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Remarks on fractional derivatives of distributions*

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Abstract

This paper investigates a new approach to studying several fractional derivatives in the distributional sense based on the products of distributions and the delta sequence with compact support. Furthermore, we consider an asymptotic expression to the fractional derivative of the delta function and show that it is the first-order approximation in the Schwartz space. At the end of paper, we provide several asymptotic formulas to more complicated fractional derivatives of distributions.

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1 Fractional derivatives of distributions

Classical fractional derivatives first mentioned in the letter from Leibniz to L'Hôpital dated 30 September 1695, can be regarded as a branch of analysis which deals with integral and differential equations often with weakly singular kernels. A lot of contributions to the theory of fractional calculus up to the middle of the 20th century were made by many famous mathematicians including Laplace, Fourier, Abel, Liouville, Riemann, Grünwald, Letnikov, Heaviside, Weyl, Erdélyi and others [1, 2, 3, 4]. After 1970s, there was a clear movement from theoretical research of fractional calculus to its applications in various fields [5, 6, 7, 8]. In the recent work of [9], we applied Caputo fractional derivatives and the following generalized Taylor's formula for $0 < \alpha < 1$

$$\varphi(t) = \sum_{i=0}^{m} \frac{({}_C\hat{D}_{0,t}^{i\alpha}\varphi)(0)}{\Gamma(i\alpha+1)} t^{i\alpha} + \frac{({}_C\hat{D}_{0,t}^{(m+1)\alpha}\varphi)(\zeta)}{\Gamma((m+1)\alpha+1)} t^{(m+1)\alpha}$$

to give meaning to the distributions $\delta^k(x)$ and $(\delta')^k(x)$ for all $k \in R$. These can be regarded as powers of Dirac delta functions and have applications to quantum theory. Up to now, fractional calculus has been found in almost every realm of science and engineering. In this paper, we use a new technique to compute fractional derivatives of complicated distributions by generalized convolutions, Heaviside functions and Faà di Bruno formula, and deliver several asymptotic formulas for them.

Let $\mathcal{D}(R)$ be the Schwartz space [10] of infinitely differentiable functions on R with compact support, and $\mathcal{D}'(R)$ be the space of distributions defined on $\mathcal{D}(R)$. Further, we shall define a sequence φ_1, $\varphi_2, \cdots, \varphi_n, \cdots$ which converges to zero in $\mathcal{D}(R)$ if all these functions vanish outside a certain fixed

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bounded interval, and converge uniformly to zero (in the usual sense) together with their derivatives of any order. The functional δ is defined as

$$(\delta(x-a), \varphi(x)) = \varphi(a)$$

where $\varphi \in \mathcal{D}(R)$. Clearly, $\delta(x-a)$ is a linear and continuous functional on $\mathcal{D}(R)$, and hence $\delta(x-a) \in \mathcal{D}'(R)$.

Define

$$\theta(x-a) = \begin{cases} 1 & \text{if } x > a, \\ 0 & \text{if } x < a \end{cases}$$

which obviously is discontinuous at $x = a$. Then

$$(\theta(x-a),\, \varphi) = \int_a^\infty \varphi(x)dx \quad \text{for} \quad \varphi \in \mathcal{D}(R),$$

which implies $\theta(x-a) \in \mathcal{D}'(R)$.

It follows from

$$(\theta'(x-a),\, \varphi) = -(\theta(x-a),\, \varphi') = -\int_a^\infty \varphi'(x)dx = \varphi(a) = (\delta(x-a),\, \varphi(x)), \quad \varphi \in \mathcal{D}(R)$$

that

$$\theta'(x-a) = \delta(x-a).$$

Consider

$$(x-a)_+^\lambda = \begin{cases} (x-a)^\lambda & \text{if } x > a, \\ 0 & \text{if } x \le a \end{cases}$$

where $\text{Re}\lambda > -1$.

Let $\mathcal{D}'(R_+)$ be the subspace of $\mathcal{D}'(R)$ (all distributions on $\mathcal{D}(R)$) with support contained in R_+. Then

$$\Phi_\lambda = \frac{x_+^{\lambda-1}}{\Gamma(\lambda)} \in \mathcal{D}'(R_+)$$

is an entire function of λ on the complex plane [10, 11], and

$$\left.\frac{x_+^{\lambda-1}}{\Gamma(\lambda)}\right|_{\lambda=-n} = \delta^{(n)}(x), \quad n = 0, 1, 2, \cdots, \tag{1}$$

$$\frac{d}{dx}\Phi_\lambda = \Phi_{\lambda-1}, \tag{2}$$

$$\Phi_\lambda * \Phi_\mu = \Phi_{\lambda+\mu} \tag{3}$$

where λ and μ are arbitrary complex numbers.

Assume λ is a complex number. We define the fractional derivative of g of order λ as the convolution

$$g_{-\lambda} = \frac{d^\lambda}{dx^\lambda} g = g * \Phi_{-\lambda}, \quad g \in \mathcal{D}'(R_+)$$

if Re$\lambda \geq 0$, and the fractional integral if Re$\lambda < 0$.

Let $m-1 < \lambda < m \in Z^+$ and $g(x)$ be a distribution in $\mathcal{D}'(R^+)$. We derive that

$$\begin{aligned} g_{-\lambda}(x) &= g(x) * \tfrac{x_+^{-\lambda-1}}{\Gamma(-\lambda)} = g(x) * \tfrac{d^m}{dx^m} \tfrac{x_+^{m-\lambda-1}}{\Gamma(m-\lambda)} \\ &= \tfrac{d^m}{dx^m} \left(g(x) * \tfrac{x_+^{m-\lambda-1}}{\Gamma(m-\lambda)} \right) = g^{(m)}(x) * \tfrac{x_+^{m-\lambda-1}}{\Gamma(m-\lambda)}, \end{aligned}$$

which indicates there is no difference between the Riemann-Liouville derivative and the Caputo derivative of the distribution $g(x)$ (both exist clearly). Based on this fact, we only call the fractional derivative of distribution for brevity.

It follows from equation (3) that

$$(g * \Phi_\lambda) * \Phi_\mu = g * (\Phi_\lambda * \Phi_\mu) = g * \Phi_{\lambda+\mu} \tag{4}$$

for any distribution $g(x)$ in $\mathcal{D}'(R_+)$.

Setting $\mu = -\lambda$, we see that differentiation and integration of the same order are mutually inverse processes, and the sequential fractional derivative law holds from equation (3)

$$\frac{d^\lambda}{dx^\lambda}\left(\frac{d^\mu g}{dx^\mu}\right) = \frac{d^{\lambda+\mu} g}{dx^{\lambda+\mu}} = \frac{d^\mu}{dx^\mu}\left(\frac{d^\lambda g}{dx^\lambda}\right)$$

for any complex numbers λ and μ.

Clearly, we may write

$$\frac{d^\lambda}{dx^\lambda}\left(\frac{x_+^\mu}{\Gamma(\mu+1)}\right) = \frac{x_+^{\mu-\lambda}}{\Gamma(\mu+1-\lambda)} \tag{5}$$

by replacing λ by $-\lambda$, μ by $\mu+1$ in equation (3). In particular, for $\mu = 0$, we get

$$\frac{d^\lambda}{dx^\lambda}\theta(x) = \frac{x_+^{-\lambda}}{\Gamma(1-\lambda)} = \Phi_{1-\lambda}.$$

Writing $\mu = -k-1$ in equation (4) for nonnegative integer k, we find

$$\frac{d^\lambda}{dx^\lambda}\delta^{(k)}(x) = \frac{x_+^{-k-\lambda-1}}{\Gamma(-k-\lambda)} = \Phi_{-k-\lambda}.$$

Setting λ by $-\lambda$ in the above, we obtain

$$\frac{d^{-\lambda}}{dx^{-\lambda}}\delta^{(k)}(x) = \frac{x_+^{-k+\lambda-1}}{\Gamma(-k+\lambda)}$$

which implies

$$\delta^{(k)}(x) = \frac{d^\lambda}{dx^\lambda}\frac{x_+^{\lambda-k-1}}{\Gamma(\lambda-k)}.$$

Let us consider the function given by

$$f(x) = \begin{cases} 1 & \text{if } 0 \le a < x < b, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$f(x) = \theta(x-a) - \theta(x-b)$$

in the distributional sense and

$$\frac{d^\lambda}{dx^\lambda} f(x) = \frac{d^\lambda}{dx^\lambda}(\theta(x-a) - \theta(x-b)) = \frac{(x-a)_+^{-\lambda}}{\Gamma(1-\lambda)} - \frac{(x-b)_+^{-\lambda}}{\Gamma(1-\lambda)}.$$

It seems impossible to define products of two arbitrary distributions in general [12, 13]. However, the product of an infinitely differentiable function $\varphi(x)$ with a distribution $f(x)$ is given by

$$(\varphi(x) f(x),\ \psi(x)) = (f(x),\ \varphi(x)\psi(x))$$

which is well defined since $\varphi(x)\psi(x) \in \mathcal{D}(R)$ if $\psi(x) \in \mathcal{D}(R)$.

It follows that

$$\varphi(x)\delta(x) = \varphi(0)\delta(x)$$

since

$$(\varphi(x)\delta(x),\ \psi(x)) = (\delta(x),\ \varphi(x)\psi(x)) = \varphi(0)\psi(0) = \varphi(0)(\delta(x), \psi(x)).$$

Theorem 1.1 Let $\varphi(x) \in C^m[0, \infty)$ and $m-1 < \lambda < m \in Z^+$. Then

$$\begin{aligned} \frac{d^\lambda}{dx^\lambda}(\theta(x)\varphi(x)) &= \varphi(0)\frac{x_+^{-\lambda}}{\Gamma(1-\lambda)} + \cdots + \varphi^{(m-1)}(0)\frac{x_+^{m-\lambda-1}}{\Gamma(m-\lambda)} \\ &\quad + \frac{1}{\Gamma(m-\lambda)}\int_0^x \varphi^{(m)}(t)(x-t)^{m-\lambda-1}dt. \end{aligned}$$

Proof. Clearly,

$$\frac{d^\lambda}{dx^\lambda}(\theta(x)\varphi(x)) = (\theta(x)\varphi(x)) * \frac{x_+^{-\lambda-1}}{\Gamma(-\lambda)} = (\theta(x)\varphi(x)) * \frac{d^m}{dx^m}\frac{x_+^{m-\lambda-1}}{\Gamma(m-\lambda)}$$
$$= \frac{d^m}{dx^m}(\theta(x)\varphi(x)) * \frac{x_+^{m-\lambda-1}}{\Gamma(m-\lambda)}$$

where $0 < m - \lambda < 1$.

First we assume that $\varphi(x) \in C^\infty[0, \infty)$. By definition, we come to

$$(\frac{d}{dx}(\theta(x)\varphi(x)),\ \psi(x)) = -(\theta(x)\varphi(x),\ \psi'(x)) = -\int_0^\infty \varphi(x)\psi'(x)dx$$
$$= -\varphi(x)\psi(x)|_0^\infty + \int_0^\infty \varphi'(x)\psi(x)dx = (\varphi(0)\delta(x) + \theta(x)\varphi'(x),\ \psi(x))$$

which implies

$$\frac{d}{dx}(\theta(x)\varphi(x)) = \varphi(0)\delta(x) + \theta(x)\varphi'(x).$$

Evidently from recursion we get

$$\begin{aligned}\frac{d^2}{dx^2}(\theta(x)\varphi(x)) &= \frac{d}{dx}(\varphi(0)\delta(x) + \theta(x)\varphi'(x)) \\ &= \varphi(0)\delta'(x) + \varphi'(0)\delta(x) + \theta(x)\varphi''(x),\end{aligned}$$

where $\theta(x)\varphi''(x)$ is defined in the distributional sense.

This claims in general

$$\frac{d^m}{dx^m}(\theta(x)\varphi(x)) = \varphi(0)\delta^{(m-1)}(x) + \cdots + \varphi^{(m-1)}(0)\delta(x) + \theta(x)\varphi^{(m)}(x).$$

Secondly, we suppose that $\varphi(x) \in C^m[0,\infty)$ and $\varphi_1(x) \in C^m(R)$ such that $\varphi_1(x) = \varphi(x)$ for $x \in [0,\infty)$. Furthermore, we let $\rho(x)$ be a fixed infinitely differentiable function on R with four properties

(i) $\rho(x) \geq 0$,

(ii) $\rho(x) = 0$ for $|x| \geq 1$,

(iii) $\rho(x) = \rho(-x)$,

(iv) $\int_{-1}^{1} \rho(x)dx = 1$.

Obviously, the Temple sequence $\delta_n(x) = n\rho(mx)$ is an infinitely differentiable sequence converging to δ in $\mathcal{D}'(R)$ as $n \to \infty$. Then the convolution given by

$$\psi_n(x) = \varphi_1^{(m)}(x) * \delta_n(x) = \int_{-\infty}^{\infty} \varphi_1^{(m)}(x-y)\delta_m(y)dy$$

is an infinitely differentiable sequence and uniformly converges to $\varphi_1^{(m)}(x)$ as $n \to \infty$ on every compact subset $L \subset R$. Indeed, $\varphi_1^{(m)}(x)$ is uniformly continuous on L since it is continuous on L. Therefore, for all $\varepsilon > 0$ there exists $\delta > 0$, such that

$$|\varphi_1^{(m)}(x-y) - \varphi_1^{(m)}(x)| < \varepsilon$$

for all $x \in L$ and $|y| < \delta$. Choosing $n > 1/\delta$, we arrive at

$$|\psi_n(x) - \varphi_1^{(m)}(x)| \leq \int_{-\infty}^{\infty} |\varphi_1^{(m)}(x-y) - \varphi_1^{(m)}(x)|\delta_m(y)dy < \varepsilon$$

holds for all $y \in L$.

It follows that

$$\frac{d^m}{dx^m}\theta(x)\psi_n(x) = \psi_n(0)\delta^{(m-1)}(x) + \cdots + \psi_n^{(m-1)}(0)\delta(x) + \theta(x)\psi_n^{(m)}(x)$$

since $\psi_n(x)$ is an infinitely differentiable sequence.

Clearly,

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\theta(x)\psi_n^{(m)}(x)\varphi(x)dx=\int_0^{\infty}\varphi^{(m)}(x)\varphi(x)dx$$

for all $\varphi(x)\in\mathcal{D}(R)$ and

$$\begin{aligned}&\lim_{n\to\infty}\psi_n(0)=\varphi_1(0)=\varphi(0),\\&\cdots\\&\lim_{n\to\infty}\psi_n^{(m-1)}(0)=\varphi_1^{(m-1)}(0)=\varphi^{(m-1)}(0).\end{aligned}$$

Therefore,

$$\frac{d^m}{dx^m}(\theta(x)\varphi(x))=\varphi(0)\delta^{(m-1)}(x)+\cdots+\varphi^{(m-1)}(0)\delta(x)+\theta(x)\varphi^{(m)}(x).$$

for all $\varphi(x)\in C^m[0,\infty)$, which implies that

$$\begin{aligned}\frac{d^\lambda}{dx^\lambda}(\theta(x)\varphi(x)) &= (\varphi(0)\delta^{(m-1)}(x)+\cdots+\varphi^{(m-1)}(0)\delta(x)+\theta(x)\varphi^{(m)}(x))*\frac{x_+^{m-\lambda-1}}{\Gamma(m-\lambda)}\\ &= \varphi(0)\frac{x_+^{-\lambda}}{\Gamma(1-\lambda)}+\cdots+\varphi^{(m-1)}(0)\frac{x_+^{m-\lambda-1}}{\Gamma(m-\lambda)}\\ &\quad+\frac{1}{\Gamma(m-\lambda)}\int_0^x\varphi^{(m)}(t)(x-t)^{m-\lambda-1}dt.\end{aligned}$$

This completes the proof of Theorem 1.1. □

In particular,

$$\begin{aligned}&\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}(\theta(x)x)=\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}x_+=\frac{2}{\sqrt{\pi}}x_+^{\frac{1}{2}},\\&\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\theta(x)=\frac{1}{\sqrt{\pi}}x_+^{-\frac{1}{2}}\end{aligned}$$

using $\Gamma(1/2)=\sqrt{\pi}$.

Remark 1: Theorem 1.1 is an extension of Theorem 4.2 given in [11], where $\varphi(x)\in C^\infty[0,\infty)$ is assumed and its proof is more complicated via integration by parts.

Assume that

$$\varphi^{(m)}(t)=\sum_{k=0}^{\infty}\frac{\varphi^{(m+k)}(0)}{k!}t^k.$$

Making the substitution $t=ux$, we have

$$\int_0^x t^k(x-t)^{m-\lambda-1}dt=x^{m-\lambda+k}\int_0^1u^k(1-u)^{m-\lambda-1}du=x^{m-\lambda+k}\frac{\Gamma(k+1)\,\Gamma(m-\lambda)}{\Gamma(m-\lambda+k+1)}.$$

Therefore,

$$\begin{aligned}\frac{d^\lambda}{dx^\lambda}(\theta(x)\varphi(x)) &= \left(\varphi(0)\frac{x_+^{-\lambda}}{\Gamma(1-\lambda)}+\cdots+\varphi^{(m-1)}(0)\frac{x_+^{m-\lambda-1}}{\Gamma(m-\lambda)}\right)+\sum_{k=0}^{\infty}\frac{\varphi^{(m+k)}(0)\,x_+^{m-\lambda+k}}{\Gamma(m-\lambda+k+1)}\\ &= \sum_{k=-m}^{\infty}\frac{\varphi^{(m+k)}(0)\,x_+^{m-\lambda+k}}{\Gamma(m-\lambda+k+1)}.\end{aligned}$$

In particular, for $\lambda = 1/2$ (thus $m = 1$) and $\varphi(x) = e^x$, we get

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}(\theta(x)e^x) = \frac{x_+^{-\frac{1}{2}}}{\sqrt{\pi}}+\sum_{k=0}^{\infty}\frac{x_+^{k+\frac{1}{2}}}{\Gamma(k+\frac{3}{2})} = \sum_{k=-1}^{\infty}\frac{x_+^{k+\frac{1}{2}}}{\Gamma(k+\frac{3}{2})}.$$

In general, we have the following theorem.

Theorem 1.2 Let $\varphi_1(x) \in C^m[a,b]$ and $\varphi_2(x) \in C^m[c,d]$, and let $f(x)$ be the function given by

$$f(x) = \begin{cases}\varphi_1(x) & \text{if } 0 \le a < x < b,\\ \varphi_2(x) & \text{if } b \le c \le x < d,\\ 0 & \text{otherwise,}\end{cases}$$

where $m-1 < \lambda < m \in Z^+$. Then

$$\begin{aligned}\frac{d^\lambda}{dx^\lambda}f(x) &= (\varphi_1(a)-\varphi_1(b))\frac{(x-a)_+^{-\lambda}-(x-b)_+^{-\lambda}}{\Gamma(1-\lambda)}+\cdots+\\ &\quad (\varphi_1^{(m-1)}(a)-\varphi_1^{(m-1)}(b))\frac{(x-a)_+^{m-\lambda-1}-(x-b)_+^{m-\lambda-1}}{\Gamma(m-\lambda)}\\ &\quad +\frac{1}{\Gamma(m-\lambda)}\int_0^x(\theta(t-a)-\theta(t-b))\varphi_1^{(m)}(t)(x-t)^{m-\lambda-1}dt\\ &\quad +(\varphi_2(c)-\varphi_2(d))\frac{(x-c)_+^{-\lambda}-(x-d)_+^{-\lambda}}{\Gamma(1-\lambda)}+\cdots+\\ &\quad (\varphi_2^{(m-1)}(c)-\varphi_2^{(m-1)}(d))\frac{(x-c)_+^{m-\lambda-1}-(x-d)_+^{m-\lambda-1}}{\Gamma(m-\lambda)}\\ &\quad +\frac{1}{\Gamma(m-\lambda)}\int_0^x(\theta(t-c)-\theta(t-d))\varphi_2^{(m)}(t)(x-t)^{m-\lambda-1}dt.\end{aligned}$$

Proof. It follows from Theorem 1.1 and identities

$$f(x) = \varphi_1(x)(\theta(x-a)-\theta(x-b))+\varphi_2(x)(\theta(x-c)-\theta(x-d)),$$

and

$$\begin{aligned}\frac{d^\lambda}{dx^\lambda}\theta(x-a) &= \frac{(x-a)_+^{-\lambda}}{\Gamma(1-\lambda)},\\ \varphi(x)\delta(x-a) &= \varphi(a)\delta(x-a).\end{aligned}$$

This completes the proof. □

Similarly, we have the following Theorem 1.3 from identity

$$f(x) = \varphi_1(x)(\theta(x-a) - \theta(x-b)) + \varphi_2(x)\theta(x-c).$$

Theorem 1.3 Let $\varphi_1(x) \in C^m[a,b]$ and $\varphi_2(x) \in C^m[c,\infty)$, where $c \geq b$, and let $f(x)$ be the function given by

$$f(x) = \begin{cases} \varphi_1(x) & \text{if } 0 \leq a < x < b, \\ \varphi_2(x) & \text{if } x \geq c, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \frac{d^\lambda}{dx^\lambda} f(x) &= (\varphi_1(a) - \varphi_1(b)) \frac{(x-a)_+^{-\lambda} - (x-b)_+^{-\lambda}}{\Gamma(1-\lambda)} + \cdots + \\ & (\varphi_1^{(m-1)}(a) - \varphi_1^{(m-1)}(b)) \frac{(x-a)_+^{m-\lambda-1} - (x-b)_+^{m-\lambda-1}}{\Gamma(m-\lambda)} \\ & + \frac{1}{\Gamma(m-\lambda)} \int_0^x (\theta(t-a) - \theta(t-b)) \varphi_1^{(m)}(t)(x-t)^{m-\lambda-1} dt \\ & + \varphi_2(c) \frac{(x-c)_+^{-\lambda}}{\Gamma(1-\lambda)} + \cdots + \\ & \varphi_2^{(m-1)}(c) \frac{(x-c)_+^{m-\lambda-1}}{\Gamma(m-\lambda)} \\ & + \frac{1}{\Gamma(m-\lambda)} \int_0^x \theta(t-c) \varphi_2^{(m)}(t)(x-t)^{m-\lambda-1} dt. \end{aligned}$$

where $m - 1 < \lambda < m \in Z^+$.

We consider the function defined as

$$f(x) = \begin{cases} e^x & \text{if } 0 < x < 1, \\ 1 & \text{if } x \geq 1. \end{cases}$$

Note that this function is even discontinuous at $x = 1$. However, we can find $f^{(\frac{1}{2})}(x)$ in the distributional sense by Theorem 1.3. Indeed,

$$\begin{aligned} f^{(\frac{1}{2})}(x) = \frac{d^{\frac{1}{2}} f(x)}{dx^{\frac{1}{2}}} &= (e^0 - e^1) \frac{x_+^{-\frac{1}{2}} - (x-1)_+^{-\frac{1}{2}}}{\Gamma(1/2)} + \frac{1}{\Gamma(1/2)} \int_0^x (\theta(t) - \theta(t-1)) e^t (x-t)^{-1/2} dt \\ & + \frac{(x-1)_+^{-\frac{1}{2}}}{\Gamma(1/2)} \\ &= (1-e) \frac{x_+^{-\frac{1}{2}}}{\sqrt{\pi}} + e \frac{(x-1)_+^{-\frac{1}{2}}}{\sqrt{\pi}} + \frac{1}{\sqrt{\pi}} \int_0^x (1 - \theta(t-1)) e^t (x-t)^{-1/2} dt. \end{aligned}$$

In particular, we arrive at for $x \geq 1$

$$\begin{aligned} f^{(\frac{1}{2})}(x) &= (1-e)\frac{x^{-\frac{1}{2}}}{\sqrt{\pi}} + e\frac{(x-1)^{-\frac{1}{2}}}{\sqrt{\pi}} + \frac{1}{\sqrt{\pi}}\int_0^1 e^t (x-t)^{-1/2} dt \\ &= (1-e)\frac{x^{-\frac{1}{2}}}{\sqrt{\pi}} + e\frac{(x-1)^{-\frac{1}{2}}}{\sqrt{\pi}} + \frac{1}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{1}{k!}\int_0^1 t^k (x-t)^{-1/2} dt \end{aligned}$$

using

$$e^t = \sum_{k=0}^{\infty} \frac{t^k}{k!}.$$

Similarly,

$$\begin{aligned} f^{(\frac{1}{2})}(x) &= (1-e)\frac{x^{-\frac{1}{2}}}{\sqrt{\pi}} + \frac{1}{\sqrt{\pi}}\int_0^x e^t (x-t)^{-1/2} dt \\ &= (1-e)\frac{x^{-\frac{1}{2}}}{\sqrt{\pi}} + \frac{1}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{1}{k!}\int_0^x t^k (x-t)^{-1/2} dt \\ &= (1-e)\frac{x^{-\frac{1}{2}}}{\sqrt{\pi}} + \frac{1}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{1}{k!}x^{k+1/2}\int_0^1 t^k (1-t)^{-1/2} dt \\ &= (1-e)\frac{x^{-\frac{1}{2}}}{\sqrt{\pi}} + \sum_{k=0}^{\infty}\frac{x^{k+1/2}}{\Gamma(k+3/2)} \end{aligned}$$

if $0 < x < 1$.

The Leibniz's Rule of differentiation in distribution is given in the following based on the generalized convolution.

Theorem 1.4 Let f be an arbitrary distribution in $\mathcal{D}'(R_+)$ and φ be an infinitely differentiable function. Then

$$\frac{d^\lambda}{dx^\lambda}(\varphi(x)f(x)) = \sum_{k=0}^{\infty}\binom{\lambda}{k}\frac{d^{\lambda-k}}{dx^{\lambda-k}}f(x)\cdot\varphi^{(k)}(x) = \sum_{k=0}^{\infty}\frac{\Gamma(\lambda+1)}{k!\,\Gamma(\lambda-k+1)}\frac{d^{\lambda-k}}{dx^{\lambda-k}}f(x)\cdot\varphi^{(k)}(x)$$

holds for $m-1 < \lambda < m \in Z^+$.

It follows from Theorem 1.4 and $\varphi(x) \in C^\infty(R)$ that

$$\begin{aligned} \frac{d^\lambda}{dx^\lambda}(\theta(x)\varphi(x)) &= \sum_{k=0}^{\infty}\binom{\lambda}{k}\varphi^{(k)}(x)\frac{d^{\lambda-k}}{dx^{\lambda-k}}\theta(x) \\ &= \sum_{k=0}^{\infty}\binom{\lambda}{k}\varphi^{(k)}(x)\frac{x_+^{k-\lambda}}{\Gamma(1-\lambda+k)}. \end{aligned}$$

Note that the product $\varphi^{(k)}(x)\dfrac{x_+^{k-\lambda}}{\Gamma(1-\lambda+k)}$ is well defined in the distributional sense since $\varphi^{(k)}(x)$ is an infinitely differentiable function.

Let $f(x)$ be the function given in Theorem 1.2. Then

$$\begin{aligned}\frac{d^\lambda}{dx^\lambda}f(x) &= \sum_{k=0}^{\infty}\binom{\lambda}{k}\varphi_1^{(k)}(x)\left(\frac{(x-a)_+^{k-\lambda}-(x-b)_+^{k-\lambda}}{\Gamma(1-\lambda+k)}\right)\\ &+ \sum_{k=0}^{\infty}\binom{\lambda}{k}\varphi_2^{(k)}(x)\left(\frac{(x-c)_+^{k-\lambda}-(x-d)_+^{k-\lambda}}{\Gamma(1-\lambda+k)}\right),\end{aligned}$$

if $\varphi_1(x)$ and $\varphi_2(x)$ are infinitely differentiable functions on their respective intervals.

Similarly, we let $\varphi_1(x)\in C^\infty[a,b]$ and $\varphi_2(x)\in C^\infty[c,\infty)$, where $c\geq b$, and let $f(x)$ be the function given by

$$f(x)=\begin{cases}\varphi_1(x) & \text{if } 0\leq a<x<b,\\ \varphi_2(x) & \text{if } x\geq c,\\ 0 & \text{otherwise.}\end{cases}$$

Then

$$\begin{aligned}\frac{d^\lambda}{dx^\lambda}f(x) &= \sum_{k=0}^{\infty}\binom{\lambda}{k}\varphi_1^{(k)}(x)\left(\frac{(x-a)_+^{k-\lambda}-(x-b)_+^{k-\lambda}}{\Gamma(1-\lambda+k)}\right)\\ &+\sum_{k=0}^{\infty}\binom{\lambda}{k}\varphi_2^{(k)}(x)\frac{(x-c)_+^{k-\lambda}}{\Gamma(1-\lambda+k)}.\end{aligned}$$

2 Fractional derivatives of composite functions

Now, we assume that $\varphi(x)$ is a composite function

$$\varphi(x)=F(h(x)).$$

The m-th order derivative of $\varphi(x)$ can be obtained with the help of Faà di Bruno formula [14]:

$$\frac{d^m}{dx^m}F(h(x))=m!\sum_{k=1}^{m}F^{(k)}(h(x))\sum\prod_{r=1}^{m}\frac{1}{a_r!}\left(\frac{h^{(r)}(x)}{r!}\right)^{a_r},$$

where the sum $\sum$ extends over all combinations of non-negative integer values of $a_1,a_2,\cdots,a_m$ such that

$$\sum_{r=1}^{m}ra_r=m \quad\text{and}\quad \sum_{r}^{m}a_r=k.$$

The following can be derived from Theorem 1.1 and Faà di Bruno formula.

Theorem 2.1 Let $F(x)$ and $h(x)$ be functions in $C^m[0,\infty)$. Then

$$\begin{aligned}\frac{d^\lambda}{dx^\lambda}(\theta(x)F(h(x))) &= F(h(0))\frac{x_+^{-\lambda}}{\Gamma(1-\lambda)}+\cdots+F^{(m-1)}(h(0))\frac{x_+^{m-\lambda-1}}{\Gamma(m-\lambda)}\\ &+\frac{1}{\Gamma(m-\lambda)}\int_0^x F^{(m)}(h(t))(x-t)^{m-\lambda-1}dt\end{aligned}$$

where $m-1<\lambda<m\in Z^+$ and

$$\frac{d^{m-1}}{dx^{m-1}}F(h(0))=(m-1)!\sum_{k=1}^{m-1}F^{(k)}(h(0))\sum\prod_{r=1}^{m-1}\frac{1}{a_r!}\left(\frac{h^{(r)}(0)}{r!}\right)^{a_r},$$

where the sum $\sum$ and coefficients a_r have the meaning explained above.

As an example, we find out $\frac{d^\lambda}{dx^\lambda}(\theta(x)\ln(1+x))$ for $m-1<\lambda<m\in Z^+$. Clearly,

$$\ln^{(m)}(1+x)=(-1)^{m-1}(m-1)!(1+x)^{-m}.$$

By Theorem 2.1 or 1.1, we have

$$\begin{aligned}\frac{d^\lambda}{dx^\lambda}(\theta(x)\ln(1+x)) &= \frac{x_+^{1-\lambda}}{\Gamma(2-\lambda)}+\cdots+(-1)^{m-2}(m-2)!\frac{x_+^{m-\lambda-1}}{\Gamma(m-\lambda)}\\ &\quad+\frac{(-1)^{m-1}(m-1)!}{\Gamma(m-\lambda)}\int_0^x\frac{(x-t)^{m-\lambda-1}}{(1+x)^m}dt.\end{aligned}$$

It follows from [15] that

$$\frac{d^m}{dx^m}\left(\frac{x}{x^2+b^2}\right)=\frac{(-1)^m m!}{(x^2+b^2)^{m+1}}\sum_{0\le 2k\le m+1}(-1)^k\binom{m+1}{2k}b^{2k}x^{m+1-2k},$$

$$\frac{d^m}{dx^m}\left(\frac{b}{x^2+b^2}\right)=\frac{(-1)^m m!}{(x^2+b^2)^{m+1}}\sum_{0\le 2k\le m}(-1)^k\binom{m+1}{2k+1}b^{2k+1}x^{m-2k}$$

we are able to get

$$\frac{d^\lambda}{dx^\lambda}\left(\theta(x)\left(\frac{x}{x^2+1}\right)\right)\quad\text{and}\quad\frac{d^\lambda}{dx^\lambda}\left(\theta(x)\left(\frac{1}{x^2+1}\right)\right).$$

from Theorem 2.1.

Remark 2: We can compute the fractional derivative

$$\frac{d^\lambda}{dx^\lambda}(\theta(x)f(x)g(x))$$

based on the classical Leibniz's rule

$$(f(x)g(x))^{(m)}=\sum_{k=0}^{m}\binom{m}{k}f^{(k)}(x)g^{(m-k)}(x).$$

Hence,

$$\begin{aligned}\frac{d^\lambda}{dx^\lambda}(\theta(x)xe^x) &= \frac{x_+^{1-\lambda}}{\Gamma(2-\lambda)}+\cdots+(m-1)\frac{x_+^{m-\lambda-1}}{\Gamma(m-\lambda)}\\ &\quad+\frac{1}{\Gamma(m-\lambda)}\int_0^x(m+t)e^t(x-t)^{m-\lambda-1}dt.\end{aligned}$$

by using

$$(xe^x)^{(m)} = (m+x)e^x.$$

Furthermore, we can carry out

$$\frac{d^\lambda}{dx^\lambda}(\theta(x)x\sin x)$$

based on the formula

$$(x\sin x)^{(m)} = x\sin(\frac{m\pi}{2}+x) - m\cos(\frac{m\pi}{2}+x).$$

We leave it to interested readers.

3 An approximation of $\frac{d^\lambda}{dx^\lambda}(\theta(x)\varphi(x))$

Let us consider the distribution

$$-\frac{\delta(x)-\delta(x+h)}{h} \quad \text{where } h>0\ ,$$

which converges to $\delta'(x)$ in $\mathcal{D}'(a,x]$, since we have for $\varphi \in \mathcal{D}[a,x]$

$$-\lim_{h\to 0}\left(\frac{\delta(x)-\delta(x+h)}{h}, \varphi(x)\right) = -\lim_{h\to 0}\frac{\varphi(0)-\varphi(-h)}{h} = -\varphi'(0)$$

where $a \le 0$ and $x > 0$.

Applying this twice gives the second-order derivative:

$$\begin{aligned}\delta''(x) &= (-1)^1 \lim_{h\to 0}\frac{\delta'(x)-\delta'(x+h)}{h}\\ &= (-1)^2 \lim_{h\to 0}\frac{1}{h}\left\{\frac{\delta(x)-\delta(x+h)}{h} - \frac{\delta(x+h)-\delta(x+2h)}{h}\right\}\\ &= (-1)^2 \lim_{h\to 0}\frac{\delta(x)-2\delta(x+h)+\delta(x+2h)}{h^2}.\end{aligned}$$

By induction,

$$\delta^{(n)}(x) = (-1)^n \lim_{h\to 0}\frac{1}{h^n}\sum_{r=0}^{n}(-1)^r\binom{n}{r}\delta(x+rh), \tag{6}$$

where

$$\binom{n}{r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$$

is the usual notation for the binomial coefficients.

Let $\lambda > 0$ and $\varphi \in \mathcal{D}[a,x]$. It follows from Podlubny [16] that

$$\varphi_h^{(\lambda)}(x) = h^{-\lambda}\sum_{r=0}^{n}(-1)^r\binom{\lambda}{r}\varphi(x-rh) \quad \text{where } nh = x-a$$

converges in the usual sense and

$$\begin{aligned}\lim_{h\to 0}\varphi_h^{(\lambda)}(x) &= \sum_{k=0}^{m}\frac{\varphi^{(k)}(a)(x-a)^{-\lambda+k}}{\Gamma(-\lambda+k+1)}\\ &+ \frac{1}{\Gamma(-\lambda+m+1)}\int_a^x (x-\tau)^{m-\lambda}\varphi^{(m+1)}(\tau)d\tau\\ &= \frac{1}{\Gamma(-\lambda+m+1)}\int_a^x (x-\tau)^{m-\lambda}\varphi^{(m+1)}(\tau)d\tau\end{aligned}$$

where m is an integer satisfying $m \le \lambda < m+1$. In particular,

$$\begin{aligned}\lim_{h\to 0}(-1)^{\lambda}\varphi_h^{(\lambda)}(0) &= \lim_{h\to 0}(-1)^{\lambda}h^{-\lambda}\sum_{r=0}^{n}(-1)^r\binom{\lambda}{r}\varphi(-rh)\\ &= \frac{(-1)^m}{\Gamma(-\lambda+m+1)}\int_a^0 \tau^{m-\lambda}\varphi^{(m+1)}(\tau)d\tau,\end{aligned}$$

where $(-1)^{\lambda} = \cos\lambda\pi + i\sin\lambda\pi$.

Let us consider the expression

$$\delta_h^{(\lambda)}(x) = (-1)^{\lambda}h^{-\lambda}\sum_{r=0}^{n}(-1)^r\binom{\lambda}{r}\delta(x+rh),$$

and clearly we have for $\varphi \in \mathcal{D}(a, x]$,

$$\lim_{h\to 0}(\delta_h^{(\lambda)}(x), \varphi(x)) = \lim_{h\to 0}(-1)^{\lambda}\varphi_h^{(\lambda)}(0) = \frac{(-1)^m}{\Gamma(-\lambda+m+1)}\int_a^0 \tau^{m-\lambda}\varphi^{(m+1)}(\tau)d\tau. \tag{7}$$

In particular, we have for $\lambda = m$ that

$$\lim_{h\to 0}(\delta_h^{(m)}(x), \varphi(x)) = (-1)^m\varphi^{(m)}(0) = (\delta^{(m)}(x), \varphi(x))$$

using $\varphi^{(m)}(a) = 0$.

On the other hand, we have

$$\delta^{(\lambda)}(x) = \frac{d^{\lambda}}{dx^{\lambda}}\delta(x) = \frac{x_+^{-\lambda-1}}{\Gamma(-\lambda)} = \frac{d^{m+1}}{dx^{m+1}}\frac{x_+^{-\lambda+m}}{\Gamma(-\lambda+m+1)}$$

where $m \le \lambda < m+1$. This implies that

$$\begin{aligned}(\delta^{(\lambda)}(x), \varphi(x)) &= \frac{(-1)^{m+1}}{\Gamma(-\lambda+m+1)}\int_0^{\infty} x_+^{-\lambda+m}\varphi^{(m+1)}(x)dx\\ &= \frac{(-1)^m}{\Gamma(-\lambda+m+1)}\int_{-\infty}^0 \tau^{m-\lambda}\varphi^{(m+1)}(\tau)d\tau\\ &= \frac{(-1)^m}{\Gamma(-\lambda+m+1)}\int_a^0 \tau^{m-\lambda}\varphi^{(m+1)}(\tau)d\tau\end{aligned}$$

by using the fact that

$$(g(x), \varphi(x)) = (g(-x), \varphi(-x))$$

if $g(x)$ is a locally integrable function.

In summary, we come to the following result.

Theorem 3.1 The expression

$$\delta_h^{(\lambda)}(x) = (-1)^\lambda h^{-\lambda} \sum_{r=0}^{n} (-1)^r \binom{\lambda}{r} \delta(x+rh) \quad \text{where } nh = x - a$$

is the first-order approximation to the distribution $\delta^{(\lambda)}(x)$ $(\lambda > 0)$ in the distributional sense.

Proof We assume that $a = 0$ for simplicity, therefore $x = nh$, where x is the point at which the fractional derivative is evaluated. Clearly,

$$\begin{aligned} \varphi_h^{(\lambda)}(x) &= h^{-\lambda} \sum_{r=0}^{n} (-1)^r \binom{\lambda}{r} \varphi(x - rh) \\ &= h^{-\lambda} \sum_{r=0}^{n} \binom{r-\lambda-1}{r} \varphi(x - rh). \end{aligned}$$

We start with the simplest function $\varphi(x) = \theta(x)$, which is not a testing function. Evidently,

$$\frac{d^\lambda}{dx^\lambda}\varphi = \frac{d^\lambda}{dx^\lambda}\theta(x) = \frac{x_+^{-\lambda}}{\Gamma(1-\lambda)}.$$

On the other hand,

$$\theta_h^{(\lambda)}(x) = h^{-\lambda} \sum_{r=0}^{n} \binom{r-\lambda-1}{r}.$$

Applying the following summation formula [16]

$$\sum_{r=0}^{n} \binom{r-\lambda-1}{r} = \binom{n-\lambda}{n}$$

and the asymptotic formula [17]

$$z^{b-a} \frac{\Gamma(z+a)}{\Gamma(z+b)} = 1 + O(z^{-1}), \tag{8}$$

we come to

$$\begin{aligned} \theta_h^{(\lambda)}(x) &= h^{-\lambda} \binom{n-\lambda}{n} = \frac{x_+^{-\lambda}}{\Gamma(1-\lambda)} \frac{n^\lambda \Gamma(n-\lambda+1)}{\Gamma(n+1)} \\ &= \frac{x_+^{-\lambda}}{\Gamma(1-\lambda)}(1 + O(h)), \end{aligned}$$

which gives the first-order approximation. Note that

$$\frac{x_+^{-\lambda}}{\Gamma(1-\lambda)}(1+O(h)) = \frac{x_+^{-\lambda}}{\Gamma(1-\lambda)} + O(h)$$

in the distributional sense since $\varphi(x) = 1$ is an infinitely differentiable function.

Let us consider $\varphi(x) = x_+^m$ for $m = 1, 2, \cdots$. Then the exact λ-th distributional derivative is

$$\frac{d^\lambda}{dx^\lambda} x_+^m = \Gamma(m+1)\frac{d^\lambda}{dx^\lambda}\frac{x_+^m}{\Gamma(m+1)} = \frac{\Gamma(m+1)}{\Gamma(m+1-\lambda)} x_+^{m-\lambda}$$

and the approximation of the exact derivative is

$$\begin{aligned} (x_+^m)_h^{(\lambda)} &= x_+^{m-\lambda} n^\lambda \sum_{r=0}^{n} \binom{r-\lambda-1}{r}\left(1-\frac{r}{n}\right)^m \\ &= x_+^{m-\lambda} \sum_{j=0}^{m} (-1)^j \binom{m}{j} n^{\lambda-j} \sum_{r=0}^{n} \binom{r-\lambda-1}{r} r^j. \end{aligned}$$

It follows from [16] that

$$\begin{aligned} S &= \sum_{r=0}^{n} \binom{r-\lambda-1}{r} r^j \\ &= \sum_{i=1}^{j} \sigma_j^{(i)} \frac{\Gamma(n-\lambda+1)}{(i-\lambda)\Gamma(-\lambda)\Gamma(n-i+1)} \end{aligned}$$

where $\sigma_j^{(i)}$ are Stirling numbers of second kind and $\sigma_j^{(j)} = 1$.

Substituting the above back we get

$$(x_+^m)_h^{(\lambda)} = \frac{x_+^{m-\lambda}}{\Gamma(-\lambda)} \sum_{j=0}^{m} (-1)^j \binom{m}{j} \sum_{i=1}^{j} \sigma_j^{(i)} \frac{n^{\lambda-j}\Gamma(n-\lambda+1)}{(i-\lambda)\Gamma(-\lambda)\Gamma(n-i+1)}.$$

Applying equation (8), we have

$$\frac{n^{\lambda-j}\Gamma(n-\lambda+1)}{\Gamma(n-i+1)} = n^{i-j}\left(n^{\lambda-i}\frac{\Gamma(n-\lambda+1)}{\Gamma(n-i+1)}\right) = n^{i-j}(1+O(n^{-1})).$$

It follows that

$$\begin{aligned} (x_+^m)_h^{(\lambda)} &= \frac{x_+^{m-\lambda}}{\Gamma(-\lambda)} \sum_{j=0}^{m} (-1)^j \binom{m}{j} \sum_{i=1}^{j} \sigma_j^{(i)} \frac{1}{i-\lambda} n^{i-j}(1+O(n^{-1})) \\ &= \frac{x_+^{m-\lambda}}{\Gamma(-\lambda)} \sum_{j=0}^{m} (-1)^j \binom{m}{j} \frac{1}{j-\lambda}(1+O(n^{-1})). \end{aligned}$$

Using the formula [16]

$$\sum_{j=0}^{m}(-1)^j\binom{m}{j}\frac{1}{j-\lambda}=\beta(-\lambda,m+1),$$

we finally get

$$(x_+^m)_h^{(\lambda)}=\frac{\Gamma(m+1)}{\Gamma(m+1-\lambda)}x_+^{m-\lambda}+O(h).$$

This claims that if $\varphi(x)$ can be written in the form of a positive x_+-term polynomial

$$\varphi(x)=\sum_{i=0}^{m}a_ix_+^i,$$

then the fractional difference gives the first-order approximation for the fractional derivative in the distributional sense. Clearly, any testing function $\varphi(x)$ can be approximated by a positive x_+-term polynomial in an arbitrary order. This completes the proof of Theorem 3.1. □

An approximate product of $\varphi(x)\in C^\infty(R)$ and $\delta_h^{(\lambda)}(x)$ follows from Theorem 3.1 that

$$\begin{aligned}\varphi(x)\delta_h^{(\lambda)}(x) &\approx (-1)^\lambda h^{-\lambda}\sum_{r=0}^{n}(-1)^r\binom{\lambda}{r}\varphi(x)\delta(x+rh)\\ &= (-1)^\lambda h^{-\lambda}\sum_{r=0}^{n}(-1)^r\binom{\lambda}{r}\varphi(-rh)\delta(x+rh) \qquad (9)\end{aligned}$$

where $nh=x-a$ and h is small and we note that

$$\varphi(x)\delta(x+rh)=\varphi(-rh)\delta(x+rh).$$

Indeed,

$$(\varphi(x)\delta(x+rh),\psi(x))=\varphi(-rh)\psi(-rh)=\varphi(-rh)(\delta(x+rh),\psi(x)).$$

Let $\varphi(x)\in C^m[0,\infty)$ and $m-1<\lambda<m\in Z^+$. Then we have from Theorem 1.1 that

$$\frac{d^\lambda}{dx^\lambda}(\theta(x)\varphi(x)) = \sum_{k=0}^{m-1}\varphi^{(k)}(0)\frac{x_+^{k-\lambda}}{\Gamma(1+k-\lambda)}+\frac{1}{\Gamma(m-\lambda)}\int_0^x\varphi^{(m)}(t)(x-t)^{m-\lambda-1}dt.$$

Clearly,

$$\frac{x_+^{k-\lambda}}{\Gamma(1+k-\lambda)}=\delta^{(\lambda-k-1)}(x)$$

from Section 1 and

$$\delta^{(\lambda-k-1)}(x)\approx(-1)^{\lambda-k-1}h^{-\lambda+k+1}\sum_{r=0}^{n}(-1)^r\binom{\lambda-k-1}{r}\delta(x+rh)$$

where $nh=x-a$ and h is small.

Thus, we arrive at the following asymptotic theorems in the first-order approximation.

Theorem 3.2 Let $\varphi(x) \in C^m[0,\infty)$ and $m-1 < \lambda < m \in Z^+$. Then

$$\begin{aligned}\frac{d^\lambda}{dx^\lambda}(\theta(x)\varphi(x)) &\approx \sum_{k=0}^{m-1}\sum_{r=0}^{n}(-1)^{\lambda-k-1+r}h^{-\lambda+k+1}\binom{\lambda-k-1}{r}\varphi^{(k)}(0)\delta(x+rh)\\ &+ \frac{1}{\Gamma(m-\lambda)}\int_0^x \varphi^{(m)}(t)(x-t)^{m-\lambda-1}dt = I_1 + I_2,\end{aligned}$$

where I_1 is the distribution given by

$$I_1 = \sum_{k=0}^{m-1}\sum_{r=0}^{n}(-1)^{\lambda-k-1+r}h^{-\lambda+k+1}\binom{\lambda-k-1}{r}\varphi^{(k)}(0)\delta(x+rh),$$

and I_2 is the continuous function defined as

$$I_2 = \frac{1}{\Gamma(m-\lambda)}\int_0^x \varphi^{(m)}(t)(x-t)^{m-\lambda-1}dt.$$

Theorem 3.3 Let $F(x)$ and $h(x)$ be functions in $C^m[0,\infty)$ and $m-1 < \lambda < m \in Z^+$. Then

$$\begin{aligned}\frac{d^\lambda}{dx^\lambda}(\theta(x)F(h(x))) &\approx \sum_{k=0}^{m-1}\sum_{r=0}^{n}(-1)^{\lambda-k-1+r}h^{-\lambda+k+1}\binom{\lambda-k-1}{r}F^{(k)}(h(0))\delta(x+rh)\\ &+ \frac{1}{\Gamma(m-\lambda)}\int_0^x F^{(m)}(h(t))(x-t)^{m-\lambda-1}dt = I_1 + I_2,\end{aligned}$$

where I_1 is the distribution given by

$$I_1 = \sum_{k=0}^{m-1}\sum_{r=0}^{n}(-1)^{\lambda-k-1+r}h^{-\lambda+k+1}\binom{\lambda-k-1}{r}F^{(k)}(h(0))\delta(x+rh),$$

and I_2 is the continuous function defined as

$$I_2 = \frac{1}{\Gamma(m-\lambda)}\int_0^x F^{(m)}(h(t))(x-t)^{m-\lambda-1}dt.$$

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