Brian Fisher; Emin Özçag; Li Chen Kuan
A commutative neutrix convolution of distributions and the exchange formula

Archivum Mathematicum, Vol. 28 (1992), No. 3-4, 187--197

Persistent URL: http://dml.cz/dmlcz/107450

Terms of use:
© Masaryk University, 1992

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.

DML-CZ: This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project DML-CZ: The Czech Digital Mathematics Library http://project.dml.cz
A COMMUTATIVE NEUTRIX CONVOLUTION OF DISTRIBUTIONS AND THE EXCHANGE FORMULA

BRIAN FISHER, EMIN ÖZÇAĞ, AND LI CHEN KUAN

Abstract. The neutrix convolution product $f \boxtimes g$ of two distributions $f$ and $g$ in $\mathcal{D}'$ is defined to be the neutrix limit of the sequence $\{(f \tau_n) \ast (g \tau_n)\}$, provided the limit exists, where $\{\tau_n\}$ is a certain sequence of functions $\tau_n$ in $\mathcal{D}$ converging to 1. The neutrix product $(Ff \Box (Fg))$ in $Z'$, where $F$ denotes the Fourier transform, is defined to be the neutrix limit of the sequence $\{F(f \tau_n).F(g \tau_n)\}$, where

$$F(f \tau_n) = F(f) \ast \delta_n, \quad F(g \tau_n) = F(g) \ast \delta_n, \quad \delta_n = F(\tau_n)$$

and $\{\delta_n\}$ is a sequence of functions in $Z$ converging to the Dirac delta function. It is proved that the exchange formula

$$F(f \boxtimes g) = F(f) \Box F(g)$$

then holds. Some examples are given.

In the following, $\mathcal{D}$ denotes the space of infinitely differentiable functions with compact support and $\mathcal{D}'$ denotes the space of distributions defined on $\mathcal{D}$.

The convolution product of certain pairs of distributions in $\mathcal{D}'$ is usually defined as follows, see for example Gel’fand and Shilov [4].

Definition 1. Let $f$ and $g$ be distributions in $\mathcal{D}'$ satisfying either of the following conditions:

(a) either $f$ or $g$ has bounded support,
(b) the supports of $f$ and $g$ are bounded on the same side.

Then the convolution product $f \ast g$ is defined by the equation

$$\langle (f \ast g)(x), \phi(x) \rangle = \langle g(y), \phi(x + y) \rangle$$

for arbitrary test function $\phi$ in $\mathcal{D}$. 

1991 Mathematics Subject Classification: 46F10.

Key words and phrases: distribution, ultradistribution, neutrix convolution product, neutrix, neutrix limit, Fourier transform, exchange formula.

Received June 14, 1991.
It follows that if the convolution product \( f * g \) exists by Definition 1 then the following equations hold:

\[
(2) \quad f * g = g * f, \\
(3) \quad (f * g)' = f * g' = f' * g.
\]

Definition 1 is rather restrictive and in order to define further convolution products of distributions, Jones in [5] gave the following definition.

**Definition 2.** Let \( f \) and \( g \) be distributions in \( \mathcal{D}' \) and let \( \tau \) be an infinitely differentiable function satisfying the following conditions:

(i) \( \tau(x) = \tau(-x) \),
(ii) \( 0 \leq \tau(x) \leq 1 \),
(iii) \( \tau(x) = 1, |x| \leq \frac{1}{2} \),
(iv) \( \tau(x) = 0, |x| \geq 1 \).

Let

\[
f_n(x) = f(x)\tau(x/n), \quad g_n(x) = g(x)\tau(x/n)
\]

for \( n = 1, 2, \ldots \). Then the **convolution product** \( f * g \) is defined as the limit of the sequence \( \{f_n * g_n\} \), providing the limit \( h \) exists in the sense that

\[
\lim_{n \to \infty} \langle f_n * g_n, \phi \rangle = \langle h, \phi \rangle
\]

for all \( \phi \) in \( \mathcal{D} \).

Note that in this definition the convolution product \( f_n * g_n \) exists by Definition 1 since \( f_n \) and \( g_n \) both have bounded supports. It is clear that if the convolution product \( f * g \) exists by this definition, then equation (2) holds. However, equations (3) need not necessarily hold since Jones proved that

\[
1 * \text{sgn} x = x = \text{sgn} x * 1
\]

and

\[
(1 * \text{sgn} x)' = 1, \quad 1' * \text{sgn} x = 0, \quad 1 * (\text{sgn} x)' = 2.
\]

Many convolution products could still not be defined by Definition 2 and the following modification of Definition 2 was given in [3]:

**Definition 3.** Let \( f \) and \( g \) be distributions in \( \mathcal{D}' \), let

\[
\tau_n(x) = \begin{cases} 
1, & |x| \leq n, \\
\tau(n^n x - n^{n+1}), & x > n, \\
\tau(n^n x + n^{n+1}), & x < -n,
\end{cases}
\]

where \( \tau \) is as in Definition 2 and let \( f_n = f \tau_n \), \( g_n = g \tau_n \). Then the **neutrix convolution product** \( f * g \) is defined to be the neutrix limit of the sequence \( \{f_n * g_n\} \), provided the limit \( h \) exists in the sense that

\[
\lim_{n \to \infty} \langle f_n * g_n, \phi \rangle = \langle h, \phi \rangle
\]
for all $\phi$ in $\mathcal{D}$, where $N$ is the neutrix, see van der Corput [1], having domain $N' = \{1, 2, \ldots, n, \ldots\}$ and range the real numbers with negligible functions finite linear sums of the functions
\[ n^\lambda \ln^{r-1} n, \quad \ln^r n \quad (\lambda > 0, \ r = 1, 2, \ldots) \]
and all functions which converge to zero as $n$ tends to infinity.

The convolution product $f_n \ast g_n$ in this definition is again in the sense of Definition 1, the supports of $f_n$ and $g_n$ being bounded. The neutrix convolution product $f \boxtimes g$ clearly satisfies equation (2) if it exists, although it does not necessarily satisfy equations (3). A non-commutative neutrix convolution product, denoted by $f \circledast g$ was defined in [2].

It was proved in [3] that if the convolution product $f \ast g$ exists by Definition 1, then the neutrix convolution product $f \boxtimes g$ exists and
\[ f \ast g = f \boxtimes g. \]

As in [4], we define the Fourier transform of a function $\phi$ in $\mathcal{D}$ by
\[ \hat{F}(\phi)(\sigma) = \tilde{\phi}(\sigma) = \int_{-\infty}^{\infty} \phi(x)e^{ix\sigma} \, dx. \]
Here $\sigma = \sigma_1 + i\sigma_2$ is a complex variable and it is well known that $\tilde{\phi}(\sigma)$ is an entire analytic function with the property
\[ |\sigma|^q |\tilde{\phi}(\sigma)| \leq C_q e^{a|\sigma_2|} \]
for some constants $C_q$ and $a$ depending on $\tilde{\phi}$. The set of all analytic functions $\mathcal{Z}$ with property (4) is in fact the space
\[ F(\mathcal{D}) = \{ \phi : \exists \tilde{\phi} \in \mathcal{D}, F(\phi) = \psi \}. \]

The Fourier transform $\hat{f}$ of a distribution $f$ in $\mathcal{D}'$ is an ultradistribution in $\mathcal{Z}'$, i.e. a continuous linear functional on $\mathcal{Z}$. It is defined by Parseval’s equation
\[ \langle \hat{f}, \tilde{\phi} \rangle = 2\pi \langle f, \phi \rangle. \]

The exchange formula is the equality
\[ F(f \ast g) = F(f) \cdot F(g). \]

It is well known that the exchange formula holds for all convolution products of distributions $f$ and $g$ satisfying Definition 1, provided $f$ and $g$ both have compact support, see for example Treves [6].
We now consider the problem of defining multiplication in $\mathcal{Z}'$. To do this we need the Fourier transform $F(\tau_n)$ of $\tau_n$ and write

$$\delta_n(\sigma) = \frac{1}{2\pi} F(\tau_n),$$

which is a function in $\mathcal{Z}$. Putting $\psi = \tilde{\phi}$, we have from Parseval’s equation

$$\langle \tau_n, \phi \rangle = \frac{1}{2\pi} \langle F(\tau_n), F(\phi) \rangle = \langle \delta_n, \psi \rangle.$$ 

Since

$$\lim_{n \to \infty} \langle \tau_n, \phi \rangle = \lim_{n \to \infty} \int_{-\infty}^{\infty} \tau_n(x) \phi(x) \, dx = \int_{-\infty}^{\infty} \phi(x) \, dx = \langle 1, \phi \rangle$$

for all $\phi$ in $\mathcal{D}$ and since $F(1) = 2\pi \delta$, we obtain

$$\lim_{n \to \infty} \langle \delta_n, \psi \rangle = \langle \delta, \psi \rangle$$

for all $\psi$ in $\mathcal{Z}$. Thus $\{\delta_n\}$ is a sequence in $\mathcal{Z}$ converging to the Dirac delta function $\delta$.

If $f$ is an arbitrary distribution in $\mathcal{D}'$, then since $\delta_n$ is a function in $\mathcal{Z}$, the convolution product $\hat{f} \ast \delta_n$ is defined by

$$(6) \quad \langle (\hat{f} \ast \delta_n)(\sigma), \psi(\sigma) \rangle = \langle \hat{f}(\nu), \langle \delta_n(\sigma), \psi(\sigma + \nu) \rangle \rangle$$

for arbitrary $\psi$ in $\mathcal{Z}$. If $\psi = \tilde{\phi}$, we have

$$\psi(\sigma + \nu) = F[e^{ix\nu} \phi(x)]$$

and it follows from Parseval’s equation that

$$(7) \quad \langle \delta_n(\sigma), \psi(\sigma + \nu) \rangle = \frac{1}{2\pi} \langle F(\tau_n)(\sigma), F(e^{ix\nu} \phi)(\sigma) \rangle = \langle \tau_n(x), e^{ix\nu} \phi(x) \rangle$$

$$= \int_{-\infty}^{\infty} \tau_n(x) e^{ix\nu} \phi(x) \, dx$$

$$\to \int_{-\infty}^{\infty} e^{ix\nu} \phi(x) \, dx = \psi(\nu).$$

Thus

$$\lim_{n \to \infty} \langle \hat{f} \ast \delta_n, \psi \rangle = \langle \hat{f}, \psi \rangle$$

for arbitrary $\psi$ in $\mathcal{Z}$ and it follows that $\{\hat{f} \ast \delta_n\}$ is a sequence of infinitely differentiable functions converging to $\hat{f}$ in $\mathcal{Z}'$.

This leads us to the following definition:
Definition 4. Let $f$ and $g$ be distributions in $\mathcal{D}'$ having Fourier transforms $\hat{f}$ and $\hat{g}$ respectively in $\mathcal{Z}'$ and let $\hat{f}_n = \hat{f} * \delta_n$ and $\hat{g}_n = \hat{g} * \delta_n$. Then the neutrix product $\hat{f} \boxtimes \hat{g}$ is defined to be the neutrix limit of the sequence $\{\hat{f}_n, \hat{g}_n\}$, provided the limit $\hat{h}$ exists in the sense that

$$\lim_{n \to \infty} \langle \hat{f}_n, \hat{g}_n, \psi \rangle = \langle \hat{h}, \psi \rangle$$

for all $\psi$ in $\mathcal{Z}$.

In this definition we use $\hat{f} \boxtimes \hat{g}$ to denote the neutrix product of $\hat{f}$ and $\hat{g}$ to distinguish it from the usual definition of the product $\hat{f}_n \hat{g}_n$ of two infinitely differentiable functions $f_n$ and $g_n$. If

$$\lim_{n \to \infty} \langle \hat{f}_n, \hat{g}_n, \psi \rangle = \langle \hat{h}, \psi \rangle$$

for all $\psi$ in $\mathcal{Z}$, we simply say that the product $\hat{f} \hat{g}$ exists and equals $\hat{h}$. We then of course have

$$\hat{f} \boxtimes \hat{g} = \hat{f} \hat{g}$$

It is immediately obvious that if the neutrix product $\hat{f} \boxtimes \hat{g}$ exists then the neutrix product is commutative.

The product of ultradistributions in $\mathcal{Z}'$ also has the following property:

Theorem 1. Let $\hat{f}$ and $\hat{g}$ be ultradistributions in $\mathcal{Z}'$ and suppose that the neutrix products $\hat{f} \boxtimes \hat{g}$ and $\hat{f} \boxtimes \hat{g}'$ (or $\hat{f} \boxtimes \hat{g}'$) exist. Then the neutrix product $\hat{f}' \boxtimes \hat{g}$ (or $\hat{f} \boxtimes \hat{g}'$) exists and

$$(\hat{f} \boxtimes \hat{g})' = \hat{f}' \boxtimes \hat{g} + \hat{f} \boxtimes \hat{g}'$$

Proof. Let $\psi$ be an arbitrary function in $\mathcal{Z}$. Then

$$\langle \hat{f} \boxtimes \hat{g}, \psi \rangle = \lim_{n \to \infty} \langle \hat{f}_n, \hat{g}_n, \psi \rangle, \quad \langle \hat{f} \boxtimes \hat{g}', \psi \rangle = \lim_{n \to \infty} \langle \hat{f}_n, \hat{g}'_n, \psi \rangle.$$ 

Further,

$$\langle (\hat{f} \boxtimes \hat{g})', \psi \rangle = -\langle \hat{f} \boxtimes \hat{g}', \psi \rangle = -\lim_{n \to \infty} \langle \hat{f}_n, \hat{g}_n', \psi \rangle$$

$$= -\lim_{n \to \infty} \langle \hat{g}_n, (\hat{f}_n, \hat{g}_n)' - \hat{f}_n', \psi \rangle$$

$$= \lim_{n \to \infty} \langle \hat{g}'_n, \hat{f}_n, \psi \rangle + \lim_{n \to \infty} \langle \hat{g}_n, \hat{f}_n', \psi \rangle$$

and so

$$\lim_{n \to \infty} \langle \hat{f}_n, \hat{g}_n, \psi \rangle = \langle (\hat{f} \boxtimes \hat{g})', \psi \rangle - \langle \hat{f} \boxtimes \hat{g}', \psi \rangle.$$ 

Hence the neutrix product $\hat{f}' \hat{g}$ exists and equation (8) follows.

It follows similarly that if $\hat{f} \boxtimes \hat{g}$ exists then $\hat{f} \boxtimes \hat{g}'$ exists.

We can now prove the exchange formula.
Theorem 2. Let \( f \) and \( g \) be distributions in \( \mathcal{D}' \) having Fourier transforms \( \hat{f} \) and \( \hat{g} \) respectively in \( \mathcal{Z}' \). Then the neutrix convolution product \( f \Box g \) exists in \( \mathcal{D}' \), if and only if the neutrix product \( \hat{f} \Box \hat{g} \) exists in \( \mathcal{Z} \) and the exchange formula

\[
F(f \Box g) = \hat{f} \Box \hat{g}
\]

is then satisfied.

Proof. We have from equation (7) that

\[
\langle \delta_n(\sigma), (\sigma + \nu) \rangle = F(\tau_n \phi)
\]

and then from equation (6) that

\[
\langle \hat{f}_n, \psi \rangle = \langle \hat{f} \ast \delta_n, \psi \rangle = \langle \hat{f}, F(\tau_n \phi) \rangle = 2\pi \langle f, \tau_n \phi \rangle = 2\pi \langle f_n, \phi \rangle = \langle F(f_n), \psi \rangle
\]

on using Parseval’s equation twice. It follows that \( F(f_n) = \hat{f}_n \). Similarly, we have \( F(g_n) = \hat{g}_n \). Now since the convolution product \( f_n \ast g_n \) exists by Definition 1 and \( f_n \) and \( g_n \) both have compact support

\[
F(f_n \ast g_n) = F(f_n) \cdot F(g_n) = \hat{f}_n \cdot \hat{g}_n
\]

and so on using Parseval’s equation again

\[
2\pi \langle f_n \ast g_n, \phi \rangle = \langle F(f_n \ast g_n), \psi \rangle = \langle \hat{f}_n \cdot \hat{g}_n, \psi \rangle.
\]

Suppose the neutrix convolution product \( f \boxtimes g \) exists. Then

\[
2\pi \langle f \boxtimes g, \phi \rangle = \lim_{n \to \infty} 2\pi \langle f_n \ast g_n, \phi \rangle = \lim_{n \to \infty} \langle F(f_n \ast g_n), \psi \rangle
\]

for arbitrary \( \phi \) in \( \mathcal{D} \) and \( F \phi \) in \( \mathcal{Z} \), proving the existence of the neutrix product \( f \Box g \) and the exchange formula.

Conversely, if the neutrix product \( \hat{f} \Box \hat{g} \) exists then the argument can be reversed to prove the existence of the neutrix convolution product \( f \boxtimes g \) and the exchange formula. This completes the proof of the theorem.

Theorem 3. The products \((\sigma + i0)^\lambda \cdot (\sigma + i0)^\mu\) and \((\sigma - i0)^\lambda \cdot (\sigma - i0)^\mu\) exist and

\[
(\sigma + i0)^\lambda \cdot (\sigma + i0)^\mu = (\sigma + i0)^{\lambda + \mu}
\]

(9)

\[
(\sigma - i0)^\lambda \cdot (\sigma - i0)^\mu = (\sigma - i0)^{\lambda + \mu}
\]

(10)

for all \( \lambda \) and \( \mu \).

Proof. It is well known that

\[
x_+^\lambda \ast x_+^\mu = B(\lambda + 1, \mu + 1)x_+^{\lambda + \mu + 1}
\]

(11)
for \( \lambda, \mu, \lambda + \mu + 1 \neq -1, -2, \ldots \), where \( B \) denotes the Beta function.

Further, see Gel’fand and Shilov [4],

\[
F(x^\lambda_+) = ie^{i\lambda \pi/2} \Gamma(\lambda + 1)(\sigma + i0)^{-\lambda - 1}
\]

for \( \lambda \neq -1, -2, \ldots \). On using the exchange formula, it follows from equations (11) and (12) that

\[
-e^{i(\lambda + \mu)\pi/2} \Gamma(\lambda + 1)\Gamma(\mu + 1)(\sigma + i0)^{-\lambda - 1}(\sigma + i0)^{-\mu - 1} = B(\lambda + 1, \mu + 1)ie^{i(\lambda + \mu + 1)\pi/2} \Gamma(\lambda + \mu + 2)(\sigma + i0)^{-\lambda - \mu - 2}
\]

for \( \lambda, \mu, \lambda + \mu + 1 \neq -1, -2, \ldots \), the product \((\sigma + i0)^{-\lambda - 1}(\sigma + i0)^{-\mu - 1}\) existing since the convolution product \(x^\lambda_+ * x^\mu_+\) exists. Equation (9) now follows for \( \lambda, \mu, \lambda + \mu \neq 0, 1, 2, \ldots \).

Now suppose that \( \lambda, \mu, \lambda + \mu > -1 \) and put

\[
(\sigma + i0)^\lambda_n = (\sigma + i0)^\lambda * \delta_n(\sigma).
\]

Then since

\[
(\sigma + i0)^\lambda = \sigma^\lambda_+ + e^{i\lambda \pi} \sigma^\lambda_-,
\]

see [4], it follows that \{\((\sigma + i0)^\lambda_n, (\sigma + i0)^\mu_n\)\} is a sequence of locally summable functions which converges to the locally summable function \((\sigma + i0)^{\lambda + \mu}\). Equation (9) follows for \( \lambda, \mu, \lambda + \mu > -1 \).

Now suppose that equation (9) holds when \( -k - 1 < \lambda < -k \), for some positive integer \( k \), and \( \lambda + \mu = 0, \pm 1, \pm 2, \ldots \). This is certainly true when \( k = 0 \). Then

\[
\lim_{n \to \infty} (\sigma + i0)^\lambda_n.(\sigma + i0)^\mu_n = (\sigma + i0)^{\lambda + \mu},
\]

by our assumption when \( -k - 1 < \lambda < -k \). It follows that

\[
\lim_{n \to \infty} [(\sigma + i0)^\lambda_n(\sigma + i0)^\mu_n]' = \\
\quad = \lim_{n \to \infty} [\lambda(\sigma + i0)^{\lambda - 1}_n(\sigma + i0)^\mu_n + \mu(\sigma + i0)^\lambda_n(\sigma + i0)^{\mu - 1}_n] \\
\quad = (\lambda + \mu)(\sigma + i0)^{\lambda + \mu - 1}
\]

and so

\[
\lim_{n \to \infty} (\sigma + i0)^{\lambda - 1}_n(\sigma + i0)^\mu_n = (\sigma + i0)^{\lambda + \mu - 1}.
\]

Equation (9) follows by induction for \( \lambda \neq -1, -2, \ldots \) and \( \lambda + \mu = 0, \pm 1, \pm 2, \ldots \).

We are finally left to prove equation (9) for the case \( \lambda = r = -1, -2, \ldots \) and \( \mu = s = 0, 1, 2, \ldots \). Since

\[
\ln(\sigma + i0) = \ln|\sigma| + i\pi H(-\sigma)
\]
and 

\[(\sigma + i0)^s = \sigma^s\]

for \(s = 0, 1, 2, \ldots\), see [4], are locally summable functions, it follows as above that if

\[\ln(\sigma + i0)_n = \ln(\sigma + i0) \ast \delta_n(\sigma),\]

then the sequence \(\{\ln(\sigma + i0)_n, (\sigma + i0)_n^s\}\) converges to the locally summable function \((\sigma + i0)^s \ln(\sigma + i0)\). Thus

\[
\lim_{n \to \infty} [\ln(\sigma + i0)_n(\sigma + i0)_n^s]' = \\
\lim_{n \to \infty} [(\sigma + i0)^{-1}_n(\sigma + i0)^s + s \ln(\sigma + i0)(\sigma + i0)^{s-1}_n] \\
= [(\sigma + i0)^s \ln(\sigma + i0)]' \\
= s(\sigma + i0)^{s-1} \ln(\sigma + i0) + (\sigma + i0)^{s-1},
\]

Equation (9) follows for \(\lambda = -1\) and \(\mu = 0, 1, 2, \ldots\). Another induction argument shows that equation (9) holds for \(\lambda = -1, -2, \ldots\) and \(\mu = 0, 1, 2, \ldots\). This completes the proof of the theorem.

**Corollary 1.**

\[(13)\]

\[\sigma^{-r} \cdot \sigma^s = \sigma^{s-r}\]

for \(r = 1, 2, \ldots\) and \(s = 0, 1, 2, \ldots\) and

\[(14)\]

\[\delta^{(r-1)}(\sigma) \cdot \sigma^s = \begin{cases} 0, & s \geq r, \\ \frac{(-1)^{r-1}y}{(r-1)!} \delta^{(r-s-1)}(\sigma), & r > s \end{cases}\]

for \(r = 1, 2, \ldots\) and \(s = 0, 1, 2, \ldots\).

**Proof.** Since

\[(\sigma + i0)^s = \sigma^s\]

for \(s = 0, 1, 2, \ldots\) and

\[(\sigma + i0)^{-r} = \sigma^{-r} + \frac{i\pi(-1)^r}{(r - 1)!} \delta^{(r-1)}(\sigma)\]

for \(r = 1, 2, \ldots\), see [4], it follows from equation (9) that

\[
(\sigma + i0)^{-r} \cdot \sigma^s = \begin{cases} \sigma^{s-r}, & s \geq r, \\ \sigma^{s-r} + \frac{i\pi(-1)^r}{(r-s-1)!} \delta^{(r-s-1)}(\sigma), & r > s \end{cases}
\]

the product clearly being distributive with respect to addition. Equating real and imaginary parts, equations (13) and (14) follow.
Corollary 2.

\begin{equation}
\sigma_+^{-r-1/2} \sigma_-^{-r-1/2} = \frac{(-1)^r \pi}{2(2r)!} \delta^{(2r)}(\sigma)
\end{equation}

for $r = 0, 1, 2, \ldots$

**Proof.** It follows from equation (9) that

\[
(\sigma + i0)^{-r-1/2} \cdot (\sigma + i0)^{-r-1/2} = (\sigma + i0)^{-2r-1}
\]

\[
= \left[\sigma_+^{-r-1/2} - i(-1)^r \sigma_-^{-r-1/2}\right] \cdot \left[\sigma_+^{-r-1/2} - i(-1)^r \sigma_-^{-r-1/2}\right]
\]

\[
= \sigma^{-2r-1} - \frac{i\pi}{(2r)!} \delta^{(2r)}(\sigma)
\]

for $r = 0, 1, 2, \ldots$. Expanding and equating the imaginary parts gives equation (15).

Corollary 3.

\begin{equation}
\sigma^{-r} \delta^{(r-1)}(\sigma) = \frac{(-1)^r(r-1)!}{2(2r-1)!} \delta^{(2r-1)}(\sigma)
\end{equation}

for $r = 1, 2, \ldots$

**Proof.** It follows from equation (9) that

\[
(\sigma + i0)^{-r} \cdot (\sigma + i0)^{-r} = (\sigma + i0)^{-2r}
\]

\[
= \left[\sigma^{-r} + \frac{i\pi(-1)^r}{(r-1)!} \delta^{(r-1)}(\sigma)\right] \cdot \left[\sigma^{-r} + \frac{i\pi(-1)^r}{(r-1)!} \delta^{(r-1)}(\sigma)\right]
\]

\[
= \sigma^{-2r} + \frac{i\pi}{(2r-1)!} \delta^{(2r-1)}(\sigma)
\]

for $r = 1, 2, \ldots$. Expanding and equating imaginary parts gives equation (16).

**Theorem 4.** The neutrix product $\sigma^\lambda_+ \square \delta^{(s)}(\sigma)$ exists and

\begin{equation}
\sigma^\lambda_+ \square \delta^{(s)}(\sigma) = 0
\end{equation}

for real $\lambda \neq 0, \pm 1, \pm 2, \ldots$ and $s = 0, 1, 2, \ldots$

**Proof.** It was proved in [3] that

\[
x_+^\lambda \ast x^s = 0, \quad x_-^\lambda \ast x^s = 0
\]

for real $\lambda \neq 0, \pm 1, \pm 2, \ldots$ and $s = 0, 1, 2, \ldots$. Thus

\[
(x - i0)^\lambda \ast x^s = (x_+^\lambda + e^{-i\lambda \pi} x_-^\lambda) \ast x^s = 0
\]
for real $\lambda \neq 0, \pm 1, \pm 2, \ldots$ and $s = 0, 1, 2, \ldots$. On applying the exchange formula to this equation we get

$$\sigma_+^{-\lambda-1} \Box \delta^{(s)}(\sigma) = 0$$

for real $\lambda \neq 0, \pm 1, \pm 2, \ldots$ and $s = 0, 1, 2, \ldots$, since

$$F[(x - i0)^{\lambda}] = \frac{2\pi e^{-i\lambda \pi / 2}}{\Gamma(-\lambda)} \sigma_+^{-\lambda-1}$$

for $\lambda \neq 0, \pm 1, \pm 2, \ldots$ and

$$F(x^s) = 2(-i)^s \pi \delta^{(s)}(\sigma)$$

for $s = 0, 1, 2, \ldots$, see Gel'fand and Shilov [4]. Equation (17) follows immediately.

**Corollary 1.** The neutrix product $\sigma_+^{\lambda} \Box \delta^{(s)}(\sigma)$ exists and

$$\sigma_+^{\lambda} \Box \delta^{(s)}(\sigma) = 0$$

for real $\lambda \neq 0, \pm 1, \pm 2, \ldots$ and $s = 0, 1, 2, \ldots$.

**Proof.** The result follows immediately from equation (17) on replacing $x$ by $-x$ in equation (17).

**Theorem 5.** The neutrix product $(\sigma - i0)^{\lambda} \Box (\sigma + i0)^{\mu}$ exists and

$$(\sigma - i0)^{\lambda} \Box (\sigma + i0)^{\mu} = \sigma_+^{\lambda + \mu} + e^{i(\mu - \lambda)} \pi \sigma_+^{\lambda + \mu}$$

for real $\lambda, \mu \neq 0, \pm 1, \pm 2, \ldots$.

**Proof.** It was proved in [3] that

$$x_+^{\lambda-1} \Box x_-^{\mu-1} = B(\lambda + \mu + 1, -\mu)x_-^{\lambda-\mu-1} + B(\lambda + \mu + 1, -\lambda)x_+^{\lambda-\mu-1}$$

for real $\lambda, \mu \neq 0, \pm 1, \pm 2, \ldots$. Applying the exchange formula to this equation and using equation (12) and the equation

$$F(x_-^{\lambda}) = -ie^{-i\lambda \pi / 2} \Gamma(\lambda + 1)(\sigma - i0)^{-\lambda-1}$$

we get

$$e^{i(\lambda - \mu) \pi / 2} \Gamma(-\mu)(\sigma - i0)^{\lambda} \Box (\sigma + i0)^{\mu} = $$

$$= e^{i(\lambda + \mu) \pi / 2} B(\lambda + \mu + 1, -\mu) \Gamma(\lambda - \mu)(\sigma - i0)^{\lambda + \mu} +$$

$$+ e^{-i(\lambda + \mu) \pi / 2} B(\lambda + \mu + 1, -\lambda) \Gamma(\lambda - \mu)(\sigma + i0)^{\lambda + \mu}$$

and so

$$(\sigma - i0)^{\lambda} \Box (\sigma + i0)^{\mu} = e^{i\mu \pi} \sin(\lambda \pi) \csc [(\lambda + \mu) \pi](\sigma - i0)^{\lambda + \mu} +$$

$$+ e^{-i\lambda \pi} \sin(\mu \pi) \csc [(\lambda + \mu) \pi](\sigma + i0)^{\lambda + \mu}$$

$$= \sigma_+^{\lambda + \mu} + e^{i(\mu - \lambda) \pi} \sigma_+^{\lambda + \mu},$$

proving equation (18) for real $\lambda, \mu \neq 0, \pm 1, \pm 2, \ldots$. 
References


B. Fisher and E. Özçağ

Department of Mathematics

The University, Leicester

LE1 7RH, England

Li Chen Kuan

Department of Mathematics and Statistics

University of Regina

Regina, S4S 0A2, Canada