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The Product and Fractional Derivative of Analytic Functionals

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Abstract

The goal of this paper is to define the product of analytic functionals in Z' by the Fourier transform and convolution in $\mathcal{D}'$. We show that the exchange formula

$$F(f \,\overline{*}\, g) = F(f) \circ F(g)$$

holds. Several interesting examples are presented using this exchange formula and the distribution $\frac{x_+^\lambda}{\Gamma(\lambda+1)}$, where λ is a complex number. Furthermore, we introduce a new definition for directly computing the product of analytic functionals and apply a new approach to defining some fractional derivatives in Z', including $\delta^{(\alpha)}(s)$, by Cauchy's integral formula, which has never been done before.

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1 Introduction

Physicists have long been using the singular function $\delta(x)$, although it cannot be properly defined within the structure of classical function theory. In elementary particle physics, one finds the need to evaluate δ^2 when calculating the transition rates of certain particle interactions [7]. Around 1950, Schwartz established the theory of distributions by treating singular functions as linearly continuous functionals on the testing function space whose elements have compact support. Although the theory is of great importance to quantum field theory and has applications to seeking solutions for differential equations, it is very difficult to define products, convolutions and compositions of distributions in general. The sequential method [9, 3, 13, 4, 5, 14, 12] and complex analysis approach [12, 2, 1], including non-standard analysis [11], have been the main tools thus far in dealing with those non-linear operations of distributions in Schwartz's space $\mathcal{D}'$ with many results.

Let $\mathcal{D}$ be the Schwartz space [8] of infinitely differentiable functions with compact support in R, and $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. Further, we shall define a sequence $\phi_1(x), \phi_2(x), \cdots, \phi_n(x), \cdots$ which converges to zero in $\mathcal{D}$ if all these functions vanish outside a certain fixed and bounded interval, and converge uniformly to zero (in the usual sense) together with their derivatives of any order. The functional δ is defined as

$$(\delta, \phi) = \phi(0)$$

where $\phi \in \mathcal{D}$. Clearly, δ is a linear and continuous functional on $\mathcal{D}$, and hence $\delta \in \mathcal{D}'$.

The convolution of certain pairs of distributions is usually defined as follows, see Gel'fand and Shilov [8] for example.

Definition 1.1 *Let f and g be distributions in $\mathcal{D}'$ satisfying either of the following conditions:*

(a) either f or g has bounded support (set of all essential points), or

(b) the supports of f and g are bounded on the same side.

Then the convolution $f * g$ is defined by the equation

$$((f * g)(x),\, \phi(x)) = (g(x),\, (f(y),\, \phi(x + y)))$$

for $\phi \in \mathcal{D}$.

Let $\delta_n(x) = n\rho(nx)$ be Temples' δ-sequence for $n = 1, 2, \cdots$, where $\rho(x)$ is a fixed, infinitely differentiable function on R with four properties:

(i) $\rho(x) \geq 0$,

(ii) $\rho(x) = 0$ for $|x| \geq 1$,

(iii) $\rho(x) = \rho(-x)$,

(iv) $\int_{-\infty}^{\infty} \rho(x)dx = 1$.

In 2007, Li [13] used the following distributional product definition to deduce several commutative products.

Definition 1.2 *Let f and g be distributions and let $f_n = f * \delta_n$ and $g_n = g * \delta_n$. We say that the commutative neutrix product $f \diamond g$ of f and g exists and is equal to h if*

$$N - \lim_{n\to\infty} \frac{1}{2}\{(f_n g, \phi) + (f g_n, \phi)\} = (h,\, \phi)$$

for all testing functions $\phi \in \mathcal{D}$, where N is the neutrix, see van der Corput [23] (use the neutrix to abandon unwanted infinite quantities from asymptotic expressions), having domain $N' = \{1, 2, \cdots\}$ and range the real numbers, with negligible functions that are finite linear sums of functions

$$n^\lambda \ln^{r-1} n, \quad \ln^r n \qquad (\lambda > 0,\ r = 1, 2, \cdots),$$

and all functions of n that converge to zero in the normal sense as n tends to infinity. If the normal limit exists, then it is simply called the commutative product. Note that f_n and g_n are two infinitely differentiable functions and hence $(f_n g, \phi)$ as well as $(f g_n, \phi)$ are well defined.

As suggested from this definition, we recently presented the following the commutative neutrix convolutional definition [16].

Definition 1.3 *Let f and g be distributions in $\mathcal{D}'$ and let*

$$\tau_n(x) = \begin{cases} 1 & \text{if } |x| \le n, \\ \tau(n^n x - n^{n+1}) & \text{if } x > n, \\ \tau(n^n x + n^{n+1}) & \text{if } x < -n, \end{cases}$$

for $n = 1, 2, \cdots$, where τ is an infinitely differentiable function satisfying the following conditions:

(i) $\tau(x) = \tau(-x)$,

(ii) $0 \le \tau(x) \le 1$,

(iii) $\tau(x) = 1 \quad$ *if* $\quad |x| \le 1/2$,

(iv) $\tau(x) = 0 \quad$ *if* $\quad |x| \ge 1$.

Then the commutative neutrix convolution $f \,\overline{*}\, g$ of f and g exists and is equal to h if

$$N - \lim_{n\to\infty} \frac{1}{2}\left\{((f\tau_n) * g, \phi) + (f * (g\tau_n), \phi)\right\} = (h,\, \phi)$$

for all testing functions $\phi \in \mathcal{D}$. If the normal limit exists, then it is simply called the commutative convolution. Clearly, this definition generalizes Definition 1.1.

To make this paper as self-contained as possible, we state Paley-Wiener-Schwartz theorem in the following, which will be used a couple of times throughout this article.

Theorem 1.4 *An entire function $f(s)$ on $\mathcal{C}^m$ is the Fourier transform of distribution $\lambda(x)$ with compact support if and only if for all $s \in \mathcal{C}^m$,*

$$|f(s)| \le Ce^{b|Ims|}(1 + |s|)^q$$

for some constants C, q and b. The distribution $\lambda(x)$ is in fact supported in the closed ball of center zero with radius b.

2 Analytic Functionals

As in [8], we define the Fourier transform of a function ϕ in $\mathcal{D}$ by

$$\psi(s) = F(\phi)(s) = \tilde{\phi}(s) = \int_{-\infty}^{\infty} \phi(x)e^{ixs}\,dx.$$

Here $s = s_1 + is_2$ is a complex variable and it is well known that $\psi(s)$ is an entire function with the following inequality for $q = 0, 1, 2, \cdots$

$$|s^q\psi(s)| \le C_q e^{a|\text{Im}s|} \tag{1}$$

for some constants C_q and a depending on $\psi(s)$. The set of all entire functions with property (1) is indeed the space

$$Z = F(\mathcal{D}) = \{\psi \mid \exists\, \phi \in \mathcal{D} \text{ such that } F(\phi) = \psi\}.$$

The definition of convergence in Z can be carried over from $\mathcal{D}$. That is, a sequence of functions $\psi_\nu(s)$ converges to zero in Z if the sequence of their inverse images $\phi_\nu(x) = F^{-1}(\psi_\nu)$ converges to zero in $\mathcal{D}$. In other words, the sequence $\psi_\nu(s)$ converges to zero in Z if for each function in this sequence we have

$$|s^q\psi_\nu(s)| \le C_q e^{a|\text{Im}s|}$$

with C_q and a independent of ν, and if the functions converge to zero uniformly on every interval of the (real) s_1 axis.

The Fourier transform $F(f) = \tilde{f}$ of a distribution f in $\mathcal{D}'$ is an ultradistribution in Z', i.e., a linear and continuous functional on Z. It is defined by Parseval's equation

$$(\tilde{f},\, \tilde{\phi}) = 2\pi(f,\, \phi).$$

Clearly, we have

$$Z' = F(\mathcal{D}') = \{F(f) \mid f \in \mathcal{D}'\}.$$

The exchange formula is the equality

$$F(f * g) = F(f) \cdot F(g).$$

It was proved in [22] that the exchange formula holds for all convolutions of distributions f and g, provided they have compact support.

We now consider the problem of defining product in Z'. To do this we need the Fourier transform $F(\tau_n)$ of τ_n and write

$$\delta_n(s) = \frac{1}{2\pi} F(\tau_n)$$

which is an element in Z since τ_n belongs to $\mathcal{D}$. Putting $\psi = \tilde{\phi}$, we have from Parseval's equation

$$(\tau_n,\, \phi) = \frac{1}{2\pi}(F(\tau_n),\, F(\phi)) = (\delta_n,\, \psi).$$

Since

$$\lim_{n\to\infty}(\tau_n, \phi) = \lim_{n\to\infty}\int_{-\infty}^{\infty} \tau_n(x)\phi(x)dx = \int_{-\infty}^{\infty} \phi(x)dx = (1,\, \phi)$$

for all $\phi \in \mathcal{D}$ and $F(1) = 2\pi\delta$, we derive that

$$\lim_{n\to\infty}(\delta_n, \psi) = (\delta, \psi)$$

for all $\psi \in Z$. Thus, $\{\delta_n\}$ is a sequence in Z converging to the Dirac delta function.

If f is an arbitrary distribution in $\mathcal{D}'$, then since δ_n is a function in Z, the convolution $\tilde{f} * \delta_n$ is defined by

$$((\tilde{f} * \delta_n)(s),\, \psi(s)) = (\tilde{f}(w),\, (\delta_n(s),\, \psi(s+w)))$$

for arbitrary ψ in Z. If $\psi = \tilde{\phi}$, we have

$$\psi(s+w) = F(e^{ixw}\phi(x))$$

and it follows from Parseval's equation that

$$\begin{aligned}(\delta_n(s),\, \psi(s+w)) &= \frac{1}{2\pi}(F(\tau_n)(s),\, F(e^{ixw}\phi)(s)) = (\tau_n(x),\, e^{ixw}\phi(x)) \\ &= \int_{-\infty}^{\infty} \tau_n(x)e^{ixw}\phi(x)dx \to \int_{-\infty}^{\infty} e^{ixw}\phi(x)dx = \psi(w).\end{aligned}$$

Thus

$$\lim_{n\to\infty}(\tilde{f} * \delta_n,\, \psi) = (\tilde{f},\, \psi)$$

for arbitrary ψ in Z and it follows that $\{\tilde{f} * \delta_n\}$ is a sequence of infinitely differentiable functions converging to $\tilde{f}$ in Z'.

Following the standard notation we let ε' be the space of distributions with compact support. Obviously, we have $\mathcal{D} \subset \varepsilon' \subset \mathcal{D}'$ and $Z = F(\mathcal{D}) \subset F(\varepsilon') \subset F(\mathcal{D}') = Z'$.

The derivative of a functional $g \in Z'$ is defined by

$$\left(\frac{dg}{ds},\, \psi\right) = -\left(g,\, \frac{d\psi}{ds}\right).$$

As is true for $\mathcal{D}'$, the generalized functions in Z' have derivatives of all orders. There is a difference, however, in that the generalized functions of Z' are not only infinitely differentiable, but also expandable in the sense that for every $g \in Z'$

$$\sum_{q=0}^{\infty} g^{(q)}(s)\frac{h^q}{q!} = g(s+h)$$

where the series on the left converges in Z', and $g(s+h)$ is the generalized function obtained from $g(s)$ by translation through h. In particular,

$$\delta(s+h) = \sum_{q=0}^{\infty} \delta^{(q)}(s)\frac{h^q}{q!}.$$

Define a multiplier space of Z as

$$\mathcal{M} = \{h(s) \,|\, h \text{ is entire and } |h(s)| \le Ce^{b|\mathrm{Im}s|}(1+|s|)^q\}$$

for some b, q and C. By Paley-Wiener-Schwartz theorem stated in the introduction, we imply that $\mathcal{M} = F(\varepsilon')$ and $Z \subset \mathcal{M} \subset Z'$. For any $g \in Z'$ and $h(s) \in \mathcal{M}$, the product $h(s)g(s)$ is well defined by

$$(h(s)g(s),\, \psi(s)) = (g(s),\, h(s)\psi(s)) \qquad (2)$$

because $h(s)\psi(s) \in Z$ (see [8]). It follows that

$$h(s)\,\delta^{(m)}(s) = \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}h^{(m-j)}(0)\delta^{(j)}(s). \qquad (3)$$

In particular,

$$\sin s\, \delta^{(m)}(s) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \sin[(m-j)\frac{\pi}{2}]\delta^{(j)}(s)$$

where $\sin s = \dfrac{1}{2i}(e^{is} - e^{-is}) \in \mathcal{M}$.

We are ready to present the following definition by noting that

$$(\tilde{f}_n \tilde{g}, \psi) + (\tilde{f}\tilde{g}_n, \psi) = (\tilde{g}, F(f\tau_n)\psi) + (\tilde{f}, F(g\tau_n)\psi)$$

is well defined since both $F(f\tau_n)$ and $F(g\tau_n)$ are in $\mathcal{M}$ and $\tilde{f}_n = F(f\tau_n)$.

Indeed,

$$\begin{aligned} (\tilde{f}_n, \psi) &= (\tilde{f} * \delta_n, \psi) = (\tilde{f}, F(\tau_n \phi)) = 2\pi(f, \tau_n \phi) \\ &= 2\pi(f\tau_n, \phi) = (F(f\tau_n), \psi). \end{aligned}$$

Definition 2.1 *Let f and g be distributions in $\mathcal{D}'$ having Fourier transforms $\tilde{f}$ and $\tilde{g}$ respectively in Z' and let $\tilde{f}_n = \tilde{f} * \delta_n$ and $\tilde{g}_n = \tilde{g} * \delta_n$. Then the neutrix product $\tilde{f} \circ \tilde{g}$ in Z' is defined to be the neutrix limit $\tilde{h}$ in the sense that*

$$N - \lim_{n\to\infty} \frac{1}{2} \left\{ (\tilde{f}_n \tilde{g}, \psi) + (\tilde{f}\tilde{g}_n, \psi) \right\} = (\tilde{h}, \psi)$$

for all ψ in Z.

In this definition, we use $\tilde{f} \circ \tilde{g}$ to denote the neutrix product of $\tilde{f}$ and $\tilde{g}$ to distinguish it from the usual definition of the product $\tilde{f}_n \tilde{g}_n$ (in terms of point wise) of two infinitely differentiable functions $\tilde{f}_n$ and $\tilde{g}_n$. If

$$\lim_{n\to\infty} \frac{1}{2} \left\{ (\tilde{f}_n \tilde{g}, \psi) + (\tilde{f}\tilde{g}_n, \psi) \right\} = (\tilde{h}, \psi)$$

for all ψ in Z, we simply say that the product $\tilde{f} \cdot \tilde{g}$ exists and equal to $\tilde{h}$. We then of course have

$$\tilde{f} \circ \tilde{g} = \tilde{f} \cdot \tilde{g}.$$

It is clear that if the neutrix product $\tilde{f} \circ \tilde{g}$ exists then the neutrix product is commutative.

The product of ultradistributions in Z' also has the following property:

Theorem 2.2 *Let $\tilde{f}$ and $\tilde{g}$ be ultradistributions in Z' and suppose that neutrix products $\tilde{f} \circ \tilde{g}$ and $\tilde{f} \circ \tilde{g}'$ (or $\tilde{f}' \circ \tilde{g}$) exists. Then the neutrix product $\tilde{f}' \circ \tilde{g}$ (or $\tilde{f} \circ \tilde{g}'$) exists and*

$$(\tilde{f} \circ \tilde{g})' = \tilde{f}' \circ \tilde{g} + \tilde{f} \circ \tilde{g}'. \tag{4}$$

Proof. Let ψ be an arbitrary function in Z, Then

$$((\tilde{f}\circ\tilde{g})',\,\psi) = -(\tilde{f}\circ\tilde{g},\,\psi')$$
$$= -N-\lim_{n\to\infty}\frac{1}{2}\Big\{(\tilde{f}_n\tilde{g},\psi')+(\tilde{f}\tilde{g}_n,\psi')\Big\}$$
$$= -N-\lim_{n\to\infty}\frac{1}{2}\Big\{(\tilde{g},(\tilde{f}_n\psi)'-\tilde{f}_n{}'\psi)+(\tilde{f},(\tilde{g}_n\psi)'-\tilde{g}_n{}'\psi)\Big\}$$
$$= N-\lim_{n\to\infty}\frac{1}{2}\Big\{(\tilde{f}_n\tilde{g}',\psi)+(\tilde{f}\tilde{g}_n{}',\psi)+(\tilde{f}_n{}'\tilde{g},\psi)+(\tilde{f}'\tilde{g}_n,\psi)\Big\}$$

on using the following identity

$$(\tilde{f}*\delta_n)' = \tilde{f}'*\delta_n.$$

This infers that

$$(\tilde{f}'\circ\tilde{g},\,\psi) = N-\lim_{n\to\infty}\frac{1}{2}\Big\{(\tilde{f}_n{}'\tilde{g},\psi)+(\tilde{f}'\tilde{g}_n,\psi)\Big\} = ((\tilde{f}\circ\tilde{g})',\,\psi)-(\tilde{f}\circ\tilde{g}',\,\psi).$$

It follows similarly that if $\tilde{f}'\circ\tilde{g}$ exists then $\tilde{f}\circ\tilde{g}'$ exists. This completes the proof of Theorem 2.2.

We can now prove the exchange formula.

Theorem 2.3 *Let f and g be distributions in $\mathcal{D}'$ having Fourier transforms $\tilde{f}$ and $\tilde{g}$ respectively in Z'. Then the neutrix convolution $f \overline{*} g$ exists in $\mathcal{D}'$, if and only if the neutrix product $\tilde{f}\circ\tilde{g}$ exists in Z' and the exchange formula*

$$F(f\,\overline{*}\,g) = F(f)\circ F(g) = \tilde{f}\circ\tilde{g}$$

is then satisfied.

Proof. We first show that

$$F(f\tau_n * g) = F(f\tau_n)F(g)$$

where $F(f\tau_n)\in\mathcal{M}$ as $f\tau_n$ has compact support. Let $\psi = F(\phi)$, then by Parseval's equation

$$(F(f\tau_n * g),\,\psi) = 2\pi(f\tau_n * g,\,\phi) = 2\pi(g(x),\,((f\tau_n)(y),\,\phi(x+y)))$$

by Definition 1.1. Clearly, the function $((f\tau_n)(y),\,\phi(x+y))$ has compact support as $f\tau_n$ and ϕ have compact support. Let $\lambda(x)\in\mathcal{D}$ such that $\lambda(x)=1$ on the support of $((f\tau_n)(y),\,\phi(x+y))$. Then

$$(F(f\tau_n * g),\,\psi) = 2\pi(g(x),\,((f\tau_n)(y),\,\phi(x+y))) = 2\pi(\lambda(x)g(x),\,((f\tau_n)(y),\,\phi(x+y)))$$
$$= (F((f\tau_n)*\lambda g),\,\psi) = (F(f\tau_n)F(\lambda g),\,\psi) = (F(f\tau_n)F(g),\,\psi)$$

by applying

$$F((f\tau_n) * \lambda g) = F(f\tau_n)F(\lambda g)$$

from [22]. Further,

$$\begin{aligned}
&(F(f \mathbin{\overline{*}} g),\, \psi) = 2\pi(f \mathbin{\overline{*}} g,\, \phi) \\
&= 2\pi(N - \lim_{n\to\infty})\frac{1}{2}\{(f\tau_n * g,\, \phi) + (f * g\tau_n,\, \phi)\} \\
&= N - \lim_{n\to\infty} \frac{1}{2}\{F(f\tau_n * g),\, \psi) + (F(f * g\tau_n),\, \psi)\} \\
&= N - \lim_{n\to\infty} \frac{1}{2}\{F(f\tau_n)F(g),\, \psi) + (F(f)F(g\tau_n),\, \psi)\} \\
&= N - \lim_{n\to\infty} \frac{1}{2}\left\{(\tilde{f}_n\tilde{g},\, \psi) + (\tilde{f}\tilde{g}_n,\, \psi)\right\} \\
&= (F(f) \circ F(g),\, \psi)
\end{aligned}$$

on using

$$F(f\tau_n) = \tilde{f}_n, \quad \text{and} \quad F(g\tau_n) = \tilde{g}_n.$$

This completes the proof of Theorem 2.3.

Let $\mathcal{D}'(R^+)$ be the subspace of $\mathcal{D}'$ with support contained in R^+. It follows from [15] that $\Phi_\lambda = \frac{x_+^{\lambda-1}}{\Gamma(\lambda)} \in \mathcal{D}'(R^+)$ is an entire function of λ on the complex plane, and

$$\left.\frac{x_+^{\lambda-1}}{\Gamma(\lambda)}\right|_{\lambda=-n} = \delta^{(n)}(x), \quad \text{for} \quad n = 0, 1, 2, \cdots. \tag{5}$$

For the functional $\Phi_\lambda = \frac{x_+^{\lambda-1}}{\Gamma(\lambda)}$, the derivative formula is simpler than that for x_+^λ. In fact,

$$\frac{d}{dx}\Phi_\lambda = \frac{d}{dx}\frac{x_+^{\lambda-1}}{\Gamma(\lambda)} = \frac{(\lambda-1)x_+^{\lambda-2}}{\Gamma(\lambda)} = \frac{x_+^{\lambda-2}}{\Gamma(\lambda-1)} = \Phi_{\lambda-1}. \tag{6}$$

Let λ and μ be arbitrary complex numbers. Then it is easy to show

$$\Phi_\lambda * \Phi_\mu = \Phi_{\lambda+\mu} \tag{7}$$

by equation (6), without any help of analytic continuation mentioned in all current books. We would like to point out that $\Phi_\lambda = \frac{x_+^{\lambda-1}}{\Gamma(\lambda)}$ plays an important role in defining fractional calculus of distributions and classical functions [20, 19, 18, 10, 17].

Theorem 2.4 *The products $(\sigma+i0)^\lambda\cdot(\sigma+i0)^\mu$ and $(\sigma-i0)^\lambda\cdot(\sigma-i0)^\mu$ exist in Z' and*

$$(\sigma+i0)^\lambda\cdot(\sigma+i0)^\mu=(\sigma+i0)^{\lambda+\mu} \tag{8}$$

$$(\sigma-i0)^\lambda\cdot(\sigma-i0)^\mu=(\sigma-i0)^{\lambda+\mu} \tag{9}$$

for all complex numbers λ and μ.

Proof. It follows from [8] that

$$F(\Phi_{\lambda+1})=ie^{i\lambda\pi/2}(\sigma+i0)^{-\lambda-1} \tag{10}$$

for any complex number λ. Applying the exchange formula, it follows from equations (7) and (10) that

$$\begin{aligned}&-e^{i(\lambda+\mu)\pi/2}(\sigma+i0)^{-\lambda-1}\cdot(\sigma+i0)^{-\mu-1}\\&=ie^{i(\lambda+\mu+1)\pi/2}(\sigma+i0)^{-\lambda-\mu-2}\end{aligned}$$

for all complex numbers λ, μ. This implies equation (8). Equation (9) follows immediately from

$$\frac{x_-^{\lambda-1}}{\Gamma(\lambda)}*\frac{x_-^{\mu-1}}{\Gamma(\mu)}=\frac{x_-^{\lambda+\mu-1}}{\Gamma(\lambda+\mu)}$$

for all complex numbers λ and μ and equation (10). This completes the proof of Theorem 2.4.

Remark 1: In [6], Fisher, Özçag and Li proved Theorem 2.4 for all real numbers λ, μ with one and half pages of complicated calculation. The current proof is much shorter and easier with a deeper result (for all complex numbers λ and μ).

Corollary 2.5

$$\sigma_+^{-r-1/2}\cdot\sigma_-^{-r-1/2}=\frac{(-1)^r\pi}{2(2r)!}\delta^{(2r)}(\sigma) \tag{11}$$

for $r=0,1,2,\cdots$.

Proof. It follows from equation (8) that

$$\begin{aligned}&(\sigma+i0)^{-r-1/2}\cdot(\sigma+i0)^{-r-1/2}=(\sigma+i0)^{-2r-1}\\&=\left[\sigma_+^{-r-1/2}-i(-1)^r\sigma_-^{-r-1/2}\right]\cdot\left[\sigma_+^{-r-1/2}-i(-1)^r\sigma_-^{-r-1/2}\right]\\&=\sigma^{-2r-1}-\frac{i\pi}{(2r)!}\delta^{(2r)}(\sigma)\end{aligned}$$

for $r=0,1,2,\cdots$. Expanding and equating the imaginary parts gives equation (11).

Corollary 2.6

$$\sigma^{-r} \cdot \delta^{(r-1)}(\sigma) = \frac{(-1)^r (r-1)!}{2(2r-1)!} \delta^{(2r-1)}(\sigma) \tag{12}$$

for $r = 1, 2, \cdots$.

Proof. It follows from equation (8) that

$$\begin{aligned}
&(\sigma + i0)^{-r} \cdot (\sigma + i0)^{-r} = (\sigma + i0)^{-2r} \\
&= \left[\sigma^{-r} + \frac{i\pi(-1)^r}{(r-1)!}\delta^{(r-1)}(\sigma)\right] \cdot \left[\sigma^{-r} + \frac{i\pi(-1)^r}{(r-1)!}\delta^{(r-1)}(\sigma)\right] \\
&= \sigma^{-2r} + \frac{i\pi}{(2r-1)!}\delta^{(2r-1)}(\sigma)
\end{aligned}$$

for $r = 1, 2, \cdots$. Expanding and equating the imaginary parts gives equation (12).

3 The Fractional Derivatives and Products in Z'

Current studies on product of analytic functionals have been mainly based on applying convolution in $\mathcal{D}'$ and the Fourier exchange formula as discussed in the previous section. The goal of this section is to introduce a new definition to compute the product of analytic functionals in Z' directly. Furthermore, we define the fractional derivatives of some generalized functions in Z', including the fractional derivative of $\delta(s)$, by Cauchy's integral formula, and obtain fresh results, which have not been seen previously.

We shall call a functional g on Z analytic if it can be written in the form

$$(g, \psi) = \int_\Gamma g(s)\psi(s)ds,$$

where $g(s)$ is a function and Γ is some contour in the complex plan $\mathcal{C}$. Thus, the delta function given by $(\delta(s - s_0), \psi(s)) = \psi(s_0)$, where $s_0 \in \mathcal{C}$, is an analytic functional, since

$$(\delta(s - s_0), \psi(s)) = \psi(s_0) = \frac{1}{2\pi i}\int_\Gamma \frac{\psi(s)}{s - s_0}ds,$$

where Γ is any contour enclosing s_0 in counterclockwise. We denote

$$\delta(s - s_0) = \left\{\frac{1}{2\pi i(s - s_0)}, \Gamma\right\}.$$

Similarly, we have

$$\delta^{(m)}(s-s_0)=\left\{\frac{(-1)^m m!}{2\pi i(s-s_0)^{m+1}},\,\Gamma\right\}.$$

Choosing a fixed function $\omega(s)\in\mathcal{M}$, we can construct two different analytic functionals for $d>0$,

$$(f_+(s),\,\psi(s))=\int_{-\infty+di}^{\infty+di}\frac{\omega(s)\psi(s)}{s^{n+1}}ds$$

and

$$(f_-(s),\,\psi(s))=\int_{-\infty-di}^{\infty-di}\frac{\omega(s)\psi(s)}{s^{n+1}}ds,$$

where n is any non-negative integer. Those two integrals are clearly convergent since $\omega(s)\psi(s)\in Z$.

The difference between them can be simplified into the form

$$(f_-(s)-f_+(s),\,\psi(s))=\oint_{|s|=1}\frac{\omega(s)\psi(s)}{s^{n+1}}ds$$

in which the integral is taken in counterclockwise along the boundary of $|s|=1$. By Cauchy's integral formula

$$(f_-(s)-f_+(s),\,\psi(s))=\frac{2\pi i}{n!}\sum_{k=0}^{n}\binom{n}{k}\omega^{(n-k)}(0)\psi^{(k)}(0).$$

Therefore,

$$f_-(s)-f_+(s)=\frac{2\pi i}{n!}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\omega^{(n-k)}(0)\delta^{(k)}(s).$$

Finally, we have the following product in Z' from equation (3)

$$h(s)(f_-(s)-f_+(s))=\frac{2\pi i}{n!}\sum_{k=0}^{n}\sum_{j=0}^{k}(-1)^j\binom{n}{k}\binom{k}{j}\omega^{(n-k)}(0)h^{(k-j)}(0)\delta^{(j)}(s)$$

where $h\in\mathcal{M}$.

Let

$$W=\{\,\eta(x)\,|\,\eta(x)\text{ is locally integrable and }F(\eta)\text{ exists}\,\}.$$

Then $h(s)*\delta_n(s)\in\mathcal{M}$, where $h(s)\in F(W)$.

Indeed, let $h(s)=F(\eta)$, where $\eta\in W$. Clearly,

$$(h*\delta_n)(s)=(\delta_n(\sigma),h(s+\sigma))=\int_{-\infty}^{\infty}\tau_n(x)e^{ixs}\eta(x)dx$$

by Parseval's equation. Since $\eta(x)$ is locally integrable and $\tau_n(x)\eta(x) \in \varepsilon'$, we claim from Paley-Wiener-Schwartz theorem that

$$\int_{-\infty}^{\infty} \tau_n(x) e^{ixs} \eta(x) dx = F(\tau_n \eta(x)) \in \mathcal{M}.$$

Now we are ready to give a new definition for the product of analytic functionals in Z'.

Definition 3.1 *Let $h(s) \in F(W)$ and let $g(s) \in Z'$. Then the product $h(s) \cdot g(s)$ is defined as*

$$(h(s) \cdot g(s),\, \psi) = \lim_{n\to\infty} (g(s),\, (h * \delta_n)\psi)$$

provided this limit exists.

In particular, if $h(s) \in \mathcal{M}$ and $g(s) \in Z'$, then $h(s) \cdot g(s) = h(s)g(s)$, which is defined in equation (2). By Definition 3.1, we only need to show that $(h * \delta_n)\psi$ converges to $h(s)\psi$ in Z. Assume that $h(s) = F(\lambda(x))$, where $\lambda(s)$ is a distribution with compact support, and $\psi = F(\phi)$ for some $\phi \in \mathcal{D}$. Then $(h * \delta_n)\psi = F((\tau_n \lambda) * \phi)$, and further it is proved in [21] that $(\tau_n \lambda) * \phi$ in $\mathcal{D}$. Therefore, $(h * \delta_n)\psi$ converges to $h(s)\psi$ in Z.

As an example demonstrating application of Definition 3.1, we let

$$\eta(x) = \begin{cases} \sin x^2 e^{-x^2} & \text{if } x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $\eta(x) \in W$. Using Definition 3.1, we are going to calculate the product $F(\eta)(s) \cdot \delta^{(m)}(s)$ for $m = 0, 1, 2, \cdots$. In fact,

$$(F(\eta) * \delta_n)(s) = \int_0^{\infty} \tau_n(x) e^{ixs} \sin x^2 e^{-x^2} dx.$$

Further,

$$\lim_{n\to\infty} (\delta^{(m)}(s),\, (F(\eta) * \delta_n)(s)\psi(s)) = (-1)^m \sum_{j=0}^{m} \binom{m}{j} \lim_{n\to\infty} (F(\eta) * \delta_n)^{(m-j)}(0)\psi^{(j)}(0)$$

$$= \sum_{j=0}^{m} (-1)^{(m-j)} \binom{m}{j} C_{m,j} \int_0^{\infty} x^{m-j} \sin x^2 e^{-x^2} dx (\delta^{(j)}(s),\, \psi(s))$$

where

$$C_{m,j} = i^{m-j} = \cos(m-j)\frac{\pi}{2} + i \sin(m-j)\frac{\pi}{2}.$$

Therefore,

$$h(s) \cdot \delta^{(m)}(s) = \sum_{j=0}^{m} (-1)^{(m-j)} \binom{m}{j} \left[\cos(m-j)\frac{\pi}{2} + i \sin(m-j)\frac{\pi}{2}\right] \int_0^{\infty} x^{m-j} \sin x^2 e^{-x^2} dx \delta^{(j)}(s).$$

We would like to point out that Definition 3.1 provides a powerful method for computing the product $h(s)\cdot\delta^{(m)}(s)$ when it is difficult to evaluate the Fourier transform $F(\eta)$, where $\eta \in W$, as indicated in the above example. This example cannot be defined by equation (3) since η does not have compact support.

In the following, we define the α-order fractional derivative of $f_-(s)-f_+(s)$ on Z for $n=0$ (for simplicity) based on Cauchy's integral formula, where $\alpha>0$

$$((f_- - f_+)^{(\alpha)}(s),\, \psi(s)) = (\cos\alpha\pi + i\sin\alpha\pi)\frac{\Gamma(\alpha+1)}{2\pi i}\oint_{|s|=1}\frac{\omega(s)\psi(s)}{s^{\alpha+1}}ds \quad (13)$$

in counterclockwise along $|s|=1$. Here we choose the fixed analytic branch $\ln 1 = 0$, $-\pi < \arg s \le \pi$ and $k=0$, such that

$$\frac{1}{s^{\alpha+1}} = s^{-(\alpha+1)} = e^{-(\alpha+1)\ln s}\cdot e^{-(\alpha+1)2k\pi i}$$

is an analytic single-valued function.

In particular, we get for $\omega(s)=1$

$$(\delta^{(\alpha)}(s),\, \psi(s)) = (\cos\alpha\pi + i\sin\alpha\pi)\frac{\Gamma(\alpha+1)}{2\pi i}\oint_{|s|=1}\frac{\psi(s)}{s^{\alpha+1}}ds.$$

Clearly, the functional $(f_- - f_+)^{(\alpha)}(s)$ is linear on Z. Let $\{\psi_m\}$ be a sequence converging to zero in Z. Then

$$\psi_m(s) = \int_{-\infty}^{\infty}\phi_m(x)e^{ixs}dx,$$

where $\{\phi_m\}$ converges to zero in $\mathcal{D}$, which implies that $\forall\,\epsilon>0$, $\exists\,N=N(\epsilon)$ such that $|\phi_m(x)|<\epsilon$ for $m>N$. Assuming $\text{supp}\phi_m \subset [-a,\, a]$ for all m, we infer that for $m>N$ and $|s|=1$

$$|\psi_m(s)| \le \int_{-a}^{a}\epsilon|e^{ixs}|dx \le 2a\epsilon e^{a}$$

by noting that $s=\cos\theta + i\sin\theta$ for $0\le\theta\le 2\pi$. Further, $\exists\,M>0$ such that $|\omega(s)|\le M$ for $|s|=1$. Therefore,

$$\lim_{n\to\infty}(\cos\alpha\pi + i\sin\alpha\pi)\frac{\Gamma(\alpha+1)}{2\pi i}\oint_{|s|=1}\frac{\omega(s)\psi_m(s)}{s^{\alpha+1}}ds = 0.$$

This means that $(f_- - f_+)^{(\alpha)}(s)$ is continuous on Z.

Assume that $\alpha \ne 0, 1, 2, \cdots$ and let $\omega(s)\psi(s) = \sum_{n=0}^{\infty}a_n s^n$ be the Taylor series which converges uniformly on $|s|=1$. Equation (13) therefore yields

$$\begin{aligned}((f_- - f_+)^{(\alpha)}(s),\, \psi(s)) &= (\cos\alpha\pi + i\sin\alpha\pi)\frac{\Gamma(\alpha+1)}{2\pi i}\sum_{n=0}^{\infty}a_n\oint_{|s|=1}\frac{s^n}{s^{\alpha+1}}ds \\ &= (\cos\alpha\pi + i\sin\alpha\pi)\frac{\Gamma(\alpha+1)}{2\pi i}\sum_{n=0}^{\infty}a_n\left(\int_{C_1}s^{n-\alpha-1}ds + \int_{C_2}s^{n-\alpha-1}ds\right)\end{aligned}$$

where C_1 and C_2 are two parts of $|s| = 1$ in the upper and lower s_1 axis respectively.

We compute on the analytic branch directly to get

$$\int_{C_1} s^{n-\alpha-1} ds = \left.\frac{s^{n-\alpha}}{n-\alpha}\right|_0^{\pi} = \frac{1}{n-\alpha}(\cos(n-\alpha)\pi + i\sin(n-\alpha)\pi - 1),$$

and

$$\int_{C_2} s^{n-\alpha-1} ds = \left.\frac{s^{n-\alpha}}{n-\alpha}\right|_{-\pi}^{0} = \frac{1}{n-\alpha}(1 - \cos(n-\alpha)\pi + i\sin(n-\alpha)\pi).$$

Adding the two terms we have

$$\int_{C_1} s^{n-\alpha-1} ds + \int_{C_2} s^{n-\alpha-1} ds = \frac{2i\sin(n-\alpha)\pi}{n-\alpha} = \frac{2i(-1)^n \sin\alpha\pi}{\alpha-n},$$

so that

$$\begin{aligned} &((f_- - f_+)^{(\alpha)}(s),\, \psi) \\ &= (\cos\alpha\pi + i\sin\alpha\pi)\frac{\Gamma(\alpha+1)}{\pi}\sin\alpha\pi\sum_{n=0}^{\infty}\frac{(-1)^n a_n}{\alpha-n}. \end{aligned}$$

Since the Taylor's series coefficient

$$a_n = \frac{(\omega\psi)^{(n)}(0)}{n!} = \frac{1}{n!}\sum_{j=0}^{n}\binom{n}{j}\omega^{(n-j)}(0)\psi^{(j)}(0) = \frac{1}{n!}\sum_{j=0}^{n}\binom{n}{j}\omega^{(n-j)}(0)(-1)^j(\delta^{(j)}(s),\, \psi),$$

we come to

$$\begin{aligned} &(f_- - f_+)^{(\alpha)}(s) \\ &= (\cos\alpha\pi + i\sin\alpha\pi)\frac{\Gamma(\alpha+1)}{\pi}\sin\alpha\pi\sum_{n=0}^{\infty}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\omega^{(n-j)}(0)\frac{\delta^{(j)}(s)}{(\alpha-n)n!} \\ &= (\cos\alpha\pi\sin\alpha\pi + i\sin^2\alpha\pi)\frac{\Gamma(\alpha+1)}{\pi}\sum_{n=0}^{\infty}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\omega^{(n-j)}(0)\frac{\delta^{(j)}(s)}{(\alpha-n)n!}. \end{aligned}$$

In particular, we choose $\omega(s) = 1$ to get

Theorem 3.2

$$\delta^{(\alpha)}(s) = (\cos\alpha\pi\sin\alpha\pi + i\sin^2\alpha\pi)\frac{\Gamma(\alpha+1)}{\pi}\sum_{n=0}^{\infty}\frac{\delta^{(n)}(s)}{(\alpha-n)n!} \tag{14}$$

for $\alpha > 0$ *and* $\alpha \neq 0, 1, 2, \cdots$.

We let $\alpha \to n$ and note that all terms on the right-hand side of equation (14) vanish except the nth term which approaches to $\delta^{(n)}(s)$. Hence

$$\lim_{\alpha \to n}(\cos \alpha\pi \sin \alpha\pi + i \sin^2 \alpha\pi)\frac{\Gamma(\alpha+1)}{\pi}\sum_{n=0}^{\infty}\frac{\delta^{(n)}(s)}{(\alpha - n)n!} = \delta^{(n)}(s).$$

This indicates that $\delta^{(\alpha)}(s)$ is an extension of the normal derivative $\delta^{(n)}(s)$.

Remark 2: The result obtained in Theorem 3.2 completely coincides with one in [21], although we use a different approach here.

On the other hand, we get the following theorem from choosing $\omega(s) = s$.

Theorem 3.3 *Let*

$$((f_- - f_+)^{(\alpha)}(s),\, \psi(s)) = (\cos \alpha\pi + i \sin \alpha\pi)\frac{\Gamma(\alpha+1)}{2\pi i}\oint_{|s|=1}\frac{\psi(s)}{s^\alpha}ds$$

in counterclockwise along $|s| = 1$. *Then,*

$$(f_- - f_+)^{(\alpha)}(s) = (\cos \alpha\pi \sin \alpha\pi + i \sin^2 \alpha\pi)\frac{\Gamma(\alpha+1)}{\pi}\sum_{n=1}^{\infty}\frac{\delta^{(n-1)}(s)}{(n-\alpha)(n-1)!} \quad (15)$$

for $\alpha > 0$ *and* $\alpha \neq 0, 1, 2, \cdots$.

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References

[1] M. Aguirre and C. Li, The distributional products of particular distributions, *Appl. Math. Comput.,* **187** (2007), 20-26.
https://doi.org/10.1016/j.amc.2006.08.098

[2] H.J. Bremermann, *Distributions, Complex Variables, and Fourier Transforms,* Addison-Wesley Publishing Co. Inc. Reading, Mass.-London, 1965.

[3] L. Cheng and C. Li, A commutative neutrix product of distributions on R^m, *Math. Nachr.,* **151** (1991), 345-355.
https://doi.org/10.1002/mana.19911510124

[4] B. Fisher, A result of the convolution of distributions, *Proc. Edinburgh Math. Soc.,* **19** (1975), 393-395.
https://doi.org/10.1017/s0013091500010506

[5] B. Fisher, On defining the convolution of distributions, *Math. Nachr.,* **106** (1982), 261-268. https://doi.org/10.1002/mana.19821060123

[6] B. Fisher, E. Özçag and C. Li, A commutative neutrix convolution of distributions and the exchange formula, *Archivum Mathematicum,* **28** (1992), 187-197.

[7] S. Gasiorowicz, *Elementary Particle Physics*, J. Wiley and Sons Inc. New York, 1966.

[8] I.M. Gel'fand and G.E. Shilov, *Generalized Functions,* Vol I, Academic Press, New York, 1964.

[9] D.S. Jones, The convolution of generalized functions, *Quart. J. Math. Oxford Ser.,* **24** (1973), 145-163. https://doi.org/10.1093/qmath/24.1.145

[10] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations,* Elsevier, New York, 2006.

[11] B.H. Li, Non-standard analysis and multiplication of distributions, *Sci. Sinica,* **21** (1978), 561-585.

[12] C. Li, The products on the unit sphere and even-dimension spaces, *J. Math. Anal. Appl.,* **305** (2005), 97-106. https://doi.org/10.1016/j.jmaa.2004.10.031

[13] C. Li, Several results on the commutative neutrix product of distributions, *Integral transforms Spec. Funct.,* **18** (2007), 559-568. https://doi.org/10.1080/10652460701366169

[14] C. Li, A review on the products of distributions, Chapter in *Mathematical Methods in Engineering,* Springer, 2007, 71-96. https://doi.org/10.1007/978-1-4020-5678-9_5

[15] C. Li, Several results of fractional derivatives in $\mathcal{D}'(R^+)$, *Fractional Calculus and Applied Analysis,* **18** (2015), 192-207. https://doi.org/10.1515/fca-2015-0013

[16] C. Li, K. Clarkson and V. Patel, The Convolution and Fractional Derivatives of Distributions, to appear in *Advances in Analysis.*

[17] C.P. Li and W. Deng, Remarks on fractional derivatives, *Appl. Math. Comput.,* **187** (2007), 777-784. https://doi.org/10.1016/j.amc.2006.08.163

[18] C. Li and C.P. Li, On defining the distributions $(\delta)^k$ and $(\delta')^k$ by fractional derivatives, *Appl. Math. Comput.,* **246** (2014), 502-513. https://doi.org/10.1016/j.amc.2014.08.068

[19] C. Li and C.P. Li, Remarks on fractional derivatives of distributions, *Tbilisi Mathematical Journal,* **10** (2017), 1-18. https://doi.org/10.1515/tmj-2017-0001

[20] C. Li, C.P. Li, B. Kacsmar, R. Lacroix and K. Tilbury, The Abel Integral Equations in Distribution, *Advances in Analysis,* **2** (2017), 88-104. https://doi.org/10.22606/aan.2017.22003

[21] C. Li, Y. Zhang, M. Aguirre and R. Tang, The product of analytic functionals in Z', *J. Korean Math. Soc.,* **45** (2008), 455-466. https://doi.org/10.4134/jkms.2008.45.2.455

[22] F. Treves, *Topological Vector Spaces, Distributions and Kernels,* Academic Press, 1970.

[23] J.G. van der Corput, Introdution to te neutrix calculus, *J. Analyse Math.,* **7** (1959), 291-398. https://doi.org/10.1007/bf02787689

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