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RESEARCH PAPER

SEVERAL RESULTS OF FRACTIONAL DERIVATIVES IN $\mathcal{D}'(R_+)$

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Abstract

In this paper, we define fractional derivative of arbitrary complex order of the distributions concentrated on R_+, based on convolutions of generalized functions with the supports bounded on the same side. Using distributional derivatives, which are generalizations of classical derivatives, we present a few interesting results of fractional derivatives in $\mathcal{D}'(R_+)$, as well as the symbolic solution for the following differential equation by Babenko's method

$$y(x) + \frac{\lambda}{\Gamma(-\alpha)} \int_0^x \frac{y(\zeta)}{(x-\zeta)^{\alpha+1}} d\zeta = \delta(x),$$

where $\operatorname{Re} \alpha > 0$.

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1. Introduction

The theory of classical derivatives of non-integer order of ordinary functions goes back to the the Leibniz's note in his letter to L'Hôpital, dated 30 September 1695, in which the meaning of the derivative of order one half was discussed. Since then, the fractional derivatives has undergone a significant and even heated development. Different from integer-order

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derivative, there are several kinds of definitions for fractional derivatives. These definitions are generally not equivalent with each other [1].

We let Φ_λ be the convolution kernel of order $\lambda > 0$ for fractional integrals, given by

$$\Phi_\lambda = \frac{x_+^{\lambda-1}}{\Gamma(\lambda)} \in L^1_{loc}(R),$$

where Γ is the well-known Gamma function, and

$$x_+^{\lambda-1} = \begin{cases} x^{\lambda-1} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

Definition **1.1**. The fractional integral (or, the Riemann-Liouville) $D_{0,x}^{-\lambda}$ of fractional order $\lambda > 0$ of function $\phi(x)$ is defined by

$$D_{0,x}^{-\lambda}\phi(x) = \Phi_\lambda * \phi(x) = \frac{1}{\Gamma(\lambda)} \int_0^x (x-\tau)^{\lambda-1}\phi(\tau)d\tau,$$

where we set the initial time to zero.

As an example, we have for $\gamma > -1$ the following:

$$D_{0,x}^{-\lambda}x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\lambda+\gamma+1)}x^{\lambda+\gamma},$$

$$D_{0,x}^{-\lambda}e^x = x^\lambda \sum_{k=0}^{\infty} \frac{1}{\Gamma(\lambda+k+1)}x^k,$$

by a simple calculation.

Φ_λ has an important convolution property [2] (or semigroup property): $\Phi_\lambda * \Phi_\mu = \Phi_{\lambda+\mu}$ holds for $\lambda > 0$ and $\mu > 0$, which implies that $D_{0,x}^{-\lambda}D_{0,x}^{-\mu} = D_{0,x}^{-\lambda-\mu}$.

Definition **1.2**. The Grünwald-Letnikov fractional derivative with fractional order λ is defined by, if $\phi(x) \in C^m[0,x]$,

$${}_{GL}D_{0,x}^{\lambda}\phi(x) = \sum_{k=0}^{m-1} \frac{\phi^{(k)}(0)x^{-\lambda+k}}{\Gamma(-\lambda+k+1)} + \frac{1}{\Gamma(m-\lambda)} \int_0^x (x-\tau)^{m-\lambda-1}\phi^{(m)}(\tau)d\tau,$$

where $m-1 \leq \lambda < m \in Z^+$.

This is not the original definition. The initial one is given by a limit, that is,

$${}_{GL}D_{0,x}^{\lambda}\phi(x) = \lim_{h\to 0,\ nh=x} h^{-\lambda} \sum_{k=0}^{n} (-1)^k \binom{\lambda}{k} \phi(x-kh).$$

The limit expression is not convenient for analysis but often used for numerical approximation.

Definition **1.3**. The Riemann-Liouville derivative of fractional order λ of a function $\phi(x)$ is defined as

$$ {}_{RL}D_{0,x}^{\lambda}\phi(x) = \frac{d^m}{dt^m} D_{0,x}^{-(m-\lambda)}\phi(x) = \frac{1}{\Gamma(m-\lambda)} \frac{d^m}{dt^m} \int_0^x (x-\tau)^{m-\lambda-1}\phi(\tau)d\tau, $$

where $m-1 \leq \lambda < m \in Z^+$.

It follows that

$$ {}_{RL}D_{0,x}^{\lambda} c = \frac{cx^{-\lambda}}{\Gamma(1-\lambda)}, $$

$$ {}_{RL}D_{0,x}^{\lambda} x^{\alpha} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\lambda+1)} x^{\alpha-\lambda}, $$

where c is a constant and $\alpha > -1$.

From Definitions **1.2** and **1.3**, one can see that ${}_{RL}D_{0,x}^{\lambda}\phi(x) = {}_{GL}\, D_{0,x}^{\lambda}\phi(x)$ if $\phi(x) \in C^m[0,x]$, which can be justified using integration by parts. This fact and the original definition of ${}_{GL}D_{0,x}^{\lambda}\phi(x)$ provide numerical methods for fractional differential equations with Riemann-Liouville derivatives [3].

The Riemann-Liouville derivative ${}_{RL}D_{0,x}^{\lambda}$ is a reasonable extension ([1]) between d^{m-1}/dt^{m-1} and d^m/dt^m. However, it has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Hence, we shall introduce a modified fractional differential operator ${}_{C}D_{0,x}^{\lambda}$ proposed by Caputo in 1967 (see [4], for example).

Definition **1.4**. The Caputo derivative of fractional order λ of function $\phi(x)$ is defined as

$$ {}_{C}D_{0,x}^{\lambda}\phi(x) = D_{0,x}^{-(m-\lambda)} \frac{d^m}{dx^m}\phi(x) = \frac{1}{\Gamma(m-\lambda)} \int_0^x (x-\tau)^{m-\lambda-1}\phi^{(m)}(\tau)d\tau, $$

where $m-1 < \lambda < m \in Z^+$.

It follows that

$$ {}_{C}D_{0,x}^{\lambda} x^p = \begin{cases} \dfrac{\Gamma(p+1)}{\Gamma(p-\lambda+1)} x^{p-\lambda}, & m-1 < \lambda < m \in Z^+,\ p > m-1,\ p \in R, \\ 0, & m-1 < \lambda < m \in Z^+,\ p \leq m-1,\ p \in Z^+. \end{cases} $$

Obviously, we get from the above definition

$$ \lim_{\lambda \to (m-1)^+} {}_{C}D_{0,x}^{\lambda}\phi(x) = \lim_{\lambda \to (m-1)^+} \left(\frac{1}{\Gamma(m-\lambda)} \int_0^x (x-\tau)^{m-\lambda-1}\phi^{(m)}(\tau)d\tau \right) $$

$$ = \int_0^x \phi^{(m)}(\tau)d\tau = \phi^{(m-1)}(x) - \phi^{(m-1)}(0), $$

and

$$\lim_{\lambda \to m^-} {}_C D_{0,x}^{\lambda} \phi(x) = \lim_{\lambda \to m^-} \left(\frac{\phi^{(m)}(0) x^{m-\alpha}}{\Gamma(m-\lambda+1)} \right.$$
$$\left. + \frac{1}{\Gamma(m-\lambda+1)} \int_0^x (x-\tau)^{m-\lambda} \phi^{(m+1)}(\tau) d\tau \right)$$
$$= \phi^{(m)}(0) + \int_0^x \phi^{(m+1)}(\tau) d\tau = \phi^{(m)}(x),$$

if $\phi(x) \in C^{m+1}[0, \infty)$.

On the other hand, integration by parts and differentiation show that

$${}_C D_{0,x}^{\lambda} \phi(x) = {}_{RL} D_{0,x}^{\lambda} \left(\phi(x) - \sum_{k=0}^{m-1} \frac{x^k}{k!} \phi^{(k)}(0) \right),$$

if $\phi(x) \in C^m[0, \infty)$ and $m - 1 < \lambda < m \in Z^+$.

2. Distributions in $\mathcal{D}'(R_+)$

In order to study fractional derivatives of certain types of distributions and proceed smoothly, we introduce the following basic definitions in detail, which can be found in [5]. Let $\mathcal{D}(R)$ be the space of infinitely differentiable function with compact support in R, and let $\mathcal{D}'(R)$ be the space of distributions defined on $\mathcal{D}(R)$. Further, we shall define a sequence $\phi_1(x)$, $\phi_2(x), \cdots, \phi_n(x), \cdots$ which converges to zero in $\mathcal{D}(R)$ if all these functions vanish outside a certain fixed bounded interval, and converge uniformly to zero (in the usual sense) together with their derivatives of any order. The distribution δ is defined as

$$(\delta, \phi) = \phi(0),$$

where $\phi \in \mathcal{D}(R)$. Clearly, δ is a linear and continuous functional on $\mathcal{D}(R)$, and hence $\delta \in \mathcal{D}'(R)$.

Define

$$\theta(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x < 0, \end{cases}$$

which clearly is discontinuous at $x = 0$. Then

$$(\theta, \phi) = \int_0^\infty \phi(x) dx \quad \text{for} \quad \phi \in \mathcal{D}(R),$$

which implies $\theta \in \mathcal{D}'(R)$.

Let $f \in \mathcal{D}'(R)$. The distributional derivative of f, denoted by f' or df/dx, is defined as

$$(f', \phi) = \left(\frac{df}{dx}, \phi \right) = -(f, \phi')$$

for $\phi \in \mathcal{D}(R)$.

Clearly, $f' \in \mathcal{D}'(R)$ and every distribution has a derivative. As a simple example, we are going to show that $\theta' = \delta$, although $\theta(x)$ is not even defined at $x = 0$. Indeed,

$$(\theta', \phi) = -(\theta, \phi') = -\int_0^\infty \phi'(x)dx = \phi(0) = (\delta, \phi),$$

which claims

$$\theta' = \delta.$$

It can be shown that the ordinary rules of differentiation apply also to distributions. For instance, the derivative of a sum is the sum of the derivatives, and a constant can be commuted with the derivative operator.

It seems impossible to define products of two arbitrary distributions in general [6]. However, the product of an infinitely differentiable function $\phi(x)$ with a distribution f is given by

$$(\phi f, \psi) = (f, \phi\psi) \tag{2.1}$$

which is well defined since $\phi\psi \in \mathcal{D}(R)$ if $\psi \in \mathcal{D}(R)$.

Let ϕ be an infinitely differentiable function. Then the product of $\phi\delta^{(n)}$ exists by equation (2.1) for $n = 0, 1, 2, \cdots$, and

$$\phi\delta^{(n)} = \sum_{i=0}^{n} \binom{n}{i} (-1)^i \phi^{(i)}(0)\, \delta^{(n-i)}. \tag{2.2}$$

In particular, we have

$$\phi\delta = \phi(0)\delta,$$
$$\phi\delta' = \phi(0)\delta' - \phi'(0)\delta.$$

Remark. How to define the products of distributions ([7] and [8]) is a very difficult and not completely understood problem, and has been studied from several points of views since Schwartz established the theory of distributions by treating singular functions as linear and continuous functionals on $\mathcal{D}(R)$.

Now, we turn our attention to the distribution x_+^λ given in [5] (λ is a complex number), which will play an important role in defining the fractional derivatives and integrals of distributions in $\mathcal{D}'(R_+)$, the subspace of $\mathcal{D}'(R)$ with the supports contained in R_+.

Consider

$$x_+^\lambda = \begin{cases} x^\lambda & \text{if } x > 0, \\ 0 & \text{if } x \le 0, \end{cases}$$

where $\mathrm{Re}\lambda > -1$. We wish to construct and study the generalized function corresponding to it. Clearly, the regular functional

$$(x_+^\lambda, \phi) = \int_0^\infty x^\lambda \phi(x)dx, \qquad \phi \in \mathcal{D}(R),$$

defined by x_+^λ for $\mathrm{Re}\lambda > -1$ can be analytically continued to $\mathrm{Re}\lambda > -2$, $\lambda \neq -1$ by means of the identity

$$\int_0^\infty x^\lambda \phi(x)dx = \int_0^1 x^\lambda[\phi(x) - \phi(0)]dx + \int_1^\infty x^\lambda \phi(x)dx + \frac{\phi(0)}{\lambda+1},$$

valid for $\mathrm{Re}\lambda > -1$. Specifically, for $\mathrm{Re}\lambda > -2$, $\lambda \neq -1$, the right-hand side exists and defines a normalization of the integral on the left.

We can proceed similarly and continue x_+^λ into the region $\mathrm{Re}\lambda > -n-1$, $\lambda \neq -1, -2, \cdots, -n$, to obtain from [5]

$$\begin{aligned} \int_0^\infty x^\lambda \phi(x)dx &= \int_0^1 x^\lambda \Big[\phi(x) - \phi(0) - \cdots - \frac{x^{n-1}}{(n-1)!}\phi^{(n-1)}(0)\Big]\,dx \\ &\quad + \int_1^\infty x^\lambda \phi(x)dx + \sum_{k=1}^n \frac{\phi^{(k-1)}(0)}{(k-1)!(\lambda+k)}. \end{aligned} \tag{2.3}$$

Here again the right-hand side regularizes the integral on the left. This defines the distribution x_+^λ for $\mathrm{Re}\lambda \neq -1, -2, \cdots$. In particular, the above equation can be written in the simpler form

$$(x_+^\lambda, \phi) = \int_0^\infty x^\lambda[\phi(x) - \phi(0) - x\phi'(0) - \cdots - \frac{x^{n-1}}{(n-1)!}\phi^{(n-1)}(0)]dx \tag{2.4}$$

in any strip $-n-1 < \mathrm{Re}\lambda < -n$, as ϕ has bounded support.

Furthermore, equation (2.3) shows that when we treat (x_+^λ, ϕ) as a function of λ, it has simple poles at $\lambda = -1, -2, \cdots$, and that its residue at $\lambda = -k$ is

$$\frac{\phi^{(k-1)}(0)}{(k-1)!} = \frac{(-1)^{k-1}}{(k-1)!}\,(\delta^{(k-1)}, \phi).$$

Therefore, we claim that the functional x_+^λ itself has a simple pole at $\lambda = -k$, and that the residue there is

$$\frac{(-1)^{k-1}}{(k-1)!}\,\delta^{(k-1)}, \qquad k = 1, 2, \cdots.$$

For $\mathrm{Re}\lambda > 0$, we have

$$(x_+^\lambda, \phi'(x)) = \int_0^\infty x^\lambda \phi'(x)dx = \int_0^\infty x^\lambda d\phi(x) = -(\lambda x_+^{\lambda-1}, \phi(x)).$$

Since both sides of the above equation can be analytically continued to the entire plane except at $\lambda = -1, -2, \cdots$, uniqueness of analytic continuation implies that the equation will hold in the entire plane. Thus,

$$\frac{dx_+^\lambda}{dx} = \lambda x_+^{\lambda-1}, \qquad \lambda \neq -1, -2, \cdots.$$

To make this paper self-contained as much as possible, we introduce the Gamma-function [5] as

$$\Gamma(\lambda) = \int_0^\infty x^{\lambda-1} e^{-x} dx,$$

which converges for $\mathrm{Re}\lambda > 0$. This integral may be thought of as representing the application of $x_+^{\lambda-1}$ to the test function equal to e^{-x} for $0 \leq x < \infty$. For $\mathrm{Re}\lambda > -n-1$, $\lambda \neq -1, -2, \cdots, -n$, we have from equation (2.3)

$$\Gamma(\lambda) = \int_0^1 x^{\lambda-1} \left[e^{-x} - \sum_{k=0}^{n} (-1)^k \frac{x^k}{k!} \right] dx + \int_1^\infty x^{\lambda-1} e^{-x} dx + \sum_{k=0}^{n} \frac{(-1)^k}{k!(\lambda+n)}.$$

For $-n-1 < \mathrm{Re}\lambda < -n$, we get from equation (2.4)

$$\Gamma(\lambda) = \int_0^\infty x^{\lambda-1} \left[e^{-x} - \sum_{k=0}^{n} (-1)^k \frac{x^k}{k!} \right] dx.$$

In the following, we are going to show that $\Phi_\lambda = \dfrac{x_+^{\lambda-1}}{\Gamma(\lambda)} \in \mathcal{D}'(R_+)$ is an entire function of λ on the complex plane.

Indeed, the values of the distribution Φ_λ at the singular points of the numerator and denominator can be obtained by taking the ratio of the corresponding residues. It follows that

$$\left. \frac{x_+^{\lambda-1}}{\Gamma(\lambda)} \right|_{\lambda=-n} = \frac{\mathrm{res}_{\lambda=-n} x_+^{\lambda-1}}{\mathrm{res}_{\lambda=-n}(x_+^{\lambda-1},\, e^{-x})} = \frac{(-1)^n \delta^{(n)} n!}{(-1)^n (\delta^{(n)}(x),\, e^{-x}) n!} = \delta^{(n)}. \quad (2.5)$$

For the functional $\Phi_\lambda = \dfrac{x_+^{\lambda-1}}{\Gamma(\lambda)}$, the derivative formula is simpler than that for x_+^λ. In fact,

$$\frac{d}{dx}\Phi_\lambda = \frac{d}{dx}\frac{x_+^{\lambda-1}}{\Gamma(\lambda)} = \frac{(\lambda-1)x_+^{\lambda-2}}{\Gamma(\lambda)} = \frac{x_+^{\lambda-2}}{\Gamma(\lambda-1)} = \Phi_{\lambda-1}. \quad (2.6)$$

3. The convolutions of distributions

The convolution of certain pairs of distributions is usually defined as follows, see for example Gel'fand and Shilov [5].

Definition **3.1**. Let f and g be distributions in $\mathcal{D}'(R)$ satisfying either of the following conditions: (a) either f or g has bounded support, (b) the supports of f and g are bounded on the same side. The the convolution $f * g$ is defined by the equation

$$((f*g)(x),\, \phi(x)) = (g(x),\, (f(y),\, \phi(x+y)))$$

for $\phi \in \mathcal{D}(R)$.

The classical definition of the convolution is as follows:

Definition **3.2**. If f and g are locally integral functions, then the convolution $f * g$ is defined by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)dt = \int_{-\infty}^{\infty} f(x-t)g(t)dt$$

for all x for which the integrals exist.

Note that if f and g are locally integrable functions satisfying either of the conditions (a) or (b) in Definition **3.1**, then Definition **3.1** is in agreement with Definition **3.2**. It also follows that if the convolution $f * g$ exists by Definition **3.1** or **3.2**, then the following equations hold

$$f * g = g * f, \tag{3.1}$$

$$(f * g)' = f * g' = f' * g, \tag{3.2}$$

where all the derivatives above are in distributional sense.

Let λ and μ be arbitrary complex numbers. Then

$$\Phi_\lambda * \Phi_\mu = \Phi_{\lambda+\mu}. \tag{3.3}$$

Let us first prove this formula for $\mathrm{Re}\lambda > 0$ and $\mathrm{Re}\mu > 0$. Since

$$\Phi_\lambda * \Phi_\mu = \frac{x_+^{\lambda-1}}{\Gamma(\lambda)} * \frac{x_+^{\mu-1}}{\Gamma(\mu)} = \int_0^x \frac{\zeta^{\lambda-1}}{\Gamma(\lambda)} \cdot \frac{(x-\zeta)^{\mu-1}}{\Gamma(\mu)} d\zeta$$

and $\Phi_{\lambda+\mu} = \dfrac{x_+^{\lambda+\mu-1}}{\Gamma(\lambda+\mu)}$, we only need prove that

$$\int_0^x \zeta^{\lambda-1}(x-\zeta)^{\mu-1}d\zeta = \frac{\Gamma(\lambda)\Gamma(\mu)}{\Gamma(\lambda+\mu)} x^{\lambda+\mu-1}.$$

We set $\zeta = xt$ on the left-hand side, so that the integral becomes

$$x^{\lambda+\mu-1}\int_0^1 t^{\lambda-1}(1-t)^{\mu-1}dt = x^{\lambda+\mu-1}B(\lambda,\mu) = x^{\lambda+\mu-1}\frac{\Gamma(\lambda)\Gamma(\mu)}{\Gamma(\lambda+\mu)}.$$

Equation (3.3) can now be proven for other values of λ, μ by analytic continuation, or equation (2.6). For example, if $-1 < \mathrm{Re}\lambda < 0$, then

$$\frac{x_+^{\lambda-1}}{\Gamma(\lambda)} = \frac{d}{dx}\frac{x_+^{\lambda+1-1}}{\Gamma(\lambda+1)}$$

and

$$\frac{x_+^{\mu-1}}{\Gamma(\mu)} * \frac{x_+^{\lambda-1}}{\Gamma(\lambda)} = \frac{d}{dx}\left(\frac{x_+^{\mu-1}}{\Gamma(\mu)} * \frac{x_+^{\lambda+1-1}}{\Gamma(\lambda+1)}\right) = \frac{d}{dx}\frac{x_+^{\mu+\lambda}}{\Gamma(\mu+\lambda+1)} = \frac{x_+^{\mu+\lambda-1}}{\Gamma(\mu+\lambda)}.$$

4. The fractional derivatives in $\mathcal{D}'(R_+)$

The Cauchy formula

$$g_n(x) = \int_0^x \int_0^{\zeta_{n-1}} \cdots \int_0^{\zeta_2} \int_0^{\zeta_1} g(\zeta) d\zeta d\zeta_1 \cdots d\zeta_{n-1} = \frac{1}{(n-1)!} \int_0^x g(\zeta)(x-\zeta)^{n-1} d\zeta$$

reduces the calculation of the n-fold primitive of a function $g(x)$ defined on R_+ to a single integral. Clearly, this formula can be written in the form

$$g_n(x) = g(x) * \frac{x_+^{n-1}}{(n-1)!} = g(x) * \frac{x_+^{n-1}}{\Gamma(n)},$$

where $g(x) = 0$ for $x < 0$.

We would like to extend this formula to the case of arbitrary complex number λ and the distribution concentrated on $x \geq 0$. Hence, we define the primitive of order λ of g as the convolution in distributional sense

$$g_\lambda(x) = g(x) * \frac{x_+^{\lambda-1}}{\Gamma(\lambda)} = g(x) * \Phi_\lambda. \tag{4.1}$$

Note that the convolution on the right-hand side is well defined since the supports of g and Φ_λ are bounded on the same side.

It follows from equation (2.5) that

$$\begin{aligned}
&g_0(x) = g(x) * \Phi_0 = g(x) * \delta(x) = g(x),\\
&g_{-1}(x) = g(x) * \Phi_{-1} = g(x) * \delta'(x) = g'(x),\\
&g_{-2}(x) = g(x) * \Phi_{-2} = g(x) * \delta''(x) = g''(x),\\
&\cdots\\
&g_1(x) = g(x) * \Phi_1 = g(x) * \theta(x) = \int_0^x g(\zeta) d\zeta,\\
&\cdots .
\end{aligned}$$

Thus equation (4.1) with various λ will give not only the derivatives but also the integrals of $g(x) \in \mathcal{D}'(R_+)$. We shall define the convolution

$$g_{-\lambda} = g(x) * \Phi_{-\lambda}$$

as the fractional derivative of the distribution $g(x)$ with order λ, writing as

$$g_{-\lambda} = \frac{d^\lambda}{dx^\lambda} g,$$

where $\mathrm{Re}\lambda \geq 0$. By the way, $\frac{d^\lambda}{dx^\lambda} g$ will be interpreted as the fractional integral if $\mathrm{Re}\lambda < 0$.

Let f and g be arbitrary distributions in $\mathcal{D}'(R_+)$. Then the fractional derivative is linear (since the convolution is linear), that is,

$$\frac{d^\lambda}{dx^\lambda}(\alpha f(x) + \beta g(x)) = \alpha \frac{d^\lambda}{dx^\lambda} f(x) + \beta \frac{d^\lambda}{dx^\lambda} g(x),$$

where α and β are constants.

Let $m - 1 < \lambda < m \in Z^+$ and $g(x)$ be a distribution in $\mathcal{D}'(R_+)$. Then we derive from equation (3.2) that

$$g_{-\lambda}(x) = g(x) * \frac{x_+^{-\lambda-1}}{\Gamma(-\lambda)} = g(x) * \frac{d^m}{dx^m} \frac{x_+^{m-\lambda-1}}{\Gamma(m-\lambda)}$$

$$= \frac{d^m}{dx^m}\left(g(x) * \frac{x_+^{m-\lambda-1}}{\Gamma(m-\lambda)} \right) = g^{(m)}(x) * \frac{x_+^{m-\lambda-1}}{\Gamma(m-\lambda)},$$

which indicates there is no difference between the Riemann-Liouville derivative and the Caputo derivative in the distributional sense.

Furthermore, we may write

$$\frac{d^\lambda}{dx^\lambda}\left(\frac{x_+^\mu}{\Gamma(\mu+1)} \right) = \frac{x_+^{\mu-\lambda}}{\Gamma(\mu+1-\lambda)} \tag{4.2}$$

by replacing λ by $-\lambda$, μ by $\mu+1$ in equation (3.3). In particular, for $\mu = 0$, we get

$$\frac{d^\lambda}{dx^\lambda}\theta(x) = \frac{x_+^{-\lambda}}{\Gamma(1-\lambda)} = \Phi_{1-\lambda}.$$

Writing $\mu = -k-1$ in equation (4.2) for nonnegative integer k, we find

$$\frac{d^\lambda}{dx^\lambda}\delta^{(k)}(x) = \frac{x_+^{-k-\lambda-1}}{\Gamma(-k-\lambda)} = \Phi_{-k-\lambda}.$$

Setting λ by $-\lambda$ in the above, we obtain

$$\frac{d^{-\lambda}}{dx^{-\lambda}}\delta^{(k)}(x) = \frac{x_+^{-k+\lambda-1}}{\Gamma(-k+\lambda)}$$

which implies

$$\delta^{(k)}(x) = \frac{d^\lambda}{dx^\lambda}\frac{x_+^{\lambda-k-1}}{\Gamma(\lambda-k)}.$$

It follows from equation (3.3) that

$$(g * \Phi_\lambda) * \Phi_\mu = g * (\Phi_\lambda * \Phi_\mu) = g * \Phi_{\lambda+\mu} \tag{4.3}$$

for any distribution $g \in \mathcal{D}'(R_+)$.

Setting $\mu = -\lambda$, we see that differentiation and integration of the same order are mutually inverse processes, and the sequential fractional derivative law holds from equation (3.3):

$$\frac{d^\lambda}{dx^\lambda}\left(\frac{d^\mu g}{dx^\mu}\right) = \frac{d^{\lambda+\mu} g}{dx^{\lambda+\mu}} = \frac{d^\mu}{dx^\mu}\left(\frac{d^\lambda g}{dx^\lambda}\right)$$

for any complex numbers λ and μ.

The following theorem can be proven easily by the inverse operation and convolution.

THEOREM **4.1**. *Let g be given in $\mathcal{D}'(R_+)$ and f be unknown in $\mathcal{D}'(R_+)$. Then the generalized Abel's integral equation given by*

$$g(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x \frac{f(\zeta)}{(x-\zeta)^\alpha} d\zeta$$

has the solution

$$f = g * \Phi_{\alpha-1},$$

where α is arbitrary. In particular, if $-m < \alpha < -m+1$ for $m \in Z^+$, then

$$f = g^{(m+1)} * \frac{x_+^{\alpha+m-1}}{\Gamma(\alpha+m)}.$$

EXAMPLE **4.1**. Let α be any complex number except all negative integers. Consider the distribution given by

$$g(x) = x_+^\alpha = \begin{cases} x^\alpha & \text{if } x > 0, \\ 0 & \text{if } x < 0. \end{cases}$$

We need to find $\dfrac{d^\lambda}{dx^\lambda} g(x)$ for $\mathrm{Re}\lambda \geq 0$ and $\lambda \neq \alpha+1, \alpha+2, \cdots$.

Clearly,

$$\frac{d^\lambda}{dx^\lambda} g(x) = \Gamma(\alpha+1) \frac{d^\lambda}{dx^\lambda} \frac{x_+^\alpha}{\Gamma(\alpha+1)} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\lambda+1)} x_+^{\alpha-\lambda}.$$

In particular, we get

$$\begin{aligned}
&\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} x_+ = \frac{2x_+^{\frac{1}{2}}}{\sqrt{\pi}}, \\
&\frac{d}{dx} x_+ = \theta(x), \\
&\frac{d}{dx} x_+^0 = \frac{d}{dx}\theta(x) = \delta(x), \\
&\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} x_+^0 = \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\theta(x) = \frac{x_+^{-\frac{1}{2}}}{\sqrt{\pi}}.
\end{aligned}$$

Using Euler's reflection formula

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)},$$

we derive that

$$\Gamma(-\frac{1}{2}) = -2\sqrt{\pi},$$
$$\Gamma(-\frac{3}{2}) = \frac{4}{3}\sqrt{\pi}.$$

Therefore, we obtain

$$\frac{d}{dx}x_+^{-1.5} = \frac{\Gamma(-0.5)}{\Gamma(-1.5)}x_+^{-2.5} = -1.5x_+^{-2.5},$$

which coincides with the derivative formula $\frac{dx_+^\lambda}{dx} = \lambda x_+^{\lambda-1}$ for $\lambda \neq -1, -2, \cdots$ in Section **2**. Note that we can also utilize the recursion $\Gamma(x+1) = x\Gamma(x)$ to get

$$\frac{\Gamma(-0.5)}{\Gamma(-1.5)} = \frac{-1.5\Gamma(-1.5)}{\Gamma(-1.5)} = -1.5.$$

Furthermore,

$$\frac{d^{0.25}}{dx^{0.25}}x_+^{-1.5} = \frac{\Gamma(-0.5)}{\Gamma(-0.75)}x_+^{-1.75} = \frac{-2\sqrt{\pi}}{\Gamma(-0.75)}x_+^{-1.75}.$$

THEOREM **4.2**. *Assume that $m - 1 < \lambda < m \in Z^+$ and f is a distribution in $\mathcal{D}'(R_+)$, given by*

$$f(x) = \begin{cases} \phi(x) & \text{if } x \geq 0, \\ 0 & \text{if } x < 0, \end{cases}$$

where

$$\phi(x) = \sum_{k=0}^{\infty} \frac{\phi^{(k)}(0)}{k!}x^k$$

for all $x \geq 0$. Then

$$\frac{d^\lambda}{dx^\lambda}f(x) = \sum_{k=-m}^{\infty} \frac{\phi^{(m+k)}(0)\, x_+^{m-\lambda+k}}{\Gamma(m-\lambda+k+1)}. \tag{4.4}$$

P r o o f. Clearly,

$$\frac{d^\lambda}{dx^\lambda}f(x) = f(x) * \frac{x_+^{-\lambda-1}}{\Gamma(-\lambda)} = f(x) * \frac{d^m}{dx^m}\frac{x_+^{m-\lambda-1}}{\Gamma(m-\lambda)} = \frac{d^m}{dx^m}f(x) * \frac{x_+^{m-\lambda-1}}{\Gamma(m-\lambda)},$$

where $0 < m - \lambda < 1$.

Obviously, we have for $\psi \in \mathcal{D}(R)$

$$\left(\frac{d^m}{dx^m}f,\, \psi\right) = (-1)^m(f,\, \psi^{(m)}) = (-1)^m \int_0^\infty \phi(x)\psi^{(m)}(x)dx$$
$$= (-1)^m \int_0^\infty \phi(x)d\psi^{(m-1)}(x) = (-1)^m \left[\phi(x)\psi^{(m-1)}(x)\Big|_0^\infty\right.$$

$$
\begin{aligned}
&+(-1)^1\int_0^\infty \phi'(x)\psi^{(m-1)}(x)dx\Big]\\
&= (-1)^m\left[-\phi(0)\psi^{(m-1)}(0)+(-1)^1\int_0^\infty \phi'(x)d\psi^{(m-2)}(x)\right]\\
&= (-1)^m\Big[-\phi(0)\psi^{(m-1)}(0)-\phi'(x)\psi^{(m-2)}(x)\Big|_0^\infty\\
&\qquad +(-1)^2\int_0^\infty \phi''(x)\psi^{(m-2)}(x)dx\Big]\\
&= (-1)^m\left[-\phi(0)\psi^{(m-1)}(0)+\phi'(0)\psi^{(m-2)}(0)+(-1)^2\int_0^\infty \phi''(x)\psi^{(m-2)}(x)dx\right]\\
&= \cdots\cdots\\
&= (-1)^{m-1}\phi(0)\psi^{(m-1)}(0)+(-1)^{m-2}\phi'(0)\psi^{(m-2)}(0)\\
&\qquad +(-1)^{m-3}\phi''(0)\psi^{(m-3)}(0)\\
&\qquad +\cdots+\phi^{(m-1)}(0)\psi(0)+\int_0^\infty \phi^{(m)}\psi(x)dx,
\end{aligned}
$$

which claims

$$
\frac{d^m}{dx^m}f(x)=\phi(0)\delta^{(m-1)}(x)+\phi'(0)\delta^{(m-2)}(x)+\cdots+\phi^{(m-1)}(0)\delta(x)+\theta(x)\phi^{(m)}(x).
$$

Thus,

$$
\begin{aligned}
\frac{d^\lambda}{dx^\lambda}f(x) &= \left[\phi(0)\delta^{(m-1)}(x)+\cdots+\phi^{(m-1)}(0)\delta(x)\right]*\frac{x_+^{m-\lambda-1}}{\Gamma(m-\lambda)}\\
&\quad +\theta(x)\phi^{(m)}(x)*\frac{x_+^{m-\lambda-1}}{\Gamma(m-\lambda)}\\
&= \phi(0)\frac{x_+^{-\lambda}}{\Gamma(1-\lambda)}+\cdots+\phi^{(m-1)}(0)\frac{x_+^{m-\lambda-1}}{\Gamma(m-\lambda)}\\
&\quad +\frac{1}{\Gamma(m-\lambda)}\int_0^x \phi^{(m)}(t)(x-t)^{m-\lambda-1}dt.
\end{aligned}
$$

Evidently,

$$
\phi^{(m)}(t)=\sum_{k=0}^\infty \frac{\phi^{(m+k)}(0)}{k!}t^k.
$$

Making the substitution $t=ux$, we have

$$
\begin{aligned}
&\int_0^x t^k(x-t)^{m-\lambda-1}dt = x^{m-\lambda+k}\int_0^1 u^k(1-u)^{m-\lambda-1}du\\
&= x^{m-\lambda+k}\frac{\Gamma(k+1)\,\Gamma(m-\lambda)}{\Gamma(m-\lambda+k+1)}.
\end{aligned}
$$

Therefore,

$$\begin{aligned}\frac{d^\lambda}{dx^\lambda} f(x) &= \left(\phi(0) \frac{x_+^{-\lambda}}{\Gamma(1-\lambda)} + \cdots + \phi^{(m-1)}(0) \frac{x_+^{m-\lambda-1}}{\Gamma(m-\lambda)} \right) \\ &\quad + \sum_{k=0}^{\infty} \frac{\phi^{(m+k)}(0)\, x_+^{m-\lambda+k}}{\Gamma(m-\lambda+k+1)} \\ &= I_1 + I_2 \\ &= \sum_{k=-m}^{\infty} \frac{\phi^{(m+k)}(0)\, x_+^{m-\lambda+k}}{\Gamma(m-\lambda+k+1)},\end{aligned}$$

where

$$I_1 = \phi(0) \frac{x_+^{-\lambda}}{\Gamma(1-\lambda)} + \cdots + \phi^{(m-1)}(0) \frac{x_+^{m-\lambda-1}}{\Gamma(m-\lambda)}$$

is the singular part (distribution), and

$$I_2 = \sum_{k=0}^{\infty} \frac{\phi^{(m+k)}(0)\, x_+^{m-\lambda+k}}{\Gamma(m-\lambda+k+1)}$$

is the regular part. □

In particular, for $\lambda = 1/2$ (thus $m = 1$) and $\phi(x) = e^x$, we get

$$f^{(\frac{1}{2})}(x) = \frac{x_+^{-\frac{1}{2}}}{\sqrt{\pi}} + \sum_{k=0}^{\infty} \frac{x_+^{k+\frac{1}{2}}}{\Gamma(k+\frac{3}{2})} = \sum_{k=-1}^{\infty} \frac{x_+^{k+\frac{1}{2}}}{\Gamma(k+\frac{3}{2})}.$$

Let

$$f(x) = \begin{cases} \sin x & \text{if } x \geq 0, \\ 0 & \text{if } x < 0, \end{cases}$$

and $m - 1 < \lambda < m \in Z^+$. Then

$$\frac{d^\lambda}{dx^\lambda} f(x) = \sum_{k=-m}^{\infty} \frac{\sin(m+k)\frac{\pi}{2}\, x_+^{m-\lambda+k}}{\Gamma(m-\lambda+k+1)}$$

as $(\sin x)^{(n)} = \sin(x + n\frac{\pi}{2})$. Similarly, we derive for $\lambda = 1/2$ (i.e., $m = 1$) that

$$f^{(\frac{1}{2})}(x) = \sum_{k=0}^{\infty} \frac{\cos\frac{k\pi}{2}\, x_+^{k+\frac{1}{2}}}{\Gamma(k+\frac{3}{2})} = \sum_{k=0}^{\infty} \frac{(-1)^k\, x_+^{2k+\frac{1}{2}}}{\Gamma(2k+\frac{3}{2})}$$

since $\cos k\pi = (-1)^k$ and $\cos k\pi/2 = 0$ if k is odd.

THEOREM **4.3**. *Let $\phi(x) \in C^\infty(R)$ and α be any complex number. Then the following generalized Abel's integral equation*

$$\phi(x)\delta^{(n)}(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x \frac{f(\zeta)}{(x-\zeta)^\alpha} d\zeta$$

has the solution

$$f(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \phi^{(k)}(0) \Phi_{\alpha-n+k-1}.$$

P r o o f. It follows from equation (2.2) that

$$\phi\delta^{(n)} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \phi^{(k)}(0) \delta^{(n-k)}.$$

By Theorem **4.1**, we have

$$\begin{aligned} f &= (\phi\delta^{(n)}) * \Phi_{\alpha-1} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \phi^{(k)}(0) \delta^{(n-k)} * \Phi_{\alpha-1} \\ &= \sum_{k=0}^{n} \binom{n}{k} (-1)^k \phi^{(k)}(0) \Phi_{\alpha-n+k-1}. \end{aligned}$$

This completes the proof of Theorem **4.3**. □

In particular, we find that the equation given by

$$x^n\delta^{(n)}(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x \frac{f(\zeta)}{(x-\zeta)^\alpha} d\zeta$$

has the solution

$$f = (-1)^n n!\, \Phi_{\alpha-1},$$

since $x^n\delta^{(n)}(x) = (-1)^n n!\, \delta(x)$.

To end this paper, let us consider the following differential equation

$$y(x) + \frac{\lambda}{\Gamma(-\alpha)} \int_0^x \frac{y(\zeta)}{(x-\zeta)^{\alpha+1}} d\zeta = \delta(x),$$

where $\operatorname{Re}\alpha > 0$. Clearly, we can write it into

$$y(x) + \lambda \frac{d^\alpha}{dx^\alpha} y(x) = \left(1 + \lambda \frac{d^\alpha}{dx^\alpha}\right) y(x) = \delta(x).$$

Following Babenko's method ([9] and [10]), we symbolically get

$$y(x) = \frac{1}{1 + \lambda \dfrac{d^\alpha}{dx^\alpha}} \delta(x) = \sum_{k=0}^{\infty} (-1)^k \lambda^k \frac{d^{k\alpha}}{dx^{k\alpha}} \delta(x).$$

Since

$$\frac{d^{k\alpha}}{dx^{k\alpha}} \delta(x) = \frac{x_+^{-k\alpha-1}}{\Gamma(-k\alpha)},$$

we finally have

$$y(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^k \, x_+^{-k\alpha-1}}{\Gamma(-k\alpha)}\,.$$

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