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THE HANKEL CONVOLUTION OF ARBITRARY ORDER

C.K. Li
Department of Mathematics and Computer Science
Brandon University
Brandon, Manitoba, R7A 6A9, CANADA
e-mail: lic@brandonu.ca

Abstract: Let $\mu > -1/2$. The classical Hankel transform is defined by

$$(h_\mu \phi)(t) = \int_0^\infty x J_\mu(xt)\phi(x)dx \quad t \in (0, \infty), \tag{1}$$

where $J_\mu(x)$ denotes the Bessel function of the first kind and order μ. The goal of this paper is to construct the Hankel transform of arbitrary order $h_{\mu,k}$ based on the two differential operators and show that $h_{\mu,k} = h_{\mu,k}^{-1}$ on $H_{\mu-\frac{1}{2}}$ for $\mu \in R$. Furthermore, the Hankel convolution of arbitrary order is introduced with the following identity

$$(h_{\mu,k} h)(t) = t^{-\mu}(h_{\mu,k}\phi)(t)(h_{\mu,k}\psi)(t)$$

on the spaces $(H_{\mu-\frac{1}{2}}, H_{\mu-\frac{1}{2}})$ and $(S_\mu, H_{\mu-\frac{1}{2}})$ respectively.

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1. Introduction

Let $R^+ = (0, \infty)$ and a weight function $\nu(x) > 0$ in R^+. We define a set of functions $L(R^+, \nu(x))$ as

$$L(R^+, \nu(x)) = \{f(x) \ | \ \int_0^\infty |f(x)|\nu(x)dx < \infty\}.$$

Different types of the Hankel transforms as well as their convolutions, including modified ones, have been investigated over the decades (for example, see [7]-

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[5] and [1]) both in the classical sense and in spaces of generalized functions. These transforms are applied to solve problems of mathematical physics and differential equations with variable coefficients. In 1995, Tuan and Saigo [12] used the following commutative Hankel convolution $h(x)$ of a function $f(x)$ with a function $g(x)$ (due to Zhitomirskii, see [14]),

$$\begin{aligned} h(x) = & \frac{2^{1-3\mu}x^{-\mu}}{\sqrt{\pi}\Gamma(\mu+1/2)}\int\int_{u+v>x,\,|u-v|<x}[x^2-(u-v)^2]^{\mu-1/2}\cdot \\ & [(u+v)^2-x^2]^{\mu-1/2}(uv)^{1-\mu}f(u)g(v)dudv \quad x\in(0,\infty), \end{aligned} \tag{2}$$

and proved Theorem 1.1 using the Hankel transform in (1).

Theorem 1.1. *Let* $Re\,\mu > 1/2$ *and* $f(x)$, $g(x) \in L(R^+,\sqrt{x})$. *Then the function* $h(x)$ *of (2) exists and the convolutional identity*

$$(h_\mu h)(t) = t^{-\mu}(h_\mu f)(t)(h_\mu g)(t)$$

holds.

Recently, Britvina studied some polyconvolutions of the Hankel transform in [2], where she defined the following functions $h_1(t)$ and $h_2(t)$, and obtained Theorem 1.2 and Theorem 1.3,

$$\begin{aligned} h_1(t) = & \; t^{\mu-1}\int_0^\infty dv\int_{|t-v|}^{t+v}u^{-\mu+1}f(u)g(v)P_1(t;u,v)du \\ & -t^{\mu-1}\int_0^\infty dv\int_{t+v}^\infty u^{-\mu+1}f(u)g(v)Q_1(t;u,v)du, \\ h_2(t) = & \; t^{-\mu}\int\int_{|t-v|<t<u+v}u^\mu f(u)g(v)P_2(t;u,v)dudv \\ & -t^{-\mu}\int_0^\infty du\int_0^{t-u}u^\mu f(u)g(v)Q_2(t;u,v)dv, \end{aligned}$$

where

$$P_j(t;u,v) = \frac{1}{\sqrt{2\pi}}v^\mu P_{\nu-1/2}^{1/2-\mu}(\cos s_j)\sin^{\mu-1/2}s_j,$$

$$Q_j(t;u,v) = \frac{\sqrt{2}}{\pi^{3/2}}\sin[(\nu-\mu)\pi]\exp\left\{\frac{(2\mu-1)\pi}{2j}\right\}v^\mu Q_{\nu-1/2}^{1/2-\mu}(\mathrm{ch}r_j)\mathrm{sh}^{\mu-1/2}r_j,$$

$j=1,2$, and P_ν^μ, Q_ν^μ are the associated Legendre functions of the first and the second kind, respectively, and

$$\begin{aligned} 2tv\cos s_1 = t^2+v^2-u^2, \quad 2tv\mathrm{ch}r_1 = u^2-t^2-v^2, \\ 2uv\cos s_2 = u^2+v^2-t^2, \quad 2uv\mathrm{ch}r_2 = t^2-u^2-v^2. \end{aligned}$$

Theorem 1.2. *Suppose that* $f(t)$, $g(t) \in L(R^+,\sqrt{t})$ *and* $Re\,\mu > 1/2$,

$Re\,\nu > Re\,\mu - 1$. Then the function $h_1(t)$ exists and the relation

$$(h_\nu h_1)(x) = x^{-\nu}(h_\mu f)(x)(h_\nu g)(x)$$

holds.

Theorem 1.3. *Assume that $f(t), g(t) \in L(R^+, \sqrt{t})$ and $Re\,\mu > 1/2$, $Re\,\nu > (2\,Re\,\mu - 3)/4$. Then the function $h_2(t)$ exists and the following relation*

$$(h_\mu h_2)(x) = x^{-\mu}(h_\nu f)(x)(h_\nu g)(x)$$

is satisfied.

Inspired by the studies of González and Negrin ([3] and [4]) on the convolution and Fourier transform, Betancor and González [1] investigated the Hankel convolution satisfying the formula

$$\mathcal{H}'_\mu(f\sharp g) = x^{-\mu-1/2}\mathcal{H}'_\mu(f)\mathcal{H}'_\mu(g)$$

on new spaces of generalized function, which are subspaces of H'_μ (see [1]). Their modified Hankel transform is given by

$$(\mathcal{H}_\mu\phi)(t) = \int_0^\infty \sqrt{xt}J_\mu(xt)\phi(x)dx, \quad t \in (0, \infty). \tag{3}$$

As outlined in the abstract, the current work is to provide a means of defining the Hankel transform $h_{\mu,k}$ for any real value of the order μ in such a way that an inverse Hankel transform $h^{-1}_{\mu,k}$ also exists and is equal to the Hankel transform itself, although Zemanian claimed that it is not true for $\mu < -1/2$ in [13]. It is the property of the inverse transform that makes this extension of the Hankel transform significant, since it serves well in constructing the Hankel convolution of arbitrary order, which has never be studied in the past as far as we know.

2. The Hankel Transform of Arbitrary Order

In order to extend the Hankel transform in (3) to generalized functions, Zemanian [13] defined the following testing space H_μ.

Definition 2.1. For any real number μ, a function $\phi(x)$ is in H_μ if and only if it is complex-valued and smooth on R^+, and for each pair of nonnegative integers m and k,

$$\gamma^\mu_{m,k}(\phi) = \sup_{x\in R^+} x^m \left|(x^{-1}D)^k x^{-\mu-1/2}\phi(x)\right| < \infty.$$

Obviously, H_μ is a linear space. Also, each $\gamma^\mu_{m,k}$ is a seminorm on H_μ, and the collection $\{\gamma^\mu_{m,k}\}^\infty_{m,k=0}$ is a multinorm because the $\gamma^\mu_{m,0}$ are norms. The topology of H_μ is that generated by $\{\gamma^\mu_{m,k}\}^\infty_{m,k=0}$.

Clearly, the mapping $\phi(x) \to x^{1/2}\phi(x)$ is an isomorphism from $H_{\mu-\frac{1}{2}}$ into H_μ. It follows from equations (1) and (3) that

$$(h_\mu \phi)(t) = t^{-1/2}\mathcal{H}_\mu(x^{1/2}\phi)(t). \tag{4}$$

Using the fact that $\mathcal{H}_\mu$ is an automorphism on H_μ for $\mu \geq -1/2$ (see [13]), we obtain

Theorem 2.1. *For $\mu > -1/2$, the Hankel transform h_μ is an automorphism on $H_{\mu-\frac{1}{2}}$.*

We now define two modified differential operators N_μ and M_μ, and a linear integral operator N_μ^{-1} by

$$N_\mu\phi(x) = (D - \mu x^{-1})\phi(x) = x^\mu D x^{-\mu}\phi(x),$$
$$M_\mu\phi(x) = (D + \mu x^{-1} + x^{-1})\phi(x) = x^{-\mu-1} D x^{\mu+1}\phi(x),$$
$$N_\mu^{-1}\phi(x) = x^\mu \int_\infty^x t^{-\mu}\phi(t)dt,$$

which will be used to prove the inverse property $h_{\mu,k} = h^{-1}_{\mu,k}$ for $\mu \in R$.

Applying identity (4) and the Zemanian's results for $\mathcal{H}_\mu$ in [13], we can easily list the following.

Lemma 2.1. *N_μ is a continuous linear mapping from $H_{\mu-\frac{1}{2}}$ into $H_{\mu+\frac{1}{2}}$ while N_μ^{-1} is a continuous linear mapping of $H_{\mu+\frac{1}{2}}$ into $H_{\mu-\frac{1}{2}}$.*

Lemma 2.2. *The mapping $\phi \to M_\mu\phi$ is linear and continuous from $H_{\mu+\frac{1}{2}}$ into $H_{\mu-\frac{1}{2}}$. Furthermore, for any integer n and for any μ, the mapping $\phi(x) \to x^n\phi(x)$ is an isomorphism from $H_{\mu-\frac{1}{2}}$ into $H_{\mu-\frac{1}{2}+n}$.*

Lemma 2.3. *Let $\mu > -1/2$. If $\phi \in H_{\mu-\frac{1}{2}}$, then*

$$h_{\mu+1}(-t\phi) = N_\mu h_\mu \phi, \tag{5}$$
$$h_{\mu+1}(N_\mu\phi) = -x h_\mu\phi, \tag{6}$$
$$h_\mu(-t^2\phi) = M_\mu N_\mu h_\mu \phi, \tag{7}$$
$$h_\mu(M_\mu N_\mu \phi) = -x^2 h_\mu \phi, \tag{8}$$
$$h_\mu(t\phi) = M_\mu h_{\mu+1}\phi, \quad \text{if } \phi \in H_{\mu+\frac{1}{2}}, \tag{9}$$
$$h_\mu(M_\mu\phi) = x h_{\mu+1}\phi, \quad \text{if } \phi \in H_{\mu+\frac{1}{2}}. \tag{10}$$

Let $\mu \in R$ and a positive integer k such that $\mu + k > -\frac{1}{2}$. Assume that

$\phi \in H_{\mu-\frac{1}{2}}$. Define the Hankel transform of arbitrary order $h_{\mu,k}$ on $H_{\mu-\frac{1}{2}}$ by

$$\Phi(x) = h_{\mu,k}(\phi(y)) = (-1)^k x^{-k} h_{\mu+k} N_{\mu+k-1} \cdots N_{\mu+1} N_\mu \phi(y),$$

and let $\Phi(x) \in H_{\mu-\frac{1}{2}}$ and define an inverse transform $h^{-1}_{\mu,k}$ on $H_{\mu-\frac{1}{2}}$ by

$$\phi(y) = h^{-1}_{\mu,k}(\Phi(x)) = (-1)^k N^{-1}_\mu N^{-1}_{\mu+1} \cdots N^{-1}_{\mu+k-1} h_{\mu+k} x^k \Phi(x).$$

From identity (4) and the Zemanian's Lemma (on p. 164 of [13]), we have the following theorem.

Theorem 2.2. *$h_{\mu,\,k}$ is an automorphism on $H_{\mu-\frac{1}{2}}$ for $\mu \in R$. Its inverse is $h^{-1}_{\mu,\,k}$, and $h_{\mu,\,k} = h_\mu$ if $\mu > -\frac{1}{2}$.*

Note that the definition of $h_{\mu,k}$ is independent of the choice of k so long as $k + \mu > -\frac{1}{2}$. Indeed if $k > p > -\mu - \frac{1}{2}$, then $h_{\mu+p,k-p} = h_{\mu+p}$ by Theorem 2.2, hence

$$\begin{aligned} h_{\mu,\,k}\phi &= (-1)^k x^{-k} h_{\mu+k} N_{\mu+k-1} \ldots N_\mu \phi \\ &= (-1)^p x^{-p} (-1)^{k-p} x^{-(k-p)} h_{\mu+p+k-p} N_{\mu+p+k-p-1} \ldots N_{\mu+p} N_{\mu+p-1} \ldots N_\mu \phi \\ &= (-1)^p x^{-p} h_{\mu+p,k-p} N_{\mu+p-1} \ldots N_\mu \phi = (-1)^p x^{-p} h_{\mu+p} N_{\mu+p-1} \ldots N_\mu \phi = h_{\mu,p}\phi. \end{aligned}$$

Zemanian claimed in [13] (on p. 165) that $\mathcal{H}_{u,\,k} \neq \mathcal{H}^{-1}_{u,\,k}$ (which implies $h_{u,\,k} \neq h^{-1}_{u,\,k}$, since $(h_{\mu,k}\phi)(t) = t^{-1/2}\mathcal{H}_{\mu,k}(x^{1/2}\phi)(t))$ when $\mu < -1/2$. However, he did not give any counterexample. F.H. Kerr [6] introduced complex fractional powers of Hankel transforms $\mathcal{H}^\alpha_\mu$ on H_μ to show that $\mathcal{H}_{\mu,k} = \mathcal{H}^{-1}_{\mu,k}$. In the present work, we are able to give a direct and simple proof that $h_{u,\,k} = h^{-1}_{u,\,k}$ for $\mu \in R$ with the help of the following identity in [13]

$$D_x x^{-\mu} J_\mu(xy) = -y x^{-\mu} J_{\mu+1}(xy). \tag{11}$$

This inverse property will play an important role in defining the Hankel convolutions in Section 3.

Lemma 2.4. *$N_\mu h_{\mu,\,k}(\phi) = h_{\mu+1,\,k}(-y\phi)$ for $\phi \in H_{\mu-\frac{1}{2}}$.*

Proof. By definition

$$\begin{aligned} h_{u,\,k}\phi &= (-1)^k x^{-k} h_{\mu+k} N_{\mu+k-1} \cdots N_{\mu+1} N_\mu \phi(y) \\ &= (-1)^k x^{-k} h_{\mu+k} y^{\mu+k} (y^{-1}D)^k y^{-\mu}\phi(y) \\ &= (-1)^k x^{-k} \int_0^\infty y J_{\mu+k}(xy) y^{\mu+k} (y^{-1}D)^k y^{-\mu}\phi(y)dy. \end{aligned}$$

It follows that

$$N_\mu h_{u,\,k}(\phi) = (-1)^k \int_0^\infty N_\mu x^{-k} y J_{\mu+k}(xy) y^{\mu+k} (y^{-1}D)^k y^{-\mu}\phi(y)dy.$$

By equation (11), we have

$$\begin{aligned} N_\mu x^{-k} J_{\mu+k}(xy) &= x^\mu D x^{-\mu-k} J_{\mu+k}(xy) \\ &= -y x^{-k} J_{\mu+1+k}(xy). \end{aligned}$$

Hence,

$$N_\mu h_{u,\,k}(\phi) = (-1)^k x^{-k} \int_0^\infty y J_{\mu+1+k}(xy) y^{\mu+1+k} (y^{-1}D)^k y^{-\mu-1}(-y\phi(y))dy$$
$$= h_{\mu+1,\,k}(-y\phi). \quad \square$$

Theorem 2.3. *Let μ be any fixed real number and let k be any positive integer such that $\mu + k > -1/2$. Then $h_{\mu,\,k} = h^{-1}_{\mu,\,k}$ on $H_{\mu-\frac{1}{2}}$.*

Proof. By Lemma 2.4 we have

$$N_\mu h_{\mu,\,k}(\phi) = h_{\mu+1,\,k}(-y\phi).$$

Applying $N_{\mu+1}$ to both sides, we obtain

$$N_{\mu+1} N_\mu h_{\mu,\,k}(\phi) = N_{\mu+1} h_{\mu+1,\,k}(-y\phi) = h_{\mu+2,\,k}\{(-1)^2 y^2 \phi\}.$$

Repeating this process, we get

$$N_{\mu+k-1} \cdots N_{\mu+1} N_\mu h_{\mu,\,k}(\phi) = h_{\mu+k,\,k}\{(-1)^k y^k \phi\}.$$

Therefore,

$$N_{\mu+k-1} \cdots N_{\mu+1} N_\mu h_{\mu,\,k}(\phi) = (-1)^k h_{\mu+k}(y^k \phi)$$

and we finally come to

$$h_{\mu,\,k}(\phi) = (-1)^k N_\mu^{-1} N_{\mu+1}^{-1} \cdots N_{\mu+k-1}^{-1} h_{\mu+k} y^k \phi(y) = h^{-1}_{\mu,\,k}(\phi).$$

This completes the proof of Theorem 2.3. $\square$

3. The Hankel Convolution of Arbitrary Order

According to the author's knowledge, the Hankel convolution of arbitrary order (in particular, $\mu \le -1/2$) has not been investigated so far on any spaces of interest. In this section, we use the inverse property to construct the Hankel convolution of arbitrary order on the spaces $(H_{\mu-\frac{1}{2}},\, H_{\mu-\frac{1}{2}})$ and $(S_\mu,\, H_{\mu-\frac{1}{2}})$ respectively.

Let $\mu \in R$ and any positive integer k such that $\mu + k > -\frac{1}{2}$. Assume that ϕ and ψ are in $H_{\mu-\frac{1}{2}}$. Define the Hankel convolution of arbitrary order $h(x)$ of ϕ and ψ by

$$h(x) = (-1)^k x^{-k} \int_0^\infty \int_0^\infty \int_0^\infty y^{\mu+k+1} (y^{-1}D_y)^k y^{-\mu} \phi(y) u^{\mu+k+1} (u^{-1}D_u)^k u^{-\mu}$$

$$\psi(u)t^{\mu+k+1}(t^{-1}D_t)^k\{t^{-2\mu-2k}J_{\mu+k}(yt)J_{\mu+k}(ut)\}J_{\mu+k}(xt)dudydt. \quad (12)$$

It is obvious to see that this convolution is commutative and we are going to prove the following theorem.

Theorem 3.1. *Let $\mu \in R$ and any positive integer k such that $\mu+k > -\frac{1}{2}$. Then the function $h(x)$ in (12) exists and the exchange formula*

$$(h_{\mu,k}h)(t) = t^{-\mu}(h_{\mu,k}\phi)(t)(h_{\mu,k}\psi)(t)$$

holds on $(H_{\mu-\frac{1}{2}}, H_{\mu-\frac{1}{2}})$.

Proof. Clearly,

$$h(x) = (-1)^k x^{-k} h_{\mu+k} \int_0^\infty \int_0^\infty y^{\mu+k+1}(y^{-1}D_y)^k y^{-\mu}\phi(y)u^{\mu+k+1}$$
$$(u^{-1}D_u)^k u^{-\mu}\psi(u)t^{\mu+k}(t^{-1}D_t)^k\{t^{-2\mu-2k}J_{\mu+k}(yt)J_{\mu+k}(ut)\}dudy.$$

The following integral, for every k

$$\int_0^\infty \int_0^\infty y^{\mu+k+1}(y^{-1}D_y)^k y^{-\mu}\phi(y)u^{\mu+k+1}(u^{-1}D_u)^k u^{-\mu}\psi(u)$$
$$(t^{-1}D_t)^k\{t^{-2\mu-2k}J_{\mu+k}(yt)J_{\mu+k}(ut)\}dudy$$

is uniformly convergent on every compact subset of R^+ with respect to t since ϕ and ψ are in $H_{\mu-\frac{1}{2}}$. Therefore,

$$h(x) = (-1)^k x^{-k} h_{\mu+k} N_{\mu+k-1} \cdots N_\mu (t^{-\mu-2k} h_{\mu+k}(y^{\mu+k}(y^{-1}D_y)^k y^{-\mu}\phi(y))$$
$$h_{\mu+k}(u^{\mu+k}(u^{-1}D_u)^k u^{-\mu}\psi(y))) = h_{\mu,k}(t^{-\mu}h_{\mu,k}\phi h_{\mu,k}\psi).$$

The result follows immediately by $h_{\mu,k} = h_{\mu,k}^{-1}$ due to Theorem 2.3. This completes the proof of Theorem 3.1. □

In particular, we get for $\mu > -1/2$

$$h(x) = h_\mu(t^{-\mu}h_\mu\phi\, h_\mu\psi) = \int_0^\infty tJ_\mu(xt)t^{-\mu}h_\mu\phi\, h_\mu\psi dt$$
$$= \int_0^\infty t^{1-\mu}J_\mu(xt)\int_0^\infty \int_0^\infty uvJ_\mu(ut)J_\mu(vt)\phi(u)\psi(v)dudvdt$$
$$= \int_0^\infty \int_0^\infty uv\phi(u)\psi(v)dudv \int_0^\infty t^{1-\mu}J_\mu(xt)J_\mu(ut)J_\mu(vt)dt.$$

This procedure is permissible due to the Fubini Theorem. Furthermore, it is well known (see [10]) that

$$\int_0^\infty t^{1-\mu}J_\mu(xt)J_\mu(ut)J_\mu(vt)dt$$

$$= \frac{2^{1-3\mu}}{\sqrt{\pi}\Gamma(\mu+1/2)}(xuv)^{-\mu}[x^2-(u-v)^2]_+^{\mu-1/2}[(u+v)^2-x^2]_+^{\mu-1/2}$$

where

$$s_+(x) = \begin{cases} s(x) & \text{if } s(x) \geq 0, \\ 0 & \text{if } s(x) < 0. \end{cases}$$

Therefore,

$$h(x) = \frac{2^{1-3\mu}x^{-\mu}}{\sqrt{\pi}\Gamma(\mu+1/2)}\int\int_{u+v>x,\,|u-v|<x}[x^2-(u-v)^2]^{\mu-1/2} \times [(u+v)^2-x^2]^{\mu-1/2}(uv)^{1-\mu}\phi(u)\psi(v)dudv$$

which coincides with equation (2) in the introduction.

In order to extend the Hankel convolution to a larger function space, we define the space S_μ for $\mu > -1/2$ with the normal addition and scalar multiplication as

$$S_\mu = \left\{\phi \in C(R^+) \;\mid\; \int_0^\infty y^{\mu+2k+1}|\phi(y)|dy < \infty \quad \text{and} \quad k = 0, 1, \cdots\right\}.$$

Clearly $H_{\mu-\frac{1}{2}}$ is a proper subset of S_μ according to Lemma 5.2.1 in [13] and we have the following theorem.

Theorem 3.2. *The function $t^{-\mu}(h_\mu\phi)(t)$ is a multiplier of $H_{\mu-\frac{1}{2}}$ for any $\phi \in S_\mu$.*

Proof. Obviously,

$$t^{-\mu}(h_\mu\phi)(t) = \int_0^\infty yt^{-\mu}J_\mu(yt)\phi(y)dy$$

and

$$(t^{-1}D)^k t^{-\mu}J_\mu(yt) = (-y)^k t^{-\mu-k}J_{\mu+k}(yt).$$

The integral

$$\int_0^\infty y(t^{-1}D)^k t^{-\mu}J_\mu(yt)\phi(y)dy = \int_0^\infty y^{\mu+k+1}(-y)^k\frac{J_{\mu+k}(yt)}{(yt)^{\mu+k}}\phi(y)dy$$

is uniformly convergent with respect to t since the term $J_{\mu+k}(yt)/(yt)^{\mu+k}$ is bounded on $0 < yt < \infty$ and $\phi \in S_\mu$. This implies that $(t^{-1}D)^k t^{-\mu}(h_\mu\phi)(t)$ is bounded and thus a multiplier of $H_{\mu-\frac{1}{2}}$. □

Theorems 3.2 and 2.1 allow us directly define the Hankel convolution $h(x)$ in $H_{\mu-\frac{1}{2}}$ as

$$h(x) = h_\mu(t^{-\mu}h_\mu\phi\, h_\mu\psi)$$

where $(\phi,\, \psi) \in (S_\mu,\, H_{\mu-\frac{1}{2}})$.

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