Math. Nachr. **157** (1992) 243–248

On the Distributions δ^k and $(\delta')^k$

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(Received October 14, 1991)

similarly, since δ exists as $\sqrt{\delta}$ and $\sqrt[3]{\delta}$ support to be meaningless

Abstract. Powers and products of distributions have not as yet been defined to hold true in general. In this paper, we choose a fixed δ-sequence and use the concept of neutrix limit to give meaning to the distributions δ^k and $(\delta')^k$ for some k. These may be regarded as powers of DIRAC delta functions.

and the product of $\frac{1}{x} \cdot \delta(x)$

1. Introduction

One of the problems in distribution theory is the lack of definitions for products and powers of distributions in general. In physics (see e.g. [1], p. 141), one finds the need to evaluate δ^2 when calculating the transition rates of certain particle interaction. In [2], a definition for product of distributions is given using delta sequences. However, δ^2 as a product of δ with itself is shown not to exist. In [3], BREMERMANN used the CAUCHY representations of distributions with compact support to define $\sqrt{\delta_+}$ and $\log \delta_+$. Unfortunately, his definition does not carry over to $\sqrt{\delta}$ and $\log \delta$. The existence of δ^k as a power of δ has never been proved although the symbol continues to appear in print (see e.g. [4], p. 128). In this paper, we define δ^k and $(\delta')^k$ for some values of k. We use a certain δ-sequence and the concept of neutrix limit due to VAN DER CORPUT [5], [6].

and the product $r^{-k}\delta(x)$ (in n-dimensional space)

2. The Distributions δ^k for $k \in (0, 1)$ and $k = 2, 3, 4, \dots$

Let D be the space of infinitely differentiable functions with compact support in R. Choosing the δ-sequence

$$\delta_n(x) = \left(\frac{n}{\pi}\right)^{1/2} e^{-nx^2}, \quad x \in R,$$

we consider the functional value of δ_n^k for $k \in (0, 1)$. For $\varphi \in D$, we have

$$(\delta_n^k(x), \varphi(x)) = \int_{-\infty}^{+\infty} \left(\frac{n}{\pi}\right)^{k/2} e^{-knx^2} \varphi(x)\, dx.$$

Setting $x = \sqrt{\frac{1}{kn}}\, y$ and $M = \sup_{x \in R} |\varphi(x)|$, we have

$$|(\delta_n^k(x), \varphi(x))| \le M \left(\frac{n}{\pi}\right)^{k/2} \sqrt{\frac{1}{kn}} \int_{-\infty}^{+\infty} e^{-y^2} dy \to 0 \quad \text{as} \quad n \to \infty .$$

Therefore $\delta^k(x) = 0$ for $0 < k < 1$.

We define $(\delta^k(x), \varphi(x)) \triangleq N - \lim_{n \to \infty} (\delta_n^k(x), \varphi(x))$ for $k = 2, 3, \dots$ where N is the neutrix having domain $N' = \{1, 2, 3, \dots\}$ and range N'' the real numbers with negligible functions finite linear sums of functions

$$n^{\lambda} ln^{r-1} n, \; ln^r n$$

for $\lambda > 0$ and $r = 1, 2, \dots$ and all functions which converge to zero in the normal sense as n tends to infinity (see [5] or [6]).

By TAYLOR's formula

$$\varphi(x) = \sum_{i=0}^{k-2} \frac{\varphi^{(i)}(0)}{i!} x^i + \frac{\varphi^{(k-1)}(0)}{(k-1)!} x^{k-1} + \frac{\varphi^{(k)}(\xi x)}{k!} x^k$$

where $0 < \xi < 1$. Evaluating the functional value

$$(\delta_n^k(x), \varphi(x)) = \sum_{i=0}^{k-2} \frac{\varphi^{(i)}(0)}{i!} \int_{-\infty}^{+\infty} \left(\frac{n}{\pi}\right)^{k/2} e^{-knx^2} x^i dx$$

$$+ \frac{\varphi^{(k-1)}(0)}{(k-1)!} \left(\frac{n}{\pi}\right)^{k/2} \int_{-\infty}^{+\infty} e^{-knx^2} x^{k-1} dx$$

$$+ \frac{1}{k!} \left(\frac{n}{\pi}\right)^{k/2} \int_{-\infty}^{+\infty} e^{-knx^2} \varphi^{(k)}(\xi x) x^k dx \triangleq I_1 + I_2 + I_3 .$$

Setting $x = \sqrt{\frac{1}{kn}}\, y$ again, we obtain

$$I_1 = \sum_{i=0}^{k-2} \frac{\varphi^{(i)}(0)}{i!} \left(\frac{n}{\pi}\right)^{k/2} \left(\frac{1}{kn}\right)^{(i+1)/2} \int_{-\infty}^{+\infty} e^{-y^2} y^i dy .$$

Hence

$$N - \lim_{n \to \infty} I_1 = 0 .$$

Similarly

$$\lim_{n \to \infty} I_3 = 0 .$$

And

$$I_2 = \frac{\varphi^{(k-1)}(0)}{(k-1)!}\left(\frac{n}{\pi}\right)^{k/2}\left(\frac{1}{kn}\right)^{k/2}\int_{-\infty}^{+\infty} e^{-y^2}y^{k-1}dy$$

$$= \frac{\varphi^{(k-1)}(0)}{(k-1)!}\left(\frac{1}{k\pi}\right)^{k/2}\int_{-\infty}^{+\infty} e^{-y^2}y^{k-1}dy.$$

Thus

$$\delta^k(x) = \frac{(-1)^{k-1}}{(k-1)!}\frac{1}{(k\pi)^{k/2}}\int_{-\infty}^{+\infty} e^{-y^2}y^{k-1}dy\cdot\delta^{(k-1)}(x).$$

It follows that

$$\delta^{2l}(x) = 0 \quad \text{for} \quad l = 1, 2, 3, \ldots$$

$$\delta^{2l+1}(x) = \frac{2}{(2l)!\,[(2l+1)\,\pi]^{\frac{2l+1}{2}}}\int_0^{+\infty} e^{-y^2}y^{2l}dy\cdot\delta^{(2l)}(x)$$

for $l = 0, 1, 2, \ldots$ We have included $l = 0$ in the latter since this is easily shown to be true.
Using formula

$$\int_0^{+\infty} x^{2n}e^{-x^2}dx = \frac{1, 3, 5, \ldots(2n-1)}{2^{n+1}}\sqrt{\pi}\;(n \in Z_0^+),$$

we obtain

$$\delta^{2l+1}(x) = C_l\delta^{(2l)}(x),$$

where

$$C_l = \frac{1}{2^{2l}l!(2l+1)^{\frac{2l+1}{2}}\pi^l} \quad \text{for} \quad l = 0, 1, 2, \ldots$$

Now we can conclude

Theorem 1. *For* $k \in (0, 1)$, $\delta^k(x) = 0$. *For* $l = 1, 2, 3, \ldots$, $\delta^{2l}(x) = 0$. *For* $l = 0, 1, 2, 3, \ldots$, $\delta^{2l+1}(x) = C_l\delta^{(2l)}(x)$ *where*

$$C_l = \frac{1}{2^{2l}l!(2l+1)^{\frac{2l+1}{2}}\pi^l}.$$

3. The Distributions $(\delta'(x))^k$ for $k \in \left(0, \frac{1}{2}\right]$ and $k = 1, 2, 3, \ldots$

Considering the derivative of the δ-sequence $\delta_n(x)$, we have

$$\delta_n'(x) = \left(\frac{n}{\pi}\right)^{1/2} e^{-nx^2}(-2nx).$$

For arbitrary $k > 0$, we assign to the complex number $(-2)^k$ the value

$$(-2)^k = e^{k \ln(-2)} \triangleq 2^k(\cos k\pi + i \sin k\pi) = 2^k e^{ik\pi}.$$

Calculating the functional value for $k \in \left(0, \frac{1}{2}\right)$

$$(\delta'^k_n(x), \varphi(x)) \triangleq 2^k e^{k\pi i} \int_{-\infty}^{+\infty} \left(\frac{n}{\pi}\right)^{k/2} n^k x^k e^{-knx^2} \varphi(x)\, dx.$$

Making the substitution $x = \sqrt{\frac{1}{kn}}\, y$ and $M = \sup_{x \in R} |\varphi(x)|$, we get

$$|(\delta'^k_n(x), \varphi(x))| \le 2^k M \left(\frac{n}{\pi}\right)^{k/2} n^k \left(\frac{1}{kn}\right)^{\frac{k+1}{2}} \int_{-\infty}^{+\infty} e^{-y^2} y^k dy \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore, $\delta'^k(x) = 0$ for $k \in \left(0, \frac{1}{2}\right)$. For $k = \frac{1}{2}$, we have

$$\begin{aligned}
(\delta'^{1/2}_n(x), \varphi(x)) &\triangleq \left(\frac{n}{\pi}\right)^{1/4} \sqrt{2} n^{1/2} \int_{-\infty}^{+\infty} (-x)^{1/2}\, e^{-\frac{nx^2}{2}} \varphi(x)\, dx \\
&= \left(\frac{n}{\pi}\right)^{1/4} \sqrt{2} n^{1/2} \int_{0}^{+\infty} i x^{1/2} e^{-nx^2/2} \varphi(x)\, dx \\
&\quad + \left(\frac{n}{\pi}\right)^{1/4} \sqrt{2} n^{1/2} \int_{0}^{+\infty} x^{1/2}\, e^{-nx^2/2} \varphi(-x)\, dx \\
&= \left(\frac{2}{\pi}\right)^{1/4} i\varphi\left(\sqrt{\frac{2}{n}}\, \xi_1\right) \Gamma\left(\frac{3}{4}\right) + \left(\frac{2}{\pi}\right)^{1/4} \varphi\left(-\sqrt{\frac{2}{n}}\, \xi_2\right) \Gamma\left(\frac{3}{4}\right)
\end{aligned}$$

where ξ_1 and $\xi_2 \in (0, 1)$. Hence $\lim_{n \to \infty} (\delta'^{1/2}_n(x), \varphi(x)) = \sqrt{2} e^{i\frac{\pi}{4}} \left(\frac{2}{\pi}\right)^{1/4} \Gamma\left(\frac{3}{4}\right) (\delta(x), \varphi(x))$ and $\delta'^{1/2}(x) = \sqrt{2} e^{i\frac{\pi}{4}} \left(\frac{2}{\pi}\right)^{1/4} \Gamma\left(\frac{3}{4}\right) \delta(x)$.

Evaluating the functional value for $k = 1, 2, \ldots$

$$(\delta'^k_n(x), \varphi(x)) = 2^k (-1)^k \int_{-\infty}^{+\infty} \left(\frac{n}{\pi}\right)^{k/2} e^{-knx^2} n^k x^k \varphi(x)\, dx.$$

By TAYLOR's formula

$$\varphi(x) = \sum_{i=0}^{2k-2} \frac{\varphi^i(0)}{i!} x^i + \frac{\varphi^{(2k-1)}(0)}{(2k-1)!} x^{2k-1} + \frac{\varphi^{(2k)}(\xi x)}{(2k)!} x^{2k}$$

we obtain

$$(\delta'^k_n(x), \varphi(x)) = 2^k(-1)^k \sum_{i=0}^{2k-2} \frac{\varphi^{(i)}(0)}{i!} \left(\frac{n}{\pi}\right)^{k/2} n^k \int_{-\infty}^{+\infty} e^{-knx^2} x^{k+i} dx$$

$$+ 2^k(-1)^k \left(\frac{n}{\pi}\right)^{k/2} n^k \frac{\varphi^{(2k-1)}(0)}{(2k-1)!} \int_{-\infty}^{+\infty} e^{-knx^2} x^{3k-1} dx$$

$$+ 2^k(-1)^k \left(\frac{n}{\pi}\right)^{k/2} n^k \frac{1}{(2k)!} \int_{-\infty}^{+\infty} e^{-knx^2} \varphi^{(2k)}(\xi x)\, x^{3k} dx$$

$$\triangleq I_1 + I_2 + I_3 .$$

Similarly we can easily show

$$N - \lim_{n\to\infty} I_1 = 0$$

$$\lim_{n\to\infty} I_3 = 0 .$$

And $I_2 = 2^k(-1)^k \dfrac{\varphi^{(2k-1)}(0)}{\pi^{k/2} k^{3k/2}(2k-1)!} \displaystyle\int_{-\infty}^{+\infty} e^{-y^2} y^{3k-1} dy$. It follows that

$$\delta'^{2l}(x) = 0 .$$

And $\delta'^{2l+1}(x) = C'_l \delta^{(4l+1)}(x)$ where $l = 1, 2, 3, \ldots$ and $C'_l = \dfrac{1, 3, 5, \ldots (6l+1)}{2^l \pi^{\frac{l}{2}} (2l+1)^{\frac{6l+3}{2}} (4l+1)!}$.

Theorem 2. $\delta'^k(x) = 0$ for $k \in \left(0, \dfrac{1}{2}\right)$,

$$\delta'^{1/2}(x) = \sqrt{2} e^{i\pi/4} \left(\frac{2}{\pi}\right)^{1/4} \Gamma\left(\frac{3}{4}\right) \delta(x)$$

and

$$\delta'^{2l}(x) = 0 \quad \text{for} \quad l = 1, 2, \ldots$$

$$\delta'^{2l+1}(x) = C'_l \delta^{(4l+1)}(x) \quad \text{for} \quad l = 0, 1, 2, \ldots$$

where C'_l is defined above.

Remarks. We have given a class of distributions δ^k and $(\delta')^k$ which are distributional limits of the k-th power of a delta-sequence and its derivatives. They may be considered powers of distributions although not in the usual sense of product of the distribution by itself k times. It remains to show that these powers can be defined for any real k.

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