Applied Mathematics and Computation 246 (2014) 502–513

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

On defining the distributions δ^k and $(\delta')^k$ by fractional derivatives ☆

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ARTICLE INFO

Keywords:
Distribution
Delta function
Delta sequence
Neutrix limit
Caputo fractional derivative and generalized Taylor's formula

ABSTRACT

How to define products and powers of distributions is a difficult and not completely understood problem, and has been investigated from several points of views since Schwartz established the theory of distributions around 1950. Many fields, such as differential equations or quantum mechanics, require such operations. In this paper, we use Caputo fractional derivatives and the following generalized Taylor's formula for $0 < \alpha < 1$

$$\phi(t) = \sum_{i=0}^{m} \frac{\left({}^{C}\hat{D}_{0,t}^{i\alpha}\phi\right)(0)}{\Gamma(i\alpha+1)} t^{i\alpha} + \frac{\left({}^{C}\hat{D}_{0,t}^{(m+1)\alpha}\phi\right)(\zeta)}{\Gamma((m+1)\alpha+1)} t^{(m+1)\alpha}$$

to give meaning to the distributions $\delta^k(x)$ and $(\delta')^k(x)$ for all $k \in R$. These can be regarded as powers of Dirac delta functions and have applications to quantum theory. At the end of this paper, the distributions $\log \delta(t)$ and $\delta(t^2)$ are given by the δ-sequence and the neutrix limit.

1. Introduction

The singular function $\delta(x)$, which is widely used in physics and mathematics, was introduced by Dirac in 1920 as follows:

(i) $\delta(x) = 0$ for $x \neq 0$,
(ii) $\delta(x) = \infty$ for $x = 0$, and
(iii) $\int_{-\infty}^{\infty} \delta(x) f(x) dx = f(0)$.

It is clear to see that the above definition of $\delta(x)$ contradicts with the integral theory in terms of Lebesgue sense, and hence it can not be properly defined within the framework of classical function theory. In elementary particle physics [1], one finds the need to evaluate $\delta^2(x)$ when calculating the transition rates of certain particle interactions. Embacher et al. [2] studied products of distributions containing the δ functions in 1992 and found applications to quantum electrodynamics. In perturbative computations of quantum-mechanical path integrals in curvilinear coordinates, people encounter Feynman diagrams involving multiple temporal integrals over products of distributions, which are undefined. In addition, there are terms proportional to powers of the δ functions at the origin coming from the measure of path integration [3]. Furthermore, products of distributions, including powers of the δ functions, are in great demand for certain types of partial differential

☆ This work was partially supported by the National Natural Science Foundation of China (11372170), the Key Program of Shanghai Municipal Education Commission (12ZZ084), the grant of "The first-class Discipline of Universities in Shanghai", and BURC.
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http://dx.doi.org/10.1016/j.amc.2014.08.068
0096-3003/

equations [4] and path integrals in quantum mechanics [5], which require complex computations. A definition for product of distributions is given using delta sequences in [6]. However, $\delta^2(x)$ as a product of $\delta(x)$ with itself is shown not to exist in mathematical sense. In [7], Bremermann used the Cauchy representations of distributions with compact support to define $\sqrt{\delta_+(x)}$ and $\log \delta_+(x)$. Unfortunately, his definition does not carry over to $\sqrt{\delta(x)}$ and $\log \delta(x)$. Koh and Li [8] adopted the neutrix limit due to Van Der Corput [9] to define the distributions $\delta^k(x)$ and $(\delta')^k(x)$ for all $k \in Z^+$, and concluded that "it remains to show that these powers can be defined for all real k". In 2001, Özçağ [10] utilized the Temple delta sequence, which plays an important role in defining non-linear operations of distributions, and the neutrix limit to show that $\delta^{-k}(x) = 0$ for all $k \in Z^+$. The technique of neglecting appropriately defined infinite quantities and resulting finite values extracted from the divergent integral is usually referred to as the Hadamard finite part. In fact Fisher's [11] method in the computation of using the neutrix limit can be regarded as a particular application of the neutrix calculus. This is a general principle for the discarding of unwanted infinite quantities from asymptotic expansions and has been exploited in context of distribution by Fisher in connection with the problem of distributional powers, multiplication ([12,13]), convolution and composition. In 2008, Aguirre [14] applied the Hankel transform to study $\delta^2(x)$ as well as $\delta^{(m)}(x)\delta^{(l)}(x)$ under his definition of product of distributions.

On the other hand, fractional calculus first mentioned in the letter from Leibniz to L'Hôpital dated 30 September 1695, can be regarded as a branch of analysis which deals with integral–differential equations often with weakly singular kernels. A lot of contributions to the theory of fractional calculus up to the middle of the 20th century were made by many famous mathematicians including Laplace, Fourier, Abel, Liouville, Riemann, Grünwald, Letnikov, Heaviside, Weyl, Erdélyi and others. After 1970, there was a clear movement from theoretical research of fractional calculus to its applications in various fields. Up to now, fractional calculus has been found in almost every realm of science and engineering. As far as we know, it is one of the best tools to characterize long-memory processes and materials, anomalous diffusion, long-range interactions, long-term behaviors, power laws, allometric scaling laws, and so on. In this current work, we use fractional derivatives to study powers of Dirac delta function.

In the following sections, we start to introduce fractional derivatives, including Riemann–Liouville and Caputo definitions, several versions of generalized Taylor's formulas and provide a couple of interesting results in computing Caputo fractional derivatives efficiently in the generalized Taylor's formula under certain conditions. Then, we will choose an infinitely differentiable δ-sequence without compact support to define the distributions $\delta^k(x)$ in Section 3, $(\delta')^k(x)$ and other distributions related to Dirac δ function in Section 4. These results are fresh and novel in distribution theory and have potential applications in elementary particle physics and quantum mechanics.

2. Fractional derivatives and generalized Taylor's formulas

Fractional calculus is the theory of integrals and derivatives of arbitrary order, which unifies and generalizes integer-order differentiation and n-fold integration. The beginning of fractional calculus is considered to be the Leibniz's letter to L'Hôspital in 1695, where the notation for differentiation of non-integer orders was discussed.

We let Y_α be the convolution kernel of order $\alpha \in R^+$ for fractional integrals, given by

$$Y_\alpha = \frac{t_+^{\alpha-1}}{\Gamma(\alpha)} \in L^1_{loc}(R^+),$$

where Γ is the well-known Euler Gamma function, and

$$t_+^{\alpha-1} = \begin{cases} t^{\alpha-1} & \text{if } t > 0, \\ 0 & \text{if } t \leqslant 0. \end{cases}$$

Definition 2.1. The fractional integral (or, the Riemann–Liouville) $D_{0,t}^{-\alpha}$ of fractional order $\alpha \in R^+$ of function $\phi(t)$ is defined by

$$D_{0,t}^{-\alpha}\phi(t) = Y_\alpha * \phi(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\phi(\tau)d\tau,$$

where we set the initial time to zero.

As an example, we have the following for $\gamma > -1$

$$D_{0,t}^{-\alpha}t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}t^{\alpha+\gamma},$$
$$D_{0,t}^{-\alpha}e^t = t^\alpha\sum_{k=0}^{\infty}\frac{1}{\Gamma(\alpha+\gamma+1)}t^k$$

by a simple calculation.

The following properties of Y_α, $D_{0,t}^{-\alpha}$, and the fractional derivatives can be found in [15,16].

Property 2.1.

(i) *The convolution property* $Y_\alpha * Y_\beta = Y_{\alpha+\beta}$ *holds for* $\alpha > 0$ *and* $\beta > 0$, *which implies that* $D_{0,t}^{-\alpha} D_{0,t}^{\beta} = D_{0,t}^{-\alpha-\beta}$.
(ii) *Consistency property with the integer-order integral:* $\lim_{\alpha\to m} D_{0,t}^{-\alpha}\phi(t) = D_{0,t}^{-m}\phi(t)$, *where* $\alpha > 0,\ m \in Z^+$ *and*

$$D_{0,t}^{-m}\phi(t) = \int_0^t \int_0^{t_{m-1}} \cdots \int_0^{t_1} \phi(\tau)d\tau dt_1 \ldots dt_{m-1} = \frac{1}{(m-1)!}\int_0^t (t-\tau)^{m-1}\phi(\tau)d\tau.$$

Definition 2.2. The Riemann–Liouville derivative of fractional order α of function $\phi(t)$ is defined as

$${}_{RL}D_{0,t}^{\alpha}\phi(t) = \frac{d^m}{dt^m} D_{0,t}^{-(m-\alpha)}\phi(t) = \frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dt^m}\int_0^t (t-\tau)^{m-\alpha-1}\phi(\tau)d\tau,$$

where $m-1 < \alpha < m \in Z^+$.

It follows that

$${}_{RL}D_{0,t}^{\alpha} c = \frac{ct^{\alpha}}{\Gamma(1-\alpha)},$$
$${}_{RL}D_{0,t}^{\alpha} t^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)} t^{\lambda-\alpha},$$

where c is a constant and $\lambda > -1$.

Furthermore, we can derive that for $\phi(t) \in C[0,\infty)$

$$\begin{aligned} {}_{RL}D_{0,t}^{\alpha} D_{0,t}^{-\alpha}\phi(t) &= \frac{1}{\Gamma(m-\alpha)\Gamma(\alpha)}\frac{d^m}{dt^m}\int_0^t (t-\tau)^{m-\alpha-1}dt \int_0^{\tau} (\tau-s)^{\alpha-1}\phi(s)ds \\ &= \frac{B(m-\alpha,\alpha)}{\Gamma(m-\alpha)\Gamma(\alpha)}\frac{d^m}{dt^m}\int_0^t (t-s)^{m-1}\phi(s)ds = \phi(t), \end{aligned}$$

where $m-1 < \alpha < m \in Z^+$.

Clearly from integration by parts and integral mean value theorem, we come to

$$\begin{aligned} {}_{RL}D_{0,t}^{\alpha}\phi(t) &= \sum_{k=0}^{m-1}\frac{\phi^{(k)}(0)t^{-\alpha+k}}{\Gamma(-\alpha+k+1)} + \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1}\phi^{(m)}(\tau)d\tau \\ &= \sum_{k=0}^{m-1}\frac{\phi^{(k)}(0)t^{-\alpha+k}}{\Gamma(-\alpha+k+1)} + \frac{\phi^{(m)}(0)t^{m-\alpha}}{\Gamma(m-\alpha+1)} + \frac{1}{\Gamma(m-\alpha+1)}\int_0^t (t-\tau)^{m-\alpha}\phi^{(m+1)}(\tau)d\tau \\ &= \sum_{k=0}^{m-1}\frac{\phi^{(k)}(0)t^{-\alpha+k}}{\Gamma(-\alpha+k+1)} + \frac{\phi^{(m)}(0)t^{m-\alpha}}{\Gamma(m-\alpha+1)} + \frac{\phi^{(m+1)}(\zeta)t^{m-\alpha+1}}{\Gamma(m-\alpha+2)}, \end{aligned}$$

where $m-1 < \alpha < m \in Z^+,\ \phi(t) \in C^{\infty}[0,\infty)$ and $0 \leqslant \zeta \leqslant t$. This shows that

$$\lim_{\alpha\to(m-1)^+} {}_{RL}D_{0,t}^{\alpha}\phi(t) = \phi^{(m-1)}(0) + \int_0^t \phi^{(m)}(\tau)d\tau = \phi^{(m-1)}(t), \text{ and}$$
$$\lim_{\alpha\to m^-} {}_{RL}D_{0,t}^{\alpha}\phi(t) = \phi^{(m)}(0) + \int_0^t \phi^{(m+1)}(\tau)d\tau = \phi^{(m)}(t),$$

in which $\Gamma(1) = 1,\ \Gamma(0) = \infty$ and $\Gamma(-k) = \infty$ for all $k \in Z^+$ are used.

Therefore, we deduce that

$${}_{RL}D_{0,t}^{m} D_{0,t}^{-m}\phi(t) = \phi(t),$$
$$D_{0,t}^{-m}\, {}_{RL}D_{0,t}^{m}\phi(t) = \phi(t) - \sum_{k=0}^{m-1}\frac{\phi^{(k)}(0)}{k!}t^k,$$

where $m \in Z^+$.

Note that from [15]

$$D_{0,t}^{-\alpha}\, {}_{RL}D_{0,t}^{\alpha}\phi(t) = \phi(t) - \sum_{k=1}^{m} {}_{RL}D_{0,t}^{\alpha-k}\phi(0)\frac{t^{\alpha-k}}{\Gamma(\alpha-k+1)},$$

where $m-1 < \alpha < m$.

From the above, the Riemann–Liouville derivative ${}_{RL}D_{0,t}^{\alpha}$ is a reasonable extension between d^{m-1}/dt^{m-1} and d^m/dt^m. However, it has certain disadvantages when tying to model real-world phenomena with fractional differential equations. Hence, we shall introduce a modified fractional differential operator ${}^{C}D_{0,t}^{\alpha}$ proposed by Caputo in 1967.

Definition 2.3. The Caputo derivative of fractional order α of function $\phi(t)$ is defined as

$$ {}^{C}D_{0,t}^{\alpha}\phi(t) = D_{0,t}^{-(m-\alpha)}\frac{d^m}{dt^m}\phi(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1}\phi^{(m)}(\tau)d\tau, $$

where $m-1<\alpha<m\in Z^+$.

It follows that

$$ {}^{C}D_{0,t}^{\alpha}t^p = \begin{cases} \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)}t^{p-\alpha}, & m-1<\alpha<m\in Z^+, \quad p>m-1, \quad p\in R, \\ 0, & m-1<\alpha<m\in Z^+, \quad p\leqslant m-1, \quad p\in Z^+. \end{cases} $$

Obviously, we get from the above definition

$$ \lim_{\alpha\to(m-1)^+} {}^{C}D_{0,t}^{\alpha}\phi(t) = \lim_{\alpha\to(m-1)^+}\left(\frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1}\phi^{(m)}(\tau)d\tau\right) = \int_0^t \phi^{(m)}(\tau)d\tau = \phi^{(m-1)}(t)-\phi^{(m-1)}(0), \text{ and} $$

$$ \lim_{\alpha\to m^-} {}^{C}D_{0,t}^{\alpha}\phi(t) = \lim_{\alpha\to m^-}\left(\frac{\phi^{(m)}(0)t^{m-\alpha}}{\Gamma(m-\alpha+1)}+\frac{1}{\Gamma(m-\alpha+1)}\int_0^t (t-\tau)^{m-\alpha}\phi^{(m+1)}(\tau)d\tau\right) = \phi^{(m)}(0)+\int_0^t \phi^{(m+1)}(\tau)d\tau = \phi^{(m)}(t) $$

if $\phi(t)\in C^{m+1}[0,\infty)$.

On the other hand, integration by parts and differentiation show that

$$ {}^{C}D_{0,t}^{\alpha}\phi(t) = {}_{RL}D_{0,t}^{\alpha}\left(\phi(t)-\sum_{k=0}^{m-1}\frac{t^k}{k!}\phi^{(k)}(0)\right) $$

if $\phi(t)\in C^m[0,\infty)$ and $m-1<\alpha<m\in Z^+$.

The ordinary Taylor's formula has been generalized by many authors. Riemann [17] had already written a formal version of the generalized Taylor's series for a real number r:

$$ \phi(t+h) = \sum_{m=-\infty}^{\infty}\frac{h^{m+r}}{\Gamma(m+r+1)}{}_{RL}D_{0,t}^{m+r}\phi(t), $$

where for $\alpha<0$, ${}_{RL}D_{0,t}^{\alpha}\phi(t) = D_{0,t}^{-\alpha}\phi(t)$ is the Riemann–Liouville fractional integral of order $-\alpha$ in Definition 2.1. Moreover, ${}_{RL}D_{0,t}^{0}\phi(t) = D_{0,t}^{0}\phi(t) = \phi(t)$.

The proof of validity of the Riemann expansion above for certain classes of functions was undertaken by Hardy [18], both for finite and infinite initial time (we set it to zero in this paper, as mentioned in Definition 2.1).

On the other hand, a variant of the generalized Taylor's series was given by Dzherbashyan and Nersesian [19,20]. For ϕ having all of the required continuous derivatives, they derived that

$$ \phi(t) = \sum_{k=0}^{m-1}\frac{\mathcal{D}^{(\alpha_k)}\phi(0)}{\Gamma(1+\alpha_k)}x^{\alpha_k}+\frac{1}{\Gamma(1+\alpha_k)}\int_0^t (t-x)^{\alpha_m-1}\mathcal{D}^{(\alpha_k)}\phi(x)dx, $$

where $t>0$, $\alpha_0,\alpha_1,\ldots,\alpha_m$ is an increasing sequence of real numbers such that $0<\alpha_k-\alpha_{k-1}\leqslant 1$, $k=1,2,\ldots,m$ and $\mathcal{D}^{(\alpha_k)}\phi = D_{0,t}^{\alpha_k-\alpha_{k-1}-1}{}_{RL}D_{0,t}^{1+\alpha_{k-1}}\phi$.

Trujillo et al. [21] established the following generalized Taylor's formula under certain conditions for ϕ and $\alpha\in[0,1]$:

$$ \phi(t) = \sum_{j=0}^{n}\frac{c_j}{\Gamma((j+1)\alpha)}t^{(j+1)\alpha-1}+R_n(t), \tag{1} $$

where

$$ R_n(t) = \frac{{}_{RL}\hat{D}_{0,t}^{(n+1)\alpha}\phi(\zeta)}{\Gamma((n+1)\alpha+1)}t^{(n+1)\alpha}, \quad 0\leqslant\zeta\leqslant t $$

and

$$ c_j = \Gamma(\alpha)[x^{1-\alpha}{}_{RL}\hat{D}_{0,t}^{j\alpha}]\phi(0^+), \quad \forall j=0,1,\ldots,n $$

and the sequential fractional derivative is denoted by

$$ {}_{RL}\hat{D}_{0,t}^{j\alpha} = {}_{RL}D_{0,t}^{\alpha}\cdots{}_{RL}D_{0,t}^{\alpha}, \quad j-\text{times and } j\in Z^+. $$

We would also like to mention that there is another version of fractional Taylor's series in the Riemann–Liouville form in [22], which is a particular case of Eq. (1).

The following theorem due to Odibat and Shawagfeh in 2007 can be found in [23].

Theorem 2.1 (*Generalized Taylor's Theorem*). *Suppose that* ${}^C\hat{D}_{0,t}^{k\alpha}\phi(t) \in C(a,b]$ *for* $k = 0, 1, 2, \ldots m+1$, *where* $0 < \alpha < 1$, *then we have*

$$\phi(t) = \sum_{i=0}^{m} \frac{(t-a)^{i\alpha}}{\Gamma(i\alpha+1)} ({}^C\hat{D}_{a,t}^{i\alpha}\phi)(a) + \frac{\left({}^C\hat{D}_{a,t}^{(m+1)\alpha}\phi\right)(\zeta)}{\Gamma((m+1)\alpha+1)}(t-a)^{(m+1)\alpha}$$

with $a \leqslant \zeta \leqslant t,\ \forall t \in (a,b]$, *where* ${}^C\hat{D}_{0,t}^{i\alpha} = {}^C D_{0,t}^{\alpha}\,{}^C D_{0,t}^{\alpha} \ldots {}^C D_{0,t}^{\alpha}$.

In particular, we have for $a = 0$,

$$\phi(t) = \sum_{i=0}^{m} \frac{t^{i\alpha}}{\Gamma(i\alpha+1)} \left({}^C\hat{D}_{0,t}^{i\alpha}\phi\right)(0) + \frac{\left({}^C\hat{D}_{0,t}^{(m+1)\alpha}\phi\right)(\zeta)}{\Gamma((m+1)\alpha+1)} t^{(m+1)\alpha}. \qquad (2)$$

Note that for $\alpha = 1$,

$$\phi(t) = \sum_{i=0}^{m} \frac{(t-a)^i}{i!}\phi^{(i)}(a) + \frac{\phi^{(m+1)}(\zeta)}{(m+1)!}(t-a)^{m+1},$$

which is the classical Taylor's formula.

Remark. One understands, in fractional calculus, that ${}^C\hat{D}_{0,t}^{i\alpha} = {}^C D_{0,t}^{\alpha}\,{}^C D_{0,t}^{\alpha} \ldots {}^C D_{0,t}^{\alpha} \neq {}^C D_{0,t}^{i\alpha}$ in general. Here is a simple example to illustrate this. Clearly,

$$\begin{aligned}
{}^C D_{0,t}^{0.6} t &= \frac{1}{\Gamma(0.4)} \int_0^t (t-\tau)^{1-0.6-1} d\tau = \frac{1}{\Gamma(1.4)} t^{0.4}, \text{ and} \\
{}^C D_{0,t}^{0.6}\,{}_C D_{0,t}^{0.6} t &= \frac{0.4}{\Gamma(1.4)\Gamma(0.4)} \int_0^t (t-\tau)^{-0.6}\tau^{-0.6} d\tau = \frac{0.4B(0.4,0.4)}{\Gamma(1.4)\Gamma(0.4)} t^{-0.2} = \frac{1}{\Gamma(0.8)} t^{-0.2}.
\end{aligned}$$

But,

$${}^C D_{0,t}^{1.2} t = \frac{1}{\Gamma(0.8)} \int_0^t (t-\tau)^{2-1.2-1} (\tau)'' d\tau = 0.$$

Generally speaking, it is easier to compute ${}^C D_{0,t}^{i\alpha}\phi(t)$ than ${}^C\hat{D}_{0,t}^{i\alpha}\phi(t)$. The following two theorems shown by Li and Deng in [24] describe, under certain circumstances, that ${}^C D_{0,t}^{i\alpha}\phi(t) = {}^C\hat{D}_{0,t}^{i\alpha}\phi(t)$.

Theorem 2.2. *If* $\phi(t) \in C^1[0,T]$ *for* $T > 0$, *then*

$${}^C D_{0,t}^{\alpha_2}\,{}^C D_{0,t}^{\alpha_1}\phi(t) = {}^C D_{0,t}^{\alpha_1}\,{}^C D_{0,t}^{\alpha_2}\phi(t) = {}^C D_{0,t}^{\alpha_1+\alpha_2}\phi(t), \quad t \in [0,T],$$

where $\alpha_1,\ \alpha_2 \in R^+$ *and* $\alpha_1 + \alpha_2 \leqslant 1$.

In particular, we obtain

$$\begin{aligned}
{}^C\hat{D}_{0,t}^{2\cdot 0.5} &= {}^C D_{0,t}^{0.5}\,{}^C D_{0,t}^{0.5}\phi(t) = \phi'(t), \quad \text{and} \\
{}^C\hat{D}_{0,t}^{2\cdot 0.3} &= {}^C D_{0,t}^{0.3}\,{}^C D_{0,t}^{0.3}\phi(t) = {}^C D_{0,t}^{0.6}\phi(t).
\end{aligned}$$

Theorem 2.3. *If* $\phi(t) \in C^m[0,T]$ *for* $T > 0$, *then*

$${}^C D_{0,t}^{\alpha}\phi(t) = {}^C D_{0,t}^{\alpha_n} \ldots {}^C D_{0,t}^{\alpha_2}\,{}^C D_{0,t}^{\alpha_1}\phi(t), \quad t \in [0,T]$$

where $\alpha = \sum_{i=1}^{n}\alpha_i,\ \alpha_i \in (0,1],\ m-1 \leqslant \alpha < m \in Z^+$ *and there exists* $i_k < n$ *such that* $\sum_{j=1}^{i_k}\alpha_j = k$ *for* $k = 1, 2, \ldots, m-1$.

Using this theorem, we get as an example,

$$\begin{aligned}
{}^C D_{0,t}^{101\cdot 0.5}\phi(t) &= {}^C D_{0,t}^{0.5}\,{}^C D_{0,t}^{0.5} \ldots {}^C D_{0,t}^{0.5}\phi(t) = {}^C\hat{D}_{0,t}^{101\cdot 0.5}\phi(t), \text{ if } \phi(t) \in C^{51}[0,T], \\
{}^C D_{0,t}^{100\cdot\frac{1}{3}}\phi(t) &= {}^C D_{0,t}^{\frac{1}{3}}\,{}^C D_{0,t}^{\frac{1}{3}} \ldots {}^C D_{0,t}^{\frac{1}{3}}\phi(t) = {}^C\hat{D}_{0,t}^{100\cdot\frac{1}{3}}\phi(t), \text{ if } \phi(t) \in C^{34}[0,T].
\end{aligned}$$

In this paper, we will adopt the generalized Taylor's formula of Eq. (2) to define powers of the distributions $\delta^k(t)$ and $(\delta')^k(t)$ for all $k \in R$ due to simplicity of the coefficients in the equation, which can be further simplified by Theorems 2.2 and 2.3 under certain conditions. Without a doubt, using other Taylor's formulas given above to define powers of the distributions will require more complicated computations for the coefficients, although it is doable.

3. The distribution $\delta^k(t)$ for all $k \in R$

Let $\mathcal{D}(R)$ be the space of infinitely differentiable function with compact support in R, and let $\mathcal{D}'(R)$ be the space of distributions defined on $\mathcal{D}(R)$. Further, we shall define a sequence $\phi_1(t), \phi_2(t), \ldots, \phi_n(t), \ldots$ converges to zero in $\mathcal{D}(R)$ if all these functions vanish outside a certain fixed bounded interval, and converge uniformly to zero (in the usual sense) together with their derivatives of any order. The function $\delta(t)$ is defined as

$$(\delta(t), \phi(t)) = \phi(0)$$

where $\phi(t) \in \mathcal{D}(R)$. Clearly, $\delta(t)$ is a linear and continuous functional on $\mathcal{D}(R)$, and hence $\delta(t) \in \mathcal{D}'(R)$.

There are two main approaches (see [22,25]) to define fractional derivatives and integrals of generalized functions (distributions). The first goes back to the Schwartz method and is based on the definition of a fractional integral as a convolution

$$Y_\alpha * f = \frac{t_+^{\alpha-1}}{\Gamma(\lambda)} * f$$

of the function $Y_\alpha = \frac{t_+^{\alpha-1}}{\Gamma(\lambda)}$ with generalized function f. This definition is suited to the case of half-line $t \geqslant 0$ since the convolution is well defined in the Schwartz sense. The second way, also commonly used, is based on using the adjoint operator (similar to Zemanian's [26] techniques to define the generalized integral transforms, such as Hankle transform). Namely,

$$(D_{0,t}^{-\alpha} f, \phi) = (f, D_{t,\infty}^{-\alpha}\phi)$$

where $D_{t,\infty}^{-\alpha}$ is the adjoint operator of $D_{0,t}^{-\alpha}$, given by

$$D_{t,\infty}^{-\alpha}\psi(x) = \frac{1}{\Gamma(\alpha)}\int_t^\infty (\tau - t)^{\alpha-1}\psi(\tau)d\tau.$$

The Lizorkin space is of particular interest and it consists of rapidly decreasing infinitely smooth functions in the space S ([22,27]), which are orthogonal to all polynomials. This space is introduced as the Fourier pre-image of a subspace of S and invariant with respect to fractional integration and differentiation operators. This is not the case for the whole space S of the rapidly decreasing test functions because the fractional integrals and derivatives of the functions from the space S dot not always belong to the space S. Hence, the Lizorkin space is a very convenient one while dealing both with the Fourier transform and with the fractional integration and differentiation operators.

The definition of the product of a distribution and an infinitely differentiable function is the following (see for example [27]).

Definition 3.1. Let f be a distribution and let g be an infinitely differentiable function. Then the product fg is defined by

$$(fg, \phi) = (f, g\phi)$$

for all testing functions $\phi \in \mathcal{D}(R)$.

It follows from Definition 3.1 that

$$t^k\delta^{(m)}(t) = \begin{cases} (-1)^k k! \binom{m}{k} \delta^{(m-k)}(t), & \text{if} \quad k \leqslant m, \\ 0, & \text{otherwise} \end{cases}$$

for $k,\ m = 0, 1, 2, \ldots$.

However, it seems impossible to define $\delta^2(t)$ since

$$(\delta^2(t), \phi(t)) = (\delta(t), \delta(t)\phi(t)) = \delta(0)\phi(0)$$

is undefined. Furthermore, $\delta(t)\phi(t) = \phi(0)\delta(t)$ is not a member of $\mathcal{D}(R)$. Indeed, $\delta(t)\phi(t) \in \mathcal{D}'(R)$.

As outlined in the introduction, powers of the δ functions (such as $\delta(t)$ and $\delta'(t)$) have been in demand in physics for computing the transition rates of certain particle interactions, although they cannot be properly defined. In this section, we shall utilize Caputo fractional derivatives and the generalized Taylor's formula in Eq. (2) to give meaning to the distribution $\delta^k(t)$ for all k, which has never appeared in research of distribution theory so far.

Choosing the following δ-sequence without compact support

$$\delta_n(t) = \left(\frac{n}{\pi}\right)^{1/2} e^{-nt^2}, \quad t \in R.$$

Obviously,

$$(\delta(t), \phi(t)) = \lim_{n\to\infty}(\delta_n(t), \phi(t)) = \phi(0). \tag{3}$$

We define for all $k \in R$

$$(\delta^k(t), \phi(t)) := N - \lim_{n\to\infty}(\delta_n^k(t), \phi(t)) = N - \lim_{n\to\infty}\int_{-\infty}^{\infty}\left(\frac{n}{\pi}\right)^{k/2} e^{-knt^2}\phi(t)dt \tag{4}$$

where N is the neutrix having domain $N' = \{1, 2, 3, \ldots\}$ and range the real numbers, with negligible functions that are finite linear sums of functions

$$n^{\lambda}\ln^{r-1}n, \quad \ln^{r}n \quad (\lambda > 0, r = 1, 2, \ldots)$$

and all functions of n that converge to zero in the normal sense as n tends to infinity (see [9,28]).

Clearly, we have from Eq. (4)

$$(\delta^0(t), \phi(t)) = N - \lim_{n\to\infty}(\delta_n^0(t), \phi(t)) = \int_{-\infty}^{\infty}\phi(t)dt = (1, \phi(t)) \text{ for } \phi(t) \in \mathcal{D}(R),$$

which implies that $\delta^0(t) = 1$.

For $k < 0$, we make the substitution $t = \sqrt{-\frac{1}{kn}}y$ in Eq. (4) and come to

$$(\delta^k(t), \phi(t)) = \lim_{n\to\infty}\left(\frac{n}{\pi}\right)^{k/2}\sqrt{-\frac{1}{kn}}\int_{\sqrt{-1/kna}}^{\sqrt{-1/knb}} e^{y^2}\phi\left(\sqrt{-\frac{1}{kn}}y\right)dy = 0,$$

where supp $\phi \in [a, b]$. Thus, $\delta^k(t) = 0$ for $k < 0$.

Setting $t = \sqrt{\frac{1}{kn}}y$ and $M = \sup_{t\in R}|\phi(t)|$, we arrive at

$$\left|(\delta_n^k(t), \phi(t))\right| \leqslant M\left(\frac{n}{\pi}\right)^{k/2}\sqrt{\frac{1}{kn}}\int_{-\infty}^{\infty} e^{-y^2}dy \to 0 \text{ as } n\to\infty$$

for $0 < k < 1$. Therefore $\delta^k(t) = 0$.

Furthermore, it follows from Eq. (3) that $\delta^1(t) = \delta(t)$. As for $k > 1$, we obtain from Eq. (4)

$$(\delta^k(t), \phi(t)) = N - \lim_{n\to\infty}\left(\int_0^{\infty}\left(\frac{n}{\pi}\right)^{k/2}e^{-knt^2}\phi(t)dt + \int_0^{\infty}\left(\frac{n}{\pi}\right)^{k/2}e^{-knt^2}\phi(-t)dt\right) := N - \lim_{n\to\infty}(I_1 + I_2).$$

By the generalized Taylor's formula from Eq. (2)

$$\begin{aligned}\phi(t) &= \sum_{i=0}^{m}\frac{t^{i\alpha}}{\Gamma(i\alpha+1)}\left({}^C\hat{D}_{0,t}^{i\alpha}\phi\right)(0) + \frac{\left({}^C\hat{D}_{0,t}^{(m+1)\alpha}\phi\right)(\zeta)}{\Gamma((m+1)\alpha+1)}t^{(m+1)\alpha}\\ &= \sum_{i=0}^{m-1}\frac{t^{i\alpha}}{\Gamma(i\alpha+1)}\left({}^C\hat{D}_{0,t}^{i\alpha}\phi\right)(0) + \frac{t^{m\alpha}}{\Gamma(m\alpha+1)}\left({}^C\hat{D}_{0,t}^{m\alpha}\phi\right)(0) + \frac{\left({}^C\hat{D}_{0,t}^{(m+1)\alpha}\phi\right)(\zeta)}{\Gamma((m+1)\alpha+1)}t^{(m+1)\alpha}\end{aligned}$$

where $m\alpha = k - 1$, $m \in Z^+$ and $0 < \alpha \leqslant 1$ (note that it reduces to the classical Taylor's formula when $\alpha = 1$).

Thus,

$$\begin{aligned}I_1 &= \sum_{i=0}^{m-1}\frac{1}{\Gamma(i\alpha+1)}\left({}^C\hat{D}_{0,t}^{i\alpha}\phi\right)(0)\left(\frac{n}{\pi}\right)^{k/2}\int_0^{\infty}e^{-knt^2}t^{i\alpha}dt + \frac{1}{\Gamma(m\alpha+1)}\left({}^C\hat{D}_{0,t}^{m\alpha}\phi\right)(0)\left(\frac{n}{\pi}\right)^{k/2}\int_0^{\infty}e^{-knt^2}t^{m\alpha}dt\\ &\quad + \frac{1}{\Gamma((m+1)\alpha+1)}\left(\frac{n}{\pi}\right)^{k/2}\int_0^{\infty}e^{-knt^2}t^{(m+1)\alpha}\left({}^C\hat{D}_{0,t}^{(m+1)\alpha}\phi\right)(\zeta)dt\\ &= I_{11} + I_{12} + I_{13}.\end{aligned}$$

Setting $t = \sqrt{\frac{1}{kn}}y$ again, we get

$$I_{11} = \sum_{i=0}^{m-1}\frac{1}{\Gamma(i\alpha+1)}\left({}^C\hat{D}_{0,t}^{i\alpha}\phi\right)(0)\left(\frac{n}{\pi}\right)^{k/2}\left(\frac{1}{kn}\right)^{\frac{i\alpha+1}{2}}\int_0^{\infty}e^{-y^2}y^{i\alpha}dy.$$

Hence

$$N - \lim_{n\to\infty}I_{11} = 0.$$

Since $\phi(t) \in \mathcal{D}(R)$, there exists a positive real number M_1 such that

$$\sup_{t\in R^+}\left|\left({}^C\hat{D}_{0,t}^{(m+1)\alpha}\phi\right)(t)\right| \leqslant M_1, \text{ for } m \in Z^+ \text{ and } 0 < \alpha \leqslant 1$$

which infers that

$$\lim_{n\to\infty}I_{13} = 0.$$

Coming to I_{12}, we use the following formula

$$\int_0^{\infty}e^{-y^2}y^{m\alpha}dy = \int_0^{\infty}e^{-y^2}y^{k-1}dy = \frac{\Gamma\left(\frac{k}{2}\right)}{2}$$

to imply that

$$I_{12} = \frac{1}{\Gamma(m\alpha+1)} \left({}^C\hat{D}^{m\alpha}_{0,t}\phi\right)(0)\left(\frac{n}{\pi}\right)^{k/2} \int_0^\infty e^{-knt^2} t^{m\alpha} dt = \frac{\Gamma(\frac{k}{2})}{2\Gamma(k)} \left(\frac{1}{k\pi}\right)^{k/2} \left({}^C\hat{D}^{m\alpha}_{0,t}\phi\right)(0) = \frac{\Gamma(\frac{k}{2})}{2\Gamma(k)} \left(\frac{1}{k\pi}\right)^{k/2} \left({}^C\hat{D}^{k-1}_{0,t}\phi\right)(0).$$

Therefore,

$$N - \lim_{n\to\infty} I_1 = \frac{\Gamma(\frac{k}{2})}{2\Gamma(k)} \left(\frac{1}{k\pi}\right)^{k/2} \left({}^C\hat{D}^{m\alpha}_{0,t}\phi\right)(0) = \frac{\Gamma(\frac{k}{2})}{2\Gamma(k)} \left(\frac{1}{k\pi}\right)^{k/2} \left({}^C\hat{D}^{k-1}_{0,t}\phi\right)(0).$$

Following the similar calculation, we derive that

$$N - \lim_{n\to\infty} I_2 = \frac{(-1)^{m\alpha}\Gamma(\frac{k}{2})}{2\Gamma(k)} \left(\frac{1}{k\pi}\right)^{k/2} \left({}^C\hat{D}^{m\alpha}_{0,t}\phi\right)(0) = \frac{(-1)^{k-1}\Gamma(\frac{k}{2})}{2\Gamma(k)} \left(\frac{1}{k\pi}\right)^{k/2} \left({}^C\hat{D}^{k-1}_{0,t}\phi\right)(0).$$

Finally,

$$(\delta^k(t), \phi(t)) := N - \lim_{n\to\infty}(I_1 + I_2) = \frac{((-1)^{m\alpha}+1)\Gamma(\frac{k}{2})}{2\Gamma(k)} \left(\frac{1}{k\pi}\right)^{k/2} \left({}^C\hat{D}^{m\alpha}_{0,t}\phi\right)(0) = \frac{((-1)^{k-1}+1)\Gamma(\frac{k}{2})}{2\Gamma(k)} \left(\frac{1}{k\pi}\right)^{k/2} \left({}^C\hat{D}^{k-1}_{0,t}\phi\right)(0).$$

In particular for $k = 1$,

$$(\delta(t), \phi(t)) = \frac{((-1)^{1-1}+1)\Gamma(\frac{1}{2})}{2\Gamma(1)} \left(\frac{1}{\pi}\right)^{1/2} \left({}^C\hat{D}^{1-1}_{0,t}\phi\right)(0) = \phi(0). \tag{5}$$

It follows that

$$\delta^{2l}(t) = 0 \text{ for } l = 1, 2, 3, \ldots$$
$$\delta^{2l+1}(t) = \frac{\Gamma(\frac{2l+1}{2})}{\Gamma(2l+1)} \left(\frac{1}{(2l+1)\pi}\right)^{(2l+1)/2} \phi^{(2l)}(0)$$

for $l = 0, 1, 2, \ldots$. We have included $l = 0$ in the latter due to Eq. (5).

Using

$$\Gamma(l + 1/2) = \frac{1\cdot 3\cdot 5\ldots(2l-1)}{2^l}\sqrt{\pi} \text{ for } l = 0, 1, 2, \ldots,$$

We have

$$\delta^{2l+1}(t) = C_l \delta^{(2l)}(t),$$

where

$$C_l = \frac{1}{2^{2l} l!(2l+1)^{(2l+1)/2}\pi^l} \text{ for } l = 0, 1, 2, \ldots.$$

Now we can summarize to get

Theorem 3.1.

$$\delta^0(t) = 1,$$
$$\delta^k(t) = 0 \text{ for } k < 1 \text{ and } k \neq 0,$$
$$(\delta^k(t), \phi(t)) = \frac{((-1)^{k-1}+1)\Gamma(\frac{k}{2})}{2\Gamma(k)} \left(\frac{1}{k\pi}\right)^{k/2} \left({}^C\hat{D}^{k-1}_{0,t}\phi\right)(0) \text{ for } k \geqslant 1$$

where $\left({}^C\hat{D}^{k-1}_{0,t}\phi\right)(0) = \left({}^C\hat{D}^{m\alpha}_{0,t}\phi\right)(0) = ({}^C D^{\alpha}_{0,t}\, {}^C D^{\alpha}_{0,t} \ldots {}^C D^{\alpha}_{0,t}\phi)(0)$*(m-times)* *and* $(-1)^k = \cos k\pi + i\sin k\pi$ for $k \in [0, \infty)$. *In particular,*

$$\delta^{2l}(t) = 0 \text{ for } l = 1, 2, 3, \ldots, \text{ and}$$
$$\delta^{2l+1}(t) = \frac{1}{2^{2l} l!(2l+1)^{(2l+1)/2}\pi^l}\delta^{(2l)}(t) \text{ for } l = 0, 1, 2, \ldots.$$

Remark 1.

(i) We would like to point out that Theorem 3.1 is a generalization of Theorem 1 obtained in [8], where the case for $k \in Z^+$ is mainly discussed.

(ii) The choice of $\alpha \in (0, 1]$ is not unique. For example, we can pick up $m = 1$, $\alpha = 0.5$ or $m = 2$, $\alpha = 0.25$ (and others) if $k = 1.5$. Generally speaking, we choose α and m in Theorem 3.1 to make $\left({}^C\hat{D}^{k-1}_{0,t}\phi\right)(0)$ as simple as possible. Hence $\left({}^C\hat{D}^{0.5}_{0,t}\phi\right)(0) = \left({}^C D^{0.5}_{0,t}\phi\right)(0)$, which is Caputo derivative of order 1/2.

It follows from Theorem 3.1 that

$$\sqrt{\delta(t)} = 0,$$
$$\delta^2(t) = 0,$$
$$\delta^3(t) = \frac{1}{12\sqrt{3}\pi}\delta''(t),$$
$$(\delta^{1.5}(t), \phi(t)) = \frac{(i+1)\Gamma(0.75)}{2\Gamma(1.5)}\left(\frac{1}{1.5\pi}\right)^{0.75}\left({}^{C}D_{0,t}^{0.5}\phi\right)(0), \quad i = \sqrt{-1}$$

As indicated in the introduction, Bremermann failed to define $\sqrt{\delta(t)}$ by the Cauchy representations [7]. We use Theorems 2.3 and 3.1 to derive

$$(\delta^{\frac{103}{3}}(t), \phi(t)) = \frac{((-1)^{\frac{100}{3}}+1)\Gamma(\frac{103}{6})}{2\Gamma(\frac{103}{3})}\left(\frac{3}{103\pi}\right)^{103/6}\left({}^{C}\hat{D}_{0,t}^{\frac{100}{3}}\phi\right)(0) = \frac{(1-i\sqrt{3})\Gamma(\frac{103}{6})}{4\Gamma(\frac{103}{3})}\left(\frac{3}{103\pi}\right)^{103/6}\left({}^{C}D_{0,t}^{\frac{100}{3}}\phi\right)(0).$$

4. The distribution $(\delta')^k(t)$ for all $k \in R$

Considering the derivative of the δ-sequence, we have

$$\delta_n'(t) = \left(\frac{n}{\pi}\right)^{1/2} e^{-nt^2}(-2nt).$$

We define for all $k \in R$

$$((\delta')^k(t), \phi(t)) := N - \lim_{n\to\infty}((\delta')_n^k(t), \phi(t)) = N - \lim_{n\to\infty} 2^k n^k \int_{-\infty}^{\infty}\left(\frac{n}{\pi}\right)^{k/2} e^{-knt^2}(-t)^k\phi(t)dt. \quad (6)$$

Clearly, we have for $k = 0$ that

$$((\delta')^0(t), \phi(t)) = \int_{-\infty}^{\infty}\phi(t)dt = (1, \phi(t))$$

which claims that $(\delta')^0(t) = 1$.

Setting $t = \sqrt{-\frac{1}{kn}}y$ in Eq. (6) for $k < 0$, we can prove that

$$(\delta')^k(t) = 0.$$

Making the substitution $t = \sqrt{\frac{1}{kn}}y$ and $M = \sup_{t\in R}|\phi(t)|$, we get

$$\left|((\delta')^k(t), \phi(t))\right| \leqslant 2^k M\left(\frac{n}{\pi}\right)^{k/2} n^k\left(\frac{1}{kn}\right)^{\frac{k+1}{2}}\int_{-\infty}^{\infty} e^{-y^2}|y|^k dy \to 0 \text{ as } n \to \infty$$

for $0 < k < 1/2$. This implies that $(\delta')^k(t) = 0$ for $k \in (-\infty, 1/2)$ and $k \neq 0$.

For $k \geqslant 1/2$, we come to

$$\begin{aligned}((\delta')^k(t), \phi(t)) &= N - \lim_{n\to\infty}((\delta')_n^k(t), \phi(t)) = N - \lim_{n\to\infty} 2^k n^k\left(\frac{n}{\pi}\right)^{k/2}\left(\int_0^{\infty} e^{-knt^2}(-t)^k\phi(t)dt + \int_0^{\infty} e^{-knt^2}t^k\phi(-t)dt\right) : \\ &= N - \lim_{n\to\infty}(I_1 + I_2).\end{aligned}$$

By the generalized Taylor's formula in Eq. (2)

$$\begin{aligned}\phi(t) &= \sum_{i=0}^{m}\frac{t^{i\alpha}}{\Gamma(i\alpha+1)}\left({}^{C}\hat{D}_{0,t}^{i\alpha}\phi\right)(0) + \frac{\left({}^{C}\hat{D}_{0,t}^{(m+1)\alpha}\phi\right)(\zeta)}{\Gamma((m+1)\alpha+1)}t^{(m+1)\alpha} \\ &= \sum_{i=0}^{m-1}\frac{t^{i\alpha}}{\Gamma(i\alpha+1)}\left({}^{C}\hat{D}_{0,t}^{i\alpha}\phi\right)(0) + \frac{t^{m\alpha}}{\Gamma(m\alpha+1)}\left({}^{C}\hat{D}_{0,t}^{m\alpha}\phi\right)(0) + \frac{\left({}^{C}\hat{D}_{0,t}^{(m+1)\alpha}\phi\right)(\zeta)}{\Gamma((m+1)\alpha+1)}t^{(m+1)\alpha}\end{aligned}$$

where $m\alpha = 2k-1,\ m \in Z^+$ and $0 < \alpha < 1$.

Following the similar calculations in Section 3 and using the formula

$$\int_0^{\infty} e^{-y^2}y^{3k-1}dy = \frac{1}{2}\Gamma\left(\frac{3k}{2}\right),$$

we obtain

$$((\delta')^k(t), \phi(t)) = \frac{((-1)^k + (-1)^{2k-1})\Gamma(\frac{3k}{2})}{2^{1-k}(k\pi)^{\frac{k}{2}}k^k\Gamma(2k)}\left({}^{C}\hat{D}_{0,t}^{2k-1}\phi\right)(0) \quad (7)$$

where $\left({}^{C}\hat{D}_{0,t}^{2k-1}\phi\right)(0) = \left({}^{C}\hat{D}_{0,t}^{m\alpha}\phi\right)(0) = ({}^{C}D_{0,t}^{\alpha}\,{}^{C}D_{0,t}^{\alpha}\ldots{}^{C}D_{0,t}^{\alpha}\phi)(0)(m\text{-times})$.

In particular for $k = 1$, we get

$$((\delta')^{1}(t), \phi(t)) = \frac{-2\Gamma(3/2)}{\sqrt{\pi}}\phi'(0) = -\phi'(0) = (\delta'(t), \phi(t)).$$

This shows that

$$(\delta')^{1}(t) = \delta'(t).$$

Similarly, we deduce that for $k = 1/2$

$$((\delta')^{1/2}(t), \phi(t)) = \sqrt{2}e^{i\frac{\pi}{4}}\left(\frac{2}{\pi}\right)^{1/4}\Gamma(3/4)\phi(0)$$

which infers that

$$(\delta')^{1/2}(t) = \sqrt{2}e^{i\frac{\pi}{4}}\left(\frac{2}{\pi}\right)^{1/4}\Gamma(3/4)\delta(t).$$

Indeed, we can directly derive the above result from Eq. (6) without using the neutrix limit.

It follows from Eq. (7) that

$$(\delta')^{2l}(t) = 0 \text{ for } l = 1, 2, \ldots,$$
$$(\delta')^{2l+1}(t) = \frac{1\cdot 3\cdot 5\ldots(6l+1)}{2^{l}\pi^{l}(2l+1)^{\frac{6l+3}{2}}(4l+1)!}\delta^{(4l+1)}(t).$$

In summary,

Theorem 4.1.

$$(\delta')^{0}(t) = 1,$$
$$(\delta')^{k}(t) = 0 \text{ for } k < 1/2 \text{ and } k \neq 0,$$
$$(\delta')^{1/2}(t) = \sqrt{2}e^{i\frac{\pi}{4}}\left(\frac{2}{\pi}\right)^{1/4}\Gamma(3/4)\delta(t),$$
$$((\delta')^{k}(t), \phi(t)) = \frac{((-1)^{k} + (-1)^{2k-1})\Gamma\left(\frac{3k}{2}\right)}{2^{1-k}(k\pi)^{\frac{k}{2}}k^{k}\Gamma(2k)}\left({}^{C}\hat{D}_{0,t}^{2k-1}\phi\right)(0) \text{ for } k > 1/2$$

where $\left({}^{C}\hat{D}_{0,t}^{2k-1}\phi\right)(0) = \left({}^{C}\hat{D}_{0,t}^{m\alpha}\phi\right)(0) = ({}^{C}D_{0,t}^{\alpha}\,{}^{C}D_{0,t}^{\alpha}\ldots{}^{C}D_{0,t}^{\alpha}\phi)(0)$*(m-times). In particular,*

$$(\delta')^{2l}(t) = 0 \text{ for } l = 1, 2, \ldots, \text{ and}$$
$$(\delta')^{2l+1}(t) = \frac{1\cdot 3\cdot 5\ldots(6l+1)}{2^{l}\pi^{l}(2l+1)^{\frac{6l+3}{2}}(4l+1)!}\delta^{(4l+1)}(t) \text{ for } l = 0, 1, 2, \ldots.$$

Remark 2.

(i) According to the authors' knowledge, no one has given meaning to the distribution $(\delta')^{k}(t)$ for all $k \in R$ previously. We should note that Theorem 4.1 is a generalization of Theorem 2 in [8], where the case for $k \in Z^{+}$ is mainly considered.

(ii) Again, the choice of $\alpha \in (0, 1]$ in Theorem 4.1 is not unique. We generally choose α and m in Theorem 4.1 to make $\left({}^{C}\hat{D}_{0,t}^{2k-1}\phi\right)(0)$ as simple as possible by Theorem 2.3.

By the way, we can define the distributions $\log\delta(t)$ mentioned in the introduction and $\delta(t^2)$, and show that

$$\log\delta(t) = -\frac{1}{2}\log\pi, \text{ and}$$
$$\delta(t^2) = 0$$

by the δ-sequence and the neutrix limit.

Indeed,

$$(\log\delta(t), \phi(t)) := N - \lim_{n\to\infty}(\log\delta_n(t), \phi(t)) = N - \lim_{n\to\infty}\left(\log\left(\frac{n}{\pi}\right)^{1/2}e^{-nt^2}, \phi(t)\right)$$
$$= N - \lim_{n\to\infty}\left(\frac{1}{2}\log n - \frac{1}{2}\log\pi - nt^2\log e, \phi(t)\right) = \left(-\frac{1}{2}\log\pi, \phi(t)\right).$$

Furthermore,

$$\log \delta^k(t) = -\frac{k}{2}\log \pi$$

for all $k \in R$.

Finally,

$$(\delta(t^2), \phi(t)) = N - \lim_{n\to\infty}\left(\frac{n}{\pi}\right)^{1/2}\int_{-\infty}^{\infty} e^{-nt^4}\phi(t)dt.$$

Setting $t = \left(\frac{1}{n}\right)^{1/4} y$, we get

$$(\delta(t^2), \phi(t)) = N - \lim_{n\to\infty}\left(\frac{n}{\pi}\right)^{1/2}\left(\frac{1}{n}\right)^{1/4}\int_{-\infty}^{\infty} e^{-y^4}\phi\left(\left(\frac{1}{n}\right)^{1/4} y\right)dy = 0.$$

To end this paper, we would like to mention that it is worth considering powers of the distributions $(t+i0)^{-n}$ and $(t-i0)^{-n}$, based on the following applications and importance.

From Gel'fand and Shilov [27], we have for $m \in Z+$

$$(t \pm i0)^{-m} = t_+^{-m} + (-1)^m t_-^{-m} \mp \frac{(-1)^{m-1} i\pi}{(m-1)!}\delta^{(m-1)}(t), \tag{8}$$

which implies that

$$(t+i0)^{-m} = t^{-m} - \frac{(-1)^{m-1} i\pi}{(m-1)!}\delta^{(m-1)}(t),$$
$$(t-i0)^{-m} = t^{-m} + \frac{(-1)^{m-1} i\pi}{(m-1)!}\delta^{(m-1)}(t).$$

In particular, for $m = 2$ we get

$$\left(\frac{1}{(t+i0)^2}, \phi(t)\right) = \int_0^{\infty}\frac{\phi(t)+\phi(-t)-2\phi(0)}{t^2}dt - i\pi\phi'(0), \quad \text{and}$$
$$\left(\frac{1}{(t-i0)^2}, \phi(t)\right) = \int_0^{\infty}\frac{\phi(t)+\phi(-t)-2\phi(0)}{t^2}dt + i\pi\phi'(0).$$

The current approach we adopt in this paper to define powers of the distributions $\delta(t)$ and $\delta'(t)$ is infeasible to give meaning to powers of high derivative orders of Dirac delta function, as there is no a simple expression for $\delta_n^{(m)}(t)$ when m is large. However, Eq. (8) may provide a workable way of defining powers of high derivative orders of the delta function with applications described in the introduction, if we are able to define powers of the distributions $(t+i0)^{-n}$ and $(t-i0)^{-n}$. Clearly, carrying out such procedures will require products of distributions and complicated computations. Furthermore, the distribution t^{-m} is related to the Cauchy principle value of $1/t^m$, which has applications to seeking weak solutions (in distributional sense) of differential equations [29].

5. Conclusions

In this paper, we mainly study arbitrary powers of the delta function $\delta(t)$ and its derivative $\delta'(t)$ using Caputo derivative and the generalized Taylor's expansion. The satisfactory results are presented in Sections 3 and 4. How to define powers of other distributions may attract attention, although it is a challenge and we hope such studies will appear somewhere in the future.

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