A COMMUTATIVE NEUTRIX CONVOLUTION PRODUCT OF DISTRIBUTIONS

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Abstract

Let $f$ and $g$ be distributions in $\mathcal{D}'$ and let

$$f_n(x) = f(x) * \eta_n(x), \quad g_n(x) = g(x) * \eta_n(x)$$

where $\eta_n(x)$ is a certain function which converges to the identity func as $n$ tends to infinity. Then the neutrix convolution product $f \ast g$ is defined as the neutrix limit of the sequence $\{f_n * g_n\}$, provided the limit $\lambda$ exists in the sense that

$$N = \lim_{n \to \infty} (f_n * g_n, \phi) = (\lambda, \phi)$$

for all $\phi \in \mathcal{D}$. The neutrix convolution products $x^\lambda \ast x^\mu$, for $\lambda, \mu \geq 0$, $\mu \neq 0$, $\pm 1, \pm 2, \ldots$ and $x^\mu \ast x^\mu$, for $\lambda \neq 0$, $\pm 1, \pm 2, \ldots$ and $s = 0, 1, 2, \ldots$ are evaluated, from which other neutrix convolution products are deduced.

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The classical definition for the convolution product of two functions $f$ and $g$ is as follows:
Definition 1. Let \( f \) and \( g \) be functions. Then the convolution product \( f \ast g \) is defined by

\[
(f \ast g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)dt
\]

for all points \( x \) for which the integral exists.

It follows easily from the definition that if \( f \ast g \) exists then \( g \ast f \) exists and

\[
(f \ast g)(x) = g \ast f
\]

and if \( (f \ast g)' \) and \( f \ast g' \) (or \( f' \ast g \)) exist, then

\[
(f \ast g)' = f \ast g' \quad \text{(or} \quad f' \ast g)
\]

The following theorem also holds and it is an immediate consequence of Hölder's inequality for integrals.

Theorem 1. Let \( f \) and \( g \) be functions in \( L^p(-\infty, \infty) \) and \( L^q(-\infty, \infty) \) respectively, where \( 1/p + 1/q = 1 \). Then the convolution product \( (f \ast g)(x) \) exists for all \( x \).

Now, suppose that the convolution product \( (f \ast g)(x) \) exists for all \( x \) and let \( \phi \) be an arbitrary test function in the space \( \mathcal{D} \) of infinitely differentiable functions with compact support. Then

\[
\langle (f \ast g)(x), \phi(x) \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)g(x-t)\phi(x-y)dtdy
\]

and for convenience we will write this as

\[
\langle (f \ast g)(x), \phi(x) \rangle = \langle f(y), (g(x), \phi(x+y)) \rangle
\]

even though the infinitely differentiable function \( (g(x), \phi(x+y)) \) does not necessarily have compact support. This equation does however suggest the following definition for the convolution product of certain distributions \( f \) and \( g \) in \( \mathcal{D}' \), see for example Gel'fand and Shilov [4].

Definition 2. Let \( f \) and \( g \) be distributions in \( \mathcal{D}' \) satisfying either of the following conditions:
(a) either $f$ or $g$ has bounded support,
(b) the supports of $f$ and $g$ are bounded on the same side.

Then the convolution product $f * g$ is defined by

$$(f * g)(x), \phi = \langle f(y), (g(x), \phi(x + y)) \rangle$$

for arbitrary $\phi$ in $D$.

Note that with this definition, if $g$ has bounded support, then $(g(x), \phi(x + y))$ is in $D$ and so $(f(y), (g(x), \phi(x + y)))$ is meaningful. If on the other hand either $f$ has bounded support or the supports of $f$ and $g$ are bounded on the same side, then the intersection of the supports of $f$ and $(g(x), \phi(x + y))$ is bounded and so $(f(y), (g(x), \phi(x + y)))$ is again meaningful.

It follows that if the convolution product $f * g$ exists by Definition 2, then equations (1) and (2) always hold.

Definition 1 and 2 are very restrictive and can only be used for a small class of distributions. In order to extend the convolution product to a larger class of distributions, Jones [5] gave the following definition.

**Definition 3.** Let $f$ and $g$ be distributions and let $\tau$ be an infinitely differentiable function satisfying the following conditions:

(i) $\tau(x) = \tau(-x)$,
(ii) $0 \leq \tau(x) \leq 1$,
(iii) $\tau(x) = 1$ for $|x| \leq 1/2$,
(iv) $\tau(x) = 0$ for $|x| \geq 1$.

Let

$$f_n(x) = f(x)\tau(x/n), \quad g_n(x) = g(x)\tau(x/n)$$

for $n = 1, 2, \ldots$. Then the convolution product $f * g$ is defined as the limit of the sequence $(f_n * g_n)$, provided the limit $h$ exists in the sense that

$$\lim_{n \to \infty} (f_n * g_n, \phi) = \langle h, \phi \rangle$$

for all test functions $\phi$ in $D$.

In this definition the convolution product $f_n * g_n$ exists by Definition 2 since $f_n$ and $g_n$ have bounded supports. It follows that if the limit of the
sequence \( \{f_n \ast g_n\} \) exists, so that the convolution product \( f \ast g \) exists, then \( g \ast f \) also exists and equation (1) holds. However equation (2) need not necessarily hold since \( \text{sign} \) proved that

\[
1 \ast \text{sign} = \text{sign} \ast 1 = x,
\]

\[
(1 \ast \text{sign})' = 1, \quad 1' \ast \text{sign} = 0, \quad 1 \ast (\text{sign})' = 2.
\]

It can be proved that if a convolution product exists by Definitions 1 and 2 then it exists by Definition 3 and defines the same distribution.

However, there were still many convolution products which did not exist by Definition 3 and in order to try and remedy this the next definition was introduced in [2].

**Definition 4.** Let \( f \) and \( g \) be distributions and let

\[
\tau_{\alpha}(x) = \begin{cases} 
1, & |x| \leq n, \\
\tau(n^n x - n^{n+1}), & z > n, \\
\tau(n^n x + n^{n+1}), & x < -n,
\end{cases}
\]

for \( n = 1, 2, \ldots \), where \( \tau \) is defined as in Definition 3. Let \( f_n(x) = f(x)\tau_{\alpha}(x) \) for \( n = 1, 2, \ldots \). Then the neutrix convolution product \( f \ast g \) is defined as the neutrix limit of the sequence \( \{f_n \ast g\} \), provided the limit \( h \) exists in the sense that

\[
N - \lim_{\alpha \to 0} (f_n \ast g, \phi) = (h, \phi)
\]

for all \( \phi \) in \( D \), where \( N \) is the neutrix, see van der Corput [1]; having domain \( N' = \{1, 2, \ldots, n, \ldots\} \) and range the real numbers with negligible functions finite linear sums of the functions

\[
n^k \ln^{k-1} n, \quad n^k, \quad (\lambda)0, \quad (\lambda)1, \quad (\lambda)2, \ldots
\]

and all functions \( \epsilon(n) \) for which \( \lim_{n \to \infty} \epsilon(n) = 0 \).

The convolution product \( f \ast g \) in this definition is again in the sense of Definition 2, the support of \( f_n \) being contained in the interval \([-n - n', n + n']\). It can be proved that if a convolution product exists by Definitions 1 or 2 then the neutrix convolution product exists and defines the same distribution.
However, the neutrix convolution product as defined in Definition 4 is in general non-commutative. For example, it was proved in [2] that

$$x_+ \circ x_- = \frac{1}{6} x^2_+ \quad \text{and} \quad x_- \circ x_+ = \frac{1}{6} x^2_-,$$

so that

$$x_+ \circ x_- \neq x_- \circ x_+.$$

In the following, we now consider a commutative neutrix convolution product. We will denote the commutative neutrix convolution product of the distributions $f$ and $g$ by $f \boxplus g$ to distinguish it from the non-commutative neutrix convolution product.

**Definition 5.** Let $f$ and $g$ be distributions and let $\tau_n$ be defined as in Definition 4. Let $f_n(x) = f(x)\tau_n(x)$ and $g_n(x)\tau_n(x)$ for $n = 1, 2, \ldots$. Then the commutative neutrix convolution product $f \boxplus g$ is defined as the neutrix limit of the sequence $\{f_n \ast g_n\}$, provided the limit $h$ exists in the sense that

$$N - \lim_{n \to \infty} (f_n \ast g_n, \phi) = (h, \phi)$$

for all $\phi$ in $D$, where $N$ is the neutrix defined above.

The convolution product $f_n \ast g_n$ in this definition is again in the sense of Definition 2 and since $f_n \ast g_n = g_n \ast f_n$, the neutrix convolution product $f \boxplus g$ is clearly commutative.

The next theorem shows that this definition generalizes Definition 1.

**Theorem 2.** Let $f$ and $g$ be functions in $L^p(-\infty, \infty)$ and $L^q(-\infty, \infty)$ respectively, where $1/p + 1/q = 1$, so that the convolution product $f \ast g$ exists by Definition 1. Then the neutrix convolution product $f \boxplus g$ exists and

$$f \boxplus g = f \ast g.$$

**Proof.** For arbitrary $\epsilon > 0$ we have

$$|f \ast g - f_n \ast g_n| = \left| \int_{-\infty}^{\infty} f(t)g(x-t)\tau(t)\tau_n(x-t)dt \right|$$

$$\leq \int_{-\infty}^{\infty} |f(t)||g(x-t)|\tau(t)\tau_n(x-t)dt$$

$$+ \int_{-\infty}^{\infty} |f_n(t)||g(x-t)|\tau(t)\tau_n(x-t)dt$$

$$\leq \int_{|t| < \epsilon} |f(t)||g(x-t)|dt + \int_{|t| \geq \epsilon} |f_n(t)||g(x-t)|dt$$

$$= \int_{|t| < \epsilon} |f(t)||g(x-t)|dt + \int_{|t| \geq \epsilon} |f_n(t)||g(t)||\tau_n(x-t)dt.$$
for all $n$ greater than some $n_0$. Thus if $\phi$ is an arbitrary function in $D$ then

$$|\langle f \ast g, \phi \rangle - \langle f_n \ast g_n, \phi \rangle| \leq \sup \{|\phi(x)|\}.$$

for $n > n_0$ and it follows that

$$\lim_{n \to \infty} \langle f_n \ast g_n, \phi \rangle = \langle f \ast g, \phi \rangle = N - \lim_{n \to \infty} \langle f_n \ast g_n, \phi \rangle.$$

The result of the theorem follows.

The next theorem shows that Definition 5 also generalizes Definition 2.

**Theorem 2.** Let $f$ and $g$ be distributions satisfying either condition (a) or condition (b) of Definition 2 so that the convolution product $f \ast g$ exists by Definition 2. Then the neutra convolution product $f \circ g$ exists and $f \circ g = f \ast g$.

**Proof.** Suppose first of all that the support of $g$ is bounded so that $g = g_n$ for some $n$ greater than some $n_0$. Then with $n > n_0$ and arbitrary $\phi$ in $D$

$$\langle f_n \ast g_n, \phi \rangle = \langle f_n, (y), (g(x), \phi(x + y)) \rangle$$

$$= \langle f_n, (y), (g(x), \phi(x + y)) \rangle$$

for large enough $n$, since the support of $(g(x), \phi(x + y))$ is bounded. It follows that

$$\lim_{n \to \infty} \langle f_n \ast g_n, \phi \rangle = \langle f(y), (g(x), \phi(x + y)) \rangle$$

$$= N - \lim_{n \to \infty} \langle f_n \ast g_n, \phi \rangle$$

and the result of the theorem follows when the support of $g$ is bounded. Now suppose that the support of $f$ is bounded. Then the result of the theorem follows as above on noting that $f_n \ast g_n = g_n \ast f_n$.

Finally, suppose that the supports of $f$ and $g$ are bounded on the same side, say on the left, so that the supports of $f$ and $g$ are contained in some half-bounded intervals $[a, \infty)$ and $[b, \infty)$ respectively. Now let $\phi$ be an arbitrary function in $D$ with its support contained in the bounded interval $[c, d]$. Then since $g(x) = 0$ if $x < b$,

$$\psi(y) = \langle g_n(x), \phi(x + y) \rangle = \langle g(x), \phi(x + y) \rangle = 0$$
if $y > d - b$. Further, since $f(y) = 0$ if $y < a$, it follows that the intersection of the supports of $\phi$ and $f$ are contained in the interval $[a, d - b]$ if $d - b > a$ and is the empty set otherwise. Thus

$$\langle f \ast g, \phi \rangle = \langle f \ast g, \phi \rangle$$

for $n > \max\{|a|, |d - b|\}$ and the result of the theorem follows as above for this third case.

**Theorem 4.** The neutrix convolution product $x_\ast \omega \ast x_\ast \omega$ exists and

$$\begin{align*}
\omega &\ast x_\ast \omega = B(-\lambda - \mu - 1, \mu + 1) x_{\lambda + \mu + 1} \\
&= B(-\lambda - \mu - 1, \lambda + 1) x_{\lambda + \mu + 1}
\end{align*}$$

for $\lambda, \mu, \lambda + \mu \neq 0$, $\pm 1, \pm 2, \ldots$, where $B$ denotes the Beta function.

**Proof.** We will first of all suppose that $\lambda, \mu > 1$, so that $x_{\lambda}$ and $x_{\mu}$ are locally summable functions. Put

$$(x_{\lambda})_n = x_{\lambda} \tau_n(x), \quad (x_{\mu})_n = x_{\mu} \tau_n(x).$$

Then the convolution product $(x_{\lambda})_n \ast (x_{\mu})_n$ exists by Definition 2 and

$$\begin{align*}
(x_{\lambda})_n \ast (x_{\mu})_n &= \left((x_{\lambda})_n \ast (x_{\mu})_n, \phi(x + y)\right) \\
&= \int_0^{+\infty} (-y)^\lambda \rho_n(y) \int_0^{+\infty} (x - y)^\mu \rho_n(x - y) dy dx + \\
&+ \int_0^{+\infty} \rho_n(x) \int_0^{+\infty} (-y)^\lambda \rho_n(y) (x - y)^\mu \rho_n(x - y) dy dx + \\
&\quad + \int_0^{+\infty} \rho_n(x) \int_0^{+\infty} (-y)^\lambda \rho_n(y) (x - y)^\mu \rho_n(x - y) dy dx
\end{align*}$$

for $n > -a$ and arbitrary $\phi$ in $D$ with support of $\phi$ contained in the interval $[a, b]$.

When $x < 0$ and $-n \leq y \leq 0$, $\tau_n(x - y) = 1$ on the support of $\phi$. Thus

$$\begin{align*}
\int_0^{+\infty} (-y)^\lambda (x - y)^\mu \rho_n(x - y) dy &= \int_0^{+\infty} (-y)^\lambda (x - y)^\mu \rho_n(x - y) dy \\
&= (-x)^{\lambda + \mu + 1} \int_0^{+\infty} u^{\lambda + \mu + 1} (1 - u)^n du \\
&= (-x)^{\lambda + \mu + 1} \int_0^{+\infty} u^{\lambda + \mu + 2} (1 - u) du - \frac{1}{n!} \int_0^{+\infty} \frac{1}{u} \int_0^{+\infty} (-y)^\lambda (x - y)^\mu \rho_n(x - y) dy du.
\end{align*}$$
\[ +z(-x)^{s+\mu+1} \sum_{r=0}^{s} \binom{s}{r} \frac{(-1)^r (\mu)_r}{r!} (-x/n)^{\lambda+r-1}, \]

for some integer \( r > \lambda + \mu + 1 \), where

\[ \lambda_s = \begin{cases} 1, & s = 0, \\ \frac{1}{x^n} \prod_{i=0}^{s-1} (\lambda - i), & s \geq 1. \end{cases} \]

It follows that

\[ N = \lim_{n \to \infty} \int_{-x}^{0} (-y)^{\lambda} (x-y)^n \tau_n(x-y) dy = \]

\[ = B(-\lambda - \mu - 1, x + 1)(-x)^{\lambda+\mu+1}, \]

see [3] or Gel'fand and Shilov [4].

When \( x > 0 \) and \(-n \leq y \leq 0\), we have

\[ \int_{-n}^{0} (-y)^{\lambda} (x-y)^n \tau_n(x-y) dy = \int_{-n}^{0} (-y)^{\lambda} (x-y)^n dy + \int_{-n}^{0} (-y)^{\lambda} (x-y)^n \tau_n(x-y) dy \]

\[ \text{(6)} \]

\[ = \int_{-n}^{0} (-y)^{\lambda} (x-y)^n \tau_n(x-y) dy \]

On making the substitution \( y = x(1 - u^{-1}) \), we have

\[ \int_{-n}^{0} (-y)^{\lambda} (x-y)^n dy = x^{\lambda+n+1} \int_{1/n}^{1} u^{-\lambda-\mu-2} (1 - u)^{\lambda+1} du \]

and it follows as above that

\[ N = \lim_{n \to \infty} \int_{-n}^{0} (-y)^{\lambda} (x-y)^n dy = B(-\lambda - \mu - 1, \lambda + 1)x^{\lambda+\mu+1}. \]

Further, with \( n \geq 2x \)

\[ \left| \int_{-n}^{0} (-y)^{\lambda} (x-y)^n \tau_n(x-y) dy \right| \leq \int_{-n}^{0} (y-x)^{\lambda} y^n dy \]

\[ = \int_{n}^{\infty} y^{\lambda+n+1} (1-x/y)^n dy \]
\[
\begin{align*}
\lambda &> 0, \\
-1(\lambda < 0),
\end{align*}
\]
and so
\[
(\text{6}) \quad \lim_{n \to \infty} \int_{x=-a}^{x=-b} (-y)^{\lambda} \tau_n(x-y) dy = 0.
\]

It now follows from equations (6), (7) and (8) that
\[
(\text{9}) \quad N - \lim_{n \to \infty} \int_{x=-a}^{x=-b} (-y)^{\lambda} \tau_n(x-y) dy = B(-\lambda - \mu - 1, \lambda + 1) x^{\lambda+\mu+1}.
\]

Next, with \( \frac{1}{2} n < a \leq x \leq b < \frac{1}{2} n \), we have
\[
\int_{x=-a}^{x=-b} (-y)^{\lambda} \tau_n(y)(x-y) dy \leq \int_{x=-a}^{x=-b} (-y)^{\lambda+n}(1-x/y)^{\mu} dy
\]
\[
\leq \begin{cases}
2^{\mu}(n + n^{-n})^{\lambda+n+1} & \mu > 0, \\
2^{-\mu}(n + n^{-n})^{\lambda+n+1} & -1 < \mu < 0
\end{cases}
\]
and so
\[
\lim_{n \to \infty} \int_{x=-a}^{x=-b} (-y)^{\lambda} \tau_n(y)(x-y)^{\mu} dy = 0.
\]

It now follows from equations (4), (5), (9), and (10) that
\[
N - \lim_{n \to \infty} \langle (x^+)_n \ast (x^+_\phi)_n \rangle = \langle B(-\lambda - \mu - 1, \mu + 1) x^{\lambda+\mu+1} + B(-\lambda - \mu - 1, \mu + 1) x^{\lambda+\mu+1}, \phi(x) \rangle
\]
and equation (3) follows for \( \lambda, \mu, \lambda + \mu + 1 \neq 0, 1, 2, \ldots \).

Now assume that equation (3) holds for \( \mu > -1, -k < \lambda < -k + 1 \) and \( \mu, \lambda + \mu + k \neq 0, 1, 2, \ldots \) where \( k \) is some positive integer. This is certainly true when \( k = 1 \). The convolution product \((x^+_\phi)_n \ast (x^+_\phi)_n\) exists by Definition 2 and so equations (2) hold. Thus if \( \phi \) is an arbitrary function in \( D \) with support contained in the interval \([a, b]\), where we may suppose that
\[
(a < 0 < b),
\]
\[
\langle (x^+)_n \ast (x^+_\phi)_n \rangle = -\langle (x^+)_n \ast (x^+_\phi)_n \rangle.
\]
\[ = -\lambda((x_n^{-1})_n \ast (x_n^+)_n, \phi(x)) + (x_n^{-1} - x_n^+)_n, \phi(x) \]

and so

\[ -\lambda((x_n^{-1})_n \ast (x_n^+)_n, \phi(x)) = (x_n^{-1})_n \ast (x_n^+)_n, \phi(x) + (x_n^{-1} - x_n^+)_n, \phi(x). \]

The support of \( x_n^{-1}r_n(x) \) is contained in the interval \([-n - n^{-a}, n]\) and so with \( n > -a > n^{-a} \), it follows as above that

\[
(x_n^{-1}r_n(x) \ast (x_n^+)_n, \phi(x)) = \int_{-n}^{n} \phi(x) \int_{-n-n^{-a}}^{-n} (-y)^\alpha r_n(x-y)^\alpha r_n(x-y)dydx
\]

\[ = \int_{-n}^{-n^{-a}} \phi(x) \int_{-n-n^{-a}}^{-n} (-y)^\alpha r_n(y)(x-y)^\alpha r_n(x-y)dydx + \int_{-n^{-a}}^{n} \phi(x) \int_{-n-n^{-a}}^{-n} (-y)^\alpha r_n(y)(x-y)^\alpha r_n(x-y)dydx, \]

where on the domain of integration \((-y)^\alpha\) and \((x-y)^\alpha\) are locally summable functions.

Putting \( M = \sup(\{|r_n(x)|\phi(x)|\}) \), we have

\[
| \int_{-n}^{-n^{-a}} \phi(x) \int_{-n-n^{-a}}^{-n} (-y)^\alpha r_n(y)(x-y)^\alpha r_n(x-y)dydx |
\]

\[ \leq Mn^{-a} \int_{-n-n^{-a}}^{-n} (-y)^\alpha r_n(y)(1 - y/x)^\alpha r_n(x-y)dydx \]

\[ \leq \begin{cases} 2^{1-\alpha}M(n + n^{-a})^{\alpha + n^{-a}} & \text{if } \alpha > 0, \\ 2^{1-\alpha}M(n + n^{-a})^{\alpha + n^{-a}} & \text{if } \alpha < 0. \end{cases} \]

and it follows that

\[
\lim_{n \to \infty} \int_{-n}^{-n^{-a}} \phi(x) \int_{-n-n^{-a}}^{-n} (-y)^\alpha r_n(y)(x-y)^\alpha r_n(x-y)dydx = 0.
\]

Integrating by parts, we have

\[
\int_{-n}^{-n^{-a}} (-y)^\alpha r_n(y)(x-y)^\alpha dy = n^\alpha(x + n)^\alpha
\]
Choosing a positive integer \( r \) greater than \( \lambda + \mu \), we see that

\[
n^\lambda (x + n)^\mu = n^{\lambda + \mu} \sum_{i=0}^{r} \left( \frac{\mu}{i!n^i} \right) x^i + o(1/n)
\]

and so

\[
\int_{a}^{\infty} n^\lambda (x + n)^\mu \phi(x)dx = n^{\lambda + \mu} \sum_{i=0}^{r} \left( \frac{\mu}{i!n^i} \right) \int_{a}^{\infty} x^i \phi(x)dx + o(1/n)
\]

where

\[
\lim_{n \to \infty} o(1/n) \int_{a}^{\infty} \phi(x)dx = 0.
\]

Putting

\[
\int x^i \phi(x)dx = \chi_i(x),
\]

for \( i = 0, 1, 2, \ldots, r \), we have

\[
\chi_i(x) = \chi_i(0) + x \chi_i'(\xi_i x),
\]

where \( 0 \leq \xi_i \leq 1 \) and so

\[
\int_{a}^{\infty} x^i \phi(x)dx = \chi_i(0) - n^{-\mu} \chi_i'(\xi_i n^{-\mu}) - \chi_i(0)
\]

for \( i = 0, 1, 2, \ldots, r \).

Thus

\[
N - \lim_{n \to \infty} n^{\lambda + \mu} \sum_{i=0}^{r} \left( \frac{\mu}{i!n^i} \right) \int_{a}^{\infty} x^i \phi(x)dx = N - \lim_{n \to \infty} n^{\lambda + \mu} \sum_{i=0}^{r} \left( \frac{\mu}{i!n^i} \right) [\chi_i(0) - \chi_i(0)]
\]
\[
\lim_{n \to \infty} n^{4+\mu-\eta} \sum_{i=0}^{\infty} \chi_i (-\xi n^{-\eta}) = 0,
\]

since \(\lambda + \mu\) is not an integer and so from equations (14) and (15) we have

\[
N - \lim_{n \to \infty} \int_0^{N^n} n^\Delta (x + n)^\Delta \phi(x) dx = 0.
\]

It now follows from equations (11), (12), (13) and (16) that

\[
N - \lim_{n \to \infty} \gamma \left( (x_j^k, x_k^j) \right) = N - \lim_{n \to \infty} \left( (x_j^k, x_k^j) \right) = (x_j^k, x_k^j),
\]

by our assumption. This proves that the neutrix product \(x_j^k \star \text{Ber}_o^2\) exists and

\[
x_j^k \star \text{Ber}_o^2 = \frac{(x_j^k)^{2 \tau}}{\lambda} = B(-\lambda - \mu, \mu + 1) x_j^k + B(-\lambda - \mu, \lambda) x_j^k.
\]

Equation (3) now follows by induction for \(\mu \geq 0, 1, 2, \ldots\) and \(\lambda, \lambda + \mu \neq 0, \pm 1, \pm 2, \ldots\)

Finally assume that equation (3) holds for \(-k(\mu - k + 1) + \lambda, \lambda + \mu \neq 0, \pm 1, \pm 2, \ldots\). This is certainly true when \(k = 1\). Then since

\[
(x_j^k, x_k^j) = (x_j^k, x_j^k),
\]

an argument similar to that given above shows us that equation (3) follows by induction for \(\lambda, \mu, \lambda + \mu + 1 \neq 0, \pm 1, \pm 2, \ldots\). This completes the proof of the theorem.

\[\square\]

**Theorem 5.** The neutrix convolution product \(x_j^k \star \text{Ber}_o^2\) exists and

\[
x_j^k \star \text{Ber}_o^2 = \frac{(-1)^{\eta} B(\lambda + 1, s + 1) x_j^k}{\lambda + \mu + 1}.
\]

for \(\lambda \neq 0, \pm 1, \pm 2, \ldots\) and \(s = 0, 1, 2, \ldots\)

**Proof.** The proof of equation (17) is exactly the same as the proof of equation (3), restricting \(\mu\) to the values \(\mu = s = 0, 1, 2, \ldots\) and noting that

\[
B(-\lambda - s - 1, s + 1) = (-1)^{s+1} B(\lambda + 1, s + 1)
\]

and

\[
B(-\lambda - s - 1, \lambda + 1) = 0.
\]
Corollary 1. The neutrix convolution product \( x_+^s \mathcal{N}x_+^t \) exists and

\[
x_+^s \mathcal{N}x_+^t = (-1)^{s+1} B(\lambda + 1, s + 1) x_+^{\lambda + s + 1}
\]

for \( \lambda \neq 0, \pm 1, \pm 2, \ldots \) and \( s = 0, 1, 2, \ldots \).

Proof. The corollary follows immediately on replacing \( x \) by \(-x\) in equation (17).

Corollary 2. The neutrix convolution product \( x_+^s \mathcal{N}x_+^t \) exists and

\[
x_+^s \mathcal{N}x_+^t = 0
\]

for \( \lambda \neq 0, \pm 1, \pm 2, \ldots \) and \( s = 0, 1, 2, \ldots \).

Proof. The convolution product \( x_+^s \ast x_+^t \) exists by Definition 2 and

\[
x_+^s \ast x_+^t = B(\lambda + 1, s + 1) x_+^{\lambda + s + 1},
\]

see [2]. Equation (19) now follows immediately from equation (17) on noting that \( x^t = x_+^t + (-1)^t x_-^t \) and that the neutrix convolution product is clearly distributive with respect to addition.

Corollary 3. The neutrix convolution product \( x_+^s \mathcal{N}x_-^t \) exists and

\[
x_+^s \mathcal{N}x_-^t = 0,
\]

for \( \lambda \neq 0, \pm 1, \pm 2, \ldots \) and \( s = 0, 1, 2, \ldots \).

Proof. The result follows immediately on replacing \( x \) by \(-x\) in equation (19).

Theorem 6. The neutrix convolution product \( x_-^r \mathcal{N}x_+^s \) exists and

\[
x_-^r \mathcal{N}x_+^s = -B(r + 1, s + 1) [-1]^s x_+^{r+s+1} + (-1)^s x_-^{r+s+1}
\]

for \( r, s = 0, 1, 2, \ldots \).
Proof. Equations (4), (5), (8) and (10) still hold with \( \lambda = r \) and \( \mu = s \) but 
\( B(\lambda, \mu) \) with \( \lambda \) a negative integer is defined as in [3], where it was proved that:

\[
B(-n, m) = (-1)^m B(m, n - m + 1)
\]

for \( m = 1, 2, \ldots, n \) and \( n = 1, 2, \ldots \). Thus equation (5) becomes

\[
N - \lim_{n \to \infty} \int_{-n}^{0} (-y)^r (x-y) y \tau_n(x-y) dy = (-1)^{r+1} B(r+1, s+1)(-x)^{r+s+1}
\]

and equation (8) becomes

\[
N - \lim_{n \to \infty} \int_{-n}^{0} (-y)^r (x-y) \tau_n(x-y) dy = (-1)^{r+1} B(r+1, s+1)(-z)^{r+s+1}.
\]

Equation (21) now follows as above.

Corollary 4. The neutrion convolution product \( z^n \otimes x^r \) exists and

\[
x^r \otimes z^n = (-1)^{r+1} B(r+1, s+1) x^{n+s+1}
\]

for \( r, s = 0, 1, 2, \ldots \).

Proof. Equation (20) holds with \( \lambda = r \) and equation (22) then follows from equations (20) and (21).

Corollary 5. The neutrion convolution product \( x^r \otimes z^n \) exists and

\[
x^r \otimes z^n = (-1)^{r+1} B(r+1, s+1) z^{n+s+1}
\]

for \( r, s = 0, 1, 2, \ldots \).

Proof. Equation (23) follows immediately on replacing \( x \) by \( -x \) in equation (22).

Corollary 6. The neutrion convolution product \( x^r \otimes z^n \) exists and

\[
x^r \otimes z^n = - B(r+1, s+1) [x^{n+s+1} + (-1)^{r+s} z^{n+s+1}]
\]

for \( r, s = 0, 1, 2, \ldots \).
Proof. Equation (24) follows immediately from equations (22) and (23).

The distributions $|x|^{\lambda}$ and $\text{sgn} x \cdot |x|^{\lambda}$ are defined by

$$|x|^{\lambda} = x_+^{\lambda} + x_-^{\lambda}, \quad \text{sgn} x \cdot |x|^{\lambda} = x_+^{\lambda} - x_-^{\lambda}.$$  

It follows that further neutrix convolution products such as

$$x_+^{\lambda} \mu(x)^{\mu}, \quad x_-^{\lambda} \mu(x)^{\mu}(\text{sgn} x \cdot |x|^{\mu}),$$

$$(\text{sgn} x \cdot |x|^{\lambda}) \mu(x)^{\mu}, \quad |x| \mu(x)^{\mu}, \quad |x|^{\lambda} \mu(x)^{\mu}$$

exist for $\lambda, \mu, \lambda + \mu \neq -1, -2, \ldots$.

References


REZIME

KOMUTATIVNA NEUTRIKS KONVOLUCIJA DISTRIBUCIJA

U ovom radu je uvedena komutativna konvolucija koja je jednaka jedinici na intervalu $[-\frac{1}{2}, \frac{1}{2}]$. Pokušano je da je dobijena konvolucija stvarno uopštena uobičajene konvolucije u $(L^1, L^1)$ kao i konvolucije distribucija u smislu Gel'fand-a Silova.