
&= -\frac{m}{(k+2)!}\sum_{i=1}^{m}S_{\psi_i}^{(k+2)}(0)
\end{aligned}\tag{3.21}$$

because the other terms vanish upon taking their N-limits.

Using Pizetti's formula, we get

$$S_{\psi_i}^{(k+2)}(0) = \begin{cases}\dfrac{(2l+2)!\triangle^{l+1}\psi_i(0)}{2^{l+1}(l+1)!m(m+2)\cdots(m+2l)} & \text{if } k=2l \text{ for } l=0,1,\ldots,\\ 0 & \text{if } k=2l-1 \text{ for } l=1,2,\ldots.\end{cases}\tag{3.22}$$

This completes the proof. □

Remark 3.2. The multiplication of x_i and $\triangle^{k+1}\delta$ in our theorem is well defined since

$$(x_i\triangle^{k+1}\delta,\phi) = (\delta,\triangle^{k+1}(x_i\phi)). \tag{3.23}$$

In particular, we have the following

$$\frac{1}{x^2}\cdot\delta'(x) = \frac{1}{6}\delta^{(3)}(x) \tag{3.24}$$

by setting $m = 1$ and $k = 1$ in the theorem, which identically coincides with equation (1.7) with $m = 1$, $r = 2$, and $p = 1$.

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