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An asymptotic product for $X^s\delta^{(k)}(r^2 - t^2)$

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Abstract: For any Schwartz testing function ϕ, the distribution $\delta^{(k)}(r^2 - t^2)$ focused on the sphere O_t of $r = t$ in R^n is defined by

$$(\delta^{(k)}(r^2 - t^2), \phi) = \frac{(-1)^k}{2t^{n-1}} \int_{O_t} \left(\frac{\partial}{2r\partial r}\right)^k (\phi r^{n-2}) dO_t,$$

which is the solution of the wave equation with the initial conditions described below in a space of odd dimension:

$$(\triangle - \frac{\partial^2}{\partial t^2})u = 0$$
$$u(x,0) = 0, \quad \frac{\partial u(x,0)}{\partial t} = (-1)^k 2\pi^{k+1}\delta(x).$$

We apply the well-known Pizzetti's formula

$$\begin{aligned} S_\phi(r) &\sim \phi(0) + \frac{1}{2!}S''_\phi(0)r^2 + \cdots + \frac{1}{(2k)!}S_\phi^{(2k)}(0)r^{2k} + \cdots \\ &= \sum_{k=0}^{\infty} \frac{\triangle^k \phi(0) r^{2k}}{2^k\, k!\, n(n+2)\cdots(n+2k-2)} \end{aligned}$$

to derive an asymptotic expansion for the distribution $\delta^{(k)}(r^2 - t^2)$ and obtain an asymptotic product for $X^s\, \delta^{(k)}(r^2 - t^2)$ based on the formula

$$\triangle^k(\phi\psi) = \sum_{m+i+l=k} 2^i \binom{m+l}{m}\binom{k}{m+l} \nabla^i \triangle^m \phi \cdot \nabla^i \triangle^l \psi.$$

This product should have potential applications in seeking certain solutions for

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the differential equations involving the gradient operator ∇ in distributional sense.

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1. Introduction

The singular function $\delta(x)$, which is widely used in physics and mathematics, was introduced by Dirac in 1920 as follows:

(i) $\delta(x) = 0$ for $x \neq 0$,

(ii) $\delta(x) = \infty$ for $x = 0$, and

(iii) $\int_{-\infty}^{\infty} \delta(x) f(x) dx = f(0)$.

It is clear to see that the above definition of $\delta(x)$ contradicts with the integral theory in terms of Lebesgue sense, and hence it can not be properly defined within the framework of classical function theory. In elementary particle physics, one finds the need to evaluate δ^2 when calculating the transition rates of certain particle interactions [1]. Schwartz [2] established the theory of distributions by treating singular functions as linearly continuous functionals on the testing function space whose elements have compact support. Although they are of great importance to quantum field theory as well as differential equations, it is difficult to define products, convolutions, and compositions of distributions in general. The sequential method ([3], [4], [5], [6], [7], [8], [9] and [10]) and complex analysis approach ([11], [12] and [13]), including nonstandard analysis [14], have been the main tools in dealing with those non-linear operations of distributions in the space $\mathcal{D}'(R^n)$ with many results. However, little progress has been made towards obtaining asymptotic products for complex distributional multiplications that cannot be carried out by any existing methods. As outlined in the abstract, we will develop an asymptotic expansion of the distribution $\delta^{(k)}(r^2 - t^2)$ by Pizzetti's formula and initiate a move to obtaining an asymptotic product for $X^s \, \delta^{(k)}(r^2 - t^2)$ that failed to be computed by other techniques.

To make this paper as self-contained as possible, we start with the concept of the gradient inner product of two functions and several theorems given in [15].

Let $\phi(x)$ and $\psi(x)$ be infinitely differentiable functions of n variables. The gradient inner product $\phi \cdot \psi$ is the operation satisfying the following properties:

(i) $\nabla^j \phi \cdot \psi = (\nabla^j \phi)\psi$;

(ii) $\phi \cdot \nabla^j \psi = \phi(\nabla^j \psi)$;

(iii) $\nabla^j \phi \cdot \nabla^j \psi = \sum_{i=1}^{n} \frac{\partial^j}{\partial x_i^j}\phi \frac{\partial^j}{\partial x_i^j}\psi$;

where $\nabla = \partial/\partial x_1 + \cdots + \partial/\partial x_n$ is the gradient operator and $j = 0, 1, 2, \cdots$

Let $X = \sum_{i=1}^{n} x_i$ and $\triangle = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2$. Then it follows from a simple calculation that

$$
\begin{aligned}
&\phi \cdot \psi = \phi\psi; \\
&\nabla X \cdot \phi = n\phi; \\
&\nabla X \cdot \nabla\phi = \nabla\phi; \\
&\nabla^2 X^2 \cdot \nabla^2\phi = 2\nabla^2\phi; \\
&\nabla^s X^s \cdot \nabla^s\phi = s!\nabla^s\phi; \\
&\triangle(\phi \cdot \psi) = \triangle(\phi\psi) = \triangle\phi \cdot \psi + \phi \cdot \triangle\psi + 2\nabla\phi \cdot \nabla\psi; \\
&\triangle(\nabla\phi \cdot \nabla\psi) = \nabla\triangle\phi \cdot \nabla\psi + \nabla\phi \cdot \nabla\triangle\psi + 2\nabla^2\phi \cdot \nabla^2\psi.
\end{aligned}
$$

for $s = 0, 1, \cdots$.

Lemma 1.1. *Let $\phi(x)$ and $\psi(x)$ be infinitely differentiable functions. Then for $k = 0, 1, 2, \cdots$*

$$\triangle^k(\phi\psi) = \sum_{m+i+l=k} 2^i \binom{m+l}{m} \binom{k}{m+l} \nabla^i \triangle^m \phi \cdot \nabla^i \triangle^l \psi \tag{1}$$

Theorem 1.2. *Let $\phi(x) \in C^\infty(R^n)$. Then the distributional product $\phi(x)$ and $\triangle^k\delta(x)$ exists and*

$$\phi(x)\,\triangle^k\delta(x) = \sum_{m+i+l=k} 2^i(-1)^i \binom{m+l}{m} \binom{k}{m+l} \nabla^i \triangle^m \phi(0) \cdot \nabla^i \triangle^l \delta(x). \tag{2}$$

for $k = 0, 1, 2, \cdots$.

It follows from Theorem 1.2 that

$$X\delta(x) = 0,$$

$$
\begin{aligned}
&X\triangle\delta(x) = -2\nabla\delta(x),\\
&X\triangle^2\delta(x) = -4\nabla\triangle\delta(x),\\
&X\triangle^k\delta(x) = -2k\nabla\triangle^{k-1}\delta(x),\\
&X^2\triangle\delta(x) = 2n\delta(x),\\
&X^2\triangle^k\delta(x) = 2nk\triangle^{k-1}\delta(x) + 2^2k(k-1)\nabla^2\triangle^{k-2}\delta(x),\\
&X^3\triangle\delta(x) = 0,\\
&X^3\triangle^k\delta(x) = -12nk(k-1)\nabla\triangle^{k-2}\delta(x) - 2^3k(k-1)(k-2)\nabla^3\triangle^{k-3}\delta(x),
\end{aligned}
$$

for $k = 0, 1, 2, \cdots$.

On the other hand, we can directly use an induction to show that

$$\triangle^k(X\phi) = 2k\nabla\triangle^{k-1}\phi + X\triangle^k\phi$$

which also claims that $X\triangle^k\delta(x) = -2k\nabla\triangle^{k-1}\delta(x)$ in the above. It is obviously true for $k = 0$. Assume it holds for the case of $k > 0$, that is

$$\triangle^k(X\phi) = 2k\nabla\triangle^{k-1}\phi + X\triangle^k\phi.$$

Therefor,

$$
\begin{aligned}
&\triangle^{k+1}(X\phi) = \triangle\triangle^k(X\phi) = \triangle(2k\nabla\triangle^{k-1}\phi + X\triangle^k\phi)\\
&= 2k\nabla\triangle^k\phi + \triangle(X\triangle^k\phi)\\
&= 2(k+1)\nabla\triangle^k\phi + X\triangle^{k+1}\phi.
\end{aligned}
$$

Similarly, we can get

$$\triangle^k(X^2\phi(x))\Big|_{x=0} = 2nk\triangle^{k-1}\phi(0) + 2^2k(k-1)\nabla^2\triangle^{k-2}\phi(0),$$

which claims

$$X^2\triangle^k\delta(x) = 2nk\triangle^{k-1}\delta(x) + 2^2k(k-1)\nabla^2\triangle^{k-2}\delta(x).$$

However, it seems very difficult or impossible to write out an explicit formula of the general product $X^s\triangle^k\delta(x)$, for any positive integer s, by a direct computation [16]. We shall employ Theorem 1.2 to give a complete proof of Theorem 1.3 to show readers a way of computing the gradient inner products, although it can be found in [15]. This product plays an important role in obtaining an asymptotic product for $X^s\,\delta^{(k)}(r^2 - t^2)$ in the next section.

Theorem 1.3. *The distributional product $X^s\triangle^k\delta(x)$ exists and*

$$X^s\triangle^k\delta(x)=\begin{cases}2^sk!s!\displaystyle\sum_{j=0}^{s/2}\frac{n^j\nabla^{s-2j}\triangle^{k-s+j}\delta(x)}{2^{2j}j!(k-s+j)!(s-2j)!} & \textit{if } s \textit{ is even,}\\ -2^sk!s!\displaystyle\sum_{j=0}^{\lfloor s/2\rfloor}\frac{n^j\nabla^{s-2j}\triangle^{k-s+j}\delta(x)}{2^{2j}j!(k-s+j)!(s-2j)!} & \textit{if } s \textit{ is odd.}\end{cases}$$

where $\triangle^{-p}=0$ for any positive integer p and $k,s=0,1,2,\cdots$.

Proof. Assume $\phi(x)=X^s$ and s is even. By Theorem 1.2,

$$\phi(x)\triangle^k\delta(x)=\sum_{m+i+l=k}2^i(-1)^i\binom{m+l}{m}\binom{k}{m+l}\nabla^i\triangle^m\phi(0)\cdot\nabla^i\triangle^l\delta(x).$$

Note that all non-zero terms in the above sum require $2m+i=s$. So,

$$\begin{aligned}&\phi(x)\triangle^k\delta(x)\\ &=2^s(-1)^s\binom{k-s}{0}\binom{k}{k-s}\nabla^s\phi(0)\cdot\nabla^s\triangle^{k-s}\delta(x)+\\ &\quad 2^{s-2}(-1)^{s-2}\binom{k-s+2}{1}\binom{k}{k-s+2}\nabla^{s-2}\triangle\phi(0)\cdot\nabla^{s-2}\triangle^{k-s+1}\delta(x)+\\ &\quad 2^{s-4}(-1)^{s-4}\binom{k-s+4}{2}\binom{k}{k-s+4}\nabla^{s-4}\triangle^2\phi(0)\cdot\nabla^{s-4}\triangle^{k-s+2}\delta(x)+\\ &\quad\cdots+\\ &\quad 2^0(-1)^0\binom{k-1}{s/2}\binom{k}{k-1}\nabla^0\triangle^{s/2}\phi(0)\cdot\nabla^0\triangle^{k-s/2}\delta(x).\end{aligned}$$

Clearly we have by the gradient inner product

$$\begin{aligned}&\nabla^s\phi(0)\cdot\nabla^s\triangle^{k-s}\delta(x)=s!\nabla^s\triangle^{k-s}\delta(x),\\ &\nabla^{s-2}\triangle\phi(0)\cdot\nabla^{s-2}\triangle^{k-s+1}\delta(x)=ns!\nabla^{s-2}\triangle^{k-s+1}\delta(x),\\ &\cdots\\ &\nabla^0\triangle^{s/2}\phi(0)\cdot\nabla^0\triangle^{k-s/2}\delta(x)=n^{s/2}s!\nabla^0\triangle^{k-s/2}\delta(x).\end{aligned}$$

Therefore,

$$\phi(x)\triangle^k\delta(x)$$

$$
\begin{aligned}
&= 2^s \frac{k!s!}{0!(k-s)!s!} \nabla^s \triangle^{k-s} \delta(x) + \\
&\quad 2^{s-2} \frac{nk!s!}{1!(k-s+1)!(s-2)!} \nabla^{s-2} \triangle^{k-s+1} \delta(x) + \\
&\quad 2^{s-4} \frac{n^2 k!s!}{2!(k-s+2)!(s-4)!} \nabla^{s-4} \triangle^{k-s+2} \delta(x) + \\
&\quad \cdots + \\
&\quad 2^0 \frac{n^{s/2} k!s!}{(s/2)!(k-s/2)!0!} \nabla^0 \triangle^{k-s/2} \delta(x) \\
&= 2^s k!s! \sum_{j=0}^{s/2} \frac{n^j \nabla^{s-2j} \triangle^{k-s+j} \delta(x)}{2^{2j} j!(k-s+j)!(s-2j)!}.
\end{aligned}
$$

The case that s is odd follows similarly. This completes the proof of the theorem.

Theorem 1.4. *Let $f(x) \in C^\infty(R)$. Then the distributional product $f(X)\triangle^k \delta(x)$ exists and*

$$
f(X)\triangle^k \delta(x) = \sum_{m+i+l=k} 2^i (-1)^i \binom{m+l}{m} \binom{k}{m+l} n^m f^{(2m+i)}(0) \nabla^i \triangle^l \delta(x).
$$

for $k = 0, 1, 2, \cdots$.

Proof. It easily follows from Theorem 1.2.

We define $S_\phi(r)$ as the mean value of $\phi(x) \in \mathcal{D}(R^n)$ on the sphere of radius r by

$$
S_\phi(r) = \frac{1}{\Omega_n} \int_\Omega \phi(x) d\Omega
$$

where $\Omega_n = 2\pi^{\frac{n}{2}}/\Gamma(\frac{n}{2})$ is the surface area of unit sphere $r = 1$ and $d\Omega$ is the hypersurface element on it. We can write out an asymptotic expression for $S_\phi(r)$ (see [17]), namely

$$
\begin{aligned}
S_\phi(r) &\sim \phi(0) + \frac{1}{2!} S''_\phi(0) r^2 + \cdots + \frac{1}{(2k)!} S^{(2k)}_\phi(0) r^{2k} + \cdots \\
&= \sum_{k=0}^{\infty} \frac{\triangle^k \phi(0) r^{2k}}{2^k\, k!\, n(n+2)\cdots(n+2k-2)}
\end{aligned}
$$

which is the well-known Pizzetti's formula and it plays an important role in the work of Li, Aguirre and Fisher ([8], [9] [10], [18], [19], [20] and [16]). Recently, it served as a foundation of building the gravity formula on the algebra (see [21]).

Remark 1. Pizzetti's formula is not a convergent series for $\phi \in \mathcal{D}(R^n)$ from the counterexample below.

$$\phi(x) = \begin{cases} \exp\{-\frac{1}{r^2(1-r^2)}\} & \text{if } 0 < r < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\phi(x) \in \mathcal{D}(R^n)$ and $S_\phi(r) \neq 0$ for $0 < r < 1$, but the series in the formula is identically equal to zero. Obviously, $S_\phi(r) \to 0$ as $r \to 0$. However, it converges in the space of analytic functions from reference [22].

Using a slightly different approach from one presented in [22], we will show that the distribution $u = \delta^{(k)}(r^2-t^2)$ is the solution of the wave equation with the initial conditions, given below, in a space of odd dimension for some value k. Following our method, we will derive an asymptotic expansion for $\delta^{(k)}(r^2-t^2)$ in terms of $\triangle^j\delta(x)$ to obtain an asymptotic product for $X^s\delta^{(k)}(r^2-t^2)$.

$$\begin{aligned} &(\triangle - \frac{\partial^2}{\partial t^2})u = 0 \\ &u(x,0) = 0, \qquad \frac{\partial u(x,0)}{\partial t} = (-1)^k 2\pi^{k+1}\delta(x). \end{aligned}$$

It follows from Gel'fand [22] that

$$\begin{aligned} &\frac{\partial}{\partial x_j}\delta^{(k)}(P) = \frac{\partial P}{\partial x_j}\delta^{(k+1)}(P), \\ &P\,\delta(P) = 0, \\ &P\,\delta'(P) + P\,\delta(P) = 0, \\ &P\,\delta''(P) + 2\delta'(P) = 0, \\ &\cdots\cdots \\ &P\,\delta^{(k)}(P) + k\,\delta^{(k-1)}(P) = 0, \\ &\cdots\cdots \end{aligned}$$

where P is a regular manifold. Therefore,

$$(r^2-t^2)\,\delta^{(k+2)}(r^2-t^2) = -(k+2)\delta^{(k+1)}(r^2-t^2). \tag{3}$$

Clearly,

$$\frac{\partial}{\partial x_j}\delta^{(k)}(r^2-t^2) = 2x_j\delta^{(k+1)}(r^2-t^2),$$

$$\frac{\partial^2}{\partial x_j^2}\delta^{(k)}(r^2-t^2) = 2\delta^{(k+1)}(r^2-t^2) + 4x_j^2\delta^{(k+2)}(r^2-t^2).$$

So that

$$\begin{aligned}\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2} &= 2n\delta^{(k+1)}(r^2-t^2) + 4(r^2-t^2)\delta^{(k+2)}(r^2-t^2) + 4t^2\delta^{(k+2)}(r^2-t^2) \\ &= (2n-4(k+2))\delta^{(k+1)}(r^2-t^2) + 4t^2\delta^{(k+2)}(r^2-t^2).\end{aligned}$$

using equation (3).

Similarly,

$$\frac{\partial^2}{\partial t_j^2}\delta^{(k)}(r^2-t^2) = -2\delta^{(k+1)}(r^2-t^2) + 4t^2\delta^{(k+2)}(r^2-t^2),$$

which implies

$$\triangle\delta^{(k)}(r^2-t^2) - \frac{\partial^2}{\partial t_j^2}\delta^{(k)}(r^2-t^2) = (2n-4k-6)\delta^{(k+1)}(r^2-t^2).$$

This disappears when $k=(n-3)/2$, so that if n is odd and $k=(n-3)/2$, the distribution $\delta^{(k)}(r^2-t^2)$ is a solution of the wave equation

$$(\triangle - \frac{\partial^2}{\partial t^2})u = 0.$$

In particular, $\delta(r^2-t^2)$ is a solution of the above equation if $n=3$. Furthermore, we note that

$$(\delta^{(k)}(r^2-t^2),\, \phi) = \frac{(-1)^k}{2}\int_\Omega \left[\left(\frac{\partial}{2r\partial r}\right)^k (\phi r^{n-2})\right]_{r=t} d\Omega,$$

which implies that the integrand for $n=2k+3$ becomes

$$\left(\frac{\partial}{2r\partial r}\right)^k (\phi r^{2k+1}).$$

Each application of the operator $r^{-1}\partial/\partial r$ reduces the power of r by two. After k such operations on ϕr^{2k+1}, we obtain a sum each of which contains r at least. We now set $r=t$ and allow t to approach zero, getting

$$\lim_{t\to 0}\delta^{(k)}(r^2-t^2) = 0.$$

To show that it satisfies the second initial condition we have

$$\frac{\partial\delta^{(k)}(r^2-t^2)}{\partial t}=-2t\delta^{(k+1)}(r^2-t^2),$$

which yields when applied to ϕ

$$2t\frac{(-1)^k}{2}\int_{\Omega}\left[\left(\frac{\partial}{2r\partial r}\right)^{k+1}(\phi r^{2k+1})\right]_{r=t}d\Omega. \tag{4}$$

On the other hand,

$$\left(\frac{\partial}{2r\partial r}\right)^{k+1}(S_\phi(r)r^{2k+1})=\frac{1}{\Omega_n}\int_{\Omega}\left(\frac{\partial}{2r\partial r}\right)^{k+1}(\phi r^{2k+1})d\Omega.$$

Obviously,

$$\left(\frac{\partial}{2r\partial r}\right)^{k+1}(S_\phi(r)r^{2k+1})=\sum_{j=0}^{k+1}\binom{k+1}{j}\left(\frac{\partial}{2r\partial r}\right)^{j}S_\phi(r)\left(\frac{\partial}{2r\partial r}\right)^{k+1-j}r^{2k+1}.$$

Hence equation (4) becomes

$$t(-1)^k\Omega_n\sum_{j=0}^{k+1}\binom{k+1}{j}\left(\frac{\partial}{2r\partial r}\right)^{j}S_\phi(r)\left(\frac{\partial}{2r\partial r}\right)^{k+1-j}r^{2k+1}\Bigg|_{r=t}$$

and let t approach zero, we get

$$(-1)^k\frac{(2k+1)!!}{2^{k+1}}\Omega_nS_\phi(0)=(-1)^k\frac{(2k+1)!!}{2^{k+1}}\Omega_n\phi(0)$$

since only the first term in the sum with $j=0$ survives. Clearly,

$$\Omega_n=\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}=\frac{2\pi^{\frac{2k+3}{2}}}{\Gamma(\frac{2k+3}{2})}=\frac{2^{k+2}}{(2k+1)!!}\pi^{k+1},$$

so that we finally come to

$$\frac{\partial\delta^{(k)}(r^2-t^2)}{\partial t}\Bigg|_{t=0}=(-1)^k2\pi^{k+1}\delta(x),$$

which completes our proof.

Remark 2. It is clear to see that the distributional product

$$(r^2-t^2)\,\delta^{(k+2)}(r^2-t^2)=-(k+2)\delta^{(k+1)}(r^2-t^2)$$

plays a key role in obtaining the solution of the wave equation with the initial conditions.

2. An Asymptotic Product

For a testing function $\phi \in \mathcal{D}(R^n)$, the distribution $\delta(r-t)$ is defined by

$$(\delta(r-t),\, \phi) = \frac{1}{t^{n-1}} \int_{O_t} \phi r^{n-1} dO_t.$$

It follows from reference [15] that

$$\delta(r-t) \sim 2\pi^{\frac{n}{2}} t^{n-1} \sum_{k=0}^{\infty} \frac{t^{2k} \triangle^k \delta(x)}{2^{2k}\, k!\, \Gamma(\frac{n}{2}+k)}, \tag{5}$$

which is equivalent to the well-known Pizzetti's formula. In particular, we obtain an asymptotic expression

$$\delta(r-1) \sim 2\pi^{\frac{n}{2}} \sum_{k=0}^{\infty} \frac{\triangle^k \delta(x)}{2^{2k}\, k!\, \Gamma(\frac{n}{2}+k)}. \tag{6}$$

From equation (5), we can write out

$$\begin{aligned} \delta(r-t) \quad &\sim \quad \Omega_n t^{n-1}\delta(x) + \frac{\Omega_n\, t^{n+1}}{2n} \triangle\delta(x) + \frac{\Omega_n t^{n+3}}{4n(2n+4)} \triangle^2\delta(x) + \cdots \\ &+ \quad \frac{2\pi^{\frac{n}{2}} t^{n-1+2k}}{2^{2k}\, k!\, \Gamma(\frac{n}{2}+k)} \triangle^k\delta(x) + \cdots . \end{aligned}$$

It follows from reference [12] that

$$\frac{\Omega_n\, \delta^{(2k)}(r)}{(2k)!} = \mathrm{res}_{\lambda=-n-2k}\, r^\lambda = \frac{\Omega_n \triangle^k\delta(x)\, \Gamma(\frac{n}{2})}{2^k\, k!\, 2^k\, \Gamma(\frac{n}{2}+k)},$$

which implies

$$\triangle^k \delta(x) = \frac{2^{2k}\, k!\, \Gamma(\frac{n}{2}+k)}{(2k)!\, \Gamma(\frac{n}{2})} \delta^{(2k)}(r). \tag{7}$$

So that

$$\begin{aligned} \delta(r-t) \quad &\sim \quad \Omega_n t^{n-1} \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} \delta^{(2k)}(r) \\ &= \quad \Omega_n t^{n-1}\delta(r) + \frac{\Omega_n t^{n+1}}{2!} \delta^{(2)}(r) + \frac{\Omega_n t^{n+3}}{4!} \delta^{(4)}(r) + \cdots . \end{aligned}$$

Note that $\delta(r)=\delta(x)$, since

$$(\delta(r),\,\phi(x)) = S_\phi(0) = \phi(0) = (\delta(x),\,\phi(x)).$$

It follows from Theorem 1.2 and equation (7) that

$$\begin{aligned}&\phi(x)\,\delta^{(2k)}(r)\\ &= \frac{(2k)!\,\Gamma(\frac{n}{2})}{2^{2k}\,k!\,\Gamma(\frac{n}{2}+k)}\sum_{m+i+l=k} 2^i(-1)^i\binom{m+l}{m}\binom{k}{m+l}\nabla^i\triangle^m\phi(0)\cdot\nabla^i\triangle^l\delta(x).\end{aligned}$$

for any $\phi(x)\in C^\infty(R^n)$ and $k=0,1,2,\cdots$.

In particular, we have for $k=0$ that

$$\phi(x)\,\delta(r) = \phi(0)\delta(x).$$

Assume that n is even and $k\le (n-2)/2$. Then

$$\begin{aligned}&\left(\frac{\partial}{2r\partial r}\right)^k r^{n-2+2j}\\ &= \frac{1}{2^k}(n-2+2j)(n-2+2j-2)\cdots(n-2+2j-2k+2)r^{n-2+2j-2k}\end{aligned}$$

for any nonnegative integer k. If $k=0$, we define

$$(n-2+2j)(n-2+2j-2)\cdots(n+2j-2k) = 1.$$

Clearly,

$$\begin{aligned}(\delta^{(k)}(r^2-t^2),\,\phi) &= \frac{(-1)^k}{2}\int_\Omega\left[\left(\frac{\partial}{2r\partial r}\right)^k(\phi r^{n-2})\right]_{r=t} d\Omega\\ &= \frac{(-1)^k\Omega_n}{2}\left(\frac{\partial}{2r\partial r}\right)^k (S_\phi(r)r^{n-2})\Big|_{r=t}.\end{aligned}$$

From

$$\begin{aligned}S_\phi(r) &\sim \phi(0)+\frac{1}{2!}S''_\phi(0)r^2+\cdots+\frac{1}{(2j)!}S_\phi^{(2j)}(0)r^{2j}+\cdots\\ &= \sum_{j=0}^{\infty}\frac{\triangle^j\phi(0)r^{2j}}{2^j\,j!\,n(n+2)\cdots(n+2j-2)},\end{aligned}$$

we get

$$(\delta^{(k)}(r^2-t^2),\,\phi)$$

$$\sim \frac{(-1)^k \Omega_n t^{n-2-2k}}{2^{k+1}} \sum_{j=0}^{\infty} \frac{(n-2+2j)\cdots(n+2j-2k)\, t^{2j}}{2^j\, j!\, n(n+2)\cdots(n+2j-2)} (\triangle^j \delta(x),\, \phi),$$

which implies

$$\delta^{(k)}(r^2-t^2) \sim \frac{(-1)^k \Omega_n t^{n-2-2k}}{2^{k+1}} \sum_{j=0}^{\infty} \frac{(n-2+2j)\cdots(n+2j-2k)\, t^{2j}}{2^j\, j!\, n(n+2)\cdots(n+2j-2)} \triangle^j \delta(x).$$

In particular, we have for $n \geq 2$

$$\begin{aligned}
\delta(r^2-t^2) &\sim \frac{\Omega_n t^{n-1}}{2t} \sum_{j=0}^{\infty} \frac{t^{2j}\, \triangle^j \delta(x)}{2^j\, j!\, n(n+2)\cdots(n+2j-2)} \\
&= \frac{\Omega_n t^{n-1}}{2t}\delta(x) + \frac{\Omega_n t^{n+1}}{4nt}\triangle\delta(x) + \frac{\Omega_n t^{n+3}}{8n(2n+4)t}\triangle^2\delta(x) + \cdots \\
&\sim \frac{1}{2t}\delta(r-t)
\end{aligned}$$

by the previous calculation. Similarly,

$$\begin{aligned}
\delta'(r^2-t^2) &\sim -\frac{\Omega_n t^{n-4}}{4} \sum_{j=0}^{\infty} \frac{(n-2+2j)\, t^{2j}\, \triangle^j \delta(x)}{2^j\, j!\, n(n+2)\cdots(n+2j-2)} \\
&= -\frac{\Omega_n (n-2) t^{n-4}}{4}\delta(x) - \frac{\Omega_n t^{n-2}}{8}\triangle\delta(x) - \frac{\Omega_n t^{n}}{32n}\triangle^2\delta(x) + \cdots,
\end{aligned}$$

for even $n \geq 4$, and

$$\begin{aligned}
\delta''(r^2-t^2) &\sim \frac{\Omega_n t^{n-6}}{8} \sum_{j=0}^{\infty} \frac{(n-2+2j)(n+2j-4)\, t^{2j}\, \triangle^j \delta(x)}{2^j\, j!\, n(n+2)\cdots(n+2j-2)} \\
&= \frac{\Omega_n (n-2)(n-4) t^{n-6}}{8}\delta(x) + \frac{\Omega_n (n-2) t^{n-4}}{16}\triangle\delta(x) + \cdots,
\end{aligned}$$

for even $n \geq 6$.

Next, we assume that $k > (n-2)/2$ and n is still even. Then

$$\delta^{(k)}(r^2-t^2) \sim \frac{(-1)^k \Omega_n t^{n-2-2k}}{2^{k+1}} \sum_{j=\frac{2k-n+2}{2}}^{\infty} \frac{(n-2+2j)\cdots(n+2j-2k)\, t^{2j}}{2^j\, j!\, n(n+2)\cdots(n+2j-2)} \triangle^j \delta(x)$$

by following the above calculation.

In particular for $k=1$ and $n=2$, we have

$$\begin{aligned}\delta'(r^2-t^2) &\sim -\frac{\Omega_2 t^{-2}}{4}\sum_{j=1}^{\infty}\frac{2j\,t^{2j}\triangle^j\delta(x)}{2^j\,j!2\cdot 4\cdot\dots\cdot 2j}\\ &= -\frac{\Omega_2}{8}\triangle\delta(x)-\frac{\Omega_2 t^2}{64}\triangle^2\delta(x)-\cdots,\end{aligned}$$

and for $k=2$ and $n=2$, we get

$$\begin{aligned}\delta''(r^2-t^2) &\sim \frac{\Omega_2 t^{-4}}{8}\sum_{j=2}^{\infty}\frac{2j(2j-2)\,t^{2j}\triangle^j\delta(x)}{2^j\,j!2\cdot 4\cdot\dots\cdot 2j}\\ &= \frac{\Omega_2}{64}\triangle^2\delta(x)+\frac{\Omega_2 t^2}{768}\triangle^3\delta(x)+\cdots.\end{aligned}$$

In summary, we come to

Theorem 2.1. *The following asymptotic expansions hold in a space of even dimension and for $k\le(n-2)/2$,*

$$\delta^{(k)}(r^2-t^2)$$
$$\sim\frac{(-1)^k\Omega_n t^{n-2-2k}}{2^{k+1}}\sum_{j=0}^{\infty}\frac{(n-2+2j)\cdots(n+2j-2k)\,t^{2j}}{2^j\,j!\,n(n+2)\cdots(n+2j-2)}\triangle^j\delta(x),$$

and for $k>(n-2)/2$

$$\delta^{(k)}(r^2-t^2)$$
$$\sim\frac{(-1)^k\Omega_n t^{n-2-2k}}{2^{k+1}}\sum_{j=\frac{2k-n+2}{2}}^{\infty}\frac{(n-2+2j)\cdots(n+2j-2k)\,t^{2j}}{2^j\,j!\,n(n+2)\cdots(n+2j-2)}\triangle^j\delta(x).$$

Note that the case that n is odd can follow similarly and we leave it to interested readers.

Remark 3. Aguirre and Marinelli [23] investigated the series expansion of $\delta^{(k)}(r^2-t^2)$ using Pizzetti's formula and the Gamma functions. However, their expansion may involve $\Gamma(x)$ for $x<0$ if k is large, which is undefined in the normal sense.

Theorem 2.2. *The asymptotic products $X^s\delta^{(k)}(r^2-t^2)$ exists in a space of even dimension n and for $k\le(n-2)/2$ and s is even,*

$$X^s\delta^{(k)}(r^2-t^2)$$

$$\sim \frac{(-1)^k \Omega_n t^{n-2-2k}}{2^{k+1}} \sum_{j=0}^{\infty} \frac{(n-2+2j)\cdots(n+2j-2k)\, t^{2j}}{2^j\, j!\, n(n+2)\cdots(n+2j-2)} X^s \triangle^j \delta(x)$$

$$= \frac{(-1)^k \Omega_n s! t^{n-2-2k}}{2^{k-s+1}} \sum_{j=0}^{\infty} \sum_{i=0}^{s/2} \frac{(n-2+2j)\cdots(n+2j-2k)\, t^{2j}}{2^j\ n(n+2)\cdots(n+2j-2)}$$

$$\cdot \frac{n^i \nabla^{s-2i} \triangle^{j-s+i} \delta(x)}{2^{2i} i!(j-s+i)!(s-2i)!},$$

and for $k > (n-2)/2$ *and* s *is odd,*

$$X^s \delta^{(k)}(r^2 - t^2)$$

$$\sim \frac{(-1)^k \Omega_n t^{n-2-2k}}{2^{k+1}} \sum_{j=\frac{2k-n+2}{2}}^{\infty} \frac{(n-2+2j)\cdots(n+2j-2k)\, t^{2j}}{2^j\, j!\, n(n+2)\cdots(n+2j-2)} X^s \triangle^j \delta(x)$$

$$= \frac{(-1)^{k+1} \Omega_n s! t^{n-2-2k}}{2^{k-s+1}} \sum_{j=\frac{2k-n+2}{2}}^{\infty} \sum_{i=0}^{\lfloor s/2 \rfloor} \frac{(n-2+2j)\cdots(n+2j-2k)\, t^{2j}}{2^j\ n(n+2)\cdots(n+2j-2)}$$

$$\cdot \frac{n^i \nabla^{s-2i} \triangle^{j-s+i} \delta(x)}{2^{2i} i!(j-s+i)!(s-2i)!}.$$

Proof. It easily follows from Theorems 2.1 and 1.3.

In particular, we have

$$X\delta(r^2 - t^2) \sim -\frac{\Omega_n t^n}{2n} \nabla\delta(x) - \frac{\Omega_n t^{n+2}}{2n(2n+4)} \nabla\triangle\delta(x) - \cdots, \quad \text{for even } n \geq 2,$$

$$X\delta'(r^2 - t^2) \sim \frac{\Omega_n t^{n-2}}{4} \nabla\delta(x) + \frac{\Omega_n t^n}{8n} \nabla\triangle\delta(x) + \cdots, \quad \text{for even } n \geq 4.$$

To end this paper, we must add that the asymptotic product $X^s\delta^{(k)}(r^2 - t^2)$ should have potential applications in seeking certain solutions for the differential equations involving the gradient operator ∇ in distributional sense. For example, we know that

$$\frac{\partial}{\partial x_j} \delta(r^2 - t^2) = 2x_j \delta'(r^2 - t^2),$$

which implies $\nabla\delta(r^2 - t^2) = 2X\delta'(r^2 - t^2)$. This product, of course, can be solved approximately using Theorem 2.2. Generally speaking, the factor $\nabla^s\delta(r^2 - t^2)$ will produce many terms containing the products $X^i\delta^{(j)}(r^2 - t^2)$ for some values of i and j, which may appear in certain types of differential equations.

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