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# SEVERAL PRODUCTS OF DISTRIBUTIONS ON MANIFOLDS

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**Abstract.** The problem of defining products of distributions on manifolds, particularly un the ones of lower dimension, has been a serious challenge since Gel'fand introduced special types of generalized functions, which are needed in quantum field. In this paper, we start with Pizetti's formula and an introduction on differential forms and distributions defined on manifolds, and then apply Pizetti's formula and a recursive structure of $\triangle^j(X^l\phi(x))$ to compute the asymptotic product $X^l\delta(r-1)$. Furthermore, we study the product

$$f(P_1, \cdots, P_k)\frac{\partial^{|\alpha|}\delta(P_1, \cdots, P_k)}{\partial P_1^{\alpha_1}\cdots\partial P_k^{\alpha_k}}$$

on smooth manifolds of lower dimension, which extends a few results obtained earlier. Several generalized functions, such as $\delta(QP_1, \cdots, QP_k)$ and $\delta(Q_1P_1, \cdots, Q_kP_k)$, are derived based on the transformation of differential form $\omega$.

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## 1. Pizetti's formula and differential forms

The simplest example of a generalized function concentrated on a manifold of dimension less than $n$ is one defined by

$$(f,\ \phi) = \int_S f(x)\phi(x)d\sigma,$$

where $S$ is the given manifold, $d\sigma$ is the induced measure on $S$, $f(x)$ is a fixed function, and $\phi \in \mathcal{D}(R^n)$.

As an example, let us consider the distribution $\delta(r-a)$, where $r^2 = \sum_{i=1}^n x_i^2$ and $a > 0$. The equation $r - a = 0$ defines the sphere $O_a$ of radius $a$. We have

$$(\delta(r-a),\ \phi) = \int_{O_a} \phi dO_a.$$

where $dO_a$ is the Euclidean element on the sphere $r - a = 0$.

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To make this paper as self-contained as possible, we begin to state Pizetti's formula and briefly introduce differential forms in the following, which are extremely helpful in defining distributions on manifolds in an invariant way. Please refer to reference [1] for detail.

Assume $d\sigma$ is the Euclidean area on the unit sphere $\Omega$ $(= O_1)$in $R^n$, and $S_\phi(r)$ is the mean value of $\phi(x) \in \mathcal{D}(R^n)$ on the sphere of radius $r$, defined by

$$S_\phi(r) = \frac{1}{\Omega_n} \int_\Omega \phi(r\sigma) d\sigma$$

where $\Omega_n = 2\pi^{\frac{n}{2}}/\Gamma(\frac{n}{2})$ is the hypersurface area of $\Omega$. We can write out an asymptotic expression for $S_\phi(r)$(see [11] ), namely

$$\begin{aligned} S_\phi(r) &\sim \phi(0) + \frac{1}{2!} S''_\phi(0) r^2 + \cdots + \frac{1}{(2k)!} S_\phi^{(2k)}(0) r^{2k} + \cdots \\ &= \sum_{k=0}^{\infty} \frac{\triangle^k \phi(0) r^{2k}}{2^k\, k!\, n(n+2)\cdots(n+2k-2)} \qquad (\triangle \text{ is the Laplacian}) \end{aligned}$$

which is the well-known Pizetti's formula and it plays an important role in the work of Li, Aguirre and Fisher [2-10].

**Remark:** Pizetti's formula is not a convergent series for $\phi \in \mathcal{D}(R^n)$ from the counterexample below.

$$\phi(x) = \begin{cases} \exp\{-\frac{1}{r^2(1-r^2)}\} & \text{if } 0 < r < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\phi(x) \in \mathcal{D}(R^n)$ and $S_\phi(r) \neq 0$ for $0 < r < 1$, but the series in the formula is identically equal to zero. Obviously, $S_\phi(r) \to 0$ as $r \to 0$. However, it converges in spaces of analytic functions from the reference [1].

A differential form of $k$th degree on an $n$-dimensional manifold with coordinates $x_1, x_2, \cdots, x_n$ is an expression of the form

$$\sum a_{i_1 i_2 \cdots i_k}(x) dx_{i_1} dx_{i_2} \cdots dx_{i_k},$$

where the sum is taken over all possible combinations of $k$ indices. The coefficients $a_{i_1 i_2 \cdots i_k}(x)$ are assumed to be infinitely differentiable functions of the coordinates. Two forms of degree $k$ are considered equal if they are transformed into each other when products of differentials are transposed according to the anti-commutation rule

$$dx_i dx_j = -dx_j dx_i$$

and all similar terms are collected.

This rule implies that if a term in a differential form has two differentials with the same index, it must be zero. It can be used to write any differential form into canonical form, in which the indices in each term appear in increasing order. Clearly, the anti-commutation rule holds for any differential forms of first degree. Indeed, let $\alpha = \sum a_j(x)dx_j$ and $\beta = \sum b_k(x)dx_k$; then

$$\alpha\beta = \sum_{j,k} a_j(x)b_k(x)dx_j dx_k = -\sum_{j,k} a_j(x)b_k(x)dx_k dx_j = -\beta\alpha.$$

Let us find how differential forms transform under an infinitely differentiable change of coordinates given by $x_i = x_i(x'_1, x'_2, \cdots, x'_n)$. We have

$$dx_i = \sum_{j=1}^{n} \frac{\partial x_i}{\partial x'_j} dx'_j$$

and

$$\sum_{i_1<\cdots<i_k} a_{i_1\cdots i_k} dx_{i_1} \cdots dx_{i_k} = \sum_{i_1<\cdots<i_k} \sum_{j} a_{i_1\cdots i_k} \frac{\partial x_{i_1}}{\partial x'_{j_1}} \cdots \frac{\partial x_{i_k}}{\partial x'_{j_k}} dx'_{j_1} \cdots dx'_{j_k}.$$

In the sum we have obtained, the terms in which the same differential occurs twice will vanish. Different terms containing the same combination of differentials can be combined using the anti-commutation rule, which holds also for the $dx'_j$. Then it follows that for $j_1 < j_2 < \cdots < j_k$, the coefficient of $dx'_{j_1} \cdots dx'_{j_k}$ is multiplied by the Jacobian

$$D \begin{pmatrix} x_{i_1} & x_{i_2} & \cdots & x_{i_k} \\ x'_{j_1} & x'_{j_2} & \cdots & x'_{j_k} \end{pmatrix}.$$

We thus arrive at

$$\sum_{i_1<\cdots<i_k} a_{i_1\cdots i_k} dx_{i_1} \cdots dx_{i_k} = \sum_{j_1<\cdots<j_k} a'_{j_1\cdots j_k} dx'_{j_1} \cdots dx'_{j_k},$$

where

$$a'_{j_1\cdots j_k} = \sum_{i_1<\cdots<i_k} D \begin{pmatrix} x_{i_1} & x_{i_2} & \cdots & x_{i_k} \\ x'_{j_1} & x'_{j_2} & \cdots & x'_{j_k} \end{pmatrix} a_{i_1\cdots i_k}.$$

The exterior derivative of a differential form

$$\alpha = \sum a_{i_1\cdots i_k} dx_{i_1} \cdots dx_{i_k}$$

is defined as the $(k+1)$st degree differential form

$$d\alpha = \sum_{i_1\cdots i_k} \left( \sum_i \frac{\partial a_{i_1\cdots i_k}}{\partial x_i} dx_i \right) dx_{i_1} \cdots dx_{i_k},$$

which, of course, can be simplified by using the anti-commutation rule. Let $a(x)$ be a scalar function. Then

$$da(x) = \sum_{i=1}^{n} \frac{\partial a(x)}{\partial x_i} dx_i.$$

It is easily shown that according to the anti-commutation rule, any differential form $\alpha$ satisfies the equation

$$dd\alpha = 0.$$

Let us assume that

$$\alpha = \sum a_{i_1 \cdots i_k} dx_{i_1} \cdots dx_{i_k}$$

and the claim holds since

$$\frac{\partial^2 a_{i_1 \cdots i_k}(x)}{\partial x_i \partial x_j} = \frac{\partial^2 a_{i_1 \cdots i_k}(x)}{\partial x_j \partial x_i}$$

and the anti-commutation rule

$$dx_i dx_j = -dx_j dx_i.$$

Let $\alpha$ be a differential form of degree $n-1$ defined on some bounded $n$-dimensional region $G$ with a piecewise smooth boundary $\Gamma$. We assume an orientation of $G$ corresponding to the positive direction of the normal to $\Gamma$. Then

$$\int_G d\alpha = \int_\Gamma \alpha$$

which is called the Gauss-Ostrogradskii formula.

As an example, consider a second degree form $\alpha$ given below in three dimensions

$$\alpha = a_1 dx_2 dx_3 + a_2 dx_3 dx_1 + a_3 dx_1 dx_2$$

and its exterior derivative is

$$d\alpha = (\frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3}) dx_1 dx_2 dx_3,$$

so that the Gauss-Ostrogradskii formula turns to be

$$\int_\Gamma a_1 dx_2 dx_3 + a_2 dx_3 dx_1 + a_3 dx_1 dx_2 = \int_G (\frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3}) dx_1 dx_2 dx_3$$

which is seen in calculus.

We consider a manifold $S$ given by $P(x_1, x_2, \cdots, x_n) = 0$, where $P$ is an infinitely differentiable function such that

$$\text{grad} P = \left\{ \frac{\partial P}{\partial x_1}, \frac{\partial P}{\partial x_2}, \cdots, \frac{\partial P}{\partial x_n} \right\} \neq 0$$

on $S$, which therefore has no singular points.

The differential form $\omega$ is defined by

$$dP \cdot \omega = dv$$

where $dv = dx_1 \cdots dx_n$, and $dP$ is the differential form of $P$. Note that if $P(x)$ is the Euclidean distance of $x$ from the $P = 0$ surface, the differential form $\omega$ on $S$ coincides with the Euclidean element of area $d\sigma$ on $S$.

Since grad$P \neq 0$ on $S$, there exists $j$ $(1 \leq j \leq n)$ such that $\partial P/\partial x_j \neq 0$. We may introduce a local coordinate system $u_1, u_2, \cdots, u_n$ to be

$$u_1 = x_1, \cdots, u_j = P(x), \cdots, u_n = x_n. \tag{1}$$

Then

$$D\binom{x}{u} = \left[D\binom{u}{x}\right]^{-1} = \frac{1}{\partial P/\partial x_j},$$

and thus we may set

$$\omega = (-1)^{j-1}\frac{dx_1 \cdots dx_{j-1}dx_{j+1} \cdots dx_n}{\partial P/\partial x_j}.$$

We naturally define the characteristic function $\theta(P)$ for the region $P \geq 0$ as

$$(\theta(P),\ \phi(x)) = \int_{P \geq 0} \phi(x)dx$$

where $\phi \in \mathcal{D}(R^n)$, and the generalized function $\delta(P)$ by

$$(\delta(P),\ \phi(x)) = \int_{P=0} \phi(x)\omega.$$

Kanwal [12] studied certain distributions defined on the surface $\Sigma(t)$ and their extensions to the whole space. The basic distribution concentrated on $\Sigma(t)$ is the Dirac delta function, whose action on a test function $\phi(x, t)$ is given by

$$\delta(\Sigma),\ \phi) = \int_{-\infty}^{+\infty} \int_{\Sigma(t)} \phi(x, t)dS(x)dt,$$

where $dS(x)$ is the surface element. Observe the special treatment of time in the above integral. The integration with respect to the space variables is surface integration while that with respect to time is ordinary integration.

According to Kanwal, the relation between $\delta(P)$ and $\delta(\Sigma)$ is given as

$$(\delta(P),\ \phi(x)) = \int_{\Sigma} \phi(y)dS(y)\frac{1}{|\text{grad}P|},$$

which implies

$$\delta(P) = \frac{\delta(\Sigma)}{|\text{grad}P|}.$$

Another way of introducing the distribution $\delta(P)$ is used by DeJager [13];

$$(\delta(P),\ \phi(x)) = \lim_{c \to 0} \frac{1}{c} \int_{0 \leq P \leq c} \phi(x)dx.$$

Similarly, its higher derivatives can be defined as

$$\delta^{(k)}(P) = \lim_{c\to 0} \frac{1}{c}[\delta^{(k-1)}(P+c) - \delta^{(k-1)}(P)], \quad k = 1, 2, \cdots .$$

It follows from DeJager [13] that

$$\begin{aligned}
(\delta(P),\ \phi(x)) &= \lim_{c\to 0} \frac{1}{c} \int_{0\le P\le c} \phi(x)\gamma dS(x) \\
&= \lim_{c\to 0} \frac{1}{c} \int_{P=0} \phi(x)\cdot c \frac{dS(x)}{|\text{grad}P|} \\
&= \int_{P=0} \phi(x) \frac{dS(x)}{|\text{grad}P|}
\end{aligned}$$

which coincides with the Kanwal's result.

It was proven in [1] that

$$\frac{\partial\theta(P)}{\partial x_j} = \frac{\partial P}{\partial x_j}\delta(P).$$

We shall first add the following identity, which has never appeared so far, according to the author's knowledge

$$\frac{\partial\theta(P)}{\partial P} = \delta(P).$$

Indeed,

$$(\frac{\partial\theta(P)}{\partial P},\ \phi(x)) = -(\theta(P),\ \frac{\partial}{\partial P}\phi(x)).$$

Since $\phi = \phi(x_1, x_2, \cdots, x_j(P), \cdots, x_n)$ by the substitution of (1), we come to

$$-(\theta(P),\ \frac{\partial}{\partial P}\phi(x)) = -(\theta(P),\ \frac{\partial\phi(x)}{\partial x_j}\frac{1}{\frac{\partial P}{\partial x_j}}) = -\int_{P\ge 0} \frac{\partial\phi(x)}{\partial x_j}\frac{1}{\frac{\partial P}{\partial x_j}}dx.$$

On the other hand,

$$(\delta(P),\ \phi(x)) = \int_{P=0} \phi(x)\omega.$$

Let us assume that $P \ge 0$ defines a bounded region. Then we may apply the Gauss-Ostrogradskii formula to the above integral over this region and to the differential form of degree $n-1$ in the integrand. We also use the fact that $P$ increases into the interior of the region to derive

$$\int_{P=0} \phi(x)\omega = -\int_{P\ge 0} d(\phi(x)\omega)$$

and

$$d(\phi(x)\omega) = \frac{\partial\phi(x)}{\partial x_j}\frac{1}{\frac{\partial P}{\partial x_j}}dx + \phi\frac{\partial}{\partial x_j}(\frac{\partial x_j}{\partial P})dx = \frac{\partial\phi(x)}{\partial x_j}\frac{1}{\frac{\partial P}{\partial x_j}}dx,$$

which implies

$$\int_{P=0} \phi(x)\omega = -\int_{P\geq 0} \frac{\partial \phi(x)}{\partial x_j} \frac{1}{\frac{\partial P}{\partial x_j}} dx.$$

Hence the identity holds on any bounded region.

If $P \geq 0$ does not define a bounded region, we replace it by its intersection $G_R$ with a sufficiently large ball $|x| \leq R$ outside of which $\phi(x)$ is known to vanish. Let $\Gamma_R$ be the boundary of $G_R$, we have

$$\int_{\Gamma_R} \phi(x)\omega = -\int_{G_R} \frac{\partial \phi(x)}{\partial x_j} \frac{1}{\frac{\partial P}{\partial x_j}} dx.$$

Now, since $\phi(x)$ vanishes outside of $|x| \leq R$, we arrive at

$$\int_{P=0} \phi(x)\omega = -\int_{P\geq 0} \frac{\partial \phi}{\partial P} dx,$$

which completes the proof.

It is well known that in one dimension every functional concentrated on a point is a linear combination of the delta function and its derivatives. For $n > 1$, we have a similar role played by generalized functions, $\delta(P),\ \delta'(P), \cdots,\ \delta^{(k)}(P)$ (the derivatives of $\delta(P)$ with respect to the argument $P$), which we shall define based on the differential forms $\omega_k(\phi)$ given by

$$\begin{aligned}
&\omega_0(\phi) = \phi \cdot \omega, \\
&d\omega_0(\phi) = dP \cdot \omega_1(\phi), \\
&\cdots\ \cdots \\
&d\omega_{k-1}(\phi) = dP \cdot \omega_k(\phi), \\
&\cdots\ \cdots
\end{aligned}$$

where $d$ denotes the exterior derivative. Now we are able to define

$$(\delta^{(k)}(P),\ \phi) = (-1)^k \int_{P=0} \omega_k(\phi)$$

for $k = 0, 1, 2, \cdots$, since the above integral over the $P = 0$ surface of any of the $\omega_k(\phi)$ is uniquely determined by $P(x)$. Furthermore, we define the generalized function $\partial\delta(P)/\partial P$ as

$$(\frac{\partial}{\partial P}\delta(P),\ \phi) = -\int_{P=0} \frac{\partial \phi}{\partial P}\omega.$$

We shall show that

$$\frac{\partial}{\partial P}\delta(P) = \delta'(P).$$

In fact,

$$(\frac{\partial}{\partial P}\delta(P),\ \phi) = -\int_{P=0} \frac{\partial\phi}{\partial P}\omega = -\int_{P=0} \omega_0(\frac{\partial\phi}{\partial P}).$$

On the other hand,

$$(\delta'(P),\ \phi) = -\int_{P=0} \omega_1(\phi) = -\int_{P=0} \frac{\partial}{\partial P}(\frac{\phi}{\partial P/\partial x_j})dx_1\cdots dx_{j-1}dx_{j+1}\cdots dx_n.$$

Since $\phi = \phi(x_1, x_2, \cdots, x_j(P), \cdots, x_n)$ and $\partial P/\partial x_j$ is not a function of $P$, we imply

$$\begin{aligned} \frac{\partial}{\partial P}(\frac{\phi}{\partial P/\partial x_j})dx_1\cdots dx_{j-1}dx_{j+1}\cdots dx_n &= \frac{\frac{\partial\phi}{\partial P}}{\partial P/\partial x_j}dx_1\cdots dx_{j-1}dx_{j+1}\cdots dx_n \\ &= \omega_0(\frac{\partial\phi}{\partial P}), \end{aligned}$$

by choosing the coordinates $u_i = x_i$, and $u_j = P$. Under these coordinates

$$\omega_k(\phi) = \frac{\partial^k}{\partial P^k}(\frac{\phi}{\partial P/\partial x_j})dx_1\cdots dx_{j-1}dx_{j+1}\cdots dx_n.$$

This completes the proof.

Similarly, we can obtain

$$\frac{\partial}{\partial P}\delta^{(k)}(P) = \delta^{(k+1)}(P) \qquad \text{for } k = 1, 2, \cdots.$$

We now prove the following recurrence relations, identities between $\delta(P)$ and its derivatives:

$$\begin{aligned} &P\delta(P) = 0 \\ &P\delta'(P) + \delta(P) = 0 \\ &P\delta''(P) + 2\delta'(P) = 0 \\ &\cdots\cdots \\ &P\delta^{(k)}(P) + k\delta^{(k-1)}(P) = 0 \\ &\cdots\cdots \end{aligned}$$

The first of these is obvious, since the integral of $P\phi$ over the $P = 0$ surface clearly vanishes. We now take the derivative with respect to $P$ to get

$$P\delta'(P) + \delta(P) = 0$$

as well as the rest similarly.

## 2. The product $X^l\delta(r-1)$

Let $X = \sum_{i=1}^n x_i$. We shall use a recursion and Pizetti's formula to derive the asymptotic product $X^l\delta(r-1)$ for any integer $l \geq 1$, which is not possible to obtain along the differential form approach, since $X$ is clearly not a function of $r$.

Setting $\psi(x) = X^l\phi(x)$ and obviously $\psi(x) \in \mathcal{D}(R^n)$. We naturally have

$$\begin{aligned}(X^l\delta(r-1),\ \phi(x)) &= (\delta(r-1),\ X^l\phi(x)) = \int_{r=1} X^l\phi(x)d\sigma \\ &= \int_{r=1}\psi(x)d\sigma = \Omega_n S_\psi(1).\end{aligned}$$

It follows from Pizetti's formula and $\psi(0) = 0\phi(0) = 0$ that

$$(X^l\delta(r-1),\ \phi(x)) \sim \Omega_n \sum_{j=1}^{\infty} \frac{\triangle^j\psi(0)}{2^j\, j!\, n(n+2)\cdots(n+2j-2)}.$$

In order to calculate $X^l\delta(r-1)$, we need to express $\triangle^j\psi(0)$ in terms of a finite combination of $\phi$ and its derivatives at $x=0$. First, we claim for $j \geq 0$ that

$$\triangle^{j+1}(X\phi) = 2(j+1)\nabla\triangle^j\phi + X\triangle^{j+1}\phi \tag{2}$$

where $\nabla = \partial/\partial x_1 + \cdots + \partial/\partial x_n$.

We use an inductive method to prove it. It is obviously true for $j=0$. Assume $j=1$, we have

$$\triangle^2(x_i\phi) = 4\frac{\partial}{\partial x_i}\triangle\phi + x_i\triangle^2\phi$$

simply by calculating the left-hand side. Hence

$$\triangle^2(X\phi) = 4\nabla\triangle\phi + X\triangle^2\phi.$$

By hypothesis, it holds for the case of $j-1$, that is

$$\triangle^j(X\phi) = 2j\nabla\triangle^{j-1}\phi + X\triangle^j\phi.$$

Hence it follows that

$$\begin{aligned}\triangle^{j+1}(X\phi) &= \triangle\triangle^j(X\phi) = \triangle(2j\nabla\triangle^{j-1}\phi + X\triangle^j\phi) \\ &= 2j\nabla\triangle^j\phi + \triangle(X\triangle^j\phi) = 2(j+1)\nabla\triangle^j\phi + X\triangle^{j+1}\phi.\end{aligned}$$

Clearly, we have from equation (2) that

$$\triangle^j(X\phi(x))\big|_{x=0} = 2j\nabla\triangle^{j-1}\phi(0) = -2j(\triangle^{j-1}\nabla\delta(x),\ \phi(x)) \tag{3}$$

for $j \geq 1$.

Next, we are going to calculate $\triangle^j (X^2\phi(x))\big|_{x=0}$ based on $\triangle^j(X\phi(x))$. Indeed,

$$\triangle^j(X^2\phi(x)) = \triangle^j(XX\phi(x)) = 2j\nabla\triangle^{j-1}(X\phi(x)) + X\triangle^j(X\phi(x)).$$

By simple calculation,

$$\nabla(X\phi(x)) = n\phi(x) + X\nabla\phi(x).$$

Hence it follows that

$$\triangle^j(X^2\phi(x))\big|_{x=0} = 2nj\triangle^{j-1}\phi(0) + 2j\triangle^{j-1}(X\nabla\phi(x))\big|_{x=0}.$$

Using equation (3), we obtain

$$\triangle^{j-1}(X\nabla\phi(x))\big|_{x=0} = 2(j-1)\nabla^2\triangle^{j-2}\phi(0).$$

Thus,

$$\triangle^j(X^2\phi(x))\big|_{x=0} = 2nj\triangle^{j-1}\phi(0) + 2^2j(j-1)\nabla^2\triangle^{j-2}\phi(0).$$

In order to construct a recursion of computing $\triangle^j(X^l\phi(x))$, we need to search for a pattern, and continue on

$$\triangle^j(X^3\phi(x)) = \triangle^j(XX^2\phi(x)) = 2j\nabla\triangle^{j-1}(X^2\phi(x)) + X\triangle^j(X^2\phi(x)).$$

Similarly,

$$\nabla(X^2\phi(x)) = 2nX\phi(x) + X^2\nabla\phi(x).$$

Therefore,

$$\begin{aligned}\triangle^j(X^3\phi(x))\big|_{x=0} &= 2j\triangle^{j-1}(2nX\phi(x) + X^2\nabla\phi(x))\big|_{x=0}\\ &= 2^2nj\triangle^{j-1}(X\phi(x))\big|_{x=0} + 2j\triangle^{j-1}(X^2\nabla\phi(x))\big|_{x=0}.\end{aligned}$$

Since,

$$\begin{aligned}&\triangle^{j-1}(X\phi(x))\big|_{x=0} = 2(j-1)\nabla\triangle^{j-2}\phi(0) \quad \text{and}\\ &\triangle^{j-1}(X^2\nabla\phi(x))\big|_{x=0} = 2n(j-1)\triangle^{j-2}\nabla\phi(0) + 2^2(j-1)(j-2)\nabla^3\triangle^{j-3}\phi(0).\end{aligned}$$

Finally, we arrive at

$$\begin{aligned}\triangle^j(X^3\phi(x))\big|_{x=0} &= 2^3nj(j-1)\nabla\triangle^{j-2}\phi(0) + 2^2nj(j-1)\triangle^{j-2}\nabla\phi(0)\\ &+ 2^3j(j-1)(j-2)\nabla^3\triangle^{j-3}\phi(0).\end{aligned}$$

In general,

$$\triangle^j(X^l\phi(x)) = \triangle^j(XX^{l-1}\phi(x)) = 2j\nabla\triangle^{j-1}(X^{l-1}\phi(x)) + X\triangle^j(X^{l-1}\phi(x)).$$

Clearly,

$$\nabla(X^{l-1}\phi(x)) = n(l-1)X^{l-2}\phi(x) + X^{l-1}\nabla\phi(x).$$

Hence,

$$\begin{aligned}&\triangle^j(X^l\phi(x))\big|_{x=0}=2j\triangle^{j-1}(n(l-1)X^{l-2}\phi(x)+X^{l-1}\nabla\phi(x))\big|_{x=0}\\&=2nj(l-1)\triangle^{j-1}(X^{l-2}\phi(x))\big|_{x=0}+2j\triangle^{j-1}(X^{l-1}\nabla\phi(x))\big|_{x=0}.\end{aligned}$$

This is obviously dependent on the two previous terms of

$$\triangle^{j-1}(X^{l-2}\phi)\big|_{x=0}\quad\text{and}\quad\triangle^{j-1}(X^{l-1}\phi)\big|_{x=0},$$

and forms a recursion for computing $\triangle^j(X^l\phi(x))$, although the author is unable to write out the explicit formula at this moment.

In particular, we have

$$\begin{aligned}X\delta(r-1)&\sim-\Omega_n\sum_{j=0}^{\infty}\frac{\triangle^j\nabla\delta(x)}{2^j\,j!\,n(n+2)\cdots(n+2j)},\\X^2\delta(r-1)&\sim\Omega_n\delta(x)+\frac{\Omega_n\triangle\delta(x)}{2(n+2)}+\frac{\Omega_n\nabla^2\delta(x)}{n(n+2)}\\&\quad+\Omega_n\sum_{j=2}^{\infty}\frac{n\triangle^j\delta(x)+2j\nabla^2\triangle^{j-1}\delta(x)}{2^j\,j!\,n(n+2)\cdots(n+2j)}.\end{aligned}$$

## 3. The product $f(P_1,\cdots,P_k)\frac{\partial^{|\alpha|}\delta(P_1,\cdots,P_k)}{\partial P_1^{\alpha_1}\cdots\partial P_k^{\alpha_k}}$

We now turn our attention to new generalized functions associated with manifolds $S$ of lower dimension defined by $k$ equations of the form

$$P_1(x_1,\cdots,x_n)=0,\;\;P_2(x_1,\cdots,x_n)=0,\cdots,P_k(x_1,\cdots,x_n)=0.$$

where $k$ is in general greater than one. Following [1], we shall make the two assumptions:

(i) The $P_i$ are infinitely differentiable functions.

(ii) The $P_i(x_1,\cdots,x_n)=\eta_i$ hypersurfaces $(i=1,2,\cdots,k)$ form a lattice such that in the neighborhood of every point of $S$ there exists a local coordinate system in which $u_i=P_i(x_1,\cdots,x_n)$ for $i=1,2,\cdots,k$ and the remaining $u_{k+1},\cdots,u_n$ can be chosen so that the Jacobian $D\binom{x}{u}>0$.

Consider the element of volume in $R^n$

$$dv=dx_1\cdots dx_n$$

a differential form of degree $n$, and let us write it as the product of the first-degree differential forms $dP_1\cdots dP_k$ with an additional differential form $\omega$ of degree $n-k$; i.e.

$$dv=dP_1\cdots dP_k\omega.$$

It was proven in [1] that such $\omega$ exists, but can not be unique, and

$$dP_1 \cdots dP_k = \sum_{i_1 < \cdots < i_k} D \begin{pmatrix} P_1 & P_2 & \cdots & P_k \\ x_{i_1} & x_{i_2} & \cdots & x_{i_k} \end{pmatrix} dx_{i_1} \cdots dx_{i_k}.$$

We define the generalized function $\delta(P_1, \cdots, P_k)$ by the equation

$$(\delta(P_1, \cdots, P_k),\ \phi) = \int_S \phi\omega.$$

It can be easily shown that this definition is independent of the particular choice of $\omega$.

Let us denote $\omega_{0,\cdots,0}(\phi) = \phi\omega$. Then we define the differential form $\omega_{1,0,\cdots,0}(\phi)$ (whose integral over $S$ will give $\partial\delta(P_1, \cdots, P_k)/\partial P_1$) as follows. We take the exterior derivative of the differential form of degree $n-1$, $dP_2 \cdots dP_k \omega_{0,\cdots,0}(\phi)$, and write it in the form

$$d(dP_2 \cdots dP_k \omega_{0,\cdots,0}(\phi)) = dP_1 \cdots dP_k \omega_{1,0,\cdots,0}(\phi).$$

We choose the local coordinate system in which the $u_i = P_i$ for $i = 1, \cdots, k$, and denote $\phi(x_1(u_1, \cdots, u_n), \cdots, x_n(u_1, \cdots, u_n))$ by $\tilde{\phi}(u_1, \cdots, u_n) = \tilde{\phi}(u)$; we then obtain

$$\begin{aligned} &\omega_{0,\cdots,0}(\phi) = \phi\omega = \tilde{\phi} D \begin{pmatrix} x \\ u \end{pmatrix} du_{k+1} \cdots du_n, \\ &dP_2 \cdots dP_k \omega_{0,\cdots,0}(\phi)\omega_{0,\cdots,0}(\phi) = \tilde{\phi} D \begin{pmatrix} x \\ u \end{pmatrix} du_2 \cdots du_n, \\ &d(dP_2 \cdots dP_k \omega_{0,\cdots,0}(\phi)\omega_{0,\cdots,0}(\phi)) = \frac{\partial}{\partial u_1} \left[ \tilde{\phi} D \begin{pmatrix} x \\ u \end{pmatrix} \right] du_1 \cdots du_n, \end{aligned}$$

which implies

$$\omega_{1,0,\cdots,0}(\phi) = \frac{\partial}{\partial u_1} \left[ \tilde{\phi} D \begin{pmatrix} x \\ u \end{pmatrix} \right] du_{k+1} \cdots du_n.$$

Any of the $k$ indices of $\omega_{0,\cdots,0}(\phi)$ can be changed from zero to one in the same way.

In general, assuming that we know $\omega_{\alpha_1,\cdots,\alpha_k}(\phi)$, we may raise its $j$th index by multiplying on the left by all the $dP_i$ with $i \neq j$, taking the exterior derivative, and writing

$$\begin{aligned} &d(dP_1 \cdots dP_{j-1} dP_{j+1} \cdots dP_k \omega_{\alpha_1,\cdots,\alpha_k}(\phi)) \\ &= (-1)^j dP_1 \cdots dP_k \omega_{\alpha_1,\cdots,\alpha_{j-1},\alpha_j+1,\alpha_{j+1},\cdots,\alpha_k}(\phi). \end{aligned}$$

This defines the $\omega_{\alpha_1,\cdots,\alpha_k}(\phi)$ for any nonnegative integral indices.

Obviously, if $\omega_{0,\cdots,0}(\phi)$ is not unique, neither are the $\omega_{\alpha_1,\cdots,\alpha_k}(\phi)$. We now define the generalized function $\frac{\partial^{|\alpha|}\delta(P_1,\cdots,P_k)}{\partial P_1^{\alpha_1}\cdots\partial P_k^{\alpha_k}}$, where $|\alpha| = \alpha_1 + \cdots + \alpha_k$, by

$$(\frac{\partial^{|\alpha|}\delta(P_1,\cdots,P_k)}{\partial P_1^{\alpha_1}\cdots\partial P_k^{\alpha_k}},\ \phi) = (-1)^{|\alpha|}\int_S \omega_{\alpha_1,\cdots,\alpha_k}(\phi),$$

which is independent of the choice of $\omega_{\alpha_1,\cdots,\alpha_k}$.

**Theorem 1.** *Let $f(u_1,\cdots u_k)$ be an infinitely differentiable function of $k$ variables. Then the product $f(P_1,\cdots,P_k)\frac{\partial^{|\alpha|}\delta(P_1,\cdots,P_k)}{\partial P_1^{\alpha_1}\cdots\partial P_k^{\alpha_k}}$ exists and*

$$f(P_1,\cdots,P_k)\frac{\partial^{|\alpha|}\delta(P_1,\cdots,P_k)}{\partial P_1^{\alpha_1}\cdots\partial P_k^{\alpha_k}} = \sum_{j_1=0}^{\alpha_1}\cdots\sum_{j_k=0}^{\alpha_k}(-1)^{|\alpha|-|j|}$$
$$\binom{\alpha_1}{j_1}\cdots\binom{\alpha_k}{j_k}\frac{\partial^{|\alpha|-|j|}}{\partial u_1^{\alpha_1-j_1}\cdots\partial u_k^{\alpha_k-j_k}}f(0,\cdots,0)\frac{\partial^{|j|}\delta(P_1,\cdots,P_k)}{\partial P_1^{j_1}\cdots\partial P_k^{j_k}}.$$

Before going into the proof, we would like to give the following products, if $f(P_1,\cdots,P_k) = P_i$, by Theorem 1.

$$P_i\delta'_{P_i}(P_1,\cdots,P_k) + \delta(P_1,\cdots,P_k) = 0,$$
$$\cdots\cdots$$
$$P_i\delta^{(m)}_{P_i,\cdots,P_i}(P_1,\cdots,P_k) + m\delta^{(m-1)}_{P_i,\cdots,P_i}(P_1,\cdots,P_k) = 0,$$

which were obtained in [1].

*Proof.* Making the substitution (without loss of generality) $u_i = P_i$ for $i = 1,\cdots,k$, and denoting

$$\tilde\phi(u) = \tilde\phi(u_1,\cdots,u_n) = \phi(x_1(u_1,\cdots,u_n),\cdots,x_n(u_1,\cdots,u_n)),$$

we come to

$$(f(P_1,\cdots,P_k)\frac{\partial^{|\alpha|}\delta(P_1,\cdots,P_k)}{\partial P_1^{\alpha_1}\cdots\partial P_k^{\alpha_k}},\ \phi) = (\frac{\partial^{|\alpha|}\delta(P_1,\cdots,P_k)}{\partial P_1^{\alpha_1}\cdots\partial P_k^{\alpha_k}},\ f(P_1,\cdots,P_k)\phi)$$
$$= (-1)^{|\alpha|}\int_S \frac{\partial^{|\alpha|}}{\partial u_1^{\alpha_1}\cdots\partial u_k^{\alpha_k}}\left[f(u_1,\cdots,u_k)\tilde\phi D\binom{x}{u}\right]du_{k+1}\cdots du_n$$
$$= (-1)^{|\alpha|}\int_S\sum_{j_1=0}^{\alpha_1}\cdots\sum_{j_k=0}^{\alpha_k}\binom{\alpha_1}{j_1}\cdots\binom{\alpha_k}{j_k}\frac{\partial^{|\alpha|-|j|}}{\partial u_1^{\alpha_1-j_1}\cdots\partial u_k^{\alpha_k-j_k}}f(u_1,\cdots,u_k)$$
$$\cdot\frac{\partial^{|j|}}{\partial u_1^{j_1}\cdots\partial u_k^{j_k}}\left[\tilde\phi D\binom{x}{u}\right]du_{k+1}\cdots du_n$$
$$= (-1)^{|\alpha|}\sum_{j_1=0}^{\alpha_1}\cdots\sum_{j_k=0}^{\alpha_k}\binom{\alpha_1}{j_1}\cdots\binom{\alpha_k}{j_k}\frac{\partial^{|\alpha|-|j|}}{\partial u_1^{\alpha_1-j_1}\cdots\partial u_k^{\alpha_k-j_k}}f(0,\cdots,0)\cdot$$
$$\cdot\int_S\frac{\partial^{|j|}}{\partial u_1^{j_1}\cdots\partial u_k^{j_k}}\left[\tilde\phi D\binom{x}{u}\right]du_{k+1}\cdots du_n$$

Using the identity

$$\int_S \frac{\partial^{|j|}}{\partial u_1^{j_1}\cdots\partial u_k^{j_k}}\left[\tilde{\phi} D\binom{x}{u}\right] du_{k+1}\cdots du_n = (-1)^{|j|}(\frac{\partial^{|j|}\delta(P_1,\cdots,P_k)}{\partial P_1^{j_1}\cdots\partial P_k^{j_k}},\ \phi),$$

we complete the proof of Theorem 1. □

To end this section, we would like to mention that Aguirre studied the following product

$$P_1^{l_1}\cdots P_k^{l_k}\frac{\partial^{|\alpha|}\delta(P_1,\cdots,P_k)}{\partial P_1^{\alpha_1}\cdots\partial P_k^{\alpha_k}},$$

which is a special case of Theorem 1 if $f(P_1,\cdots,P_k) = P_1^{l_1}\cdots P_k^{l_k}$.

## 4. The generalized function $\delta(Q_1P_1,\cdots,Q_kP_k)$

Assuming that $Q$ is a nonvanishing function and $P$ is a manifold of dimension $n-1$, we have for any $m \geq 0$ that

$$\delta^{(m)}(QP) = Q^{-(m+1)}\delta^{(m)}(P). \tag{4}$$

This is a powerful formula which can be used to derive some products, such as $X^l\delta(r^2-1)$, since

$$\delta(r^2-1) = \frac{1}{2}\delta(r-1).$$

We are interested in extending equation (4) to smooth manifolds of lower dimension. First of all, we would like to see how the differential form $\omega$ and functional $\delta(P_1,\cdots,P_k)$ change while making the substitution

$$W_j(x) = \sum_{i=1}^{k}\alpha_{ij}(x)P_i(x).$$

Here the $\alpha_{ij}(x)$ are assumed to be infinitely differentiable functions and the matrix they form is assumed nonsingular. The defining equations for the initial differential form $\omega$ and for the new one $\tilde{\omega}$ are

$$dP_1\cdots dP_k\omega = dv = dW_1\cdots dW_k\tilde{\omega}$$
$$= (\sum\alpha_{i1}dP_i)\cdots(\sum\alpha_{ik}dP_i)\tilde{\omega}.$$

By expanding the terms in parentheses and using the anti-commutation rule $dP_idP_j = -dP_jdP_i$, we write $\det||\alpha_{ij}||dP_1\cdots dP_k\,\tilde{\omega} = dv$, which implies

$$\tilde{\omega} = \frac{1}{\det||\alpha_{ij}||}\omega.$$

Hence

$$(\delta(W_1,\cdots,W_k),\ \phi) = (\delta(P_1,\cdots,P_k),\ \frac{\phi}{\det||\alpha_{ij}||}).$$

Let us find the generalized function $\delta(QP_1, \cdots, QP_k)$ , where $Q \neq 0$. By the substitution $W_1 = QP_1, \cdots, W_k = QP_k$, we arrive at $\det ||\alpha_{ij}|| = Q^{-k}(x)$. This indicates

$$\delta(QP_1, \cdots, QP_k) = Q^{-k}(x)\delta(P_1, \cdots, P_k). \tag{5}$$

In particular, we obtain for $k = 1$ that $\delta(QP_1) = Q^{-1}\delta(P_1)$, which coincides with equation (4) for $m = 0$.

It follows that

$$\delta^{(m)}(QP_1, \cdots, QP_k) = Q^{-(k+m)}(x)\delta^{(m)}(P_1, \cdots, P_k)$$

by differentiating both sides of equation (5) $m$ times with respect to some $P_i$.

Similarly,

$$\delta(Q_1P_1, \cdots, Q_kP_k) = \frac{1}{Q_1 \cdots Q_k}\delta(P_1, \cdots, P_k)$$

where the $Q_i$ are nonzero and infinitely differentiable functions. Let $|\alpha| = \alpha_1 + \cdots + \alpha_k$, then

$$\delta^{|\alpha|}(Q_1P_1, \cdots, Q_kP_k) = \frac{1}{Q_1^{1+\alpha_1} \cdots Q_k^{1+\alpha_k}}\delta^{|\alpha|}(P_1, \cdots, P_k).$$

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