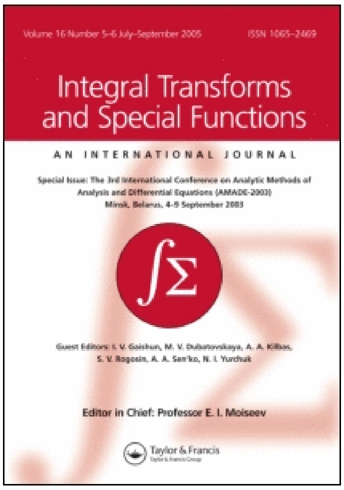

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A kernel theorem from the Hankel transform in Banach spaces

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A kernel theorem from the Hankel transform in Banach spaces

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One of the cornerstones in distribution theory was the kernel theorem of Schwartz in 1957, which showed that every bilinear continuous functional $f(\varphi, \psi)$ on the space $D(\Omega_1) \times D(\Omega_2)$ can be represented by a linear continuous functional g on the space $D(\Omega_1 \times \Omega_2)$. Zemanian [Zemanian, A.H., 1972, *Realizability Theory for Continuous Linear Systems* (New York: Academic Press).] extended the theorem to a more general type of product space $D_{R^n} \times V$ where V is a Fréchet space. His work was based on the fact that the space D_{R^n} is an inductive limit space and the convolution product is well defined in D_{K_j}. In this paper, we study a new product space $H_\mu \times A$, where H_μ is the testing space for the classical Hankel transform and A is a Banach space, and derive the kernel theorem which is considered as a unified form for integral transforms such as Mellin, Laplace, Hankel and the K-transform by choosing particular Banach spaces for A. Using the Hankel transform of arbitrary order and pseudo-integrals, we find a generalized solution in H'_μ for the following differential equation:

$$\frac{d^2}{dx^2}u - \left(1 + \frac{4\mu^2 - 1}{4x^2}\right)u = -\sqrt{x}J_\mu(x) \tag{1}$$

where $J_\mu(x)$ is the Bessel function of first kind and order $\mu \neq -1, -2, -3, \ldots$.

Keywords: Hankel transform; Zemanian space; Kernel theorem; Generalized function

2000 Mathematics Subject Classification: Primary: 46F10

1. Introduction

The natural framework for a realizability theory of continuous linear systems in physics is distribution theory. Since the signals in the systems of interest take their values in Banach spaces, Zemanian introduced Banach-space-valued distributions in ref. [2] for this purpose. This is more general than that of scalar distributions.

Let m be an n-tuple each of whose components is either a non-negative integer or ∞. Also, let K be a compact subset in R^n and A is a Banach space. The space $D_K^m(A)$ denotes the linear space of all functions ϕ from R^n into A such that supp $\phi \subset K$, and for every integer vector $k \in R^n$ with $0 \leq k \leq m$, $\phi^{(k)}$ is continuous. $D_K^m(A)$ is assigned the topology generated by the

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collection $\{\gamma_k \mid 0 \le k \le m\}$ of seminorms, where

$$\gamma_k(\phi) \triangleq \sup_{t \in K} \|\phi^{(k)}(t)\|_A.$$

When all the components of m are ∞, we denote $D_K^m(A)$ by $D_K(A)$. Moreover, we set $D_K^m(C) = D_K^m$ where C is a complex plane and $D_K(C) = D_K$. Now let $\{K_j\}_{j=1}^{\infty}$ be compact subsets of R^n such that $K_j \subset K_{j+1}$, $\cup_{j=1}^{\infty} K_j = R^n$, and every compact subset $J \subset R^n$ is contained in some K_j. We define

$$D^m(A) = D^m_{R^n}(A) = \bigcup_{j=1}^{\infty} D^m_{K_j}(A).$$

This space, which is independent of choices of K_j, possesses the inductive-limit topology. Furthermore, it has the closure property since $D^m_{K_j}(A)$ is complete.

Given any two topological vector spaces A and B, $[A; B]$ denotes the linear space of all continuous linear mappings of A into B. The element of B assigned by $f \in [A; B]$ to $\phi \in A$ is denoted by $\langle f, \phi \rangle$. $[A; B]$ is supplied with the topology of uniform convergence on bounded sets in A. $\|\cdot\|_B$ denotes the norm in any Banach space B and I is the open interval $(0, \infty)$. Other notations will be introduced as the need arises.

Applying the interpolation theory, Zemanian described the following local structure property.

THEOREM 1.1 *Let* $f \in [D^m(A); B]$ *and* K *be a compact interval in* R^n. *Then there exists an integer* $p \in R^n$ *with* $0 \le p \le m$ *and a continuous* $[A; B]$*-valued function* h *on* K *such that, for all* $\phi \in D_K^{m+[2]}(A)$,

$$\langle f, \phi \rangle = \int_K h(t) D^{p+[2]} \phi(t)\, \mathrm{d}t.$$

In general, p *and* h *depend on* f *and* K.

Then the following kernel theorem in proved.

THEOREM 1.2 *If* $\mathcal{T}^m$ *and* $\mathcal{T}^m(A)$ *are normal spaces (i.e.* D *is dense in* $\mathcal{T}^m$ *and* $D(A)$ *is* $\mathcal{T}^m(A)$*, respectively), then there exists a bijetion from* $[\mathcal{T}^m(A); B]$ *onto* $[\mathcal{T}^m; [A : B]]$ *defined by*

$$\langle g, \psi \rangle a \triangleq \langle f, \psi a \rangle \quad \psi \in \mathcal{T}^m, \;\; a \in A$$

where $g \in [\mathcal{T}^m; [A; B]]$ *and* $f \in [\mathcal{T}^m(A); B]$.

Tiwari [3] followed the method of Zemanian in defining Banach-space-valued distributions for which a Mellin transform can be used. Several properties including a Mellin-type convolution theorem were proved. These results are similar to those of Zemanian [2].

To make this paper as self-contained as possible, we introduce a dense subspace ${}_{\mu}D_I(A)$ (which is proposed by Koh and Li [8, 9]) of $H_\mu(A)$. It does not have an inductive-limit topology. The local structure theorem is no longer discussed in $[H_\mu(A); B]$. However, with a different method, we show that there is still a bijection from $[H_\mu(A); B]$ onto $[H_\mu; [A; B]]$ from which we derive a kernel theorem as a 'root' of a wide range of integral transforms by applying two lemmas given in ref. [2]. Furthermore, we provide a direct and simple proof for the inverse Hankel transform which states that for any fixed real number μ and any positive integer k

such that $\mu + k \geq -1/2$, we have $h_{\mu,\,k} = h^{-1}_{\mu,\,k}$. Finally we solve equation (1) in the abstract by the Hankel transform of arbitrary order and pseudo-integrals, and show that

$$h_{-1/2}(M_{-1/2}\phi) = -\left(\frac{2}{\pi}\right)^{1/2} \phi(0^+) + yh_{1/2}(\phi)$$

which re-describes a formula for the case $\mu = -1/2$ in Zemanian's book [1].

2. The spaces $[H_\mu(A); B]$ and $[H_\mu; [A; B]]$

In order to extend the classical Hankel transform of Zemanian to Banach-space-valued generalized functions, we define $H_\mu(A)$ as follows.

DEFINITION 2.1 *Let A be a Banach space and x be a real variable restricted to I. For each real number μ, we say any $\phi(x) \in H_\mu(A)$ iff it is a smooth (infinitely differentiable) mapping from I into A, and for each pair of non-negative integers m and k*

$$\gamma^\mu_{m,k}(\phi) \triangleq \sup_{x\in I} (1+x^2)^m \left\| \left(\frac{1}{x}D\right)^k [x^{-\mu-1/2}\phi(x)] \right\|_A < \infty.$$

Obviously, $H_\mu(A)$ is a linear space. The topology of $H_\mu(A)$ is that generated by $\{\gamma^\mu_{m,\,k}\}^\infty_{m,\,k=0}$.

Let

$${}_\mu D_I(A) = \{\phi \in H_\mu(A) | \text{supp}\phi \text{ bounded}\} \subset H_\mu(A).$$

THEOREM 2.1 *The subspace ${}_\mu D_I(A)$ is dense in $H_\mu(A)$ for all $\mu \in R$.*

Proof Let $\lambda(x) \in D_I$ such that $\lambda(x) = 1$ for $0 < x \leq 1$ and $\lambda(x) = 0$ for $x \geq 2$ (obviously this function can be constructed by a convolution). For arbitrary $\phi(x) \in H_\mu(A)$ and each pair of non-negative integers m and k, we consider

$$\begin{aligned} &x^m(x^{-1}D)^k x^{-\mu-(1/2)} \left[\lambda\left(\frac{x}{N}\right)\phi(x) - \phi(x)\right] \\ &\quad = x^{m+1} \sum_{\nu=0}^{k} \binom{k}{\nu} (x^{-1}D)^{k-\nu} x^{-\mu-(1/2)}\phi \frac{(x^{-1}D)^\nu[\lambda(x/N) - 1]}{x}. \end{aligned}$$

Therefore,

$$\begin{aligned} &\sup_{x\in I} \left\| x^m(x^{-1}D)^k x^{-\mu-(1/2)} \left(\lambda\left(\frac{x}{N}\right)\phi(x) - \phi(x)\right) \right\|_A \\ &\quad \leq \sum_{\nu=0}^{k} \binom{k}{\nu} \sup_{x\in I} \left\| x^{m+1}(x^{-1}D)^{k-\nu} x^{-\mu-(1/2)}\phi \right\|_A \cdot \sup_{x\geq N} \left| \frac{(x^{-1}D)^\nu[\lambda(x/N) - 1]}{x} \right|. \end{aligned}$$

It follows from $\phi \in H_\mu(A)$ that $\sup_{x\in I} \|(x^{-1}D)^{k-\nu}x^{-\mu-(1/2)}\phi\|_A$ is bounded.

Since $\lambda(x)$ and its derivatives are bounded, it follows that

$$\sup_{x\geq N} \left| \frac{(x^{-1}D)^\nu[\lambda(x/N) - 1]}{x} \right| \longrightarrow 0 \quad \text{as } N \longrightarrow \infty,$$

for fixed k and $0 \leq \nu \leq k$, whence our assertion. ∎

We should point out that Zemanian in ref. [4] introduced the testing function spaces $B_{\mu,b}$ and B_μ while defining the Hankel transform on the space Y_μ. Indeed, $B_\mu = {}_\mu D_I(C)$ in terms of set equality.

$H_\mu(A)$ is not a ρ-type testing function space in the sense of Zemanian [2]. To see this, we choose $\phi(x) = x^{\mu+(1/2)}e^{-x^2}a_0$, $a_0 \in A$ and $a_0 \neq 0$. Then for all ψ which is smooth from I into A with compact support contained in I, $\gamma_{0,0}^{\mu}(\phi - \psi) \geq \|a_0\|/2 > 0$. This means the balloon

$$\left\{\theta | \theta \in H_\mu(A) \quad \text{and} \quad \gamma_{0,0}^{\mu}(\phi - \psi) \leq \frac{\|a_0\|}{3}\right\}$$

does not contain any element of $D^m(A)$. Thus our result is true.

The following lemmas will be used subsequently (see refs. [1] and [2]).

LEMMA 2.1 *Let V, W be locally convex spaces, and Γ and P generate families of seminorms for topologies of V and W, respectively. Let f be a linear mapping of V into W. Then the following assertions are equivalent:*

(1) *f is continuous.*
(2) *f is continuous at the origin.*
(3) *For every continuous seminorm ρ on W, there exists a continuous seminorm γ on V such that $\rho(f(\theta)) \leq \gamma(\theta)$ for all θ.*
(4) *For every $\rho \in P$, there exists a constant $M > 0$ and a finite collection $\{\gamma_1, \gamma_2, \ldots, \gamma_m\} \subset \Gamma$ such that*

$$\rho(f(\theta)) \leq M \max_{0 \leq k \leq m} \gamma_k(\theta)$$

for all $\theta \in V$.

LEMMA 2.2 *For $\mu \geq -1/2$, the conventional Hankel transform h_μ is an automorphim on $H_\mu(A)$.*

Proof Consider all integrals in a Banach space and the rest is as in ref. [1, Theorem 5.4-1, p. 141]. ■

THEOREM 2.2 *Every $f \in [H_\mu(A); B]$ uniquely defines a $g \in [H_\mu; [A; B]]$ by the equation*

$$\langle g, \theta \rangle a \triangleq \langle f, \theta a \rangle \quad \theta \in H_\mu, \ \ a \in A$$

for all $\mu \in R$.

Proof Fixing upon some $\theta \in H_\mu$ we define a mapping j_θ of A into B by $j_\theta a = \langle f, \theta a \rangle$ for all $a \in A$. It readily follows that j_θ is linear. By Lemma 2.1 (4), there exist positive integers

m_0, k_0 and constant $M > 0$ such that

$$\|j_\theta a\|_B = \|\langle f, \theta a\rangle\|_B \leq M \max_{\substack{0\leq k\leq k_0 \\ 0\leq m\leq m_0}} \gamma^{\mu}_{m,k}(\theta a),$$

where

$$\gamma^{\mu}_{m,k}(\theta a) = \sup_{x\in I} \left\| x^m (x^{-1}D)^k x^{-\mu-(1/2)} \theta a \right\|_A = \|a\|_A \sup_{x\in I} \left| x^m (x^{-1}D)^k x^{-\mu-(1/2)} \theta \right|.$$

Hence

$$\|j_\theta a\|_B \leq M \|a\|_A \max_{\substack{0\leq k\leq k_0 \\ 0\leq m\leq m_0}} \gamma^{\mu}_{m,k}(\theta)$$

and

$$\|j_\theta\|_{[A;B]} \leq M \max_{\substack{0\leq k\leq k_0 \\ 0\leq m\leq m_0}} \gamma^{\mu}_{m,k}(\theta). \tag{2}$$

Next, set $\langle g, \theta\rangle \stackrel{\Delta}{=} j_\theta$. This uniquely defines g as a mapping from H_μ into $[A; B]$. g is linear because, for any $a \in A,\ \alpha,\ \beta \in C$ and $\theta,\ \psi \in H_\mu$

$$\begin{aligned} \langle g, \alpha\theta + \beta\psi\rangle a &= \langle f, \alpha\theta a + \beta\psi a\rangle = \alpha\langle f, \theta a\rangle + \beta\langle f, \psi a\rangle \\ &= (\alpha\langle f, \theta\rangle + \beta\langle g, \psi\rangle)a. \end{aligned}$$

Moreover, inequality (2) implies that g is continuous. ■

We let

$${}_\mu D_I \bigodot A = \left\{ \sum_{k=1}^{r} \theta_k a_k | \theta_k \in {}_\mu D_I, \quad a_k \in A \text{ and } r \text{ is finite} \right\}.$$

Obviously, ${}_\mu D_I \bigodot A \subset H_\mu(A)$ and further it leads to Theorem 2.3.

THEOREM 2.3 *The space ${}_\mu D_I \bigodot A$ is dense in $H_\mu(A)$ for $\mu \geq -1/2$.*

Proof Let $\lambda(x)$ be defined as in the proof of Theorem 2.1. For $\phi \in {}_\mu D_I(A)$, we first show that

$$\lambda\left(\frac{x}{N}\right) h_\mu(\phi) \longrightarrow h_\mu \quad \text{in } H_\mu(A) \quad \text{as } N \longrightarrow \infty$$

for all $\mu \in R$.

The following equation will be used in the proof (see ref. [1]):

$$(-1)^{m+k}y^m(y^{-1}D)^k y^{-\mu-(1/2)}h_\mu(\phi)(y) = \int_0^\infty x^{2\mu+2k+m+1}[(x^{-1}D)^m x^{-\mu-(1/2)}\phi(x)]\frac{J_{\mu+k+m}(xy)}{(xy)^{\mu+k}}\mathrm{d}x. \quad (3)$$

Hence

$$\sup_{x\in I}\left\|x^m(x^{-1}D)^k x^{-\mu-(1/2)}h_\mu(\phi)\left[\lambda\left(\frac{x}{N}\right)-1\right]\right\|_A \leq \sum_{\nu=0}^{k}\binom{k}{\nu}\sup_{x\geq N}\left|\frac{(x^{-1}D)^\nu[\lambda(\frac{x}{N})-1]}{x}\right|\sup_{x\in I}\left\|x^{m+1}(x^{-1}D)^{k-\nu}x^{-\mu-(1/2)}h_\mu(\phi)\right\|_A.$$

By Theorem 2.1,

$$\sup_{x\geq N}\left|\frac{(x^{-1}D)^\nu[\lambda(x/N)-1]}{x}\right| \longrightarrow 0 \quad \text{as } N\longrightarrow\infty$$

for fixed k and $0\leq\nu\leq k$.

On using equation (3) and noting that $J_{\mu+k-\nu+m+1}/(xy)^{\mu+k-\nu}$ is bounded, say by $B_{k,\nu,m}$, we get

$$\sup_{x\in I}\|x^{m+1}(x^{-1}D)^{k-\nu}x^{-\mu-(1/2)}h_\mu(\phi)\|_A = \sup_{x\in I}\left\|\int_0^\infty y^{2\mu+2(k-\nu)+m+2}[(y^{-1}D)^{m+1}y^{-\mu-1/2}\phi(y)]\frac{J_{\mu+k-\nu+m+1}(xy)}{(xy)^{\mu+k-\nu}}\mathrm{d}y\right\|_A.$$

Choose a positive integer n such that

$$y^{2\mu+2(k-\nu)+m+2}\leq(1+y^2)^n \quad \text{for all } y\in I,$$

we have

$$\sup_{y\in I}\|y^{2\mu+2(k-\nu)+m+2}[(y^{-1}D)^{m+1}y^{-\mu-(1/2)}\phi(y)]\|_A \leq \sup_{y\in I}\|(1+y^2)^n[(y^{-1}D)^{m+1}y^{-\mu-(1/2)}\phi(y)]\|_A.$$

Since $\phi\in {}_\mu D_I(A)$, there exists $b\in I$ such that $\phi(x)=0$ for $x\in[b,\infty)$. It follows that

$$\sup_{x\in I}\|x^{m+1}(x^{-1}D)^{k-\nu}x^{-\mu-(1/2)}h_\mu(\phi)\|_A \leq B_{k,\nu,m}b\sup_{y\in I}\|(1+y^2)^n[(y^{-1}D)^{m+1}y^{-\mu-(1/2)}\phi(y)]\|_A$$

is finite. Therefore,

$$\lambda\left(\frac{x}{N}\right)h_\mu(\phi)\longrightarrow h_\mu(\phi) \text{ in } H_\mu(A) \quad \text{as } N\longrightarrow\infty.$$

Secondly, we prove that ${}_\mu D_I\odot A$ is dense in $H_\mu(A)$ for $\mu\geq-1/2$. For positive integer m_1, we have

$$\sqrt{xy}J_\mu(xy) = \sum_{j=0}^{m_1}\frac{(xy)^{1/2}(-1)^j(xy/2)^{\mu+2j}}{j!\,\Gamma(\mu+j+1)} + \sum_{j=m_1+1}^{+\infty}\frac{(xy)^{1/2}(-1)^j(xy/2)^{\mu+2j}}{j!\,\Gamma(\mu+j+1)}.$$

For every $\phi \in {}_{\mu}D_I(A)$, the term

$$T_{N,m_1} = \lambda\left(\frac{x}{N}\right)\int_0^{+\infty}\phi(t)\sum_{j=0}^{m_1}\frac{(xt)^{1/2}(-1)^j(xt/2)^{\mu+2j}}{j!\,\Gamma(\mu+j+1)}\,\mathrm{d}t \quad N,\ m_1 = 1,2,\ldots$$

belongs to ${}_{\mu}D_I \bigodot A$ since $\mu \geq -1/2$. Now,

$$\begin{aligned}
T_{N,m_1} &- \int_0^{\infty}\phi(t)\sqrt{xt}\,J_\mu(xt)\,\mathrm{d}t\\
&= T_{N,m_1} - \lambda\left(\frac{x}{N}\right)\int_0^{\infty}\phi(t)\sqrt{xt}\,J_\mu(xt)\,\mathrm{d}t\\
&\quad + \lambda\left(\frac{x}{N}\right)\int_0^{\infty}\phi(t)\sqrt{xt}\,J_\mu(xt)\,\mathrm{d}t - \int_0^{\infty}\phi(t)\sqrt{xt}\,J_\mu(xt)\,\mathrm{d}t.
\end{aligned}$$

By what we have proved, for arbitrary $\epsilon > 0$, there exists an N_1 such that for $N \geq N_1$, we have

$$\sup_{x\in I}\left\|x^m(x^{-1}D)^k x^{-\mu-(1/2)}\left[\lambda\left(\frac{x}{N}\right)h_\mu(\phi) - h_\mu(\phi)\right]\right\|_A < \frac{\epsilon}{2}.$$

Fixing $N(\geq N_1)$, then

$$\lambda\left(\frac{x}{N}\right)\left[\sum_{j=0}^{m_1}\frac{(xt)^{1/2}(-1)^j(xt/2)^{\mu+2j}}{j!\,\Gamma(\mu+j+1)} - \sqrt{xt}\,J_\mu(xt)\right]$$

and its derivatives with respect to x converge to zero uniformly on every compact subset of I. It has a uniformly bounded support. Therefore it converges in the sense of Schwartz, whose topology is stronger than that of H_μ (see ref. [1]). It follows that there exists an $L \in I$ such that as long as $m_1 \geq L$, for all $t \leq b$,

$$\sup_{x\in I}\left|x^m(x^{-1}D)^k x^{\mu-(1/2)}\lambda\left(\frac{x}{N}\right)\left[\sum_{j=0}^{m_1}\frac{(xt)^{1/2}(-1)^j(xt/2)^{\mu+2j}}{j!\,\Gamma(\mu+j+1)} - \sqrt{xt}\,J_\mu(xt)\right]\right| \leq \frac{\epsilon}{2M_1},$$

where $M_1 = b\sup_{t\in I}\|\phi(t)\|_A$. If $M_1 = 0$, then there is nothing to be proved. Therefore,

$$\sup_{x\in I}\left\|x^m(x^{-1}D)^k x^{-\mu-(1/2)}\left[T_{N,m_1} - \int_0^{\infty}\phi(t)\sqrt{xt}\,J_\mu(xt)\,\mathrm{d}t\right]\right\|_A < \epsilon$$

provided $N \geq N_1$ and $m_1 \geq L$.

Since h_μ is an automorphism on $H_\mu(A)$ for $\mu \geq -1/2$ by Lemma 2.2, and the fact that ${}_{\mu}D_I(A)$ is dense in $H_\mu(A)$, it follows that $h_\mu({}_{\mu}D_I(A))$ is dense in $H_\mu(A)$. Our assertion follows directly from the fact that ${}_{\mu}D_I \bigodot A$ is dense in $h_\mu({}_{\mu}D_I(A))$. ■

THEOREM 2.4 *There is a bijection from* $[H_\mu(A); B]$ *onto* $[H_\mu; [A; B]]$ *defined by*

$$\langle g, \theta\rangle a = \langle f, \theta a\rangle$$

where $a \in A$, $g \in [H_\mu; [A; B]]$ *and* $f \in [H_\mu(A); B]$, $\theta \in H_\mu$ *for* $\mu \geq -1/2$.

Proof By Theorem 2.2, every $f \in [H_\mu(A); B]$ uniquely defines a $g \in [H_\mu; [A; B]]$ by the equation

$$\langle g, \theta \rangle a \triangleq \langle f, \theta a \rangle \quad \text{for all } \mu \in R.$$

Let us consider the converse. For every $\phi \in {}_\mu D_I \bigodot A$, we define

$$\langle f, \phi \rangle = \sum \langle g, \theta_k \rangle a_k \quad \text{for } \phi = \sum \theta_k a_k.$$

It follows from the definition (well-defined) that f is linear on ${}_\mu D_I \bigodot A$. We show that f is continuous. Indeed, for arbitrary $\epsilon > 0$, as long as $\theta a (\theta \in {}_\mu D_I$ and $a \in A)$ belongs to the balloon $\{\phi\colon \gamma^\mu_{m,k}(\phi) < \epsilon/M, m = 0, 1, 2, \ldots, m_0, k = 0, 1, 2, \ldots, k_0\}$. M, m_0, k_0 are defined as follows. We infer that

$$\|\langle f, \theta a \rangle\|_B = \|\langle g, \theta \rangle a\|_B \le \|a\|_A \cdot \|\langle g, \theta \rangle\|_{[A;B]}.$$

By Lemma 2.1 (4), there exist $M > 0$, positive integers $m_0,\ k_0$ such that

$$\|\langle f, \theta a \rangle\|_B \le \|a\|_A \cdot M \max_{\substack{0 \le k \le k_0 \\ 0 \le m \le m_0}} \gamma^\mu_{m,k}(\theta) < M \cdot \frac{\epsilon}{M} = \epsilon.$$

Therefore, f is continuous at the origin. By Lemma 2.1 (2), f is continuous on ${}_\mu D_I \bigodot A$. According to Theorem 2.3, ${}_\mu D_I \bigodot A$ is dense in $H_\mu(A)$ for $\mu \ge -1/2$. Thus our assertion is true. ■

3. The Hankel transform on $H_\mu(A)$

We shall use the following differential and integral operators proposed by Zemanian [1].

$$N_\mu \phi(x) \triangleq x^{\mu+(1/2)} D x^{-\mu-(1/2)} \phi(x)$$

$$M_\mu \phi(x) \triangleq x^{-\mu-(1/2)} D x^{\mu+(1/2)} \phi(x)$$

$$N_\mu^{-1} \phi(x) \triangleq x^{\mu+(1/2)} \int_\infty^x t^{-\mu-(1/2)} \phi(t)\, \mathrm{d}t.$$

LEMMA 3.1 *N_μ is a continuous linear mapping of $H_\mu(A)$ into $H_{\mu+1}(A)$.*
Indeed, $\gamma^{\mu+1}_{m,k}(N_\mu \phi) = \gamma^\mu_{m,k+1}(\phi)$ for every $\phi \in H_\mu(A)$ and every choice of m and k.

LEMMA 3.2 *N_μ^{-1} is a continuous linear mapping of $H_{\mu+1}(A)$ into $H_\mu(A)$.*

Proof It follows from

$$\begin{aligned}(x^{-1}D)^k x^{-\mu-(1/2)} N_\mu^{-1} \phi(x) &= (x^{-1}D)^k x^{-\mu-(1/2)} x^{\mu+(1/2)} \int_\infty^x t^{-\mu-(1/2)} \phi(t)\, \mathrm{d}t \\ &= (x^{-1}D)^{k-1} x^{-\mu-(3/2)} \phi(x),\end{aligned}$$

where $\phi(x) \in H_{\mu+1}(A)$ and k is a fixed positive integer. ■

Let $\mu \in R$ and positive integer k such that $\mu + k \geq -1/2$. Assume that $\phi \in H_\mu(A)$. Define $h_{\mu,k}$ on $H_\mu(A)$ by

$$\Phi(x) = h_{\mu,k}[\phi(y)] \triangleq (-1)^k x^{-k} h_{\mu+k} N_{\mu+k-1} \cdots N_{\mu+1} N_\mu \phi(y).$$

Let $\Phi(x) \in H_\mu(A)$ and define $h_{\mu,k}^{-1}$ on $H_\mu(A)$ by

$$\phi(y) = h_{\mu,k}^{-1}[\Phi(x)] \triangleq (-1)^k N_\mu^{-1} N_{\mu+1}^{-1} \cdots N_{\mu+k-1}^{-1} h_{\mu+k} x^k \Phi(x).$$

THEOREM 3.1 *$h_{\mu,k}$ is an automorphism on $H_\mu(A)$. Its inverse is $h_{\mu,k}^{-1}$, and $h_{\mu,k} = h_\mu$ if $\mu \geq -1/2$.*

Proof By Lemma 3.1 and Lemma 3.2, $\phi \to N_\mu N_{\mu+1} \cdots N_{\mu+k-1}\phi$ is an isomorphism from $H_\mu(A)$ onto $H_{\mu+k}(A)$.

By Lemma 2.2, $h_{\mu+k}$ is an automorphism on $H_{\mu+k}(A)$ for $\mu + k \geq -1/2$. It follows from $\gamma_{m,k}^{\mu}(x^{-k}\phi) = \gamma_{m,k}^{\mu+k}(\phi)$ that $\phi \to x^{-k}\phi$ is an isomorphism from $H_{\mu+k}(A)$ onto $H_\mu(A)$. Therefore $h_{\mu,k}$ is an automorphism on $H_\mu(A)$. Similarly, $h_{\mu,k}^{-1}$ is an automorphism on $H_\mu(A)$, and is inverse to $h_{\mu,k}$ because $h_{\mu+k}^{-1} = h_{\mu+k}$ and the inverse of $N_{\mu+k-1} \cdots N_\mu$ is $N_\mu^{-1} \cdots N_{\mu+k-1}^{-1}$.

To prove the last statement, let $\phi(y) \in H_\mu(A)$, $\mu \geq -1/2$ and consider $k = 1$;

$$\begin{aligned} h_{\mu,1}\phi &= -x^{-1} h_{\mu+1} N_\mu \phi = -x^{-1} \int_0^\infty y^{\mu+(1/2)} [D_y y^{-\mu-(1/2)} \phi(y)] \sqrt{xy} J_{\mu+1}(xy)\, \mathrm{d}y \\ &= -x^{-1} \sqrt{xy} J_{\mu+1}(xy)\phi(y)|_0^\infty + \int_0^\infty \phi(y) \sqrt{xy} J_\mu(xy)\, \mathrm{d}y. \end{aligned}$$

Since $\phi(y)$ is of rapid descent and $\sqrt{xy} J_{\mu+1}(xy)$ is bounded as $y \to \infty$, while $\phi(y) = O(y^{\mu+(1/2)})$ and $\sqrt{xy} J_{\mu+1}(xy) = O(y^{\mu+3/2})$ as $y \to 0^+$, the limit terms are zero for $\mu \geq -1/2$. Thus $h_{\mu,1}\phi = h_\mu \phi$. By induction, $h_{\mu,k} = h_\mu$ for $\mu \geq -1/2$. ■

Note that the definition of $h_{\mu,k}$ is independent of the choice of k so long as $k + \mu \geq -1/2$. Indeed if $k > p \geq -\mu - (1/2)$, then $h_{\mu+p,k-p} = h_{\mu+p}$ by Theorem 3.1, hence

$$\begin{aligned} h_{\mu,k}\phi &= (-1)^k x^{-k} h_{\mu+k} N_{\mu+k-1} \cdots N_\mu \phi \\ &= (-1)^p x^{-p} (-1)^{k-p} x^{-(k-p)} h_{\mu+p+k-p} N_{\mu+p+k-p-1} \cdots N_{\mu+p} N_{\mu+p-1} \cdots N_\mu \phi \\ &= (-1)^p x^{-p} h_{\mu+p,k-p} N_{\mu+p-1} \cdots N_\mu \phi \\ &= (-1)^p x^{-p} h_{\mu+p} N_{\mu+p-1} \cdots N_\mu \phi \\ &= h_{\mu,p}\phi. \end{aligned}$$

Zemanian claimed in ref. [1] that $h_{u,k} \neq h_{u,k}^{-1}$ when $\mu < -1/2$. However, he did not give any counterexample. Kerr [5] introduced complex fractional powers of Hankel transforms h_μ^α in H_μ to show that $h_\mu = h_\mu^{-1}$. In the present work, we are able to give a direct and simple proof that $h_{u,k} = h_{u,k}^{-1}$ for $\mu \in R$ with the help of the following identity [1]:

$$D_x x^{-\mu} J_\mu(xy) = -y x^{-\mu} J_{\mu+1}(xy) \tag{4}$$

LEMMA 3.3 *$N_\mu h_{\mu,k}(\phi) = h_{\mu+1,k}(-y\phi)$ for $\phi \in H_\mu(A)$.*

Proof By definition,

$$\begin{aligned} h_{u,\,k}\phi &= (-1)^k x^{-k} h_{\mu+k} N_{\mu+k-1}\cdots N_{\mu+1}N_\mu\phi(y) \\ &= (-1)^k x^{-k} h_{\mu+k} y^{\mu+k+(1/2)}(y^{-1}D)^k y^{-\mu-(1/2)}\phi(y) \\ &= (-1)^k x^{-k}\int_0^\infty \sqrt{xy}J_{\mu+k}(xy)y^{\mu+k+(1/2)}(y^{-1}D)^k y^{-\mu-(1/2)}\phi(y)\,\mathrm{d}y. \end{aligned}$$

It follows that

$$N_\mu h_{u,k}(\phi) = (-1)^k\int_0^\infty N_\mu x^{-k}\sqrt{xy}J_{\mu+k}(xy)y^{\mu+k+(1/2)}(y^{-1}D)^k y^{-\mu-(1/2)}\phi(y)\,\mathrm{d}y.$$

By equation (4), we have

$$\begin{aligned} N_\mu x^{-k}(xy)^{1/2}J_{\mu+k}(xy) &= x^{\mu+(1/2)}Dx^{-\mu-k}J_{\mu+k}(xy)y^{1/2} \\ &= x^{-k}\sqrt{xy}J_{\mu+1+k}(xy)(-y). \end{aligned}$$

Hence,

$$\begin{aligned} N_\mu h_{u,k}(\phi) &= (-1)^k x^{-k}\int_0^\infty \sqrt{xy}J_{\mu+1+k}(xy)y^{\mu+1+k+(1/2)}(y^{-1}D)^k y^{-\mu-1-(1/2)}[-y\phi(y)]\,\mathrm{d}y \\ &= h_{\mu+1,k}(-y\phi). \end{aligned}$$

■

THEOREM 3.2 *Let μ be any fixed real number and let k be any positive integer such that $\mu+k \geq -1/2$. Then $h_{\mu,k} = h^{-1}_{\mu,k}$.*

Proof By Lemma 3.3, we have

$$N_\mu h_{\mu,k}(\phi) = h_{\mu+1,k}(-y\phi).$$

Applying $N_{\mu+1}$ to both sides, we obtain

$$N_{\mu+1}N_\mu h_{\mu,k}(\phi) = N_{\mu+1}h_{\mu+1,k}(-y\phi) = h_{\mu+2,k}[(-1)^2y^2\phi].$$

Repeating this process, we get

$$N_{\mu+k-1}\cdots N_{\mu+1}N_\mu h_{\mu,k}(\phi) = h_{\mu+k,k}[(-1)^k y^k\phi].$$

Since $\mu+k\geq -1/2$, $h_{\mu+k,k}[(-1)^k y^k\phi] = h_{\mu+k}[(-1)^k y^k\phi]$. Therefore,

$$N_{\mu+k-1}\cdots N_{\mu+1}N_\mu h_{\mu,k}(\phi) = (-1)^k h_{\mu+k}(y^k\phi)$$

and we finally come to

$$h_{\mu,k}(\phi) = (-1)^k N_\mu^{-1}N_{\mu+1}^{-1}\cdots N_{\mu+k-1}^{-1}h_{\mu+k}y^k\phi(y) = h^{-1}_{\mu,k}(\phi).$$

This completes the proof. ■

DEFINITION 3.1 *Let $\mu \in R$, k positive integer such that $\mu + k \geq -1/2$. For any $f \in [H_\mu(A); B]$, the generalized Hankel transform $h'_\mu f$ is defined by*

$$\langle h'_\mu f, \phi\rangle = \langle f, h_{\mu,k}\phi\rangle, \quad \phi \in H_\mu(A).$$

by Theorems 3.1 *and* 3.2 *and the fact that h'_μ is the adjoint operator of $h_{\mu,k}$ on $H_\mu(A)$, we have the following corollary.*

COROLLARY 3.1 *h'_μ is an automorphism on $[H_\mu(A); B]$ for all $\mu \in R$.*

Applying operator $T \triangleq N_{\mu+k-1} \cdots N_\mu$, we have Theorem 3.3.

THEOREM 3.3 *Let A, B be two Banach spaces. There is a bijection from $[H_\mu(A); B]$ onto $[H_\mu; [A; B]]$ defined by*

$$\langle g, \theta\rangle a = \langle f, \theta a\rangle$$

where $a \in A, \theta \in H_\mu$, $g \in [H_\mu; [A; B]]$ and $f \in [H_\mu(A); B]$, $\mu \in R$.

Proof For any $\mu \in R$, we choose positive integer k such that $\mu + k \geq -1/2$. The operator T is an isomorphism from ${}_\mu D_I \bigodot A$ onto ${}_{\mu+k} D_I \bigodot A$ which is dense in $H_{\mu+k}(A)$. Also T is an isomorphism from $H_\mu(A)$ onto $H_{\mu+k}(A)$. Therefore, ${}_\mu D_I \bigodot A$ is dense in $H_\mu(A)$. By Theorems 2.3 and 2.4, there is a bijection from $[H_\mu(A); B]$ onto $[H_\mu; [A; B]]$ satisfying the preceeding equation. ■

4. The kernel theorem

The following lemma can be found in ref. [2]

LEMMA 4.1 *Let W be locally convex space and let Γ be a generating family of seminorms for the topology of W. Let V_1 and V_2 be Fréchet spaces. Let μ_1 and μ_2 be dense linear subspaces of V_1 and V_2, respectively. Supply $V_1 \times V_2$, with the product topology and $\mu_1 \times \mu_2$ with the induced topology. Assume that f is a continuous sesquilinear*[†] *mapping of $\mu_1 \times \mu_2$ into W. The continuity property is equivalent to the condition that, given any $\rho \in \Gamma$, there is a constant $M > 0$ and two continuous seminorms γ_1 and γ_2 on V_1 and V_2, respectively, for which*

$$\rho[f(\varphi_1, \varphi_2)] \leq M\gamma_1(\varphi_1)\gamma_2(\varphi_2), \quad \varphi_1 \in \mu_1, \ \varphi_2 \in \mu_2. \tag{5}$$

We can conclude that there exists a unique continuous sesquilinear mapping g of $V_1 \times V_2$ into W such that $g(\varphi_1, \varphi_2) = f(\varphi_1, \varphi_2)$ for all $\varphi_i \in \mu_i$. Moreover, inequality (5) holds again for f replaced by g and for all $\varphi_1 \in V_1$ and $\varphi_2 \in V_2$.

In particular, Lemma 4.1 holds for bilinear f. Our main result is stated as follows:

THEOREM 4.1 *Corresponding to every continuous bilinear mapping f of $H_\mu \times A$ into B, i.e $f \in [H_\mu \times A; B]$, there exists one and only one $g \in [H_\mu(A); B]$ such that*

$$f(\varphi, \psi) = g(\varphi\psi) \tag{6}$$

for all $\varphi \in H_\mu$ and $\psi \in A$.

[†]A function $f(x, y)$ is said to be sesquilinear if $f(\alpha x + \beta y, z) = \alpha f(x, z) + \beta f(y, z)$ and $f(x, \alpha y + \beta z) = \overline{\alpha} f(x, y) + \overline{\beta} f(x, z)$.

Proof First of all, let us consider the converse. Since g is linear, by equation (6), f is bilinear. Let $\varphi_n \to \varphi$ in H_μ, $\psi_n \to \psi$ in A. Then

$$\begin{aligned}\gamma^{\mu}_{m,k}(\varphi_n\psi_n - \varphi\psi) &\triangleq \sup_{x\in I}\left\|x^m(x^{-1}D)^k x^{-\mu-(1/2)}(\varphi_n\psi_n - \varphi\psi)\right\|_A \\ &\le \sup_{x\in I}|x^m(x^{-1}D)^k x^{-\mu-(1/2)}\varphi_n|\cdot\|\psi_n-\psi\| \\ &\quad + \sup_{x\in I}|x^m(x^{-1}D)^k x^{-\mu-(1/2)}(\varphi_n-\varphi)|\cdot\|\psi\|_A \longrightarrow 0 \quad \text{as } n\longrightarrow\infty\end{aligned}$$

for $\sup_{x\in I}|x^m(x^{-1}D)^k x^{-\mu-(1/2)}\varphi_n|$ is bounded by a constant which does not depend on n.

Since g is continuous on $H_\mu(A)$, it follows that f is continuous on $H_\mu \times A$.

Let f be given as in Theorem 4.1. For $\varphi \in {}_\mu D_I \bigodot A$, we define

$$g(\varphi) \triangleq \sum_{k=1}^{r} f(\theta_k, a_k) \quad \text{for } \varphi = \sum_{k=1}^{r}\theta_k a_k.$$

To justify this definition, we have to show that the right-hand side does not depend on the choice of the representation for φ. Let $\varphi = \sum_{i=1}^{s} h_i b_i$ where $h_i \in {}_\mu D_I$, $b_i \in A$, be another representation. Now, we find l linearly independent elements $e_1, e_2, \ldots, e_l \in A$, such that for each k and i,

$$a_k = \sum_{j=1}^{l}\alpha_{k_j}e_j \quad \text{and} \quad b_i = \sum_{j=1}^{l}\beta_{i_j}e_j,$$

where $\alpha_{k_j}, \beta_{i_j} \in C$. On substituting these sums into the two representations of φ and invoking the linear independence of e_j, we obtain

$$\sum_{k=1}^{r}\theta_k\alpha_{k_j} = \sum_{i=1}^{s}h_i\beta_{i_j}.$$

Hence

$$\begin{aligned}\sum_{k=1}^{r} f(\theta_k, a_k) &= \sum_{k=1}^{r} f\left(\theta_k, \sum_{j=1}^{l}\alpha_{k_j}e_j\right) = \sum_{k=1}^{r}\sum_{j=1}^{l}\alpha_{k_j} f(\theta_k, e_j) \\ &= \sum_{j=1}^{l} f\left(\sum_{k=1}^{r}\alpha_{k_j}\theta_k, e_j\right) = \sum_{i=1}^{s} f\left(h_i, \sum_{j=1}^{l}\beta_{i_j}e_j\right) \\ &= \sum_{i=1}^{s} f(h_i, b_i).\end{aligned}$$

Furthermore, g is linear. Indeed, let $\varphi_1, \varphi_2 \in {}_\mu D_I \bigodot A$ such that $\varphi_1 = \sum_{k=1}^{r}\theta_k a_k$, $\varphi_2 = \sum_{i=1}^{s}h_i b_i$. Then $\varphi_1 + \varphi_2 = \sum_{k=1}^{r+s}\theta'_k a'_k$, where $\theta'_k = \theta_k$, $a'_k = a_k$ for $1 \le k \le r$, $\theta'_{r+i} = h_i$, $a'_{r+i} = b_i$ for $1 \le i \le s$.

$$\begin{aligned}g(\varphi_1+\varphi_2) &\triangleq \sum_{k=1}^{r+s} f(\theta'_k, a'_k) = \sum_{k=1}^{r} f(\theta'_k, a'_k) + \sum_{k=r+1}^{r+s} f(\theta'_k, a'_k) \\ &= g(\varphi_1) + g(\varphi_2).\end{aligned}$$

Obviously, $g(\alpha\varphi) = \alpha g(\varphi)$ for $\alpha \in C$.

Now, we show that g is uniformly continuous on ${}_{\mu}D_I \bigodot A$. Indeed, for any $\epsilon > 0$, as long as $\varphi\psi$ ($\varphi \in {}_{\mu}D_I$ and $\psi \in A$) belongs to the balloon $\{\varphi\colon \gamma^{\mu}_{m,k}(\varphi) < \epsilon/M,\ m = 0, 1, \ldots, m_0,\ k = 0, 1, \ldots, k_0\}$, then there exist $M > 0$, positive integer m_0, k_0 such that

$$\|g(\varphi\psi)\|_B \le \|f(\varphi, \psi)\|_B \le M\gamma^{\mu}_{m_0,k_0}(\varphi)\|\psi\|_A < \epsilon.$$

This follows from Lemma 4.1. Thus g is uniformly continuous at the origin. By Lemma 2.1 (3), g is uniformly continuous on ${}_{\mu}D_I \bigodot A$. Since ${}_{\mu}D_I \bigodot A$ is dense in $H_\mu(A)$, we are able to extend g to $H_\mu(A)$.

For any $\varphi \in H_\mu$, Theorem 2.3 enables us to construct $\varphi_n \in {}_{\mu}D_I$ such that $\varphi_n \to \varphi$ in H_μ. Therefore from

$$g(\varphi_n\psi) = f(\varphi_n, \psi) \quad \psi \in A$$

and letting $n \to +\infty$, we get $g(\varphi\psi) = f(\varphi, \psi)$. Such g is unique. This completes the proof. ■

By applying Theorems 3.3 and 4.1, we establish the kernel theorem.

THEOREM 4.2 *Corresponding to every continuous bilinear mapping f of $H_\mu \times A$ into B, i.e $f \in [H_\mu \times A; B]$, there exists one and only one $g \in [H_\mu; [A; B]]$ such that*

$$f(\varphi, \psi) = \langle g, \varphi\rangle\psi,$$

where $\varphi \in H_\mu,\ \psi \in A$.

5. A root of integral transforms

We always take $B = C$ in the following examples.

Example 1 (Laplace transform) We choose $A = L^p(0, +\infty)$ in Theorem 4.2. Since $[L^p(0, +\infty); C] = L^q(0, +\infty)$ where p, q are conjugate numbers satisfying $1/p + 1/q = 1$. By applying the theorem, we know that for any $f \in [H_\mu \times L^p; C]$, there exists a unique $g \in [H_\mu; L^q]$ such that $f(\varphi, \psi) = \langle g, \varphi\rangle\psi$ where $\varphi \in H_\mu, \psi \in L^p$.

Define a family of function $g_s (s \in I)$ on H_μ by $\langle g_s, \varphi\rangle = \varphi(\sqrt{sx}), x \in I$. Then $g_s \in [H_\mu; L^q]$. In fact,

$$\int_0^\infty |\varphi(\sqrt{sx})|^q \,\mathrm{d}x = \int_0^\infty |\varphi(u)|^q \frac{2u}{s} \,\mathrm{d}u < \infty$$

since $\varphi \in H_\mu$. The topology of H_μ is stronger than that of L^q. Hence the assertion follows.

Therefore,

$$f(\varphi, \psi) = \langle g, \varphi\rangle\psi = \int_0^\infty \varphi(\sqrt{sx})\psi(x) \,\mathrm{d}x.$$

Set $\mu = -1/2$, then $\varphi = \mathrm{e}^{-t^2} \in H_{-1/2}$, and

$$f(e^{-t^2}, \psi) = \int_0^\infty \mathrm{e}^{-sx}\psi(x) \,\mathrm{d}x$$

which is the Laplace transform on L^p.

Example 2 (A discrete transform) We take $A = l^p$ in Theorem 4.2. By using the fact $[l^p; C] = l^q$, it follows that for $f \in [H_\mu \times l^p; C]$, there exists a unique $g \in [H_\mu; l^q]$ such that $f(\varphi, \psi) = \langle g, \varphi\rangle\psi$ where $\varphi \in H_\mu,\ \psi \in l^p$.

We define $\langle g_s, \varphi\rangle = \{i^s\varphi(i)\}_{i=1}^{+\infty}$ for $s \in R$. Then $g_s \in [H_\mu; l^q]$ since $\varphi(x)$ is a rapid decent function. From Theorem 4.2, we have

$$f(\varphi, \psi) = \sum_{i=1}^{\infty} i^s\varphi(i)y_i$$

where $\psi = \{y_i\}_{i=1}^{\infty} \in l^p$.

Example 3 (Mellin transform) Set $A = \{\psi \in C_I^\infty | \exists$ polynomial P_ψ such that $|x\psi| \leq P_\psi\}$ and the norm is defined as $\|\psi\| = \sup_{x\in I} |e^{-x}x\psi(x)|$. It is easily verified that A is a Banach space. We define $\langle g, \varphi\rangle\psi = \int_0^\infty \varphi(x)\psi(x)\,\mathrm{d}x$, where $\psi \in A$.

In particular, $\psi_s = x^{s-1} \in A$ for $s > 0$. We get a Mellin transform on $H_\mu(\mu \geq -1/2)$

$$f(\varphi, \psi_s) = \int_0^\infty \varphi(x)x^{s-1}\,\mathrm{d}x,$$

where $s > 0$.

Example 4 (Hankel transform) Set $A = \{\psi(x) \in C_I^\infty | \psi$ is bounded$\}$ and the norm is defined as $\|\psi\| = \sup_{x\in I} |\psi(x)|$. It follows that A is a Banach space. We define $\langle g, \varphi\rangle\psi = \int_0^\infty \varphi(x)\psi(x)\,\mathrm{d}x$, where $\psi(x) \in A$.

In particular, $\psi_y(x) = \sqrt{xy}J_\mu(xy) \in A$ for $y > 0$. We have the Hankel transform

$$f(\varphi, \sqrt{xy}J_\mu(xy)) = \int_0^\infty \varphi(x)\sqrt{xy}J_\mu(xy)\,\mathrm{d}x.$$

The K-transform can follow similarly.

6. An approach for equation (1)

By direct computation, we have

$$M_\mu N_\mu = \frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{4\mu^2 - 1}{4x^2} = M_{-\mu}N_{-\mu}.$$

Obviously, differential equation (1) can be converted to

$$u - M_\mu N_\mu u = \sqrt{x}J_\mu(x).$$

Clearly,

$$\begin{aligned}\langle h_\mu\delta(y-1), \phi(x)\rangle &= \langle\delta(y-1),\ h_\mu\phi\rangle \\ &= \int_0^\infty \sqrt{x}J_\mu(x)\phi(x)\,\mathrm{d}x = \langle\sqrt{x}J_\mu(x), \phi(x)\rangle\end{aligned}$$

which leads to

$$h_\mu[\sqrt{x}J_\mu(x)] = \delta(y-1)$$

since $h_\mu = h_\mu^{-1}$ for $\mu \geq -1/2$.

Applying the Hankel transform h_μ on both sides of $u - M_\mu N_\mu u = \sqrt{x} J_\mu(x)$, we get

$$(1 + y^2)h_\mu(u) = h_\mu[\sqrt{x} J_\mu(x)] = \delta(y - 1) \tag{7}$$

where the generalized function $\delta(y - a)$ for $a > 0$ is defined on H_μ by

$$\langle \delta(y - a), \phi(x) \rangle = \phi(a), \quad \phi \in H_\mu.$$

It follows from equation (7) that

$$h_\mu(u) = \frac{\delta(y - 1)}{1 + y^2} = \frac{1}{2}\delta(y - 1). \tag{8}$$

Now applying the Hankel inverse to equation (8), we have

$$u = \frac{1}{2}h_\mu^{-1}[\delta(y - 1)] = \frac{1}{2}h_\mu[\delta(y - 1)] = \frac{1}{2}\sqrt{x} J_\mu(x),$$

since $h_\mu^{-1} = h_\mu$ for $\mu \geq -1/2$. Therefore $u = (1/2)\sqrt{x} J_\mu(x)$ is a solution in H_μ for differential equation (1). For $\mu < -1/2$, we need the the following two identities in ref. [1, 6]

$$D_x x^\mu J_\mu(xy) = y x^\mu J_{\mu-1}(xy) \tag{9}$$

$$\sqrt{xy} J_\mu(xy) \sim \sqrt{\frac{2}{\pi}} \cos\left(xy - \frac{\mu\pi}{2} - \frac{\pi}{4}\right) \quad x \longrightarrow \infty \tag{10}$$

as well as the following lemmma.

Lemma 6.1 *For any $\phi \in H_\mu$ and $\psi(x) \triangleq (x^{-1}D)^k x^{-\mu-(1/2)}\phi(x)$, the following two statements are satisfied for each non-negative integer k:*

(1) *The limit $\lim_{x \to 0^+} \psi(x)$ exists (and hence is finite). In particular, $\lim_{x\to 0^+} \phi(x) = \phi(0^+)$ for $\mu = -1/2$ and $k = 0$.*
(2) *$\psi(x)$ is of rapid descent as $x \to \infty$ [i.e. $\psi(x) \to 0$ faster than any power of $1/x$ as $x \to \infty$].*

Proof Left for interested readers. ∎

Assume that $\mu \neq -1, -2, \ldots$ and let k be any positive integer such that $\mu + k \geq -1/2$ (note that $\mu + k - j \neq 0$ for any $0 \leq j \leq k$). We symbolically compute $h_{\mu,\,k}\ \delta(y - 1)$ using integration by parts and abandon all divergent terms $x^{\mu+k-j} J_{\mu+k-j}(x)$ (if any) as $x \to 0^+$, according to pseudo-integrals defined in Zemanian's book [7],

$$\begin{aligned}\langle h_{\mu,k}\delta(y - 1), \phi(x)\rangle &= \langle \delta(y - 1), h_{\mu,k}\phi(x)\rangle \\ &= (-1)^{k+1}\int_0^\infty x^{\mu+k} J_{\mu+k-1}(x)(x^{-1}D)^{k-1}x^{-\mu-(1/2)}\phi(x)\,\mathrm{d}x.\end{aligned}$$

This procedure is permissible since the limit terms are equal to zero using identities (9), (10) and Lemma 6.1. Repeating this process $k - 1$ times, we get

$$\langle h_{\mu,k}\delta(y - 1), \phi(x)\rangle = \int_0^\infty \sqrt{x} J_\mu(x)\phi(x)\,\mathrm{d}x = \langle \sqrt{x} J_\mu(x),\ \phi(x)\rangle.$$

Hence, for any $\mu \in R$,

$$h_{\mu,k}\delta(y-1) = \sqrt{x}J_\mu(x). \tag{11}$$

Changing $h_{\mu,k}$ to h_μ and following the previous steps for $\mu \geq -1/2$, we get

$$h_{\mu,k}(u) = \frac{1}{2}\delta(y-1)$$

and therefore, from Theorem 3.2,

$$u = \frac{1}{2}h_{\mu,k}^{-1}\delta(y-1) = \frac{1}{2}h_{\mu,k}\delta(y-1) = \frac{1}{2}\sqrt{x}J_\mu(x).$$

To see that $(1/2)\sqrt{x}J_\mu(x)$ is a solution of differential equation (1), we notice that

$$M_\mu N_\mu \sqrt{x}J_\mu(x) = -\sqrt{x}J_\mu(x).$$

Indeed, $N_\mu\sqrt{x}J_\mu(x) = -x^{1/2}J_{\mu+1}(x)$. It follows that

$$\begin{aligned} M_\mu(-x^{1/2}J_{\mu+1}) &= -x^{-\mu-(1/2)}D[x^{\mu+1}J_{\mu+1}(x)] \\ &= -x^{-\mu-(1/2)}x^{\mu+1}J_\mu(x) \\ &= -\sqrt{x}J_\mu(x) \end{aligned}$$

using identity (9). Hence $u = (1/2)\sqrt{x}J_\mu(x)$ is a solution in H_μ for differential equation (1) when $\mu \neq -1, -2, -3, \ldots$. We leave interested readers the case for $\mu = -1, -2, -3, \ldots$, which shall produce one δ function term in equation (11).

In conclusion, we point out a very minor error in Zemanian's book, in which he constructed the following operational formula

$$h_\mu(M_\mu\phi) = yh_{\mu+1}\phi$$

for $\mu \geq -1/2$. However, it is not quite correct for $\mu = -1/2$. Indeed,

$$\begin{aligned} h_{-1/2}(M_{-1/2}\phi) &= \sqrt{y}\int_0^\infty \phi'(x)\sqrt{x}J_{-1/2}(xy)\,\mathrm{d}x \\ &= \phi(x)\ \sqrt{xy}J_{-1/2}(xy)\big|_{0^+}^{\infty} + y\int_0^\infty \phi(x)\sqrt{xy}J_{1/2}(xy)\,\mathrm{d}x \\ &= -\sqrt{\frac{2}{\pi}}\phi(0^+) + yh_{1/2}(\phi), \end{aligned}$$

since

$$\lim_{x\to 0^+}\sqrt{xy}J_{-1/2}(xy) = \sqrt{\frac{2}{\pi}} \quad \text{for } y \in I,$$

which is not equal to zero. Note that there exists $\phi \in H_{-1/2}$ such that $\phi(0^+) \neq 0$. For example, $\phi(x) = e^{-x^2} \in H_{-1/2}$ and $\phi(0^+) = 1$.

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