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The Products of Distributions on Manifolds and Invariant Theorem

C.K. Li*

Abstract. The problem of defining products of distributions on manifolds has been very difficult since there is a serious lack of definitions for products and powers of generalized functions overall, although they are in great demand for quantum field theory. Few results have been obtained so far in the area of interest. In this paper, we initially study the products, such as $f(P)\,\delta^{(k)}(P)$ and $f(P,Q)\,\delta(PQ)$, on regular manifolds, where

$$(\delta^{(k)}(P),\,\phi) = (-1)^k \int_{P=0} \omega_k(\phi).$$

Then utilizing Pizetti's formula, we compute the product $X\ \delta(P)$, which one is unable to obtain along the differential form approach. Furthermore, we use the delta sequence and the convolution given on the $P = 0$ to derive an invariant theorem, that powerfully converts the products of distributions on manifolds into the well-defined products of a single variable. Several examples, including the product of $P_+^r(x)$ and $\delta^{(k)}(P(x))$, are presented by the invariant theorem.

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1. Differential Forms

To make this paper as self-contained as possible, we begin to state Pizetti's formula and briefly introduce differential forms, which are extremely helpful in defining distributions on manifolds in an invariant way. Please refer to reference [1] for detail.

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Assume $d\sigma$ is the Euclidean area on the unit sphere in R^n, and $S_\phi(r)$ is the mean value of $\phi(x) \in \mathcal{D}(R^n)$ on the sphere of radius r, defined by

$$S_\phi(r) = \frac{1}{\Omega_n} \int_\Omega \phi(r\sigma) d\sigma$$

where $\Omega_n = 2\pi^{\frac{n}{2}}/\Gamma(\frac{n}{2})$ is the hypersurface area of the unit sphere Ω. We can write out the Taylor's series for $S_\phi(r)$, namely

$$\begin{aligned} S_\phi(r) &= \phi(0) + \frac{1}{2!} S''_\phi(0) r^2 + \cdots + \frac{1}{(2k)!} S_\phi^{(2k)}(0) r^{2k} + \cdots \\ &= \sum_{k=0}^{\infty} \frac{\triangle^k \phi(0) r^{2k}}{2^k\, k!\, n(n+2)\cdots(n+2k-2)} \qquad \triangle \text{ is the Laplacian} \end{aligned}$$

which is the well-known Pizetti's formula and it plays an important role in the work of Li, Aguirre and Fisher [2-8].

A differential form of kth degree on an n-dimensional manifold with coordinates $x_1, x_2, \cdots, x_n$ is an expression of the form

$$\sum a_{i_1 i_2 \cdots i_k}(x) dx_{i_1} dx_{i_2} \cdots dx_{i_k},$$

where the sum is taken over all possible combinations of k indices. The coefficients $a_{i_1 i_2 \cdots i_k}(x)$ are assumed to be infinitely differentiable functions of the coordinates. Two forms of degree k are considered equal if they are transformed into each other when products of differentials are transposed according to the anti-commutation rule

$$dx_i dx_j = -dx_j dx_i$$

and all similar terms are collected.

This rule implies that if a term in a differential form has two differentials with the same index, it must be zero. It can be used to write any differential form into canonical form, in which the indices in each term appear in increasing order. Clearly, the anti-commutation rule holds for any differential forms of first degree. Indeed, let $\alpha = \sum a_j(x) dx_j$ and $\beta = \sum b_k(x) dx_k$; then

$$\alpha\beta = \sum_{j,k} a_j(x) b_k(x) dx_j dx_k = -\sum_{j,k} a_j(x) b_k(x) dx_k dx_j = -\beta\alpha.$$

Let us find how differential forms transform under an infinitely differentiable change of coordinates given by $x_i = x_i(x'_1, x'_2, \cdots, x'_n)$. We have

$$dx_i = \sum_{j=1}^{n} \frac{\partial x_i}{\partial x'_j} dx'_j$$

and

$$\sum_{i_1<\cdots<i_k} a_{i_1\cdots i_k} dx_{i_1}\cdots dx_{i_k} = \sum_{i_1<\cdots<i_k}\sum_j a_{i_1\cdots i_k}\frac{\partial x_{i_1}}{\partial x'_{j_1}}\cdots\frac{\partial x_{i_k}}{\partial x'_{j_k}} dx'_{j_1}\cdots dx'_{j_k}.$$

In the sum we have obtained, terms in which the same differential occurs twice will vanish. Different terms containing the same combination of differentials can be combined using the anti-commutation rule, which holds also for the dx'_j. Then it follows that for $j_1 < j_2 < \cdots < j_k$, the coefficient of $dx'_{j_1}\cdots dx'_{j_k}$ is multiplied by the Jacobian

$$D(x)_{i_1}\ x_{i_2}\ \cdots\ x_{i_k} x'_{j_1}\ x'_{j_2}\ \cdots\ x'_{j_k}.$$

We thus arrive at

$$\sum_{i_1<\cdots<i_k} a_{i_1\cdots i_k} dx_{i_1}\cdots dx_{i_k} = \sum_{j_1<\cdots<j_k} a'_{j_1\cdots j_k} dx'_{j_1}\cdots dx'_{j_k},$$

where

$$a'_{j_1\cdots j_k} = \sum_{i_1<\cdots<i_k} D(x)_{i_1}\ x_{i_2}\ \cdots\ x_{i_k} x'_{j_1}\ x'_{j_2}\ \cdots\ x'_{j_k} a_{i_1\cdots i_k}.$$

The exterior derivative of a differential form

$$\alpha = \sum a_{i_1\cdots i_k} dx_{i_1}\cdots dx_{i_k}$$

is defined as the $(k+1)$st degree differential form

$$d\alpha = \sum_{i_1\cdots i_k}\left(\sum_i \frac{\partial a_{i_1\cdots i_k}}{\partial x_i} dx_i\right) dx_{i_1}\cdots dx_{i_k},$$

which, of course, can be simplified by using the anti-commutation rule. Let $a(x)$ be a scalar function. Then

$$da(x) = \sum_{i=1}^{n}\frac{\partial a(x)}{\partial x_i} dx_i.$$

It is easily shown that according to the anti-commutation rule, any differential form α satisfies the equation

$$dd\alpha = 0.$$

Let us assume that

$$\alpha = \sum a_{i_1\cdots i_k} dx_{i_1}\cdots dx_{i_k}$$

and the claim holds since

$$\frac{\partial^2 a_{i_1\cdots i_k}(x)}{\partial x_i \partial x_j} = \frac{\partial^2 a_{i_1\cdots i_k}(x)}{\partial x_j \partial x_i}$$

and the anti-commutation rule

$$dx_i dx_j = -dx_j dx_i.$$

Let α be a differential form of degree $n-1$ defined on some bounded n-dimensional region G with a piecewise smooth boundary Γ. We assume an orientation of G corresponding to the positive direction of the normal to Γ. Then

$$\int_G d\alpha = \int_\Gamma \alpha$$

which is called the Gauss-Ostrogradskii formula.

As an example, consider a second degree form α given below in three dimensions

$$\alpha = a_1 dx_2 dx_3 + a_2 dx_3 dx_1 + a_3 dx_1 dx_2$$

and its exterior derivative is

$$d\alpha = (\frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3}) dx_1 dx_2 dx_3,$$

so that the Gauss-Ostrogradskii formula turns to be

$$\int_\Gamma a_1 dx_2 dx_3 + a_2 dx_3 dx_1 + a_3 dx_1 dx_2 = \int_G (\frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3}) dx_1 dx_2 dx_3$$

which is seen in calculus.

2. The Products on Manifolds

We consider a manifold S given by $P(x_1, x_2, \cdots, x_n) = 0$, where P is an infinitely differentiable function such that

$$\text{grad}P = \left\{ \frac{\partial P}{\partial x_1}, \frac{\partial P}{\partial x_2}, \cdots, \frac{\partial P}{\partial x_n} \right\} \neq 0$$

on S, which therefore has no singular points.

The differential form ω is defined by

$$dP \cdot \omega = dv$$

where $dv = dx_1 \cdots dx_n$, and dP is the differential form of P. Note that if $P(x)$ is the Euclidean distance of x from the $P = 0$ surface, the differential form ω on S coincides with the Euclidean element of area $d\sigma$ on S.

Since grad$P \neq 0$ on S, there exists j $(1 \leq j \leq n)$ such that $\partial P/\partial x_j \neq 0$. We may introduce a local coordinate system $u_1, u_2, \cdots, u_n$ to be

$$u_1 = x_1, \cdots, u_j = P(x), \cdots, u_n = x_n. \tag{1}$$

Then

$$D\binom{x}{u} = \left[D\binom{u}{x}\right]^{-1} = \frac{1}{\partial P/\partial x_j},$$

and thus we may set

$$\omega = (-1)^{j-1}\frac{dx_1 \cdots dx_{j-1}dx_{j+1}\cdots dx_n}{\partial P/\partial x_j}.$$

We naturally define the characteristic function $\theta(P)$ for the region $P \geq 0$ as

$$(\theta(P),\ \phi(x)) = \int_{P\geq 0} \phi(x)dx$$

where $\phi \in \mathcal{D}(R^n)$, and the generalized function $\delta(P)$ by

$$(\delta(P),\ \phi(x)) = \int_{P=0} \phi(x)\omega.$$

As an example, consider the generalized function $\delta(r-c)$, where $r^2 = \sum_{i=1}^{n} x_i^2$, and $c > 0$. The equation $r - c = 0$ defines the sphere O_c of radius c. Since $P - c = 0$ is the Euclidean distance from the surface of the sphere, at $r = c$ the differential form ω coincides with the Euclidean element of area dO_c on the sphere, so that

$$(\delta(r-c),\ \phi) = \int_{O_c} \phi dO_c.$$

It was proved in [1] that

$$\frac{\partial\theta(P)}{\partial x_j} = \frac{\partial P}{\partial x_j}\delta(P).$$

We shall first add the following identity, which has never appeared so far, according to the author's knowledge

$$\frac{\partial\theta(P)}{\partial P} = \delta(P).$$

Indeed,

$$(\frac{\partial\theta(P)}{\partial P},\ \phi(x)) = -(\theta(P),\ \frac{\partial}{\partial P}\phi(x)).$$

Since $\phi = \phi(x_1, x_2, \cdots, x_j(P), \cdots, x_n)$ by the substitution of (1), we come to

$$-(\theta(P),\ \frac{\partial}{\partial P}\phi(x)) = -(\theta(P),\ \frac{\partial\phi(x)}{\partial x_j}\frac{1}{\frac{\partial P}{\partial x_j}}) = -\int_{P\geq 0}\frac{\partial\phi(x)}{\partial x_j}\frac{1}{\frac{\partial P}{\partial x_j}}dx.$$

On the other hand,

$$(\delta(P),\ \phi(x)) = \int_{P=0}\phi(x)\omega.$$

Let us assume that $P \geq 0$ defines a bounded region. Then we may apply the Gauss-Ostrogradskii formula to the above integral over this region and to the differential form of degree $n-1$ in the integrand. We also use the fact that P increases into the interior of the region to derive

$$\int_{P=0}\phi(x)\omega = -\int_{P\geq 0}d(\phi(x)\omega)$$

and

$$d(\phi(x)\omega) = \frac{\partial\phi(x)}{\partial x_j}\frac{1}{\frac{\partial P}{\partial x_j}}dx + \phi\frac{\partial}{\partial x_j}(\frac{\partial x_j}{\partial P})dx = \frac{\partial\phi(x)}{\partial x_j}\frac{1}{\frac{\partial P}{\partial x_j}}dx,$$

which implies

$$\int_{P=0}\phi(x)\omega = -\int_{P\geq 0}\frac{\partial\phi(x)}{\partial x_j}\frac{1}{\frac{\partial P}{\partial x_j}}dx.$$

Hence the identity holds on any bounded region.

If $P \geq 0$ does not define a bounded region, we replace it by its intersection G_R with a sufficiently large ball $|x| \leq R$ outside of which $\phi(x)$ is known to vanish. Let Γ_R be the boundary of G_R, we have

$$\int_{\Gamma_R}\phi(x)\omega = -\int_{G_R}\frac{\partial\phi(x)}{\partial x_j}\frac{1}{\frac{\partial P}{\partial x_j}}dx.$$

Now since $\phi(x)$ vanishes outside of $|x| \leq R$, we arrive at

$$\int_{P=0}\phi(x)\omega = -\int_{P\geq 0}\frac{\partial\phi}{\partial P}dx,$$

which completes the proof. □

It is well known that in one dimension every functional concentrated on a point is a linear combination of the delta function and its derivatives. For $n > 1$, we have a similar role played by generalized functions,

$\delta(P)$, $\delta'(P), \cdots$, $\delta^{(k)}(P)$ (the derivatives of $\delta(P)$ with respect to the argument P), which we shall define based on the differential forms $\omega_k(\phi)$ given by

$$\begin{aligned}
&\omega_0(\phi) = \phi \cdot \omega, \\
&d\omega_0(\phi) = dP \cdot \omega_1(\phi), \\
&\cdots\ \cdots \\
&d\omega_{k-1}(\phi) = dP \cdot \omega_k(\phi), \\
&\cdots\ \cdots
\end{aligned}$$

where d denotes the exterior derivative. Now we are able to define

$$(\delta^{(k)}(P),\ \phi) = (-1)^k \int_{P=0} \omega_k(\phi)$$

for $k = 0, 1, 2, \cdots$, since the above integral over the $P = 0$ surface of any of the $\omega_k(\phi)$ is uniquely determined by $P(x)$. Furthermore, we define generalized function $\partial\delta(P)/\partial P$ as

$$(\frac{\partial}{\partial P}\delta(P),\ \phi) = -\int_{P=0} \frac{\partial\phi}{\partial P}\omega.$$

We shall show that

$$\frac{\partial}{\partial P}\delta(P) = \delta'(P).$$

In fact,

$$(\frac{\partial}{\partial P}\delta(P),\ \phi) = -\int_{P=0} \frac{\partial\phi}{\partial P}\omega = -\int_{P=0} \omega_0(\frac{\partial\phi}{\partial P}).$$

On the other hand,

$$\begin{aligned}
(\delta'(P),\ \phi) &= -\int_{P=0} \omega_1(\phi) \\
&= -\int_{P=0} \frac{\partial}{\partial P}(\frac{\phi}{\partial P/\partial x_j})dx_1 \cdots dx_{j-1}dx_{j+1} \cdots dx_n.
\end{aligned}$$

Since $\phi = \phi(x_1, x_2, \cdots, x_j(P), \cdots, x_n)$ and $\partial P/\partial x_j$ is not a function of P, we imply

$$\begin{aligned}
\frac{\partial}{\partial P}(\frac{\phi}{\partial P/\partial x_j})dx_1 \cdots dx_{j-1}dx_{j+1} \cdots dx_n &= \frac{\frac{\partial\phi}{\partial P}}{\partial P/\partial x_j}dx_1 \cdots dx_{j-1}dx_{j+1} \\
&\quad \cdots dx_n \\
&= \omega_0(\frac{\partial\phi}{\partial P}),
\end{aligned}$$

by choosing the coordinates $u_i = x_i$, and $u_j = P$. Under these coordinates

$$\omega_k(\phi) = \frac{\partial^k}{\partial P^k}(\frac{\phi}{\partial P/\partial x_j})dx_1 \cdots dx_{j-1}dx_{j+1}\cdots dx_n.$$

This completes the proof. □

Similarly, we can obtain

$$\frac{\partial}{\partial P}\delta^{(k)}(P) = \delta^{(k+1)}(P) \qquad \text{for } k = 1, 2, \cdots.$$

We now prove the following recurrence relations, identities between $\delta(P)$ and its derivatives:

$$\begin{aligned}
&P\delta(P) = 0\\
&P\delta'(P) + \delta(P) = 0\\
&P\delta''(P) + 2\delta'(P) = 0\\
&\cdots\cdots\\
&P\delta^{(k)}(P) + k\delta^{(k-1)}(P) = 0\\
&\cdots\cdots
\end{aligned}$$

The first of these is obvious, since the integral of $P\phi$ over the $P = 0$ surface clearly vanishes. We now take the derivative with respect to P to get

$$P\delta'(P) + \delta(P) = 0$$

as well as the rest similarly.

Remark. Gel'fand [1] derived the same relations by applying the identity

$$\frac{\partial}{\partial x_j}\delta(P) = \delta'(P)\frac{\partial P}{\partial x_j},$$

which is slightly more complicated than our approach.

Let us now construct $\delta^{(k)}(r - c)$, where $r^2 = \sum_{i=1}^n x_i^2$. We use the spherical coordinates

$$\begin{aligned}
x_1 &= r\cos\theta_1,\\
x_2 &= r\sin\theta_1\cos\theta_2,\\
x_3 &= r\sin\theta_1\sin\theta_2\cos\theta_3,\\
&\cdots\ \cdots\\
x_{n-1} &= r\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}\cos\theta_{n-1}\\
x_n &= r\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}\sin\theta_{n-1}
\end{aligned}$$

and $u_1 = r,\ u_2 = \theta_1, \cdots, u_n = \theta_{n-1}$. We obtain

$$\omega = r^{n-1} d\Omega$$

where $d\Omega = \sin^{n-2}\theta_1 \sin^{n-3}\theta_2 \cdots \sin\theta_{n-2} d\theta_1 \cdots d\theta_{n-1}$ is the element of area on the unit sphere $r = 1$. This gives

$$\omega_0 = \phi\omega = \phi r^{n-1} d\Omega \text{ and } \omega_k(\phi) = \frac{\partial^k}{\partial r^k}(\phi r^{n-1}) d\Omega,$$

so that

$$\int_{R^n} \delta^{(k)}(r-c)\phi dx = (-1)^k \int_{r=1} \frac{\partial^k}{\partial r^k}(\phi r^{n-1}) d\Omega = \frac{(-1)^k}{c^{n-1}} \int_{O_c} \frac{\partial^k}{\partial r^k}(\phi r^{n-1}) dO_c$$

where O_c is the sphere $r - c = 0$, and dO_c is the Euclidean element of area of it.

Theorem 2.1. *Let f be an infinitely differentiable function. Then the product $f(P)\delta^{(k)}(P)$ exists and*

$$\begin{aligned} f(P)\delta^{(k)}(P) &= \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f^{(k-j)}(0)\delta^{(j)}(P) \\ &= \sum_{j=0}^{k} (-1)^{j} \binom{k}{j} f^{(j)}(0)\delta^{(k-j)}(P). \end{aligned}$$

Before going into the proof, however, we would like to give out the following product if $f(P) = P^l$ by Theorem 2.1.

$$P^l \delta^{(k)}(P) = \begin{cases} (-1)^l\, l!\, \binom{k}{l} \delta^{(k-l)}(P) & \text{if } l \le k, \\ 0 & \text{otherwise} \end{cases}$$

which, of course, generalizes the identity

$$P\delta^{(k)}(P) + k\delta^{(k-1)}(P) = 0 \quad \text{for all } k \ge 0.$$

Proof. By the definition

$$\begin{aligned} & (f(P)\delta^{(k)}(P),\ \phi(x)) = (\delta^{(k)}(P),\ f(P)\phi(x)) \\ = \ & (-1)^k \int_{P=0} \omega_k(f(P)\phi(x)) \\ = \ & (-1)^k \int_{P=0} \frac{\partial^k}{\partial P^k}\left(\frac{f(P)\phi(x)}{\partial P/\partial x_j}\right) dx_1 \cdots dx_{j-1} dx_{j+1} \cdots dx_n \end{aligned}$$

$$= (-1)^k \int_{P=0} \sum_{j=0}^{k} \binom{k}{j} f^{(k-j)}(P) \frac{\partial^j}{\partial P^j} (\frac{\phi(x)}{\partial P/\partial x_j}) dx_1 \cdots dx_{j-1} dx_{j+1} \cdots dx_n$$

$$= \sum_{j=0}^{k} \binom{k}{j} (-1)^k f^{(k-j)}(0) \int_{P=0} \frac{\partial^j}{\partial P^j} (\frac{\phi(x)}{\partial P/\partial x_j}) dx_1 \cdots dx_{j-1} dx_{j+1} \cdots dx_n$$

$$= \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} f^{(k-j)}(0) (\delta^{(j)}(P),\ \phi(x))$$

which completes the proof. □

Remark. Since every functional g of the form

$$(g,\ \phi) = \sum_j \int_{P=0} a_j(x) D^j \phi(x) d\sigma$$

where

$$j = (j_1, j_2, \cdots, j_n) \quad \text{and} \quad D^j = \frac{\partial^{j_1 + \cdots + j_n}}{\partial x_1^{j_1} \cdots \partial x_n^{j_n}},$$

can be written as the sum of multiplet layers (see [1]), given by

$$g = \sum_k b_k(x) \delta^{(k)}(P).$$

We can easily obtain the product $f(P)g$ by Theorem 2.1

We now consider two functions $P(x)$ and $Q(x)$ such that the $P = 0$ and $Q = 0$ hypersurfaces have, as before, no singular points. Assume further that these surfaces fail to intersect and that the $PQ = 0$ surface also has no singular points. We have

Theorem 2.2. *Let f be an infinitely differentiable function of two variables. Then the product $f(P, Q)\delta(PQ)$ exists and*

$$f(P, Q)\delta(PQ) = \frac{f(0, Q)}{Q} \delta(P) + \frac{f(P, 0)}{P} \delta(Q).$$

In particular, we get

$$P\delta(PQ) = \delta(Q) \quad \text{and} \quad Q\delta(PQ) = \delta(P),$$
$$\delta(PQ) = Q^{-1}\delta(P) + P^{-1}\delta(Q).$$

Proof. Let ω_P be the differential form corresponding to the $P = 0$ surface, ω_Q be the differential form corresponding to the $Q = 0$ surface and ω be the differential form corresponding to the $PQ = 0$ surface. Then

$$\begin{aligned} &\omega_P dP = dv \quad \text{on} \quad P = 0 \\ &\omega_Q dP = dv \quad \text{on} \quad Q = 0 \\ &\omega d(PQ) = \omega(PdQ + QdP) = dv \quad \text{on} \quad PQ = 0 \end{aligned}$$

Now on the $P = 0$ surface we arrive at

$$\omega Q dP = dv = \omega_P dP$$

which implies $\omega = Q^{-1}\omega_P$. Similarly, on the $Q = 0$ surface $\omega = P^{-1}\omega_Q$. This leads directly to

$$\begin{aligned} &(f(P,Q)\delta(PQ),\ \phi(x)) = (\delta(PQ),\ f(P,Q)\phi(x)) \\ &= \int_{P=0} f(P,Q)\phi\omega + \int_{Q=0} f(P,Q)\phi\omega \\ &= \int_{P=0} f(0,Q)\phi Q^{-1}\omega_P + \int_{Q=0} f(P,0)\phi P^{-1}\omega_Q \\ &(\frac{f(0,Q)}{Q}\delta(P) + \frac{f(P,0)}{P}\delta(Q),\ \phi) \end{aligned}$$

which completes the proof. □

We assume Q is a nonvanishing function. From $Q\delta(PQ) = \delta(P)$ we obtain $\delta(PQ) = Q^{-1}\delta(P)$. Then the derivative with respect to P gives

$$Q\delta'(PQ) = Q^{-1}\delta'(P) \quad \text{implying} \quad \delta'(PQ) = Q^{-2}\delta'(P).$$

In a similar way, we have for any $k \geq 0$ and $Q \neq 0$ that

$$\delta^{(k)}(PQ) = Q^{-(k+1)}\delta^{(k)}(P). \tag{2}$$

To end this section, we shall apply Pizetti's formula to compute the product $X\delta(P)$ (obviously different from all the above products), where $X = \sum_{i=1}^{n} x_i$ and P is the unit sphere given by $r = 1$.

Setting $\psi(x) = X\phi(x)$, we use the fact that $\omega = d\sigma$ (Euclidean area) on P to get

$$\begin{aligned} &(X\delta(P),\ \phi(x)) = (\delta(P),\ X\phi(x)) = \int_{P=0} X\phi(x)\omega \\ &= \int_{P=0} \psi(x)d\sigma = \Omega_n S_\psi(1). \end{aligned}$$

By Pizetti's formula given in Section 1, we come to

$$S_\psi(1) = \psi(0) + \frac{1}{2!}\psi''(0) + \cdots + \frac{1}{(2k)!}\psi^{(2k)}(0) + \cdots \quad \text{and}$$

$$\psi^{(2k)}(0) = \frac{(2k)!\Delta^k\psi(0)}{2^k\, k!\, n(n+2)\cdots(n+2k-2)}.$$

It follows from $\psi(0) = 0\phi(0) = 0$ that

$$(X\delta(P),\ \phi(x)) = \Omega_n \sum_{k=1}^{\infty} \frac{\Delta^k\psi(0)}{2^k\, k!\, n(n+2)\cdots(n+2k-2)}.$$

In order to calculate $\Delta^k\psi(0)$, we claim for $k \geq 0$ that

$$\Delta^{k+1}(X\phi) = 2(k+1)\nabla\Delta^k\phi + X\Delta^{k+1}\phi$$

where $\nabla = \partial/\partial x_1 + \cdots + \partial/\partial x_n$.

We use an inductive method to prove it. It is obviously true for $k = 0$. Assume $k = 1$, we have

$$\Delta^2(x_i\phi) = 4\frac{\partial}{\partial x_i}\Delta\phi + x_i\Delta^2\phi$$

simply by calculating the left-hand side. Hence

$$\Delta^2(X\phi) = 4\nabla\Delta\phi + X\Delta^2\phi.$$

By hypothesis, it holds for the case of $k-1$, that is

$$\Delta^k(X\phi) = 2k\nabla\Delta^{k-1}\phi + X\Delta^k\phi.$$

Hence it follows that

$$\begin{aligned}
&\Delta^{k+1}(X\phi) = \Delta\Delta^k(X\phi) = \Delta(2k\nabla\Delta^{k-1}\phi + X\Delta^k\phi) \\
&= 2k\nabla\Delta^k\phi + \Delta(X\Delta^k\phi) \\
&= 2(k+1)\nabla\Delta^k\phi + X\Delta^{k+1}\phi.
\end{aligned}$$

By the claim we have

$$\Delta^k\psi(0) = 2k\nabla\Delta^{k-1}\phi(0) = -2k(\Delta^{k-1}\nabla\delta(x),\ \phi(x)).$$

Finally,

$$X\delta(P) = -\Omega_n \sum_{k=0}^{\infty} \frac{\Delta^k\nabla\delta(x)}{2^k\, k!\, n(n+2)\cdots(n+2k)}.$$

Remark. By equation (2) we can easily derive the product $X\delta(r^2-1)$ since

$$\delta(r^2-1) = \frac{1}{2}\delta(r-1).$$

In 1991, Aguirre expressed distribution $\delta^{(k)}(r-1)$ in terms of an infinite sum of linear combinations of $\Delta^l\delta$ (see [9]). Therefore, it would not be hard to compute the product $X\,\delta^{(k)}(r-1)$. However, an interesting problem is how to deduce general products $X^l\delta^{(k)}(r-1)$ for integers l and $k \geq 0$. The author leaves this for interested readers.

3. Invariant Theorem

In order to obtain more complicated products defined on manifolds, such as the product of P_+^l and $\delta^{(k)}(P)$, where

$$(P_+^l,\, \phi) = \int_{P\geq 0} P^l(x)\phi(x)dx,$$

we need the following invariant theorem, which computes the products of distributions on manifolds based on the products of a single variable.

Let $\rho(x)$ be a fixed infinitely differentiable function on R with four properties

(i) $\rho(x) \geq 0$,

(ii) $\rho(x) = 0$ for $|x| \geq 1$,

(iii) $\rho(x) = \rho(-x)$,

(iv) $\int_{-1}^{1} \rho(x)dx = 1$

Obviously, Temple sequence $\delta_m(x) = m\rho(mx)$ is an infinitely differentiable sequence converging to δ in $\mathcal{D}'(R)$. Let f be an arbitrary distribution in $\mathcal{D}'(R)$, we define

$$f_m(x) = (f * \delta_m)(x) = (f(t),\, \delta_m(x-t))$$

for $m = 1, 2, \cdots$. It follows that $\{f_m(x)\}$ is a regular sequence converging to the distribution f in $\mathcal{D}'(R)$. The definition of the product of a distribution and an infinitely differentiable function is the following (see for example [1]).

Definition 3.1. Let f be a distribution in $\mathcal{D}'(R)$ and let g be an infinitely differentiable function. Then the product fg is defined by

$$(fg,\, \phi) = (f,\, g\phi)$$

for all functions ϕ in $\mathcal{D}(R)$.

We use the following definition [10] for the commutative neutrix products of distributions in a single variable.

Definition 3.2. Let f and g be distributions in $\mathcal{D}'(R)$ and let $f_m(x) = (f * \delta_m)(x)$ and $g_m(x) = (g * \delta_m)(x)$. We say that the commutative neutrix product $f \diamond g$ of f and g exists and is equal to h if

$$N - \lim_{m\to\infty} \frac{1}{2}\{(f_m g,\ \phi) + (f g_m,\ \phi)\} = (h,\ \phi)$$

for all function $\phi \in \mathcal{D}(R)$, where N is the neutrix [11] having domain $N' = \{1, 2, \cdots\}$ and range the real numbers, with negligible functions that are finite linear sums of functions

$$m^\lambda \ln^{r-1} m, \quad \ln^r m \quad (\lambda > 0,\ r = 1, 2, \cdots)$$

and all functions of m that converge to zero in the normal sense as m tends to infinity. If the normal limit exists, then it is simply called the commutative product.

To see Definition 3.2 extends Definition 3.1, we let g be a C^∞ function. Clearly, $g_m\phi$ has an uniform support and converges to $g\phi$ in $\mathcal{D}(R)$. For any f in $\mathcal{D}'(R)$, we imply that

$$\begin{aligned}(f \diamond g,\ \phi) &= N - \lim_{m\to\infty} \frac{1}{2}\{(f_m g,\ \phi) + (f g_m,\ \phi)\} \\ &= N - \lim_{m\to\infty} \frac{1}{2}\{(f_m,\ g\phi) + (f,\ g_m\phi)\} = (f,\ g\phi) = (fg,\ \phi).\end{aligned}$$

Let $f(t)$ be a distribution of one variable and P be given as in Section 2. We define for $\phi \in \mathcal{D}(R^n)$ (see [12]) that

$$(f(P),\ \phi(x)) = (f(t),\ \psi(t))$$

where

$$\psi(t) = \int_{P(x)=t} \phi(x)\,\omega \quad \text{and} \quad dP \cdot \omega = dv.$$

Clearly $\psi(t) \in \mathcal{D}(R)$, since there is at least one unbounded x_j when t is large, which implies that ϕ vanishes.

As an example, we consider the functional r^λ defined by

$$(r^\lambda,\ \phi) = \int_{R^n} r^\lambda \phi(x) dx \qquad (3)$$

for $\mathrm{Re}\lambda > -n$. Using the spherical coordinates in Equation (3), we write

$$(r^\lambda,\ \phi) = \int_0^\infty r^\lambda \{ \int_\Omega \phi(r\omega) r^{n-1} d\Omega \} dr$$
$$= \int_0^\infty r^\lambda \{ \int_\Omega \phi(r\omega)\omega \} dr = (r^\lambda,\ \int_\Omega \phi(r\omega)\omega),$$

where Ω is the unit sphere.

As we will see, the sequence $\delta_m(P)$ plays an important role in obtaining the invariant theorem. First, we claim that $\lim_{m\to\infty} \delta_m(P(x)) = \delta(P(x))$. Indeed,

$$\lim_{m\to\infty} (\delta_m(P(x)),\ \phi(x)) = \lim_{m\to\infty} (\delta_m(t),\ \psi(t)) = (\delta(t),\ \psi(t)) = (\delta(P(x)),\ \phi(x)).$$

Let f be a distribution of one variable. The convolution $f(P(x)) * \phi$ is defined by

$$f(P(x)) * \phi = \int_{-\infty}^{\infty} f(t)dt \int_{P(z)=t} \phi(z-x)\omega,$$

where $\phi \in \mathcal{D}(R^n)$.

Influenced by Aguirre's work in [12], we shall show that $\delta(P(x)) * \psi = \psi$ holds locally for $\psi \in \mathcal{D}(R^n)$, under the condition that $\partial P/\partial x_i = 1$ for some i $(1 \le i \le n)$. In fact,

$$(\delta_m(P(x)) * \psi,\ \phi) = (\int_{-\infty}^{\infty} \delta_m(t)dt \int_{P(z)=t} \psi(z-x)\omega,\ \phi(x))$$
$$= \int_{R^n} \int_{-\infty}^{\infty} \delta_m(t)dt \int_{P(z)=t} \psi(z-x)\omega\phi(x)dx$$
$$= \int_{-\infty}^{\infty} \delta_m(t)dt \int_{P(z)=t} \int_{R^n} \psi(z-x)\phi(x)dx\omega$$

which is permissible by Fubini's theorem. Hence,

$$(\delta_m(P(x)) * \psi,\ \phi) = \int_{-\infty}^{\infty} \delta_m(t)dt \int_{P(z)=t} (\phi * \psi)\omega$$
$$= (\delta_m(P(x)),\ (\psi * \phi)(x_1, \cdots, x_n)).$$

Making substitution $u_1 = x_1, \cdots, u_i = P(x), \cdots, u_n = x_n$ locally and using $\partial P/\partial x_i = 1$ (thus the Jacobian $= 1$), we arrive at

$$(\delta_m(P(x)),\ (\psi * \phi)(x_1, \cdots, x_n)) = (\delta_m(u_i),\ (\psi_1 * \phi_1)(u_1, \cdots, u_n))$$
$$= (\delta_m(u_i) * \psi_1(u_1, \cdots, u_n),\ \phi_1(u_1, \cdots, u_n)).$$

Since $\lim_{m\to\infty} \delta_m(u_i) * \psi_1(u_1, \cdots, u_n) = \psi_1(u_1, \cdots, u_n)$, we imply

$$\lim_{m\to\infty} (\delta_m(P(x)) * \psi,\ \phi) = (\psi_1(u_1, \cdots, u_n),\ \phi_1(u_1, \cdots, u_n))$$
$$= (\psi(x_1, \cdots, x_n),\ \phi(x_1, \cdots, x_n)),$$

which completes the proof. □

In particular, we have

$$\delta(x_i + P_1(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)) * \psi(x) = \psi(x) \quad \text{for} \quad i = 1, 2, \cdots, n$$

where P_1 is any infinitely differentiable function.

Next, we shall prove that $\lim_{m\to\infty} \delta_m(P(x)) * f(P(x)) = f(P(x))$ if f is a distribution of a single variable and P is regular. Consider

$$(\delta_m(P(x)) * f(P(x)),\ \phi(x))$$
$$= \int_{R^n} \int_{-\infty}^{\infty} f(t)dt \int_{P(z)=t} \delta_m(z - x)\, \omega\, \phi(x)dx$$
$$= \int_{-\infty}^{\infty} f(t)dt \int_{P(z)=t} \int_{R^n} \delta_m(z - x)\phi(x)dx\, \omega.$$

Since the sequence

$$\int_{R^n} \delta_m(z - x)\phi(x)dx$$

converges to $\phi(z)$ in $\mathcal{D}(R^n)$ as $m \to \infty$ by the four properties of $\rho(x)$. Therefore,

$$\lim_{m\to\infty} (\delta_m(P(x)) * f(P(x)),\ \phi(x)) = \int_{-\infty}^{\infty} f(t)dt \int_{P(z)=t} \phi(z)\omega$$
$$= (f(P(x)),\ \phi(x)),$$

which completes the proof. □

Definition 3.3. Let $f(t)$ and $g(t)$ be distributions of one variable and let $P(x)$ be a regular $(n-1)$ dimensional manifold. Then the commutative neutrix product $f(P(x)) \diamond g(P(x))$ of $f(P(x))$ and $g(P(x))$ is defined as

$$(f(P(x)) \diamond g(P(x)),\ \phi)$$
$$= N - \lim_{m\to\infty} \frac{1}{2}\{((f(P(x)) * \delta_m(P(x)))g(P(x)),\ \phi)$$
$$+ (f(P(x))(g(P(x)) * \delta_m(P(x))),\ \phi)\}$$

if the left-hand side limit exists for all function $\phi \in \mathcal{D}(R^n)$. If the normal limit exists, then it is simply called the commutative product.

Theorem 3.1 (Invariant Theorem.) *Assume $P(x)$ is a regular $(n-1)$ dimensional manifold and the commutative neutrix product $h(t) = f(t) \circ g(t)$ exists. Then the commutative neutrix product $f(P(x)) \circ g(P(x))$ also exists and*

$$f(P(x)) \circ g(P(x)) = h(P(x)).$$

Proof. It follows from Definition 3.3 that

$$\begin{aligned}
&(f(P(x)) \circ g(P(x)),\ \phi) \\
&= N - \lim_{m\to\infty} \frac{1}{2}\{(((f(P(x)) * \delta_m(P(x)))g(P(x)),\ \phi) \\
&+(f(P(x))(g(P(x)) * \delta_m(P(x))),\ \phi)\} \\
&= N - \lim_{m\to\infty} \frac{1}{2}\{(f(t)(g(t) * \delta_m(t)),\ \psi(t)) \\
&+(g(t)(f(t) * \delta_m(t)),\ \psi(t))\} \\
&= (h(t), \psi(t)) = (h(P(x)),\ \phi(x)),
\end{aligned}$$

where

$$\psi(t) = \int_{P(x)=t} \phi(x)\,\omega \quad \text{for} \quad \phi(x) \in \mathcal{D}(R^n),$$

which completes the proof. □

As a simple example to use the invariant theorem, we let $P(x) = x_1 + x_2 + 1$, which is obviously regular. The functional $\theta(P)$ is given by

$$(\theta(P),\ \phi(x)) = \int_{x_1+x_2+1\geq 0} \phi(x_1, x_2)dx = \int_{x_1+x_2\geq -1} \phi(x_1, x_2)dx$$

and the functional $\delta(P)$ is defined as

$$(\delta(P),\ \phi(x)) = \int_{x_1+x_2+1=0} \phi(x_1, x_2)\omega = \int \phi(-1-x_2, x_2)dx_2.$$

It was proved in [7] that

$$\theta(t) \circ \delta(t) = \frac{1}{2}\delta(t),$$

which implies

$$\theta(P) \circ \delta(P) = \frac{1}{2}\delta(P)$$

by the invariant theorem. Note that it seems infeasible to compute this product along the differential form approach discussed in Section 2.

Using the invariant theorem, we can easily get Theorem 2.1, since by Definition 3.1

$$\begin{aligned} f(t)\,\delta^{(k)}(t) &= \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} f^{(k-j)}(0)\delta^{(j)}(t) \\ &= \sum_{j=0}^{k}(-1)^{j}\binom{k}{j} f^{(j)}(0)\delta^{(k-j)}(t), \end{aligned}$$

if $f(t)$ is an infinitely differentiable function of one variable, and the fact Definition 3.2 extends Definition 3.1.

Applying Definition 3.2, one can derive that the product $x_+^r \diamond \delta^{(r+k)}(x)$ exists and

$$x_+^r \diamond \delta^{(r+k)}(x) = \frac{(-1)^r (r+k)!}{2\,k!}\delta^{(k)}(x) \tag{4}$$

for $r, k = 0, 1, 2, \cdots$. In particular, we have $\theta(x) \diamond \delta^{(k)}(x) = \delta^{(k)}(x)/2$ since $x_+^0 = \theta(x)$.

Let $P(x)$ be a regular $(n-1)$ dimensional manifold. It follows from equation (4) and the invariant theorem that

$$\begin{aligned} & P_+^r(x) \diamond \delta^{(r+k)}(P(x)) = \frac{(-1)^r (r+k)!}{2\,k!}\delta^{(k)}(P(x)), \\ & \theta(P(x)) \diamond \delta^{(k)}(P(x)) = \delta^{(k)}(P(x))/2. \end{aligned}$$

However, the differential form method is clearly unable to carry out these products since $P_+^r(x)$ is discontinuous with respect to the argument P.

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Department of Mathematics and Computer Science
Brandon University
Brandon, Manitoba
Canada R7A 6A9
E-mail: lic@brandonu.ca

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