The distributional products on spheres and Pizetti's formula

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ABSTRACT

The distribution $\delta^{(k)}(r - a)$ concentrated on the sphere O_a with $r - a = 0$ is defined as

$$(\delta^{(k)}(r - a), \phi) = \frac{(-1)^k}{a^{n-1}} \int_{O_a} \frac{\partial^k}{\partial r^k}(\phi r^{n-1}) d\sigma.$$

Taking the Fourier transform of the distribution and the integral representation of the Bessel function, we obtain asymptotic expansions of $\delta^{(k)}(r - a)$ for $k = 0, 1, 2, \ldots$ in terms of $\triangle^j \delta(x_1, \ldots, x_n)$, in order to show the well-known Pizetti formula by a new method. Furthermore, we derive an asymptotic product of $\phi(x_1, \ldots, x_n) \, \delta^{(k)}(r - a)$, where ϕ is an infinitely differentiable function, based on the formula of $\triangle^m(\phi\psi)$, and hence we are able to characterize the distributions focused on spheres, which can be written as the sums of multiplet layers in the Gel'fand sense.

1. Introduction

The sequential method [1] and the complex analysis approach [2], including non-standard analysis [3], have been the main tools used for dealing with products, powers and convolutions of distributions, such as δ^2, which is needed when calculating the transition rates of certain particle interactions in physics [4]. Fisher (see [5–10], for example) has actively used the Jones δ-sequence $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots$, where $\rho(x)$ is a fixed infinitely differentiable function on R with the following properties:

(i) $\rho(x) \geq 0$,
(ii) $\rho(x) = 0$ for $|x| \geq 1$,
(iii) $\rho(x) = \rho(-x)$,
(iv) $\int_{-1}^{1} \rho(x) dx = 1$,

and the concept of the neutrix limit of van der Corput [11] to deduce numerous products, powers, convolutions, and compositions of distributions on R since 1969. The technique of neglecting appropriately defined infinite quantities and the resulting finite value extracted from the divergent integral is usually referred to as the Hadamard finite part. In fact, Fisher's method of computation can be regarded as a particular application of the neutrix calculus. This is a general principle for the discarding of unwanted infinite quantities from asymptotic expansions and has been exploited in the context of distributions by Fisher in connection with the problem of distributional multiplication, convolution and composition. To extend such an approach from the one-dimensional case to the n-dimensional case, Li et al. [12–15] constructed several workable δ-sequences on R^n for non-commutative neutrix products such as $r^{-k} \cdot \nabla\delta$ as well as $r^{-k} \cdot \triangle^l \delta$, where $\triangle$ denotes the Laplacian. Aguirre [16] used the Laurent series expansion of r^λ and derived a more general product $r^{-k} \cdot \nabla(\triangle^l \delta)$ by calculating the residue of r^λ. His approach represents another interesting example of using complex analysis to obtain products of distributions on R^n.

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The problem of defining products of distributions on a manifold (the unit sphere is a particular example) has been a serious challenge since Gel'fand [17] introduced generalized functions of special types, such as P_+^λ and $\delta^{(k)}(P)$, where

$$(\delta^{(k)}(P), \phi) = (-1)^k \int_{P=0} \omega_k(\phi).$$

Li [18] studied the products $f(P)\delta^{(k)}(P)$ and $f(P, Q)\delta(PQ)$ on regular manifolds along the differential form line. Furthermore, he used the delta sequence and the convolution given for $P = 0$ to derive an invariant theorem, that powerfully converts the products of distributions on manifolds into well defined products of a single variable. Several examples, including the products of $P_+^r(x)$ and $\delta^{(k)}(P(x))$, are presented using the invariant theorem. Aguirre [19] employed the Taylor expansion of the distribution $\delta^{(k-1)}(m^2+P)$ and gave a meaning to the product $\delta^{(k-1)}(m^2+P) \cdot \delta^{(l-1)}(m^2+P)$. In [20], Li obtained a regular product $f(r) \cdot \delta^{(k)}(r-1)$ on Ω $(=O_1)$, as well as computing several new products related to $\delta(x)$ on even-dimension spaces by a complex analysis method. Recently, Li [21] applied Pizetti's formula and a recursive structure of $\triangle^j(X^l\phi(x))$ to compute the product $X^l\delta(r-1)$. As outlined in the abstract, the goal of this work is to attempt to obtain a generalized product of $\phi(x_1, \ldots, x_n)\delta^{(k)}(r-a)$, where ϕ is an infinitely differentiable function, based on the following formula:

$$\triangle^k(\phi\psi) = \sum_{m+i+l=k} 2^i \binom{m+l}{m} \binom{k}{m+l} \nabla^i \triangle^m \phi \nabla^i \triangle^l \psi.$$

This enables us to expand every functional f of the form

$$(f, \phi) = \int_{r=a} \sum_j a_j(x)D^j\phi(x)\mathrm{d}\sigma$$

as an infinite expansion in the distributional sense.

2. Pizetti's formula

We let $\mathcal{D}(R^n)$ be the Schwartz space of the testing functions with bounded support in R^n and let $r^2 = \sum_{i=1}^n x_i^2$. The distribution $\delta(r-a)$ concentrated on the sphere O_a with $r-a=0$ is defined as

$$(\delta(r-a), \phi) = \int_{O_a} \phi \mathrm{d}\sigma$$

where $\mathrm{d}\sigma$ is the Euclidean element on the sphere $r-a=0$.

We define $S_\phi(r)$ as the mean value of $\phi(x) \in \mathcal{D}(R^n)$ on the sphere of radius r by

$$S_\phi(r) = \frac{1}{\Omega_n} \int_\Omega \phi(r\sigma)\mathrm{d}\sigma$$

where $\Omega_n = 2\pi^{\frac{n}{2}}/\Gamma(\frac{n}{2})$ is the area of the unit sphere Ω $(=O_1)$. We can write out an asymptotic expression for $S_\phi(r)$, namely

$$S_\phi(r) \sim \phi(0) + \frac{1}{2!}S_\phi''(0)r^2 + \cdots + \frac{1}{(2k)!}S_\phi^{(2k)}(0)r^{2k} + \cdots$$

$$= \sum_{k=0}^\infty \frac{\triangle^k \phi(0)r^{2k}}{2^k k! n(n+2)\cdots(n+2k-2)} \quad (\triangle \text{ is the Laplacian})$$

which is the well-known Pizetti formula and it plays an important role in the work of Li et al. [12,22–24]. Recently, it served as a foundation for building the gravity formula for the algebra (see [25]).

To the authors' knowledge, Pizetti's formula has not been proved as a convergent series for $\phi \in \mathcal{D}(R^n)$ since it appeared in [26]. Now we are going to show that it does converge by using the Fourier transform and the following formula which can be found in [27]:

$$J_\nu(x) = \frac{1}{2^\nu \sqrt{\pi} \Gamma\left(\nu + \frac{1}{2}\right)} \int_0^\pi \mathrm{e}^{ix\cos\theta} x^\nu \sin^{2\nu}\theta \mathrm{d}\theta. \tag{1}$$

The Fourier transform of $\delta(r-a)$ is defined as

$$F(\delta(r-a)) = (\delta(r-a), \mathrm{e}^{i(x,\sigma)}) = \int_{O_a} \mathrm{e}^{i(x,\sigma)}\mathrm{d}x.$$

In spherical coordinates ($r = |x| = a$, $\rho = |\sigma|$ and θ is the angle between the x and σ vectors) this becomes

$$F(\delta(r-a)) = \int \mathrm{e}^{ia\rho\cos\theta} a^{n-1} \sin^{n-2}\theta \mathrm{d}\theta \mathrm{d}\omega$$

$$= a^{n-1}\Omega_{n-1} \int_0^\pi \mathrm{e}^{ia\rho\cos\theta} \sin^{n-2}\theta \mathrm{d}\theta,$$

where $\mathrm{d}\omega$ is the element of area on the unit sphere in the $(n-1)$-dimensional subspace orthogonal to ρ.

It is known that the integral on the right-hand side can be expressed in terms of Bessel function by Eq. (1), so we obtain

$$N_n F(\delta(r-a)) = a^{n-1} \Omega_{n-1} (a\rho)^{1-\frac{1}{2}n} J_{\frac{1}{2}(n-2)}(a\rho)$$
$$= \Omega_{n-1} a^{\frac{1}{2}n} \rho^{1-\frac{1}{2}n} J_{\frac{1}{2}(n-2)}(a\rho)$$

where

$$N_n = \frac{2^{1-\frac{1}{2}n}}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)} \quad \text{and}$$

$$J_\mu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{\mu+2k}}{k!\Gamma(\mu+k+1)}.$$

Therefore,

$$F(\delta(r-a)) = 2\pi^{\frac{n}{2}} a^{n-1} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{a\rho}{2}\right)^{2k}}{k!\Gamma\left(\frac{n}{2}+k\right)}$$

in $\mathcal{Z}'(R^n) = F(\mathcal{D}'(R^n))$.

Using the identity in [17]

$$F(\triangle^k \delta(x_1,\ldots,x_n)) = (-1)^k \rho^{2k}$$

we arrive at

$$\delta(r-a) = 2\pi^{\frac{n}{2}} a^{n-1} \sum_{k=0}^{\infty} \frac{a^{2k}}{2^{2k} k!\Gamma\left(\frac{n}{2}+k\right)} \triangle^k \delta(x_1,\ldots,x_n). \tag{2}$$

Remark 1. The above formula (2) was first obtained by Aguirre and Marinelli in [24] under the condition of Pizetti's formula being a convergent series in the Schwartz space.

It follows from Ref. [16] that

$$\frac{\Omega_n \delta^{(2k)}(r)}{(2k)!} = \text{res}_{\lambda=-n-2k} r^\lambda = \frac{\Omega_n \triangle^k \delta(x_1,\ldots,x_n)\Gamma\left(\frac{n}{2}\right)}{2^k k! 2^k \Gamma\left(\frac{n}{2}+k\right)},$$

which implies

$$\triangle^k \delta(x_1,\ldots,x_n) = \frac{2^{2k} k!\Gamma\left(\frac{n}{2}+k\right)}{(2k)!\Gamma\left(\frac{n}{2}\right)} \delta^{(2k)}(r). \tag{3}$$

Hence

$$\delta(r-a) = \frac{2\pi^{\frac{n}{2}} a^{n-1}}{\Gamma\left(\frac{n}{2}\right)} \sum_{k=0}^{\infty} \frac{\delta^{(2k)}(r)}{(2k)!} a^{2k}.$$

Since

$$(\delta(r-a),\phi) = \int_{O_a} \phi d\sigma = a^{n-1} \int_\Omega \phi(r\sigma) d\sigma,$$

we come to

$$S_\phi(r) = \frac{1}{a^{n-1}\Omega_n}(\delta(r-a),\phi)$$
$$= \frac{1}{a^{n-1}\Omega_n}\left(\frac{2\pi^{\frac{n}{2}} a^{n-1}}{\Gamma\left(\frac{n}{2}\right)} \sum_{k=0}^{\infty} \frac{\delta^{(2k)}(r)}{(2k)!} r^{2k}, \phi\right)$$
$$= \sum_{k=0}^{\infty} \frac{(\delta^{(2k)}(r),\phi)}{(2k)!} r^{2k} = \sum_{k=0}^{\infty} \frac{S_\phi^{(2k)}(0)}{(2k)!} r^{2k}$$

where

$$S_\phi^{(2k)}(0) = (\delta^{(2k)}(r),\phi) = \frac{(2k)!\Gamma\left(\frac{n}{2}\right)}{2^{2k} k!\Gamma\left(\frac{n}{2}+k\right)} \triangle^k \phi(0)$$

from Eq. (3). This completes the proof of Pizetti's formula as a convergent series in the Schwartz space.

3. The generalized products on spheres

Following the Aguirre approach, we apply the Fourier transform and the following formula:

$$\int_{O_a} \delta^{(k)}(r-a)\phi \mathrm{d}x = \frac{(-1)^k}{a^{n-1}}\int_{O_a}\frac{\partial^k}{\partial r^k}(\phi r^{n-1})\mathrm{d}\sigma$$

to derive an expansion of $\delta^{(k)}(r-a)$, which will be used to study the generalized product of $\phi(x)$ and $\delta^{(k)}(r-a)$ on spheres in R^n later on.

The Fourier transform of $\delta^{(k)}(r-a)$ is defined as

$$F(\delta^{(k)}(r-a)) = (\delta^{(k)}(r-a), \mathrm{e}^{\mathrm{i}(x,\sigma)}) = \int_{O_a}\delta^{(k)}(r-a)\mathrm{e}^{\mathrm{i}(x,\sigma)}\mathrm{d}x.$$

Employing the spherical coordinates of the previous section, we come to

$$F(\delta^{(k)}(r-a)) = (-1)^k\Omega_{n-1}\int_0^\pi\frac{\partial^k}{\partial r^k}(\mathrm{e}^{\mathrm{i}r\rho\cos\theta}r^{n-1})\bigg|_{r=a}\sin^{n-2}\theta\mathrm{d}\theta.$$

It follows from Eq. (1) that

$$2^{\frac{n-2}{2}}\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)\rho^{1-\frac{n}{2}}\frac{\partial^k}{\partial r^k}(r^{\frac{n}{2}}J_{\frac{n-2}{2}}(r\rho))\bigg|_{r=a} = \int_0^\pi\frac{\partial^k}{\partial r^k}(\mathrm{e}^{\mathrm{i}r\rho\cos\theta}r^{n-1})\bigg|_{r=a}\sin^{n-2}\theta\mathrm{d}\theta.$$

Therefore,

$$F(\delta^{(k)}(r-a)) = (-1)^k 2^{\frac{n}{2}}\pi^{\frac{n}{2}}\rho^{1-\frac{n}{2}}\frac{\partial^k}{\partial r^k}(r^{\frac{n}{2}}J_{\frac{n-2}{2}}(r\rho))\bigg|_{r=a}.$$

Since

$$z(z-1)\cdots(z-k+1) = \frac{\Gamma(z+1)}{\Gamma(z-k+1)},$$

we can directly compute the factor $\frac{\partial^k}{\partial r^k}(r^{\frac{n}{2}}J_{\frac{n-2}{2}}(r\rho))|_{r=a}$ to obtain

$$\frac{\partial^k}{\partial r^k}(r^{\frac{n}{2}}J_{\frac{n-2}{2}}(r\rho))\bigg|_{r=a} = \begin{cases} \dfrac{2a^{n-1-k}\rho^{\frac{n}{2}-1}}{2^{\frac{n}{2}}}\displaystyle\sum_{j=0}^\infty\dfrac{(-1)^j\rho^{2j}\Gamma(n+2j)a^{2j}}{2^{2j}j!\Gamma\left(\frac{n}{2}+j\right)\Gamma(n+2j-k)} & \text{if } k \le n-1, \\[4mm] \dfrac{2a^{n-1-k}\rho^{\frac{n}{2}-1}}{2^{\frac{n}{2}}}\displaystyle\sum_{j=\lceil\frac{k-n+1}{2}\rceil}^\infty\dfrac{(-1)^j\rho^{2j}\Gamma(n+2j)a^{2j}}{2^{2j}j!\Gamma\left(\frac{n}{2}+j\right)\Gamma(n+2j-k)} & \text{if } k > n-1, \end{cases}$$

where $\lceil x\rceil$ represents the ceiling number of x; for example $\lceil 3.5\rceil = 4$. This implies

$$F(\delta^{(k)}(r-a)) = \begin{cases} (-1)^k 2\pi^{\frac{n}{2}}a^{n-1-k}\displaystyle\sum_{j=0}^\infty\dfrac{(-1)^j\rho^{2j}\Gamma(n+2j)a^{2j}}{2^{2j}j!\Gamma\left(\frac{n}{2}+j\right)\Gamma(n+2j-k)} & \text{if } k \le n-1, \\[4mm] (-1)^k 2\pi^{\frac{n}{2}}a^{n-1-k}\displaystyle\sum_{j=\lceil\frac{k-n+1}{2}\rceil}^\infty\dfrac{(-1)^j\rho^{2j}\Gamma(n+2j)a^{2j}}{2^{2j}j!\Gamma\left(\frac{n}{2}+j\right)\Gamma(n+2j-k)} & \text{if } k > n-1. \end{cases}$$

Again using the identity in [17]

$$F(\triangle^j\delta(x_1,\ldots,x_n)) = (-1)^j\rho^{2j}$$

we come to

$$\delta^{(k)}(r-a) = \begin{cases} (-1)^k 2\pi^{\frac{n}{2}}a^{n-1-k}\displaystyle\sum_{j=0}^\infty\dfrac{\triangle^j\delta(x_1,\ldots,x_n)\Gamma(n+2j)a^{2j}}{2^{2j}j!\Gamma\left(\frac{n}{2}+j\right)\Gamma(n+2j-k)} & \text{if } k \le n-1, \\[4mm] (-1)^k 2\pi^{\frac{n}{2}}a^{n-1-k}\displaystyle\sum_{j=\lceil\frac{k-n+1}{2}\rceil}^\infty\dfrac{\triangle^j\delta(x_1,\ldots,x_n)\Gamma(n+2j)a^{2j}}{2^{2j}j!\Gamma\left(\frac{n}{2}+j\right)\Gamma(n+2j-k)} & \text{if } k > n-1. \end{cases}$$

In particular, we have for $k=0$

$$\delta(r-a) = 2\pi^{\frac{n}{2}}a^{n-1}\sum_{j=0}^\infty\frac{a^{2j}}{2^{2j}j!\Gamma\left(\frac{n}{2}+j\right)}\triangle^j\delta(x_1,\ldots,x_n)$$

which coincides with Eq. (2) in Section 2.

It follows from Eq. (3) that

$$
\delta^{(k)}(r-a) = \begin{cases} \dfrac{(-1)^k 2\pi^{\frac{n}{2}} a^{n-1-k}}{\Gamma\left(\frac{n}{2}\right)} \displaystyle\sum_{j=0}^{\infty} \dfrac{\delta^{(2j)}(r)\,\Gamma(n+2j)\,a^{2j}}{(2j)!\,\Gamma(n+2j-k)} & \text{if } k \le n-1, \\[4mm] \dfrac{(-1)^k 2\pi^{\frac{n}{2}} a^{n-1-k}}{\Gamma\left(\frac{n}{2}\right)} \displaystyle\sum_{j=\lceil\frac{k-n+1}{2}\rceil}^{\infty} \dfrac{\delta^{(2j)}(r)\,\Gamma(n+2j)\,a^{2j}}{(2j)!\,\Gamma(n+2j-k)} & \text{if } k > n-1. \end{cases}
$$

Remark 2. The above expansions of $\delta^{(k)}(r-a)$ were initially investigated by Aguirre in [28], where he derived several implicit expressions without the cases $k \le n-1$ and $k > n-1$, but with certain additional conditions on the Gamma function, such as $\Gamma(n) = \infty$ if $n \le 0$.

Clearly, we get from $k = 0$

$$
\delta(r-a) = \frac{2\pi^{\frac{n}{2}} a^{n-1}}{\Gamma\left(\frac{n}{2}\right)} \sum_{j=0}^{\infty} \frac{\delta^{(2j)}(r)}{(2j)!} a^{2j}
$$

which is the same result as was obtained in Section 2.

In order to study the generalized product of an infinitely differentiable function $\phi(x)$ and $\delta^{(k)}(r-a)$, we need the following important lemma.

Lemma 3.1. Let $\phi(x)$ and $\psi(x)$ be infinitely differentiable functions. Then for $k = 0, 1, 2, \ldots$,

$$
\triangle^k(\phi\psi) = \sum_{m+i+l=k} 2^i \binom{m+l}{m}\binom{k}{m+l} \nabla^i \triangle^m \phi \nabla^i \triangle^l \psi \tag{4}
$$

where $\nabla = \partial/\partial x_1 + \cdots + \partial/\partial x_n$ is the gradient operator.

Proof. We use induction to prove the formula. Assume that $k = 0$; it is clearly true since both sides are equal to $\phi\psi$. Suppose it holds for some integer $k > 0$ and we need to consider the $k+1$ case. Obviously,

$$
\triangle^{k+1}(\phi\psi) = \sum_{m+i+l=k} 2^i \binom{m+l}{m}\binom{k}{m+l} \triangle(\nabla^i \triangle^m \phi \nabla^i \triangle^l \psi)
$$

and

$$
\triangle(\nabla^i \triangle^m \phi \nabla^i \triangle^l \psi) = \nabla^i \triangle^{m+1} \phi \nabla^i \triangle^l \psi + \nabla^i \triangle^m \phi \nabla^i \triangle^{l+1} \psi + 2\nabla^{i+1} \triangle^m \phi \nabla^{i+1} \triangle^l \psi
$$
$$
\stackrel{\triangle}{=} I_1 + I_2 + I_3
$$

by simple calculation.

Replacing $m + 1$ by m, we calculate

$$
\sum_{m+i+l=k} 2^i \binom{m+l}{m}\binom{k}{m+l} I_1 = \sum_{m+i+l=k} 2^i \binom{m+l}{m}\binom{k}{m+l} \nabla^i \triangle^{m+1} \phi \nabla^i \triangle^l \psi
$$
$$
= \sum_{m+i+l=k+1} 2^i \binom{m-1+l}{m-1}\binom{k}{m-1+l} \nabla^i \triangle^m \phi \nabla^i \triangle^l \psi.
$$

Similarly,

$$
\sum_{m+i+l=k} 2^i \binom{m+l}{m}\binom{k}{m+l} I_2 = \sum_{m+i+l=k} 2^i \binom{m+l}{m}\binom{k}{m+l} \nabla^i \triangle^m \phi \nabla^i \triangle^{l+1} \psi
$$
$$
= \sum_{m+i+l=k+1} 2^i \binom{m+l-1}{m}\binom{k}{m+l-1} \nabla^i \triangle^m \phi \nabla^i \triangle^l \psi.
$$

As for I_3,

$$
\sum_{m+i+l=k} 2^i \binom{m+l}{m}\binom{k}{m+l} I_3 = \sum_{m+i+l=k} 2^{i+1} \binom{m+l}{m}\binom{k}{m+l} \nabla^{i+1} \triangle^m \phi \nabla^{i+1} \triangle^l \psi
$$
$$
= \sum_{m+i+l=k+1} 2^i \binom{m+l}{m}\binom{k}{m+l} \nabla^i \triangle^m \phi \nabla^i \triangle^l \psi.
$$

By direct calculation,

$$\binom{m-1+l}{m-1}\binom{k}{m-1+l} + \binom{m+l-1}{m}\binom{k}{m+l-1} + \binom{m+l}{m}\binom{k}{m+l} = \binom{m+l}{m}\binom{k+1}{m+l}.$$

This completes the proof of the lemma. □

Theorem 3.1. *Let $\phi(x_1, \ldots, x_n) \in C^\infty(R^n)$. Then the generalized product $\phi(x_1, \ldots, x_n)$ and $\triangle^k \delta(x_1, \ldots, x_n)$ exists and*

$$\phi(x_1, \ldots, x_n) \triangle^k \delta(x_1, \ldots, x_n) = \sum_{m+i+l=k} 2^i (-1)^i \binom{m+l}{m}\binom{k}{m+l} \nabla^i \triangle^m \phi(0) \nabla^i \triangle^l \delta(x_1, \ldots, x_n).$$

Proof. Clearly, $\phi(x)\psi(x) \in \mathcal{D}(R^n)$ if $\psi(x) \in \mathcal{D}(R^n)$ and $\phi(x) \in C^\infty(R^n)$. Hence

$$(\phi(x_1, \ldots, x_n) \triangle^k \delta(x_1, \ldots, x_n), \psi(x_1, \ldots, x_n)) = (\triangle^k \delta(x_1, \ldots, x_n), \phi(x_1, \ldots, x_n)\psi(x_1, \ldots, x_n))$$

$$= \sum_{m+i+l=k} 2^i \binom{m+l}{m}\binom{k}{m+l} \nabla^i \triangle^m \phi(0, \ldots, 0) \nabla^i \triangle^l \psi(0, \ldots, 0).$$

The result follows from

$$\nabla^i \triangle^l \psi(0, \ldots, 0) = (-1)^i (\nabla^i \triangle^l \delta(x_1, \ldots, x_n), \psi(x_1, \ldots, x_n)).$$

By Eq. (3), we come to

$$\phi(x_1, \ldots, x_n)\delta^{(2k)}(r) = \frac{(2k)!\Gamma\left(\frac{n}{2}\right)}{2^{2k}k!\Gamma\left(\frac{n}{2}+k\right)} \sum_{m+i+l=k} 2^i (-1)^i \binom{m+l}{m}\binom{k}{m+l} \nabla^i \triangle^m \phi(0) \nabla^i \triangle^l \delta(x_1, \ldots, x_n).$$

In particular, we have the following products by Theorem 3.1:

$$X\delta(x_1, \ldots, x_n) = 0,$$
$$X\triangle\delta(x_1, \ldots, x_n) = -2n\nabla\delta(x_1, \ldots, x_n),$$
$$X\triangle^2 \delta(x_1, \ldots, x_n) = -4n\nabla\triangle\delta(x_1, \ldots, x_n),$$
$$X^2\triangle\delta(x_1, \ldots, x_n) = 2n\delta(x_1, \ldots, x_n)$$

where $X = \sum_1^n x_i = x_1 + x_2 + \cdots + x_n$.

Furthermore, we have the following generalized products from Theorem 3.1:

$$\phi(x_1, \ldots, x_n)\delta^{(k)}(r-a) = \begin{cases} (-1)^k 2\pi^{\frac{n}{2}} a^{n-1-k} \sum_{j=0}^{\infty} \dfrac{\phi(x_1, \ldots, x_n) \triangle^j \delta(x_1, \ldots, x_n) \Gamma(n+2j)a^{2j}}{2^{2j}j!\Gamma\left(\frac{n}{2}+j\right)\Gamma(n+2j-k)} \\ = (-1)^k 2\pi^{\frac{n}{2}} a^{n-1-k} \sum_{j=0}^{\infty} \dfrac{\Gamma(n+2j)a^{2j}}{2^{2j}j!\Gamma\left(\frac{n}{2}+j\right)\Gamma(n+2j-k)} \\ \quad \times \sum_{m+i+l=j} 2^i (-1)^i \binom{m+l}{m}\binom{j}{m+l} \nabla^i \triangle^m \phi(0)\nabla^i \triangle^l \delta(x_1, \ldots, x_n) \\ \text{if } k \leq n-1, \\ (-1)^k 2\pi^{\frac{n}{2}} a^{n-1-k} \sum_{j=\lceil \frac{k-n+1}{2} \rceil}^{\infty} \dfrac{\phi(x_1, \ldots, x_n) \triangle^j \delta(x_1, \ldots, x_n) \Gamma(n+2j)a^{2j}}{2^{2j}j!\Gamma\left(\frac{n}{2}+j\right)\Gamma(n+2j-k)} \\ = (-1)^k 2\pi^{\frac{n}{2}} a^{n-1-k} \sum_{j=\lceil \frac{k-n+1}{2} \rceil}^{\infty} \dfrac{\Gamma(n+2j)a^{2j}}{2^{2j}j!\Gamma\left(\frac{n}{2}+j\right)\Gamma(n+2j-k)} \\ \quad \times \sum_{m+i+l=j} 2^i (-1)^i \binom{m+l}{m}\binom{j}{m+l} \nabla^i \triangle^m \phi(0)\nabla^i \triangle^l \delta(x_1, \ldots, x_n) \\ \text{if } k > n-1. \quad \square \end{cases}$$

4. The characterization of multiplet layers

We consider a manifold S given by $P(x_1, x_2, \ldots, x_n) = 0$, where P is an infinitely differentiable function such that

$$\text{grad}P = \left\{ \frac{\partial P}{\partial x_1}, \frac{\partial P}{\partial x_2}, \ldots, \frac{\partial P}{\partial x_n} \right\} \neq 0$$

on S, which therefore has no singular points.

Take two functions $P(x)$ and $Q(x)$ such that the $P = 0$ and $Q = 0$ hypersurfaces have no singular points. We now assume that these surfaces fail to intersect and that the $PQ = 0$ surface also has no singular points. We have the following theorem from [18].

Theorem 4.1. *Let f be an infinitely differentiable function of two variables. Then the product $f(P, Q)\delta(PQ)$ exists and*

$$f(P, Q)\delta(PQ) = \frac{f(0, Q)}{Q}\delta(P) + \frac{f(P, 0)}{P}\delta(Q).$$

In particular, we get

$$P\delta(PQ) = \delta(Q) \quad and \quad Q\delta(PQ) = \delta(P),$$
$$\delta(PQ) = Q^{-1}\delta(P) + P^{-1}\delta(Q).$$

If Q is non-vanishing function, we obtain $\delta(PQ) = Q^{-1}\delta(P)$ from $Q\delta(PQ) = \delta(P)$. Then taking the derivative with respect to P gives

$$Q\delta'(PQ) = Q^{-1}\delta'(P) \quad implying \quad \delta'(PQ) = Q^{-2}\delta'(P).$$

In a similar way, we have for any $k \geq 0$ and $Q \neq 0$ that

$$\delta^{(k)}(PQ) = Q^{-(k+1)}\delta^{(k)}(P),$$

which appeared in [18].

We can easily obtain an infinite expansion for $\delta^{(k)}((r - a)Q)$ since $Q^{-(k+1)}$ is an infinitely differentiable function from the previous result.

According to Gel'fand, a functional of the form $\mu(x)\delta^{(k-1)}(P)$, or

$$\int \mu(x)\delta^{(k-1)}(P) = \int_{P=0} \omega_{k-1}(\mu\phi),$$

is called a k-fold layer or distribution on the $P = 0$ hypersurface, where

$$\omega_k(\phi) = \frac{\partial^k}{\partial P^k}\left(\frac{\phi}{\partial P/\partial x_j}\right) dx_1 \cdots dx_{j-1}dx_{j+1} \cdots dx_n.$$

In particular, a singlet or simple layer ($k = 1$) is given by

$$(\mu(x)\delta(P), \phi) = \int_{P=0} \mu\phi\omega = \int_{P=0} \omega_0(\mu\phi)$$

while a doublet or double layer ($k = 2$) is given by

$$(\mu\delta'(P), \phi) = \int_{P=0} \omega_1(\mu\phi).$$

The function $\mu(x)$ in these expressions is called the density of the corresponding layer.

The definition that we have given would not be consistent if it were to depend on the form in which the $P = 0$ equation is written. It is found, however, that the statement that some functional f is a k-fold layer is independent of the form of this equation, and that if it is transformed from $P = 0$ to $a(x)P = 0$, where $a(x)$ is some non-vanishing function, only the expression for $\mu(x)$ will change. This can be seen from the following:

$$\mu(x)\delta^{(k-1)}(aP) = \mu(x)a(x)\delta^{(k-1)}(P) = \mu_1(x)\delta^{(k-1)}(P).$$

It is well-known that in one dimension every functional concentrated on a point is a linear combination of the delta function and its derivatives. For $n > 1$, we have a similar role played by generalized functions, $\delta(P), \delta'(P), \ldots, \delta^{(k)}(P)$ (the derivatives of $\delta(P)$ with respect to the argument P). We wish to show that every functional f of the form

$$(f, \phi) = \int_{P=0} \sum_j a_j(x)D^j\phi(x)d\sigma$$

can be written as the sum of multiplet layers. From the above, we may use any convenient form to specify the $P = 0$ surface. Let us assume that we have written it in a way that $P(x)$ is the distance from x to the surface, so that the associated differential form coincides with the Euclidean element of area of $d\sigma$. Then

$$(f, \phi) = \int_{P=0} \sum_j a_j(x)D^j\phi(x)\omega = \int_{P=0} \omega_0\left(\sum_j a_j(x)D^j\phi(x)\right)$$

$$= \left(\delta(P), \sum a_j(x) D^j \phi(x) \right) = \sum_j (-1)^j (D^j a_j(x) \delta(P), \phi(x))$$

$$= \sum_j (-1)^j \left(\sum_k a_{jk}(x) \delta^{(k)}(P), \phi \right) = \left(\sum_k b_k(x) \delta^{(k)}(P) \phi \right),$$

where

$$b_k(x) = \sum_j (-1)^j a_{jk}(x).$$

Hence

$$f = \sum_k b_k(x) \delta^{(k)}(P).$$

In particular, if $P = r - a$ we can obtain the product of $b_k(x)\delta^{(k)}(r-a)$ since $b_k(x) \in C^\infty(R^n)$ from the previous section. Therefore, we could write out an infinite series for such an f in the distributional sense. □

Acknowledgements

References

[1] P. Antosik, J. Mikusiński, R. Sikorski, Theory of Distributions, The Sequential Approach, PWN—Polish Scientific Publishers, Warsaw, 1973.
[2] J.H. Bremermann, Distributions, Complex Variables, and Fourier Transforms, Addison-Wesley, Reading, MA, 1965.
[3] B.H. Li, Non-standard analysis and multiplication of distributions, Sci. Sin. 21 (1978) 561–585.
[4] S. Gasiorowicz, Elementary Particle Physics, J. Wiley and Sons, Inc., NY, 1966.
[5] B. Fisher, The product of distributions, Q. J. Math. Oxford (2) 22 (1971) 291–298.
[6] B. Fisher, On defining the product of distributions, Math. Nachr. 99 (1980) 239–249.
[7] B. Fisher, On defining the convolution of distributions, Math. Nachr. 106 (1982) 261–269.
[8] B. Fisher, K. Taş, The convolution of functions and distributions, J. Math. Anal. Appl. 306 (2005) 364–374.
[9] B. Fisher, K. Taş, On the composition of the distributions x_+^λ and x_+^μ, J. Math. Anal. Appl. 318 (2006) 102–111.
[10] B. Fisher, E. Özçağ, Ü. Gülen, A theorem on the commutative neutrix product of distributions, Sarajevo J. Math. 14 (2005) 235–242.
[11] J.G. van der Corput, Introduction to the neutrix calculus, J. Anal. Math. 7 (1959–1960) 291–398.
[12] C.K. Li, B. Fisher, Examples of the neutrix product of distributions on R^m, Radiat. Mat. 6 (1990) 129–137.
[13] L.Z. Cheng, C.K. Li, A commutative neutrix product of distributions on R^m, Math. Nachr. 151 (1991) 345–356.
[14] C.K. Li, The product of r^{-k} and $\nabla\delta$, Internat. J. Math. Math. Sci. 24 (2000) 361–369.
[15] C.K. Li, An approach for distributional products on R^m, Integral Transforms Spec. Funct. 16 (2005) 139–151.
[16] M.A. Aguirre, A convolution product of (2j)-th derivative of Dirac's delta in r and multiplicative distributional product between r^{-k} and $\nabla(\triangle^j \delta)$, Internat. J. Math. Math. Sci. 13 (2003) 789–799.
[17] I.M. Gel'fand, G.E. Shilov, Generalized Functions, vol. I, Academic Press, 1964.
[18] C.K. Li, The products of distributions on manifolds and invariant theorem, J. Anal. Appl. 6 (2008) 77–95.
[19] M.A. Aguirre, The expansion in series (of Taylor Types) of $(k-1)$ derivative of Dirac's delta in $m^2 + P$, Integral Transforms Spec. Func. 14 (2003) 117–127.
[20] C.K. Li, The products on the unit sphere and even-dimension spaces, J. Math. Anal. Appl. 305 (2005) 97–106.
[21] C.K. Li, Several products of distributions on manifolds, Novi Sad J. Math. 39 (2009) 31–46.
[22] C.K. Li, A review on the products of distributions, in: Mathematical Methods in Engineering, Springer, 2007, pp. 71–96.
[23] M.A. Aguirre, C.K. Li, The distributional products of particular distributions, Appl. Math. Comput. 187 (2007) 20–26.
[24] M.A. Aguirre, C. Marinelli, The series expansion of $\delta^{(k)}(r-c)$, Math. Notae 35 (1991) 53–61.
[25] A.H. Cook, The external gravity field of a rotating spheroid to the order of e^3, Geophys. J. Int. 2 (2007) 199–214.
[26] R. Courant, D. Hilbert, Methods of Mathematical Physics, vol. II, Interscience, New York, 1962.
[27] I.S. Gradshteyn, I.M. Ryzhik, Tables of Integrals, Series, and Products, Academic Press, 1980.
[28] M.A. Aguirre, The convolution product of $\delta^{(k)}(|x|-c) * \delta^{(l)}(|x|-c)$, Thai J. Math. (2009) (in press).