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A NON-COMMUTATIVE NEUTRIX PRODUCT OF DISTRIBUTIONS ON R^m

Brian Fisher
Department of Mathematics, The University,
Leicester,LE1 7RH, England.

Li Chen Kuan
Department of Mathematics and Statistics,
University of Regina, S4S OA2, Canada.

Abstract

We let $\rho(r)$ be a fixed infinitely differentiable function of $r = (x_1^2 + \ldots + x_m^2)^{1/2}$ satisfying the properties (i) $\rho \geq 0$, (ii) $\rho(r) = 0,\ r \geq 1$, (iii) $\int_{R^m} \rho(r)\,d\mathbf{x} = 1$. The function $\delta_n(\mathbf{x})$, with $\mathbf{x}$ in R^m, is then defined by $\delta_n(\mathbf{x}) = n^m \rho(nr)$ for $n = 1, 2, \ldots$. The product $f \circ g$ of two distributions f and g is then defined to be the neutrix limit of the sequence $\{f g_n\}$, where $g_n = g * \delta_n$. Some results are given.

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1. Introduction

The product $g\phi$ of an infinitely differentiable function g and a test function ϕ in the space $\mathcal{D}$ of infinitely differentiable functions with compact support is itself in $\mathcal{D}$. This leads to the following definition $fg = gf$ of a distribution f in $\mathcal{D}'$ and an infinitely differentiable function g.

Definition 1. *Let f be a distribution in $\mathcal{D}'$ and let g be an infinitely differentiable function. The product $fg = gf$ is defined by*

$$\langle fg, \phi \rangle = \langle gf, \phi \rangle = \langle f, g\phi \rangle$$

for all ϕ in $\mathcal{D}$.

It follows easily by induction that we then have

$$f^{(s)}g = \sum_{i=0}^{s} \binom{s}{i} (-1)^i \left[fg^{(i)} \right]^{(s-i)}$$

for $s = 1, 2, \ldots$, where

$$\binom{s}{i} = \frac{s!}{i!(s-i)!}.$$

This suggests the following definition for the product of two distributions, see [2].

Definition 2. *Let f and g be distributions in $\mathcal{D}'$ for which on the interval (a, b) f is the s-th derivative of a locally summable function F in $L^p(a, b)$ and $g^{(s)}$ is a locally summable function in $L^q(a, b)$ with $1/p + 1/q = 1$. Then the product $fg = gf$ is defined by*

$$fg = gf = \sum_{i=0}^{s} \binom{s}{i} (-1)^i \left[Fg^{(i)} \right]^{(s-i)}.$$

Now let ρ be a fixed infinitely differentiable function with the properties:

(a) $\rho(x) = 0, \ |x| \geq 1$,

(b) $\rho(x) \geq 0$,

(c) $\rho(x) = \rho(-x)$,

(d) $\int_{-1}^{1} \rho(x)\, dx = 1$.

The function δ_n is defined by $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots$. It is obvious that $\{\delta_n\}$ is a sequence of infinitely differentiable functions converging to the Dirac delta-function δ.

For an arbitrary distribution g in $\mathcal{D}'$ the function g_n is defined by

$$g_n(x) = (g * \delta_n)(x) = \langle g(x - t), \delta_n(t) \rangle.$$

It follows that $\{g_n\}$ is a sequence of infinitely differentiable functions converging to g.

The next definition for the neutrix product $f \circ g$ of two distributions f and g in $\mathcal{D}'$ generalizes Definition 2 and was given in [4].

Definition 3. *Let f and g be distributions in $\mathcal{D}'$ and let $g_n = g * \delta_n$. We say that the neutrix product $f \circ g$ of f and g exists and is equal to h on the interval (a, b) if*

$$\mathop{\text{N}-\lim}_{n\to\infty}\langle f g_n, \phi\rangle = \langle h, \phi\rangle$$

for all test functions ϕ in $\mathcal{D}$ with support contained in the interval (a, b), where N is the neutrix, see van der Corput [1], having domain $N' = \{1, 2, \ldots n, \ldots\}$ and range N'' the real numbers with negligible functions finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \quad \ln^r n$$

for $\lambda > 0$ and $r = 1, 2, \ldots$ and all functions which converge to zero in the normal sense as n tends to infinity.

Note that if

$$\lim_{n\to\infty}\langle f g_n, \phi\rangle = \langle h, \phi\rangle$$

for all ϕ in $\mathcal{D}$, we simply say that the product $f \circ g$ exists and equals h, see [3].

The following two theorems hold, see [3] and [4].

Theorem 1. *Let f and g be distributions in $\mathcal{D}'$. If the product fg exists on the interval (a, b) then the products (and so the neutrix products) $f \circ g$ and $g \circ f$ exist and*

$$f \circ g = g \circ f = fg$$

on this interval.

Theorem 2. *Let f and g be distributions in $\mathcal{D}'$ and suppose that the neutrix products $f \circ g$ and $f \circ g'$ (or $f' \circ g$) exist on the interval (a, b). Then the neutrix product $f' \circ g$ (or $f \circ g'$) exists and*

$$(f \circ g)' = f' \circ g + f \circ g'$$

on this interval.

In the following, we now give a definition for the neutrix product $f \circ g$ of two distributions f and g in $\mathcal{D}'_m$ the space of distributions defined on the space $\mathcal{D}_m$ of infinitely differentiable functions of $\mathbf{x} = (x_1, \ldots, x_m)$ with compact support.

2. The function $\delta_n(\mathbf{x})$ in $\mathcal{D}_m$

We first of all generalize the infinitely differentiable function ρ defined above. From now on we let $\rho(r)$, where

$$r = (x_1^2 + \ldots + x_m^2)^{1/2},$$

be a fixed function which is infinitely differentiable with respect to r and with respect to $x_1, \ldots, x_m$ having the properties:

(i) $\rho(r) \geq 0$,

(ii) $\rho(r) = 0, \; r \geq 1$,

(iii) $\int_{R^m} \rho(r)\, d\mathbf{x}$.

Property (iii) is represented in spherical polar coordinates by

(iv) $\Omega_m \int_0^1 \rho(r) r^{m-1}\, dr = 1$,

where Ω_m is the surface area of the unit sphere in R^m.

The function

$$\rho(r) = \begin{cases} ke^{1/(r^2-1)}, & r < 1, \\ 0, & r \geq 1, \end{cases}$$

where

$$k^{-1} = \int_{R^m} e^{1/(r^2-1)}\, d\mathbf{x}$$

is an example of a function satisfying the above conditions.

We now define the function $\delta_n(\mathbf{x})$, with $\mathbf{x}$ in R^m, by

$$\delta_n(\mathbf{x}) = n^m \rho(nr)$$

for $n = 1, 2, \ldots$. It is obvious that $\{\delta_n\}$ is a sequence of infinitely differentiable functions converging to δ in the sense that

$$\lim_{n\to\infty} \langle \delta_n(\mathbf{x}), \phi(\mathbf{x}) \rangle = \langle \delta(\mathbf{x}), \phi(\mathbf{x}) \rangle = \phi(0)$$

for all functions ϕ in $\mathcal{D}_m$.

Note that we can not only consider $\delta_n(\mathbf{x})$ as an infinitely differentiable function of the variables $x_1, \ldots, x_m$ but also as an infinitely differentiable function of the variable r, which is very useful in calculating the product of distributions in $\mathcal{D}'_m$.

For an arbitrary distribution g in $\mathcal{D}'_m$ the function g_n is defined by

$$g_n(\mathbf{x}) = (g * \delta_n)(\mathbf{x}) = \langle g(\mathbf{x} - \mathbf{t}), \delta_n(\mathbf{t}) \rangle.$$

It again follows that $\{g_n\}$ is a sequence of infinitely differentiable functions converging to g.

3. The definition of the product in $\mathcal{D}'_m$

The following definition was introduced by Gel'fand and Shilov in [5] as a natural extension of Definition 1.

Definition 4. *Let f be a distribution in $\mathcal{D}'_m$ and let g be an infinitely differentiable function of* $\mathbf{x}$. *The product $fg = gf$ is defined by*

$$\langle fg, \phi \rangle = \langle gf, \phi \rangle = \langle f, g\phi \rangle$$

for all ϕ in $\mathcal{D}_m$.

We denote the partial derivative with respect to the variable x_i by D_i. D_i is defined in the obvious way on $\mathcal{D}'_m$ by

$$\langle D_i f, \phi \rangle = -\langle f, D_i \phi \rangle$$

for all ϕ in $\mathcal{D}_m$. It then follows by induction that if

$$\mathbf{s} = (s_1, \ldots, s_m), \quad \mathbf{i} = (i_1, \ldots, i_m),$$

$$(-1)^{\mathbf{i}} = (-1)^{i_1 + \cdots + i_m}, \quad \mathbf{D}^{\mathbf{s}} = D_1^{s_1} \ldots D_m^{s_m},$$

$$\binom{\mathbf{s}}{\mathbf{i}} = \binom{s_1}{i_1} \cdots \binom{s_m}{i_m}, \quad \sum_{\mathbf{i}=0}^{\mathbf{s}} = \sum_{i_1=0}^{s_1} \cdots \sum_{i_m=0}^{s_m},$$

we have

$$(\mathbf{D}^{\mathbf{s}} f)g = \sum_{\mathbf{i}=0}^{\mathbf{s}} \binom{\mathbf{s}}{\mathbf{i}} (-1)^{\mathbf{i}} \mathbf{D}^{\mathbf{s}-\mathbf{i}}(f\mathbf{D}^{\mathbf{i}} g)$$

for any differentiable function g and $s_1, \ldots, s_m = 0, 1, 2, \ldots$.

This suggests the following generalization of Definition 2.

Definition 5. *Let f and g be distributions in $\mathcal{D}'_m$ for which on the interval $(\mathbf{a}, \mathbf{b})$, where $\mathbf{a} = (a_1, \ldots, a_m)$ and $\mathbf{b} = (b_1, \ldots, b_m)$, $f = \mathbf{D}^{\mathbf{s}} F$ where F is a locally summable function in $L^p(\mathbf{a}, \mathbf{b})$ and $\mathbf{D}^{\mathbf{s}} g$ is a locally summable function in $L^q(\mathbf{a}, \mathbf{b})$ with $1/p + 1/q = 1$. Then the product $fg = gf$ is defined by*

$$fg = gf = \sum_{\mathbf{i}=0}^{\mathbf{s}} \binom{\mathbf{s}}{\mathbf{i}} (-1)^{\mathbf{i}} \mathbf{D}^{\mathbf{s}-\mathbf{i}}(F(\mathbf{D}^{\mathbf{i}} g).$$

The next definition is a generalization of Definition 3.

Definition 6. *Let f and g be distributions in $\mathcal{D}'_m$ and let $g_n = g * \delta_n$. We say that the neutrix product $f \circ g$ of f and g exists and is equal to h on the interval $(\mathbf{a}, \mathbf{b})$ if*

$$\operatorname*{N-lim}_{n \to \infty} \langle f g_n, \phi \rangle = \langle h, \phi \rangle$$

for all ϕ in $\mathcal{D}_m$ with support contained in the interval $(\mathbf{a}, \mathbf{b})$.

Note that if

$$\lim_{n \to \infty} \langle f g_n, \phi \rangle = \langle h, \phi \rangle$$

for all ϕ, we again simply say that the product $f \circ g$ exists and equals h.

The proofs of Theorems 1 and 2 can be modified to give the following two theorems.

Theorem 3. *Let f and g be distributions in $\mathcal{D}'_m$. If the product fg exists on the interval $(\mathbf{a}, \mathbf{b})$ then the products (and so the neutrix products) $f \circ g$ and $g \circ f$ exist and*

$$f \circ g = g \circ f = fg$$

on this interval.

Theorem 4. *Let f and g be distributions in $\mathcal{D}'_m$ and suppose that the neutrix products $f \circ g$ and $f \circ D_i g$ (or $D_i f \circ g$) exist on the interval $(\mathbf{a}, \mathbf{b})$. Then the neutrix product $D_i f \circ g$ (or $f \circ D_i g$) exists and*

$$D_i(f \circ g) = D_i f \circ g + f \circ D_i g$$

on this interval.

4. Some results

Theorem 5. *The neutrix products $\delta(\mathbf{x}) \circ r^{-k}$ and $r^{-k} \circ \delta(\mathbf{x})$ exist for $k = 1, 2, \ldots, m-1$ and*

(1) $$\delta(\mathbf{x}) \circ r^{-k} = 0,$$

for $k = 1, 2, \ldots, m-1$,

(2) $$r^{1-2k} \circ \delta(\mathbf{x}) = 0,$$

for $k = 1, 2, \ldots, [\frac{1}{2}m]$ and

(3) $$r^{-2k} \circ \delta(\mathbf{x}) = \frac{\Delta^k \delta(\mathbf{x})}{2^k k! m(m+2) \ldots (m+2k-2)}$$

for $k = 1, 2, \ldots, [\frac{1}{2}m - 1]$, where Δ denotes the Laplace operator.

Proof. We note that r^{-k} is a locally summable function on R^m for $k = 1, 2, \ldots, m-1$ and so on putting

$$(r^{-k})_n = r^{-k} * \delta_n$$

we have

$$\begin{aligned}
\langle \delta(\mathbf{x})(r^{-k})_n, \phi(\mathbf{x}) \rangle &= \phi(0) \int_{R^m} r^{-k} \delta_n(\mathbf{x})\, d\mathbf{x} \\
&= n^m \phi(0) \int_{R^m} r^{-k} \rho(nr)\, d\mathbf{x} \\
&= n^m \Omega_m \phi(0) \int_0^{1/n} r^{m-k-1} \rho(nr)\, dr \\
&= n^k \Omega_m \phi(0) \int_0^1 t^{m-k-1} \rho(t)\, dt
\end{aligned}$$

on changing to spherical polar coordinates, see Gel'fand and Shilov [5], and then putting $nr = t$. It follows that

$$\mathrm{N}\!-\!\lim_{n\to\infty} \langle \delta(\mathbf{x})(r^{-k})_n, \phi(\mathbf{x})\rangle = 0 = \langle 0, \phi(\mathbf{x})\rangle$$

for arbitrary ϕ in $\mathcal{D}_m$ and equation (1) follows.

Next we have

$$\begin{aligned} \langle r^{-k}\delta_n(\mathbf{x}), \phi(\mathbf{x})\rangle &= n^m \int_{R^m} r^{-k}\rho(nr)\phi(\mathbf{x})\,d\mathbf{x} \\ &= n^m\Omega_m \int_0^{1/n} r^{m-k-1}\rho(nr)S_\phi(r)\,dr \end{aligned}$$

on changing to spherical polar coordinates, where $S_\phi(r)$ is the mean value of ϕ on the sphere $x_1^2 + \ldots + x_m^2 = r^2$, see [5].

By Taylor's theorem we have

$$S_\phi(r) = \sum_{i=0}^{k-1} \frac{S_\phi^{(i)}(0)}{i!} r^i + \frac{S_\phi^{(k)}(0)}{k!} + \frac{S_\phi^{(k+1)}(\xi r)}{(k+1)!} r^{k+1}, \tag{4}$$

where $0 < \xi < 1$. Hence

$$\begin{aligned} \langle r^{-k}\delta_n(\mathbf{x}), \phi(\mathbf{x})\rangle &= n^m\Omega_m \sum_{i=0}^{k-1} \frac{S_\phi^{(i)}(0)}{i!} \int_0^{1/n} r^{m+i-k-1}\rho(nr)\,dr + \\ &+ \frac{n^m\Omega_m S_\phi^{(k)}(0)}{k!} \int_0^{1/n} r^{m-1}\rho(nr)\,dr + \frac{n^m\Omega}{(k+1)!} \int_0^{1/n} r^m\rho(nr)S_\phi^{(k+1)}(\xi r)\,dr \\ &= I_1 + I_2 + I_3. \end{aligned}$$

On making the substitution $nr = t$ we have

$$I_1 = \Omega_m \sum_{i=0}^{k-1} \frac{n^{k-i}S_\phi^{(i)}(0)}{i!} \int_0^1 t^{m+i-k-1}\rho(t)\,dt$$

and so

$$\mathrm{N}\!-\!\lim_{n\to\infty} I_1 = 0.$$

Similarly

$$I_2 = \frac{\Omega_m S_\phi^{(k)}(0)}{k!} \int_0^1 t^{m-1}\rho(t)\,dt = \frac{S_\phi^{(k)}(0)}{k!}$$

on using (iv) and so

$$\operatorname*{N-lim}_{n\to\infty} I_2 = \frac{S_\phi^{(k)}(0)}{k!}.$$

Finally, on putting

$$M_\phi = \max\left\{\left|S_\phi^{(k+1)}(r)\right| \,:\, r \geq 0\right\}$$

we have on again putting $nr = t$

$$|I_3| \leq \frac{M_\phi \Omega_m}{n(k+1)!}\int_0^1 t^m \rho(t)\,dt$$

and so

$$\operatorname*{N-lim}_{n\to\infty} I_3 = 0.$$

It follows that

$$\operatorname*{N-lim}_{n\to\infty} \langle r^{-k}\delta_n(\mathbf{x}), \phi(\mathbf{x})\rangle = \frac{S_\phi^{(k)}(0)}{k!}$$

for arbitrary ϕ in $\mathcal{D}_m$.

On using the following equations given in [5]

$$S_\phi^{(2k-1)}(0) = 0, \quad S_\phi^{(2k)}(0) = \frac{(2k)!\langle \Delta^k \delta(\mathbf{x}), \phi(\mathbf{x})\rangle}{2^k k! m(m+2)\dots(m+2k-2)}$$

for $k = 1, 2, \dots$, equations (2) and (3) follow. This completes the proof of the theorem. □

Theorem 6. *The neutrix product $r^{-k} \circ \delta(\mathbf{x})$ exists and*

$$\langle r^{-k} \circ \delta(\mathbf{x}), \phi(\mathbf{x})\rangle = \frac{S_\phi^{(k)}(0)}{k!}$$

for $k = m, m+1, \dots$ and arbitrary ϕ in $\mathcal{D}_m$.

Proof. Since r^{-k} is not locally summable on R^m for $k = m, m+1, \dots$, it was normalized by Gel'fand and Shilov [5] using the following equation

$$\langle r^{-k}, \phi(\mathbf{x})\rangle = \Omega_m \int_0^\infty r^{m-k-1}\left[S_\phi(r) - \sum_{i=0}^{k-m-1}\frac{S_\phi^{(i)}(0)}{i!}r^i - \frac{S_\phi^{(k-m)}(0)}{(k-m)!}H(1-r)r^{k-m}\right]dr,$$

where H denotes Heaviside's function. Using Definition 4 we therefore have

$$\langle r^{-k}\delta_n(\mathbf{x}), \phi(\mathbf{x}) = \langle r^{-k}, \delta_n(\mathbf{x})\phi(\mathbf{x})\rangle =$$

$$= \Omega_m \int_0^\infty r^{m-k-1}\left[S_{\psi_n}(r) - \sum_{i=0}^{k-m-1}\frac{S_{\psi_n}^{(i)}(0)}{i!}r^i - \frac{S_{\psi_n}^{(k-m)}(0)}{(k-m)!}H(1-\right.$$

$$\left.-r)\, r^{k-m}\right] dr = \Omega_m \int_0^{1/n} r^{m-k-1}\left[S_{\psi_n}(r) - \sum_{i=0}^{k-m}\frac{S_{\psi_n}^{(i)}(0)}{i!}r^i\right] dr$$

$$-\Omega_m \sum_{i=0}^{k-m-1}\int_{1/n}^\infty \frac{S_{\psi_n}^{(i)}(0)}{i!}r^{m+i-k-1}\, dr - \Omega_m \int_{1/n}^1 \frac{S_{\psi_n}^{(k-m)}(0)}{(k-m)!}r^{-1}\, dr =$$

$$= I_4 - I_5 - I_6,$$

where

$$\psi_n(\mathbf{x}) = \delta_n(\mathbf{x})\phi(\mathbf{x}) = n^m\rho(nr)\phi(\mathbf{x})$$

and so

$$S_{\psi_n}(r) = n^m\rho(nr)S_\phi(r),$$

$$S_{\psi_n}^{(i)}(0) = n^m \sum_{j=0}^{i}\binom{i}{j} n^j\rho^{(j)}(0)S_\phi^{(i-j)}(0) \tag{5}$$

for $i = 0, 1, 2\dots$.

Using equation (4) again we have

$$I_4 = \Omega_m \int_0^{1/n} r^{m-k-1}\left\{n^m\rho(nr)\left[\sum_{i=0}^{k-1}\frac{S_\phi^{(i)}(0)}{i!}r^i + \frac{S_\phi^{(k)}(0)}{k!}r^k +\right.\right.$$

$$\left.\left. + \frac{S_\phi^{(k+1)}(\xi r)}{(k+1)!}r^{k+1}\right] - \sum_{i=0}^{k-m}\frac{S_{\psi_n}^{(i)}(0)}{i!}r^i\right\} dr =$$

$$= \Omega_m \int_0^{1/n}\left[n^m\rho(nr)\sum_{i=0}^{k-1}\frac{S_\phi^{(i)}(0)}{i!}r^{m+i-k-1} - \sum_{i=0}^{k-m}\frac{S_{\psi_n}^{(i)}(0)}{i!}r^{m+i-k-1}\right] dr+$$

$$+ I_2 + I_3 =$$

$$= \Omega_m \int_0^1 \left[n^{m-1}\rho(t)\sum_{i=0}^{k-1}\frac{S_\phi^{(i)}(0)}{i!}(t/n)^{m+i-k-1}\right.$$

$$- \sum_{i=0}^{k-m} \frac{S_{\psi_n}^{(i)}(0)}{i!}(t/n)^{m+i-k-1}\Bigg] dt + I_2 + I_3 =$$

$$= I_7 + I_2 + I_3$$

on making the substitution $nr = t$. On using equation (5) it follows that

$$\underset{n\to\infty}{\mathrm{N-lim}}\, I_7 = 0$$

and as above

$$\underset{n\to\infty}{\mathrm{N-lim}}\, I_2 = \frac{S_{\phi}^{(k)}(0)}{k!}, \quad \underset{n\to\infty}{\mathrm{N-lim}}\, I_3 = 0.$$

Thus

$$\underset{n\to\infty}{\mathrm{N-lim}}\, I_4 = \frac{S_{\phi}^{(k)}(0)}{k!}.$$

Next we have

$$I_5 = -\Omega_m \sum_{i=0}^{k-m-1} \frac{S_{\psi_n}^{(i)}(0)}{i!(m+i-k)} n^{k-m-i}$$

and on using equation (5) it follows that

$$\underset{n\to\infty}{\mathrm{N-lim}}\, I_5 = 0.$$

Finally we have

$$I_6 = \frac{\Omega_m S_{\psi_n}^{(k-m)}(0)}{(k-m)!} \ln n$$

and again on using equation (5) it follows that

$$\underset{n\to\infty}{\mathrm{N-lim}}\, I_6 = 0.$$

We have therefore proved that

$$\underset{n\to\infty}{\mathrm{N-lim}} \langle r^{-k}\delta_n(\mathbf{x}), \phi(\mathbf{x})\rangle = \frac{S_{\phi}^{(k)}(0)}{k!}$$

and the result of the theorem follows.

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REZIME

O NEKOMUTATIVNOM NEUTRIKS PROIZVODU DISTRIBUCIJA NA R^m

U radu se definiše nekomutativni proizvod distribucija u R^m pomoću fiksirane beskonačno diferencijabilne radijalne funkcije $\rho : R^m \to R$, $\rho = \rho(r) = \rho((x_1^2 + x_2^2 + ... + x_m^2)^{\frac{1}{2}})$ i pojma neutriksa iz [1]. Pored osnovnih osobina, eksplicitno se izračunava proizvod $r^{-k} \circ \delta(x)$.

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