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The distributional products of particular distributions

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Dedicated to Professor H.M. Srivastava on the occasion of his 65th birthday

Abstract

Let f be a C^∞ function on R and P be a quadratic form defined by $P(x) = P(x_1, x_2, \ldots, x_m) = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2$ with $p + q = m$. In this paper, we mainly show that

$$f(P) \cdot \delta^{(k)}(P) = \sum_{i=0}^{k} \binom{k}{i} f^{(i)}(0)\delta^{(k-i)}(P),$$

where $\delta^{(k)}(P)$ is given by

$$(\delta^{(k)}(P), \phi) = (-1)^k \int_0^\infty \left[\left(\frac{\partial}{2r\partial r}\right)^k \left\{r^{p-2}\frac{\psi(r,s)}{2}\right\}\right]_{r=s} s^{q-1}\,\mathrm{d}s.$$

In particular, we have

$$P^n \cdot \delta^{(k)}(P) = \begin{cases} n!\binom{k}{n}\delta^{(k-n)}(P) & \text{if } k \geqslant n, \\ 0 & \text{if } k < n, \end{cases}$$

which solves a problem raised by Li in 2004.

Keywords: Particular distribution; δ-function; Product and quadratic form

1. Introduction

Physicists have long been using so-called singular functions such as δ, although these can not be properly defined within the framework of classical function theory. In elementary particle physics, one [14] finds the need to evaluate δ^2 when calculating the transition rates of certain particle interactions. In [6], a definition

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for product of distributions is given using delta sequences. However, δ^2 as a product of δ with itself is shown not to exist. In [7], Bremermann used the Cauchy representations of distributions with compact support to define $\sqrt{\delta_+}$ and $\log \delta_+$. Unfortunately, his definition does not carry over to $\sqrt{\delta}$ and $\log \delta$. Fisher, with his collaborators [8–13], has actively used Jones' δ-sequence and Van der Corput's neutrix limit [19] (in order to abandon unwanted infinite quantities from asymptotic expansions) to deduce numerous products, powers, convolutions and compositions of distributions on R since 1969.

To extend multiplications from one-dimensional to m-dimensional, Li [16,17] constructed a workable δ-sequence on R^m by $\delta_n(x) = c_m n^m \rho(n^2 r^2)$, where $\rho(s)$ is a fixed infinitely differentiable function defined on $R^+ = [0, \infty)$ having the properties:

(i) $\rho(s) \geqslant 0$,
(ii) $\rho(s) = 0$ for $s \geqslant 1$,
(iii) $\int_{R^m} \delta_n(x)\,\mathrm{d}x = 1$

and obtained non-commutative neutrix products such as $r^{-k} \cdot \nabla\delta$ as well as $r^{-k} \cdot \Delta^l\delta$, where Δ denotes the Laplacian. Aguirre [1] used the Laurent series expansion of r^λ and derived a more general product $r^{-k} \cdot \nabla(\Delta^l\delta)$ by calculating the residue of r^λ. His approach is an interesting example of using complex analysis to obtain products of distribution on R^m [2–5].

The problem of defining products of distributions on a manifold (such as the unit sphere) has been a serious challenge since Gel'fand introduced special types of generalized functions, such as P_+^λ and $\delta^{(k)}(P)$. Aguirre [3] employed the Taylor expansion of distribution $\delta^{(k-1)}(m^2 + P)$ and gave a meaning of the product $\delta^{(k-1)}(m^2 + P) \cdot \delta^{(l-1)}(m^2 + P)$. Li [18] applied the expansion formula stated below:

$$\int_\Omega \frac{\partial^k}{\partial r^k} \phi(r\omega)\,\mathrm{d}\omega = (-1)^k \left(\sum_{i=0}^{k} \binom{k}{i} C(m,i) \delta^{(k-i)}(r-1), \phi(x) \right)$$

to evaluate the product of $f(r)$ and $\delta^{(k)}(r-1)$ on the unit sphere of R^m with the condition $k \leqslant m-1$.

The objective of this paper is to use a much simpler method of deriving the product of $f(r) \cdot \delta^{(k)}(r-1)$ for all k and further study a more general product $f(H) \cdot \delta^{(k)}(H)$, where H is a regular hypersurface. In Section 4, we find the product $P^n \cdot \delta^{(k)}(P)$, which is an open problem in [18], as well as a general product $f(P) \cdot \delta^{(k)}(P)$ where f is a C^∞ function on R.

2. The product $f(r) \cdot \delta^{(k)}(r-1)$

Let $r = (x_1^2 + x_2^2 + \cdots + x_m^2)^{\frac{1}{2}}$. The distribution $\delta^{(k)}(r-1)$ focused on the unit sphere Ω of R^m is defined by

$$(\delta^{(k)}(r-1), \phi) = (-1)^k \int_\Omega \frac{\partial^k}{\partial r^k} [\phi(r\omega) r^{m-1}]\,\mathrm{d}\omega$$

where ϕ is a Schwartz testing function.

Theorem 1. *Let $f(x)$ be a smooth function at $x = 1$. Then the product $f(r) \cdot \delta^{(k)}(r-1)$ exists and*

$$f(r) \cdot \delta^{(k)}(r-1) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} f^{(j)}(1) \delta^{(k-j)}(r-1)$$

for any non-negative integer k.

Proof. Obviously, we have for any testing function ϕ

$$(f(r) \cdot \delta^{(k)}(r-1), \phi) = (-1)^k \int_\Omega \frac{\partial^k}{\partial r^k} [\phi(r\omega) f(r) r^{m-1}]\,\mathrm{d}\omega.$$

It follows that

$$\frac{\partial^k}{\partial r^k}[\phi(r\omega)f(r)r^{m-1}] = \sum_{j=0}^{k}\binom{k}{j}\left[\frac{\partial^j}{\partial r^j}f(r)\right]\left[\frac{\partial^{k-j}}{\partial r^{k-j}}(\phi(r\omega)r^{m-1})\right].$$

Thus,

$$\begin{aligned}(f(r)\cdot\delta^{(k)}(r-1),\phi) &= \sum_{j=0}^{k}(-1)^j\binom{k}{j}f^{(j)}(1)(-1)^{k-j}\int_\Omega \frac{\partial^{k-j}}{\partial r^{k-j}}[\phi(r\omega)r^{m-1}]\,\mathrm{d}\omega\\ &= \sum_{j=0}^{k}(-1)^j\binom{k}{j}f^{(j)}(1)(\delta^{(k-j)}(r-1),\phi),\end{aligned}$$

which completes the proof of theorem. □

In particular, we have

$$(r-1)^n\cdot\delta^{(k)}(r-1) = \begin{cases} -1^n n!\binom{k}{n}\delta^{(k-n)}(r-1) & \text{if } k \geqslant n,\\ 0 & \text{otherwise,}\end{cases}$$

which is a nicer and simpler result than the one in [18].

Choosing $f(r) = \sin r$, we get

$$\sin r\cdot\delta^{(k)}(r-1) = \sum_{j=0}^{k}(-1)^j\binom{k}{j}\sin\left(1+j\frac{\pi}{2}\right)\delta^{(k-j)}(r-1).$$

Clearly for $k = 0,1$, we have

$$\begin{aligned}&f(r)\cdot\delta(r-1) = f(1)\delta(r-1) \quad \text{and}\\ &f(r)\cdot\delta'(r-1) = f(1)\delta'(r-1) - f'(1)\delta(r-1),\end{aligned}$$

respectively.

If $f(r) = 1/r$, we arrive at

$$\frac{1}{r}\cdot\delta^{(k)}(r-1) = \sum_{j=0}^{k}\binom{k}{j}j!\delta^{(k-j)}(r-1).$$

To end this section, we would like to point out that following a similar approach to that of Theorem 1 one can carry out the product of $f(r)$ and $\delta^{(k)}(r^2-1)$, where

$$(\delta^{(k)}(r^2-1),\phi) = \frac{(-1)^k}{2}\int_\Omega\left(\frac{\partial}{2r\partial r}\right)^k(\phi r^{m-2})\,\mathrm{d}\omega.$$

3. The product $f(H)\cdot\delta^{(k)}(H)$

Let $H(x_1, x_2, \ldots, x_m)$ be any sufficiently smooth function such that on $H = 0$ we have

$$\mathrm{grad}H \neq 0,$$

which means that there are no singular points on $H = 0$. Then the generalized function $\delta(H)$ can be defined in the following way:

$$(\delta(H),\phi) = \int_{P=0}\psi(0,u_2,\ldots,u_m)\mathrm{d}u_2\cdots\mathrm{d}u_m,$$

where $\phi_1(u_1,\ldots,u_m) = \phi(x_1,\ldots,x_m)$ and $\psi = \phi_1(u)D\binom{x}{u}$.

Similarly, we shall define

$$(\delta^{(k)}(H),\phi) = (-1)^k \int_{P=0} \psi_{u_1}^{(k)}(0,u_2,\ldots,u_m)\mathrm{d}u_2\cdots\mathrm{d}u_m.$$

As an example, we consider the generalized function $\delta(\alpha_1 x_1 + \cdots + \alpha_m x_m)$, where $\sum_{i=1}^m \alpha_i^2 = 1$. The equation

$$\alpha_1 x_1 + \cdots + \alpha_m x_m = 0$$

determines a hypersurface which passes through the origin and is orthogonal to the unit vector α. Making the substitution

$$u_1 = \alpha_1 x_1 + \cdots + \alpha_m x_m, \quad u_2 = x_2,\ldots,u_m = x_m,$$

we thus arrive at

$$(\delta(\alpha_1 x_1 + \cdots + \alpha_m x_m),\phi) = \int_{\sum \alpha_i x_i = 0} \phi\,\mathrm{d}u_2\cdots\mathrm{d}u_m.$$

Theorem 2. *Let f be a C^∞ function and let H be defined as above. Then the product $f(H)\cdot\delta^{(k)}(H)$ exists and*

$$f(H)\cdot\delta^{(k)}(H) = \sum_{i=0}^k \binom{k}{i}(-1)^i f^{(i)}(0)\delta^{(k-i)}(H).$$

Proof. Using the substitutions $u_1 = H(x_1,\ldots,x_m)$, $u_2 = x_2,\ldots,u_m = x_m$, we arrive at

$$(f(H)\cdot\delta^{(k)}(H),\phi) = (-1)^k \int_{H=0} \frac{\partial^k}{\partial u_1^k}\left\{f(u_1)\phi_1 D\binom{x}{u}\right\}\bigg|_{u_1=0} \mathrm{d}u_2\cdots\mathrm{d}u_m$$

and

$$\frac{\partial^k}{\partial u_1^k}\left\{f(u_1)\phi_1 D\binom{x}{u}\right\}\bigg|_{u_1=0} = \sum_{i=0}^k \binom{k}{i} f^{(i)}(0) D_{u_1}^{k-i}\phi_1 D\binom{x}{u}\bigg|_{u_1=0}.$$

Hence,

$$\begin{aligned}(f(H)\cdot\delta^{(k)}(H),\phi) &= (-1)^k \sum_{i=0}^k \binom{k}{i} f^{(i)}(0)\int_{H=0} \frac{\partial^{k-i}}{\partial u_1^{k-i}}\phi_1 D\binom{x}{u}\bigg|_{u_1=0} \mathrm{d}u_2\cdots\mathrm{d}u_m \\ &= \sum_{i=0}^k \binom{k}{i}(-1)^i f^{(i)}(0)(\delta^{(k-i)}(H),\phi),\end{aligned}$$

which completes the proof of theorem. □

In particular, we have

$$H\cdot\delta'(H) = -\delta(H),$$
$$H^2\cdot\delta'(H) = 0.$$

4. The product $P^n\cdot\delta^{(k)}(P)$

Assume that both $p > 1$ and $q > 1$. Let P be a quadratic form defined by $P(x) = P(x_1,x_2,\ldots,x_m) = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2$ with $p+q = m$, then the $P = 0$ hypersurface is a hypercone with a singular point (the vertex) at the origin.

We start by assuming that $\phi(x)$ vanishes in a neighborhood of the origin. The distribution $\delta^{(k)}(P)$ is defined by

$$(\delta^{(k)}(P),\phi) = (-1)^k \int \frac{\partial^k}{\partial P^k}\left\{\frac{1}{2}\phi(r^2-P)^{\frac{1}{2}(q-2)}\right\}\bigg|_{P=0} r^{p-1}\,\mathrm{d}r\,\mathrm{d}\Omega^{(p)}\,\mathrm{d}\Omega^{(q)},$$

which is convergent.

Furthermore, if we transform from P to $s=\sqrt{r^2-P}$ we note that $\partial/\partial P=-(2s)^{-1}\partial/\partial s$, and we may write this in the form

$$(\delta^{(k)}(P),\phi)=\int\left[\left(\frac{\partial}{2s\partial s}\right)^k\left\{s^{q-2}\frac{\phi}{2}\right\}\right]_{s=r} r^{p-1}\,\mathrm{d}r\,\mathrm{d}\Omega^{(p)}\,\mathrm{d}\Omega^{(p)}.$$

Let us now define

$$\psi(r,s)=\int\phi\,\mathrm{d}\Omega^{(p)}\,\mathrm{d}\Omega^{(p)}.$$

Hence,

$$(\delta^{(k)}(P),\phi)=\int_0^\infty\left[\left(\frac{\partial}{2s\partial s}\right)^k\left\{s^{q-2}\frac{\psi(r,s)}{2}\right\}\right]_{s=r} r^{p-1}\,\mathrm{d}r.$$

Theorem 3. *The product P^n and $\delta^{(k)}(P)$ exists and*

$$P^n\cdot\delta^{(k)}(P)=\begin{cases} n!\binom{k}{n}\delta^{(k-n)}(P) & \text{if } k\geqslant n,\\ 0 & \text{if } k<n.\end{cases}$$

Proof. We start with

$$\begin{aligned}(P^n\cdot\delta^{(k)}(P),\phi)&=(-1)^k\int\frac{\partial^k}{\partial P^k}\left\{P^n\frac{1}{2}\phi(r^2-P)^{\frac{1}{2}(q-2)}\right\}\Big|_{P=0} r^{p-1}\,\mathrm{d}r\,\mathrm{d}\Omega^{(p)}\,\mathrm{d}\Omega^{(q)}\\ &=\int_0^\infty\left[\left(\frac{\partial}{2s\partial s}\right)^k\left\{(r^2-s^2)^n s^{q-2}\frac{\psi(r,s)}{2}\right\}\right]_{s=r} r^{p-1}\,\mathrm{d}r.\end{aligned}$$

Making the substitutions $u=r^2$ and $v=s^2$, we have

$$\frac{\partial}{2s\partial s}=\frac{1}{2s}2s\frac{\partial}{\partial v}=\frac{\partial}{\partial v},$$

which leads us to

$$(P^n\cdot\delta^{(k)}(P),\phi)=\frac{1}{4}\int_0^\infty\left[\left(\frac{\partial}{\partial v}\right)^k\left\{(u-v)^n v^{\frac{q-2}{2}}\psi_1(u,v)\right\}\right]_{u=v} u^{\frac{p-2}{2}}\,\mathrm{d}u.$$

Clearly,

$$\begin{aligned}\frac{\partial^k}{\partial v^k}\left\{(u-v)^n v^{\frac{q-2}{2}}\psi_1(u,v)\right\}\Big|_{u=v}&=\sum_{i=0}^{k}\binom{k}{i}D_v^i(u-v)^n D_v^{k-i}\left\{v^{\frac{q-2}{2}}\psi_1(u,v)\right\}\Big|_{u=v}\\ &=\sum_{i<n}\binom{k}{i}D_v^i(u-v)^n D_v^{k-i}\left\{v^{\frac{q-2}{2}}\psi_1(u,v)\right\}\Big|_{u=v}\\ &\quad+\sum_{i=n}\binom{k}{i}D_v^i(u-v)^n D_v^{k-i}\left\{v^{\frac{q-2}{2}}\psi_1(u,v)\right\}\Big|_{u=v}\\ &\quad+\sum_{i>n}\binom{k}{i}D_v^i(u-v)^n D_v^{k-i}\left\{v^{\frac{q-2}{2}}\psi_1(u,v)\right\}\Big|_{u=v}=I_1+I_2+I_3,\end{aligned}$$

where $D_v^i = \partial/\partial v^i$. It follows that

$$I_1 = I_2 = 0$$

since $i \neq n$. As for I_2, we arrive at

$$I_2 = \begin{cases} (-1)^n n! \binom{k}{n} D_v^{k-n}\left\{v^{\frac{q-2}{2}}\psi_1(u,v)\right\}\Big|_{u=v} & \text{if } k \geqslant n, \\ 0 & \text{if } k < n. \end{cases}$$

Substituting I_2 back and using

$$(\delta^{(k-n)}(P), \phi) = (-1)^{k-n}\int \frac{\partial^{k-n}}{\partial P^{k-n}}\left\{\frac{1}{2}\phi(r^2-P)^{\frac{1}{2}(q-2)}\right\}\Big|_{P=0} r^{p-1}\,\mathrm{d}r\,\mathrm{d}\Omega^{(p)}\,\mathrm{d}\Omega^{(q)},$$

we obtain

$$P^n \cdot \delta^{(k)}(P) = \begin{cases} n!\binom{k}{n}\delta^{(k-n)}(P) & \text{if } k \geqslant n, \\ 0 & \text{if } k < n, \end{cases}$$

which completes the proof of theorem. □

Remark. We assumed that ϕ disappears in a neighborhood of the origin, so that the integrals in the proof of Theorem 3 converge for any k. However, if $k < \frac{1}{2}(p+q-2)$, these integrals will converge for any ϕ. If, on the other hand, $k \geqslant \frac{1}{2}(p+q-2)$, we can apply an identical approach on $(\delta_1^{(k)}(P), \phi)$ and $(\delta_2^{(k)}(P), \phi)$ (see [15]) and the results still follow.

Obviously, we can extend Theorem 3 to a more general product in the following:

Theorem 4. *Let f be a C^∞ function on R. Then the product f(P) and $\delta^{(k)}(P)$ exists and*

$$f(P)\cdot\delta^{(k)}(P) = \sum_{i=0}^{k}\binom{k}{i} f^{(i)}(0)\delta^{(k-i)}(P).$$

Proof. It follows by replacing P^n by $f(P)$ in the proof of Theorem 3 and noting that

$$\begin{aligned}\frac{\partial^k}{\partial v^k}\left\{f(u-v)v^{\frac{q-2}{2}}\psi_1(u,v)\right\}\Big|_{u=v} &= \sum_{i=0}^{k}\binom{k}{i} D_v^i f(u-v) D_v^{k-i}\left\{v^{\frac{q-2}{2}}\psi_1(u,v)\right\}\Big|_{u=v} \\ &= \sum_{i=0}^{k}\binom{k}{i}(-1)^i f^{(i)}(0) D_v^{k-i}\left\{v^{\frac{q-2}{2}}\psi_1(u,v)\right\}\Big|_{u=v}.\end{aligned}$$

In particular, we have

$$\sin P\cdot\delta^{(k)}(P) = \sum_{i=0}^{k}\binom{k}{i}\sin\frac{\mathrm{i}\pi}{2}\delta^{(k-i)}(P),$$

$$e^P\cdot\delta^{(k)}(P) = \sum_{i=0}^{k}\binom{k}{i}\delta^{(k-i)}(P). \qquad \square$$

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