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# A COMMUTATIVE NEUTRIX CONVOLUTION PRODUCT OF DISTRIBUTIONS

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**Abstract**

Let $f$ and $g$ be distributions in $\mathcal{D}'$ and let

$$f_n(x) = f(x)\tau_n(x), \qquad g_n(x) = g_n(x)\tau_n(x)$$

where $\tau_n(x)$ is a certain function which converges to the identity func as $n$ tends to infinity. Then the neutrix convolution product $f \boxast g$ is defined as the neutrix limit of the sequence $\{f_n * g_n\}$, provided the limit $h$ exists in the sense that

$$N-\lim_{n\to\infty}\langle f_n * g_n, \phi\rangle = \langle h, \phi\rangle$$

for all $\phi$ in $\mathcal{D}$. The neutrix convolution products $x_-^\lambda \boxast x_+^\mu$ for $\lambda, \mu, \lambda + \mu \neq 0, \pm 1, \pm 2, \ldots$ and $x_-^\lambda \boxast x_+^s$ for $\lambda \neq 0, \pm 1, \pm 2, \ldots$ and $s = 0, 1, 2, \ldots$ are evaluated, from which other neutrix convolution products are deduced.

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The classical definition for the convolution product of two functions $f$ and $g$ is as follows:

**Definition 1.** *Let $f$ and $g$ be functions. Then the convolution product $f * g$ is defined by*

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)dt$$

*for all points $x$ for which the integral exists.*

It follows easily from the definition that if $f * g$ exists then $g * f$ exists and

(1) $$f * g = g * f$$

and if $(f * g)'$ and $f * g'$ (or $f' * g$) exist, then

(2) $$(f * g)' = f * g' \quad (\text{or } f' * g)$$

The following theorem also holds and it is an immediate consequence of Hölder's inequality for integrals.

**Theorem 1.** *Let $f$ and $g$ be functions in $L^p(-\infty,\infty)$ and $L^q(-\infty,\infty)$ respectively, where $1/p + 1/q = 1$. Then the convolution product $(f * g)(x)$ exists for all $x$.*

Now, suppose that the convolution product $(f * g)(x)$ exists for all $x$ and let $\phi$ be an arbitrary test function in the space $\mathcal{D}$ of infinitely differentiable functions with compact support. Then

$$\begin{aligned} \langle (f * g)(x), \phi(x)\rangle &= \int_{-\infty}^{\infty} \phi(x) \int_{-\infty}^{\infty} f(t)g(x-t)dtdx \\ &= \int_{-\infty}^{\infty} f(y) \int_{-\infty}^{\infty} g(x)\phi(x+y)dxdy \end{aligned}$$

and for convenience we will write this as

$$\langle (f * g)(x), \phi(x)\rangle = \langle f(y), \langle g(x), \phi(x+y)\rangle\rangle$$

even though the infinitely differentiable function $\langle g(x), \phi(x+y)\rangle$ does not necessarily have compact support. This equation does however suggest the following definition for the convolution product of certain distributions $f$ and $g$ in $\mathcal{D}'$, see for example Gel'fand and Shilov [4].

**Definition 2.** *Let $f$ and $g$ be distributions in $\mathcal{D}'$ satisfying either of the following conditions:*

*(a) either $f$ or $g$ has bounded support,*

*(b) the supports of $f$ and $g$ are bounded on the same side.*

*Then the convolution product $f * g$ is defined by*

$$\langle (f*g)(x), \phi \rangle = \langle f(y), \langle g(x), \phi(x+y) \rangle \rangle$$

*for arbitrary $\phi$ in $\mathcal{D}$.*

Note that with this definition, if $g$ has bounded support, then $\langle g(x), \phi(x+y)\rangle$ is in $\mathcal{D}$ and so $\langle f(y), \langle g(x), \phi(x+y)\rangle\rangle$ is meaningful. If on the other hand either $f$ has bounded support or the supports of $f$ and $g$ are bounded on the same side, then the intersection of the supports of $f$ and $\langle g(x), \phi(x+y)\rangle$ is bounded and so $\langle f(y), \langle g(x), \phi(x+y)\rangle\rangle$ is again meaningful.

It follows that if the convolution product $f * g$ exists by Definition 2, then equations (1) and (2) always hold.

Definition 1 and 2 are very restrictive and can only be used for a small class of distributions. In order to extend the convolution product to a larger class of distributions, Jones [5] gave the following definition.

**Definition 3.** *Let $f$ and $g$ be distributions and let $\tau$ be an infinitely differentiable function satisfying the following conditions:*

*(i)* $\tau(x) = \tau(-x)$,

*(ii)* $0 \le \tau(x) \le 1$,

*(iii)* $\tau(x) = 1$ *for* $|x| \le 1/2$,

*(iv)* $\tau(x) = 0$ *for* $|x| \ge 1$.

*Let*

$$f_n(x) = f(x)\tau(x/n), \quad g_n(x) = g(x)\tau(x/n)$$

*for $n = 1, 2, \ldots$ Then the convolution product $f * g$ is defined as the limit of the sequence $\{f_n * g_n\}$, provided the limit $h$ exists in the sense that*

$$\lim_{n\to\infty} \langle f_n * g_n, \phi \rangle = \langle h, \phi \rangle$$

*for all test functions $\phi$ in $\mathcal{D}$.*

In this definition the convolution product $f_n * g_n$ exists by Definition 2 since $f_n$ and $g_n$ have bounded supports. It follows that if the limit of the sequence $\{f_n * g_n\}$ exists, so that the convolution product $f * g$ exists, then $g * f$ also exists and equation (1) holds. However equation (2) need not necessarily hold since Jones proved that

$$1 * sgnx = sgnx * 1 = x,$$

$$(1 * sgnx)' = 1, \quad 1' * sgnx = 0, \quad 1 * (sgnx)' = 2.$$

It can be proved that if a convolution product exists by Definitions 1 and 2 then it exists by Definition 3 and defines the same distribution.

However, there were still many convolution products which did not exist by Definition 3 and in order to try and remedy this the next definition was introduced in [2].

**Definition 4.** *Let $f$ and $g$ be distributions and let*

$$\tau_n(x) = \begin{cases} 1, & |x| \leq n, \\ \tau(n^n x - n^{n+1}), & x > n, \\ \tau(n^n x + n^{n+1}), & x < -n, \end{cases}$$

*for $n = 1, 2, \ldots$, where $\tau$ is defined as in Definition 3. Let $f_n(x) = f(x)\tau_n(x)$ for $n = 1, 2, \ldots$. Then the neutrix convolution product $f \boxplus g$ is defined as the neutrix limit of the sequence $\{f_n * g\}$, provided the limit $h$ exists in the sense that*

$$N - \lim_{n\to\infty} \langle f_n * g, \phi \rangle = \langle h, \phi \rangle$$

*for all $\phi$ in $\mathcal{D}$, where $N$ is the neutrix, see van der Corput [1], having domain $N' = \{1, 2, \ldots, n, \ldots\}$ and range the real numbers with negligible functions finite linear sums of the functions*

$$n^\lambda \ln^{r-1} n, \quad \ln^r n, \quad (\lambda \rangle 0; \ r = 1, 2, \ldots))$$

*and all functions $\epsilon(n)$ for which $\lim_{n\to\infty} \epsilon(n) = 0$.*

The convolution product $f_n * g$ in this definition is again in the sense of Definition 2, the support of $f_n$ being contained in the interval $[-n - n^n, n + n^{-n}]$. It can be proved that if a convolution product exists by Definitions 1 or 2 then the neutrix convolution product exists and defines the same distribution.

However, the neutrix convolution product as defined in Definition 4 is in general non-commutative. For example, it was proved in [2] that

$$x_- \boxast x_+ = \frac{1}{6} x_-^3, \quad x_+ \boxast x_- = \frac{1}{6} x_+^3$$

so that

$$x_- \boxast x_+ \neq x_+ \boxast x_-.$$

In the following, we now consider a commutative neutrix convolution product. We will denote the commutative neutrix convolution product of the distributions $f$ and $g$ by $f \boxast g$ to distinguish it from the non-commutative neutrix convolution product.

**Definition 5.** *Let $f$ and $g$ be distributions and let $\tau_n$ be defined as in Definition 4. Let $f_n(x) = f(x)\tau_n(x)$ and $g_n(x)\tau_n(x)$ for $n = 1, 2, \ldots$. Then the commutative neutrix convolution product $f \boxast g$ is defined as the neutrix limit of the sequence $\{f_n * g_n\}$, provided the limit $h$ exists in the sense that*

$$N - \lim_{n\to\infty} \langle f_n * g_n, \phi \rangle = \langle h, \phi \rangle$$

*for all $\phi$ in $\mathcal{D}$, where $N$ is the neutrix defined above.*

The convolution product $f_n * g_n$ in this definition is again in the sense of Definition 2 and since $f_n * g_n = g_n * f_n$, the neutrix convolution product $f \boxast g$ is clearly commutative.

The next theorem shows that this definition generalizes Definition 1.

**Theorem 2.** *Let $f$ and $g$ be functions in $L^p(-\infty, \infty)$ and $L^q(-\infty, \infty)$ respectively, where $1/p + 1/q = 1$, so that the convolution product $f * g$ exists by Definition 1. Then the neutrix convolution product $f \boxast g$ exists and*

$$f \boxast g = f \boxast g.$$

*Proof.* For arbitrary $\epsilon > 0$ we have

$$\begin{aligned}
|f * g - f_n * g_n| &= |\textstyle\int_{-\infty}^{\infty} f(t)g(x-t)[1 - \tau_n(t)\tau_n(x-t)]dt| \\
&\leq \textstyle\int_{-\infty}^{\infty} |f(t)g(x-t)[1 - \tau_n(t)]\tau_n(x-t)|dt \\
&+ \textstyle\int_{-\infty}^{\infty} |f(t)g(x-t)[1 - \tau_n(x-t)]|dt \\
&\leq \textstyle\int_{|t|\geq} |f(t)g(x-t)|dt + \int_{|x-t|\geq} |f(t)g(x-t)|dt \\
&= \textstyle\int_{|t|\geq n} |f(t)g(x-t)|dt + \int_{|t|\geq n} |f(x-t)g(t)|dt \langle \epsilon
\end{aligned}$$

for all $n$ greater than some $n_0$. Thus if $\phi$ is an arbitrary function in $\mathcal{D}$ then

$$|\langle f * g, \phi\rangle - \langle f_n * g_n, \phi\rangle| \leq \sup\{|\phi(x)|\}\epsilon$$

for $n > n_0$ and it follows that

$$\lim_{n\to\infty} \langle f_n * g_n, \phi\rangle = \langle f * g, \phi\rangle = N - \lim_{n\to\infty} \langle f_n * g_n, \phi\rangle.$$

The result of the theorem follows.

The next theorem shows that Definition 5 also generalizes Definition 2.

**Theorem 3.** *Let $f$ and $g$ be distributions satisfying either condition (a) or condition (b) of Definition 2 so that the convolution product $f * g$ exists by Definition 2. Then the neutrix convolution product $f \boxplus g$ exists and*

$$f \boxplus g = f * g.$$

*Proof.* Suppose first of all that the support of $g$ is bounded so that $g = g_n$ for some $n$ greater than some $n_0$. Then with $n > n_0$ and arbitrary $\phi$ in $\mathcal{D}$

$$\begin{aligned} \langle f_n * g_n, \phi\rangle &= \langle f_n * g, \phi\rangle \\ &= \langle f_n(y), \langle g(x), \phi(x + y)\rangle\rangle \\ &= \langle f(y), \langle g(x), \phi(x + y)\rangle\rangle \end{aligned}$$

for large enough $n$, since the support of $\langle g(x), \phi(x + y)\rangle$ is bounded. It follows that

$$\begin{aligned} \lim_{n\to\infty}\langle f_n * g_n, \phi\rangle &= \langle f(y), \langle g(x), \phi(x + y)\rangle\rangle \\ &= N - \lim_{n\to\infty}\langle f_n * g_n, \phi\rangle \end{aligned}$$

and the result of the theorem follows when the support of $g$ is bounded.

Now suppose that the support of $f$ is bounded. Then the result of the theorem follows as above on noting that $f_n * g_n = g_n * f_n$.

Finally, suppose that the supports of $f$ and $g$ are bounded on the same side, say on the left, so that the supports of $f$ and $g$ are contained in some half - bounded intervals $[a, \infty)$ and $[b, \infty)$ respectively. Now let $\phi$ be an arbitrary function in $\mathcal{D}$ with its support contained in the bounded interval $[c, d]$. Then since $g(x) = 0$ if $x < b$,

$$\psi(y) = \langle g_n(x), \phi(x + y)\rangle = \langle g(x), \phi(x + y)\rangle = 0$$

if $y > d-b$. Further, since $f(y)=0$ if $y<a$, it follows that the intersection of the supports of $\psi$ and $f$ are contained in the interval $[a, d-b]$ if $d-b>a$ and is the empty set otherwise. Thus

$$\langle f_n * g_n, \phi\rangle = \langle f * g, \phi\rangle$$

for $n > \max\{|a|, |d-b|\}$ and the result of the theorem follows as above for this third case.

**Theorem 4.** *The neutrix convolution product $x_-^\lambda \boxtimes x_+^n$ exists and*

$$\begin{aligned} x_-^\lambda \boxtimes x_+^n &= B(-\lambda-\mu-1, \mu+1)x^{\lambda+\mu+1} \\ &= B(-\lambda-\mu-1, \lambda+1)x^{\lambda+\mu+1} \end{aligned} \tag{3}$$

*for $\lambda, \mu, \lambda+\mu \neq 0, \pm 1, \pm 2, \ldots$, where $B$ denotes the Beta function.*

*Proof.* We will first of all suppose that $\lambda, \mu\rangle - 1$, so that $x_-^\lambda$ and $x_+^n$ are locally summable functions. Put

$$(x_-^\lambda)_n = x_-^\lambda \tau_n(x), \quad (x_+^\mu)_n = x_+^\mu \tau_n(x).$$

Then the convolution product $(x_-^\lambda)_n * (x_+^\mu)_n$ exists by Definition 2 and

(4)

$$\begin{aligned} \langle (x_-^\lambda)_n * (x_+^\mu)_n = & \ \langle (y_-^\lambda)_n, \langle (x_+^\mu)_n, \phi(x+y)\rangle\rangle \\ = & \ \int_{-n-n^{-n}}^0 (-y)^\lambda \tau_n(y) \int_a^b (x-y)_+^\mu \tau_n(x-y)\,dy\,dx + \\ = & \ \int_a^b \phi(x) \int_{-n}^0 (-y)^\lambda (x-y)_+^\mu \tau_n(x-y)\,dx\,dy + \\ + & \ \int_a^b \phi(x) \int_{-n-n^{-n}}^{-n} (-y)^\lambda \tau_n(y)(x-y)_+^\mu \tau_n(x-y)\,dy\,dx \end{aligned}$$

for $n > -a$ and arbitrary $\phi$ in $\mathcal{D}$ with support of $\phi$ contained in the interval $[a, b]$.

When $x<0$ and $-n \le y \le 0$, $\tau_n(x-y)=1$ on the support of $\phi$. Thus with $x<0$ and $-n \le y \le 0$, we have on making the substitution $y = xu^{-1}$,

$$\int_{-n}^0 (-y)^\lambda (x-y)_+^\mu \tau_n(x-y)\,dy = \int_{-n}^x (-y)^\lambda (x-y)^\mu\,dy$$

$$= (-x)^{\lambda+\mu+1} \int_{-x/n}^1 u^{-\lambda-\mu-2}(1-u)^\mu\,du$$

$$= (-x)^{\lambda+\mu+1} \int_{-x/n}^1 u^{-\lambda-\mu-2}\Big[(1-u)^\mu - \sum_{i=0}^r \frac{(-1)^i(\mu)_i}{i!}u^i\Big]du +$$

$$+(-x)^{\lambda+\mu+1}\sum_{i=0}^{r}\frac{(-1)^i(\mu)_i}{i!(i-\lambda-\mu-1)}[1-(-x/n)^{i-\lambda-\mu-1}],$$

for some integer $r > \lambda+\mu+1$, where

$$(\lambda)_s = \begin{cases} 1, & s=0 \\ \prod_{i=0}^{s-1}(\lambda-1), & s \geq 1. \end{cases}$$

It follows that

$$N-\lim_{n\to\infty}\int_{-n}^{0}(-y)^{\lambda}(x-y)_+^{\mu}\tau_n(x-y)dy =$$

$$= B(-\lambda-\mu-1,\ \mu+1)(-x)^{\lambda+\mu+1}, \tag{5}$$

see [3] or Gel'fand and Shilov [4].

When $x > 0$ and $-n \leq y \leq 0$, we have

$$\int_{-n}^{0}(-y)^{\lambda}(x-y)_+^{\mu}\tau_n(x-y)dy = \int_{x-n}^{0}(-y)^{\lambda}(x-y)^{\mu}dy+$$

$$+\int_{x-n-n^{-n}}^{x-n}(-y)^{\lambda}(x-y)^{\mu}\tau_n(x-y)dy \tag{6}$$

On making the substitution $y = x(1-u^{-1})$, we have

$$\int_{x-n}^{0}(-y)^{\lambda}(x-y)^{\mu}dy = x^{\lambda+\mu+1}\int_{x/n}^{1}u^{-\lambda-\mu-2}(1-u)^{\lambda}du$$

and it follows as above that

$$N-\lim_{n\to\infty}\int_{x-n}^{0}(-y)^{\lambda}(x-y)^{\mu}dy = B(-\lambda-\mu-1,\ \lambda+1)x^{\lambda+\mu+1}. \tag{7}$$

Further, with $n > 2x$

$$\left|\int_{x-n-n^{-n}}^{x-n}(-y)^{\lambda}(x-y)^{\mu}\tau_n(x-y)dy\right| \leq \int_{n}^{n+n^{-n}}(y-x)^{\lambda}y^{\mu}dy$$

$$= \int_{n}^{n+n^{-n}}y^{\lambda+\mu}(1-x/y)^{\lambda}dy$$

$$\leq \begin{cases} (n+n^{-n})^{\lambda+\mu}n^{-n}, & \lambda > 0, \\ 2^{-\lambda}(n+n^{-n})^{\lambda+\mu}n^{-n}, & -1 \langle \lambda < 0, \end{cases}$$

and so

$$\lim_{n\to\infty} \int_{x-n-n^{-n}}^{x-n} (-y)^{\lambda}(x-y)^{\mu}\tau_n(x-y)dy = 0. \tag{8}$$

It now follows from equations (6), (7) and (8) that
(9)

$$N - \lim_{n\to\infty} \int_{-n}^{0} (-y)^{\lambda}(x-y)_+^{\mu}\tau_n(x-y)dy = B(-\lambda-\mu-1, \lambda+1)x^{\lambda+\mu+1}.$$

Next, with $\frac{1}{2}n < a \leq x \leq b < \frac{1}{2}n$, we have

$$\left| \int_{-n-n^{-n}}^{-n} (-y)^{\lambda}\tau_n(y)(x-y)^{\mu}\tau_n(x-y)dy \right| \leq \int_{-n-n^{-n}}^{-n} (-y)^{\lambda+\mu}(1-x/y)^{\mu}dy$$

$$\leq \begin{cases} 2^{\mu}(n+n^{-n})^{\lambda+\mu}n^{-n}, & \mu > 0, \\ 2^{-\mu}(n+n^{-n})^{\lambda+\mu}n^{-n}, & -1 < \mu < 0 \end{cases}$$

and so

$$\lim_{n\to\infty} \int_{-n-n^{-n}}^{-n} (-y)^{\lambda}\tau_n(y)(x-y)^{\mu}dy = 0. \tag{10}$$

It now follows from equations (4), (5), (9) and (10) that

$$N - \lim_{n\to\infty}\langle (x_-^{\lambda})_n * (x_+^{\mu})_n, \phi\rangle = \langle B(-\lambda-\mu-1, \mu+1)x_-^{\lambda+\mu+1}$$

$$+B(-\lambda-\mu-1, \mu+1)x_+^{\lambda+\mu+1}, \phi(x)\rangle$$

and equation (3) follows for $\lambda, \mu\rangle - 1$ and $\lambda, \mu, \lambda+\mu+1 \neq 0, 1, 2, \ldots.$

Now assume that equation (3) holds for $\mu > -1$, $-k < \lambda < -k+1$ and $\mu, \lambda+\mu+k \neq 0, 1, 2, \ldots$, where $k$ is some positive integer. This is certainly true when $k = 1$. The convolution product $(x_-^{\lambda})_n * (x_+^{\mu})_n$ exists by Definition 2 and so equations (2) hold. Thus if $\phi$ is an arbitrary function in $\mathcal{D}$ with support contained in the interval $[a, b]$, where we may suppose that $a < 0 < b$,

$$\langle [(x_-^{\lambda})_n * (x_+^{\mu})_n]', \phi(x)\rangle = -\langle (x_-^{\lambda})_n * (x_+^{\mu})_n, \phi'(x)\rangle$$

$$= -\lambda\langle (x_-^{\lambda-1})_n * (x_+^\mu)_n, \phi(x)\rangle + \langle [x_-^\lambda - \tau_n'(x)] * (x_+^\mu)_n, \phi(x)\rangle$$

and so

$$-\lambda\langle (x_-^{\lambda-1})_n * (x_+^\mu)_n, \phi(x)\rangle = \langle (x_-^\lambda)_n * (x_+^\mu)_n, \phi'(x)\rangle + \langle [x_-^\lambda \tau_n'(x)] * (x_+^\mu)_n, \phi(x)\rangle.$$

The support of $x_-^\lambda \tau_n'(x)$ is contained in the interval $[-n - n^{-n}, n]$ and so with $n > -a > n^{-n}$, it follows as above that

$$\langle [x_-^\lambda \tau_n'(x)] * (x_+^\mu)_n, \phi(x)\rangle = \int_a^b \phi(x) \int_{-n-n^{-n}}^{-n} (-y)^\lambda \tau_n'(y)(x-y)^\mu \tau_n(x-y)\,dy\,dx$$

$$= \int_a^{-n^{-n}} \phi(x) \int_{-n-n^{-n}}^{-n} (-y)^\lambda \tau_n'(y)(x-y)^\mu\,dy\,dx +$$

$$(11) \qquad + \int_{-n^{-n}}^{n^{-n}} \phi(x) \int_{-n-n^{-n}}^{-n} (-y)^\lambda \tau_n'(y)(x-y)^\mu \tau_n(x-y)\,dy\,dx,$$

where on the domain of integration $(-y)^\lambda$ and $(x-y)^\mu$ are locally summable functions.

Putting $M = \sup\{|\tau'(x)\phi(x)|\}$, we have

$$\left| \int_{-n^{-n}}^{n^{-n}} \phi(x) \int_{-n-n^{-n}}^{-n} (-y)^\lambda \tau_n'(y)(x-y)^\mu \tau_n(x-y)\,dy\,dx \right|$$

$$\leq Mn^n \int_{-n^{-n}}^{n^{-n}} \int_{-n-n^{-n}}^{-n} (-y)^{\lambda+\mu}(1 - x/y)^\mu\,dy\,dx$$

$$\leq \begin{cases} 2^{1+\mu} M(n+n^{-n})^{\lambda+\mu} n^{-n}, & \mu > 0, \\ 2^{1-\mu} M(n+n^{-n})^{\lambda+\mu} n^{-n}, & -1 < \mu < 0 \end{cases}$$

and it follows that

$$(12) \qquad \lim_{n\to\infty} \int_{-n-n^{-n}}^{n^{-n}} \phi(x) \int_{-n-n^{-n}}^{-n} (-y)^\lambda \tau_n'(x-y)^\mu \tau_n(x-y)\,dy\,dx = 0.$$

Integrating by parts, we have

$$\int_{-n-n^{-n}}^{-n} (-y)^\lambda \tau_n'(y)(x-y)^\mu\,dy = n^\lambda (x+n)^\mu$$

$$+\int_{-n-n^{-n}}^{-n}[\lambda(-y)^{\lambda-1}(x-y)^{\mu}+\mu(-y)^{\lambda}(x-y)^{\mu-1}]\tau_n(y)dy. \tag{13}$$

Choosing a positive integer $r$ greater than $\lambda+\mu$, we see that

$$n^{\lambda}(x+n)^{\mu}=n^{\lambda+\mu}\sum_{i=0}^{r}\frac{(\mu)_i x^i}{i!n^i}+o(1/n)$$

and so

$$\int_a^{-n^{-n}} n^{\lambda}(x+n)^{\mu}\phi(x)dx = n^{\lambda+\mu}\sum_{i=0}^{r}\frac{(\mu)_i}{i!n^i}\int_a^{-n^{-n}} x^i\phi(x)dx$$

$$+o(1/n)\int_a^{-n^{-n}}\phi(x)dx \tag{14}$$

where

$$\lim_{n\to\infty} o(1/n)\int_a^{-n^{-n}}\phi(x)dx=0. \tag{15}$$

Putting

$$\int x^i\phi(x)dx=\chi_i(x),$$

for $i=0,1,2,...,r$, we have

$$\chi_i(x)=\chi_i(0)+x\chi_i'(\xi_i x),$$

where $0\le\xi_i\le 1$ and so

$$\int_a^{-n^{-n}} x^i\phi(x)dx=\chi_i(0)-n^{-n}\chi_i'(\xi_i n^{-n})-\chi_i(a)$$

for $i=0,1,2,...,r$.

Thus

$$N-\lim_{n\to\infty} n^{\lambda+\mu}\sum_{i=0}^{r}\frac{(\mu)_i}{i!n^i}\int_a^{-n^{-n}} x^i\phi(x)dx$$

$$=N-\lim_{n\to\infty} n^{\lambda+\mu}\sum_{i=0}^{r}\frac{(\mu)_i}{i!n^i}[\chi_i(0)-\chi_i(a)]$$

$$+\lim_{n\to\infty} n^{\lambda+\mu-n}\sum_{i=0}^{r}\chi_i'(-\xi_i n^{-n}) = 0,$$

since $\lambda+\mu$ is not an integer and so from equations (14) and (15) we have

$$N-\lim_{n\to\infty}\int_a^{n^{-n}} n^{\lambda}(x+n)^{\mu}\phi(x)dx = 0. \tag{16}$$

It now follows from equations (11), (12), (13) and (16) that

$$\begin{aligned} N-\lim_{n\to\infty}\lambda\langle(x_-^{\lambda-1})_n * (x_+^{\mu})_n,\phi(x)\rangle &= N-\lim_{n\to\infty}\langle(x_-^{\lambda})_n * (x_+^{\mu})_n,\phi'(x)\rangle \\ &= \langle x_-^{\lambda} \boxast x_+^{\mu},\phi'(x)\rangle \end{aligned}$$

by our assumption. This proves that the neutrix product $x_-^{\lambda-1} \boxast x_+^{\mu}$ exists and

$$\begin{aligned} x_-^{\lambda-1} \boxast x_+^{\mu} &= -\frac{(x_-^{\lambda} \boxast x_+^{\mu})'}{\lambda} \\ &= B(-\lambda-\mu,\mu+1)x_-^{\lambda+\mu} + B(-\lambda-\mu,\lambda)x_+^{\lambda+\mu}. \end{aligned}$$

Equation (3) now follows by induction for $\mu\rangle-1$, $\mu\neq 0,1,2,\ldots$ and $\lambda,\lambda+\mu\neq 0,\ \pm1,\pm2,\ldots$.

Finally assume that equation (3) holds for $-k\langle\mu\langle-k+1$ and $\lambda,\lambda+\mu\neq 0,\ \pm1,\pm2,\ldots$. This is certainly true when $k=1$. Then since

$$(x_-^{\lambda})_n * (x_+^{\mu})_n = (x_+^{\mu})_n * (x_-^{\lambda})_n,$$

an argument similar to that given above shows us that equation (3) follows by unduction for $\lambda,\mu,\lambda+\mu+1\neq 0,\ \pm1,\pm2,\ldots$. This completes the proof of the theorem. □

**Theorem 5.** *The neutrix convolution product $x_-^{\lambda} \boxast x_+^{s}$ exists and*

$$x_-^{\lambda} \boxast x_+^{s} = (-1)^{s+1}B(\lambda+1,s+1)x_-^{\lambda+s+1} \tag{17}$$

*for $\lambda\neq 0,\pm1,\pm2,\ldots$ and $s=0,1,2,\ldots$.*

*Proof.* The proof of equation (17) is exactly the same as the proof of equation (3), restricting $\mu$ to the values $\mu=s=0,1,2,\ldots$, and noting that

$$B(-\lambda-s-1,s+1) = (-1)^{s+1}B(\lambda+1,s+1)$$

and

$$B(-\lambda-s-1,\lambda+1) = 0.$$

**Corollary 1.** *The neutrix convolution product $x_+^\lambda \boxtimes x_-^s$ exists and*

$$x_+^\lambda \boxtimes x_-^s = (-1)^{s+1} B(\lambda+1, s+1) x_+^{\lambda+s+1} \tag{18}$$

*for $\lambda \neq 0, \pm 1, \pm 2, \ldots$ and $s = 0, 1, 2, \ldots$.*

*Proof.* The corollary follows immediately on replacing $x$ by $-x$ in equation (17).

**Corollary 2.** *The neutrix convolution product $x_-^\lambda \boxtimes x^s$ exists and*

$$x_-^\lambda \boxtimes x^s = 0 \tag{19}$$

*for $\lambda \neq 0, \pm 1, \pm 2, \ldots$ and $s = 0, 1, 2, \ldots$.*

*Proof.* The convolution product $x_-^\lambda * x_-^s$ exists by Definition 2 and

$$x_-^\lambda * x_-^s = B(\lambda+1, s+1) s_-^{\lambda+s+1}, \tag{20}$$

see [2]. Equation (19) now follows immediately from equation (17) on noting that $x^s = x_+^s + (-1)^s x_-^s$ and that the neutrix convolution product is clearly distributive with respect to addition.

**Corollary 3.** *The neutrix convolution product $x_+^\lambda \boxtimes x^s$ exists and*

$$x_+^\lambda \boxtimes x^s = 0,$$

*for $\lambda \neq 0, \pm 1, \pm 2, \ldots$ and $s = 0, 1, 2, \ldots$.*

*Proof.* The result follows immediately on replacing $x$ by $-x$ in equation (19).

**Theorem 6.** *The neutrix convolution product $x_-^r \boxtimes x_+^s$ exists and*

$$x_-^r \boxtimes x_+^s = -B(r+1, s+1)[(-1)^r x_+^{r+s+1} + (-1)^s x_-^{r+s+1}] \tag{21}$$

*for $r, s = 0, 1, 2, \ldots$.*

*Proof.* Equations (4), (5), (9) and (10) still hold with $\lambda = r$ and $\mu = s$ but $B(\lambda, \mu)$ with $\lambda$ a negative integer is defined as in [3], where it was proved that

$$B(-n, m) = (-1)^m B(m, n - m + 1)$$

for $m = 1, 2, ..., n$ and $n = 1, 2, ....$ Thus equation (5) becomes

$$N - \lim_{n\to\infty} \int_{-n}^{0} (-y)^r (x-y)_+^s \tau_n(x-y)dy = (-1)^{s+1} B(r+1, s+1)(-x)^{r+s+1}$$

and equation (9) becomes

$$N - \lim_{n\to\infty} \int_{-n}^{0} (-y)^r (x-y)_+^s \tau_n(x-y)dy = (-1)^{r+1} B(r+1, s+1)(-x)^{r+s+1}.$$

Equation (21) now follows as above

**Corollary 4.** *The neutrix convolution product* $x_-^r \circledast x^s$ *exists and*

$$x_-^r \circledast x^s = (-1)^{r+1} B(r+1, s+1) x_+^{r+s+1} \tag{22}$$

*for* $r, s = 0, 1, 2, ....$

*Proof.* Equation (20) holds with $\lambda = r$ and equation (22) then follows from equations (20) and (21).

**Corollary 5.** *The neutrix convolution product* $x_+^r \circledast x^s$ *exists and*

$$x_+^r \circledast x^s = (-1)^{r+s+1} B(r+1, s+1) x_-^{r+s+1} \tag{23}$$

*for* $r, s = 0, 1, 2, ....$

*Proof.* Equation (23) follows immediately on replacing $x$ by $-x$ in equation (22).

**Corollary 6.** *The neutrix convolution product* $x^r \circledast x^s$ *exists and*

$$x^r \circledast x^s = -B(r+1, s+1)[x_+^{r+s+1} + (-1)^{r+s} x_-^{r+s+1}] \tag{24}$$

*for* $r, s = 0, 1, 2, ....$

*Proof.* Equation (24) follows immediately from equations (22) and (23).

The distributions $|x|^\lambda$ and $sgnx \cdot |x|^\lambda$are defined by

$$|x|^\lambda = x_+^\lambda + x_-^\lambda, \; sgnx \cdot |x|^\lambda = x_+^\lambda - x_-^\lambda.$$

It follows that further neutrix convolution products such as

$$x_-^\lambda \boxed{*} |x|^\mu, \;\; x_+^\lambda \boxed{*} |x|^\mu (sgnx \cdot |x|^\mu),$$

$$(sgnx \cdot |x|^\lambda) \boxed{*} x_+^\mu, \;\; |x|^\lambda \boxed{*} |x|^\mu, \;\; |x|^\lambda \boxed{*} x_-^\mu$$

exist for $\lambda, \mu, \lambda + \mu \neq -1, -2, ....$

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**REZIME**

KOMUTATIVNA NEUTRIKS KONVOLUCIJA DISTRIBUCIJA

U ovom radu je uvedena komutativna konvolucija koja je jednaka jedinici na intervalu $[-\frac{1}{2}, \frac{1}{2}]$. Pokazano je da je dobijena konvolucija stvarno uopštenje uobičajene konvolucije u $(L^p, L^q)$ kao i konvolucije distribucija u smislu Gel'fand- Šilova.

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