

Article

Solutions to Abel's Integral Equations in Distributions

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Received: 10 August 2018; Accepted: 31 August 2018; Published: 2 September 2018



Abstract: The goal of this paper is to study fractional calculus of distributions, the generalized Abel's integral equations, as well as fractional differential equations in the distributional space $\mathcal{D}'(R^+)$ based on inverse convolutional operators and Babenko's approach. Furthermore, we provide interesting applications of Abel's integral equations in viscoelastic systems, as well as solving other integral equations, such as $\int_{\theta}^{\pi/2} \frac{y(\varphi)}{\cos^{\beta} \varphi (\cos \theta - \cos \varphi)^{\alpha}} d\varphi = f(\theta)$, and $\int_0^{\infty} x^{1/2} g(x) y(x+t) dx = f(t)$.

Keywords: distribution; fractional calculus; Mittag-Leffler function; Abel's integral equation; convolution

MSC: 46F10; 26A33

1. Introduction

Fractional modeling is an emergent tool which uses fractional differential and integral equations to describe non-local dynamic processes associated with complex systems [1–8]. Integral and fractional differential equations arise in numerous physical problems [9–12], in the fields of chemistry, biology, electronics, noncommutative quantum field theories [13], and quantum mechanics [14]. Mathematical models of systems and processes in the mentioned areas of engineering [15] and scientific disciplines involve integrals of unknown functions and derivatives of fractional order. As far as we know, fractional calculus provides an excellent tool to construct certain electro-chemical problems and characterizes long-term behaviors [16,17], allometric scaling laws, hereditary properties of various materials and so on [18]. This is the main advantage of fractional differential equations, in comparison with classical integer-order models in practice. Recently, Srivastava et al. presented the model under-actuated mechanical system with fractional order derivative [19]. Many initial and boundary value problems associated with ordinary (or partial) differential equations, can be converted into Volterra integral equations [1,20]. The Volterra's population growth model, biological species living together, and the heat change can all be characterized by integral equations. For example, Gorenflo and Mainardi [21] provided applications of Abel's integral equations, of the first and second kind, in solving the partial differential equation which describes the problem of the heating (or cooling) of a semi-infinite rod by influx (or efflux) of heat across the boundary into (or from) its interior. In 1985, Hatcher [22] worked on a nonlinear Hilbert problem of a power type, solved in closed form by representing a sectionally holomorphic function by means of an integral with power kernel, and transformed the problem to one of solving a generalized Abel's integral equation. The development of integral equations has led to the construction of many real world problems, such as mathematical physics models [23,24], scattering in quantum mechanics and water waves. There have been lots of techniques, such as numerical analysis and integral transforms [25–27], thus far to studying fractional differential and integral equations, including Abel's equations, with many applications [1,20,28–42].

Kilbas et al. [43] presented a solution in a closed form of multi-dimensional integral equations of the first kind with the Gauss hypergeometric function in the kernel over special pyramidal domains.

Raina et al. [44] later on investigated the solvability of the one-dimensional Abel-type hypergeometric integral equation, given by

$$\frac{(x - a)^{-\alpha}}{\Gamma(\gamma)} \int_a^x (x - t)^{\gamma-1} F\left(\alpha, \beta; \gamma; \frac{x - t}{x - a}\right) \phi(t) dt = f(x)$$

where $x > a$ with $\alpha, \beta \in R$ and $0 < \gamma < 1$, as well as the multidimensional Abel-type hypergeometric integral equation over a pyramidal domain in R^n . The generalized fractional integral and differential operators are introduced and their properties are investigated systematically based on the results obtained.

Srivastava and Buschman [20] presented the comprehensive theory and numerous applications of the integral equations of convolution type, and of certain classes of integro-differential and non-linear integral equations, including Abel’s integral equations, in the classical sense.

We start with the necessary concepts and definitions of fractional calculus of distributions in $\mathcal{D}'(R^+)$ based on the generalized convolution in the Schwartz space. Using inverse convolutional operators and Babenko’s approach, we study and solve several Abel’s integral (for all $\alpha \in R$) and fractional differential equations, as convergent series or in terms of the Mittag–Leffler functions. Many of the results derived can not be archived in the classical sense including numerical analysis methods, or by the Laplace transform. Applications are presented at the end in viscoelastic systems, and for solving other types of integral equations which can be converted into Abel’s ones.

2. Abel’s Integral Equations in Distribution

In order to study Abel’s integral and fractional differential equations distributionally, we briefly introduce the following basic concepts in distribution. Let $\mathcal{D}(R)$ be the Schwartz space (testing function space) [45] of infinitely differentiable functions with compact support in R , and $\mathcal{D}'(R)$ the (dual) space of distributions defined on $\mathcal{D}(R)$. A sequence $\phi_1, \phi_2, \dots, \phi_n, \dots$ goes to zero in $\mathcal{D}(R)$ if and only if these functions vanish outside a certain fixed bounded set, and converge to zero uniformly together with their derivatives of any order. We further assume that $\mathcal{D}'(R^+)$ is the subspace of $\mathcal{D}'(R)$ with support contained in R^+ .

The functional $\delta^{(n)}(x - x_0)$ for $x_0 \in R$ is defined as

$$(\delta^{(n)}(x - x_0), \phi(x)) = (-1)^n \phi^{(n)}(x_0)$$

where $\phi \in \mathcal{D}(R)$. Clearly, $\delta^{(n)}(x - x_0)$ is a linear and continuous functional on $\mathcal{D}(R)$, and hence $\delta^{(n)}(x - x_0) \in \mathcal{D}'(R)$.

Define

$$f(x) = \begin{cases} \sin x & \text{if } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then, $f(x)$ is a locally integrable function on R (clearly not continuous) and

$$(f(x), \phi(x)) = \int_0^1 \sin x \phi(x) dx \quad \text{for } \phi \in \mathcal{D}(R), \tag{1}$$

defines a regular distribution $f(x) \in \mathcal{D}'(R^+)$.

Let $f \in \mathcal{D}'(R)$. The distributional derivative of f , denoted by f' or df/dx , is defined as

$$(f', \phi) = -(f, \phi')$$

for $\phi \in \mathcal{D}(R)$.

Assume f is a distribution in $\mathcal{D}'(R)$ and g is a function in $C^\infty(R)$. Then the product fg is well defined by

$$(fg, \phi) = (f, g\phi)$$

for all functions $\phi \in \mathcal{D}(R)$ as $g\phi \in \mathcal{D}(R)$.

Clearly, $f' \in \mathcal{D}'(R)$ and every distribution has a derivative.

It can be shown that the ordinary rules of differentiation also apply to distributions. For instance, the derivative of a sum, is the sum of the derivatives, and a constant can be commuted with the derivative operator.

It follows from [45] that $\Phi_\lambda = \frac{x_+^{\lambda-1}}{\Gamma(\lambda)} \in \mathcal{D}'(R^+)$ is an entire function of λ on the complex plane, and

$$\left. \frac{x_+^{\lambda-1}}{\Gamma(\lambda)} \right|_{\lambda=-n} = \delta^{(n)}(x), \quad \text{for } n = 0, 1, 2, \dots \tag{2}$$

Clearly, the Laplace transform of Φ_λ is given by

$$\mathcal{L}\{\Phi_\lambda(x)\} = \int_0^\infty e^{-sx} \Phi_\lambda(x) dx = \frac{1}{s^\lambda}, \quad \text{Re}\lambda > 0, \quad \text{Res} > 0$$

which plays an important role in solving integral equations [46].

For the functional $\Phi_\lambda = \frac{x_+^{\lambda-1}}{\Gamma(\lambda)}$, the (distributional) derivative formula is simpler than that for x_+^λ .

In fact,

$$\frac{d}{dx} \Phi_\lambda = \frac{d}{dx} \frac{x_+^{\lambda-1}}{\Gamma(\lambda)} = \frac{(\lambda-1)x_+^{\lambda-2}}{\Gamma(\lambda)} = \frac{x_+^{\lambda-2}}{\Gamma(\lambda-1)} = \Phi_{\lambda-1}. \tag{3}$$

The convolution of certain pairs of distributions is usually defined as follows, see Gel'fand and Shilov [45] for example.

Definition 1. Let f and g be distributions in $\mathcal{D}'(R)$ satisfying either of the following conditions:

- (a) either f or g has bounded support (set of all essential points), or
- (b) the supports of f and g are bounded on the same side.

Then the convolution $f * g$ is defined by the equation

$$((f * g)(x), \phi(x)) = (g(x), (f(y), \phi(x + y)))$$

for $\phi \in \mathcal{D}(R)$.

The classical definition of the convolution is as follows:

Definition 2. If f and g are locally integrable functions, then the convolution $f * g$ is defined by

$$(f * g)(x) = \int_{-\infty}^\infty f(t)g(x - t)dt = \int_{-\infty}^\infty f(x - t)g(t)dt$$

for all x for which the integrals exist.

Note that if f and g are locally integrable functions satisfying either of the conditions in (a) or (b) in Definition 1, then Definition 1 is in agreement with Definition 2. It also follows that if the convolution $f * g$ exists by Definitions 1 or 2, then the following equations hold:

$$f * g = g * f \tag{4}$$

$$(f * g)' = f * g' = f' * g \tag{5}$$

where all the derivatives above are in the distributional sense.

Let λ and μ be arbitrary complex numbers. Then it is easy to show

$$\Phi_\lambda * \Phi_\mu = \Phi_{\lambda+\mu} \tag{6}$$

by Equation (3), without any help of analytic continuation mentioned in all current books.

Let λ be an arbitrary complex number and $g(x)$ be the distribution concentrated on $x \geq 0$. We define the primitive of order λ of g as convolution in the distributional sense

$$g_\lambda(x) = g(x) * \frac{x_+^{\lambda-1}}{\Gamma(\lambda)} = g(x) * \Phi_\lambda. \tag{7}$$

Note that the convolution on the right-hand side is well defined since supports of g and Φ_λ are bounded on the same side.

Thus Equation (7) with various λ will not only give the fractional derivatives, but also the fractional integrals of $g(x) \in \mathcal{D}'(R^+)$ when $\lambda \notin Z$, and it reduces to integer-order derivatives or integrals when $\lambda \in Z$. We shall define the convolution

$$g_{-\lambda} = g(x) * \Phi_{-\lambda}$$

as the fractional derivative of the distribution $g(x)$ with order λ , writing it as

$$g_{-\lambda} = \frac{d^\lambda}{dx^\lambda} g$$

for $\text{Re}\lambda \geq 0$. Similarly, $\frac{d^\lambda}{dx^\lambda} g$ is interpreted as the fractional integral if $\text{Re}\lambda < 0$.

In 1996, Matignon [47] also studied fractional derivatives in the distributional sense using the kernel distribution $\Phi_{-\lambda}$, and defined the fractional derivative of order λ of a continuous (in the normal sense) causal (zero for $t < 0$) function g , as $g_{-\lambda} = g(x) * \Phi_{-\lambda}$, and further obtained a relation between the distributional derivative and the classical one for a smooth function. Mainardi [42] extended Matignon’s work and formally defined the fractional derivative of order $\lambda > 0$ of a causal function (not necessarily continuous) as

$$\frac{d^\lambda}{dx^\lambda} g(x) = \Phi_{-\lambda} * g = \frac{1}{\Gamma(-\lambda)} \int_0^x \frac{g(\zeta) d\zeta}{(x-\zeta)^{1+\lambda}}, \quad \lambda \in R^+.$$

The limit case $\lambda = 0$ is defined as

$$\frac{d^0}{dx^0} g(x) = \Phi_0 * g = \delta * g = g.$$

In addition, Podlubny [46] investigated fractional calculus of generalized functions by the distributional convolution and derived the following identities of fractional derivatives and integrals

$$\begin{aligned} \frac{d^\lambda}{dx^\lambda} \theta(x-a) &= \frac{(x-a)_+^{-\lambda}}{\Gamma(-\lambda+1)}, \\ \frac{d^\lambda}{dx^\lambda} \delta^{(k)}(x-a) &= \frac{(x-a)_+^{-k-\lambda-1}}{\Gamma(-k-\lambda)} \end{aligned}$$

where $\lambda \in C$ and k is a nonnegative integer.

The following theorem can be obtained from [41] with a minor change in the proof.

Theorem 1. Let $G(x)$ be a given distribution and y be an unknown distribution in $\mathcal{D}'(R^+)$. Then the generalized Abel's integral equation of the first kind

$$G(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x - \zeta)^{\alpha-1} y(\zeta) d\zeta \tag{8}$$

has the solution

$$y(x) = G(x) * \Phi_{-\alpha}$$

where α is any real number in R . In particular, if $G(x) = \theta(x - x_0)g(x)$, where $g(x)$ is an infinitely differentiable function on $[0, \infty]$ and $x_0 \geq 0$, then we have four different cases depending on the value of α .

(i) If $m < \alpha < m + 1$ for $m = 0, 1, \dots$, then

$$\begin{aligned} y(x) &= \frac{d^{m+1}}{dx^{m+1}} \theta(x - x_0)g(x) * \frac{x_+^{-\alpha+m}}{\Gamma(-\alpha + m + 1)} \\ &= g(x_0) \frac{(x - x_0)_+^{-\alpha}}{\Gamma(-\alpha + 1)} + \dots + g^{(m)}(x_0) \frac{(x - x_0)_+^{-\alpha+m}}{\Gamma(-\alpha + m + 1)} \\ &\quad + \frac{1}{\Gamma(-\alpha + m + 1)} \int_{x_0}^x g^{(m+1)}(\zeta) (x - \zeta)^{-\alpha+m} d\zeta \end{aligned}$$

for $x \geq x_0$.

(ii) If $\alpha = 1, 2, \dots$, then

$$y(x) = g(x_0)\delta^{(\alpha-1)}(x - x_0) + \dots + g^{(\alpha-1)}(x_0)\delta(x - x_0) + \theta(x - x_0)g^{(\alpha)}(x).$$

(iii) If $\alpha = 0$, then $y(x) = \theta(x - x_0)g(x)$.

(iv) If $\alpha < 0$, then for $x \geq x_0$

$$y(x) = \frac{1}{\Gamma(-\alpha)} \int_{x_0}^x g(\zeta) (x - \zeta)^{-\alpha-1} d\zeta$$

which is well defined.

Example 1. Let k be a nonnegative integer, and $n \in N, s \in R$ with $s \neq -1, -2, \dots$. Then, the integral equation

$$x_+^s + \delta^{(k)}(x) = \int_0^x y(\tau) (x - \tau)^{n/2-1} d\tau \tag{9}$$

has the solution in the space $\mathcal{D}'(R^+)$

$$y(x) = \frac{2^{(n-1)/2}}{(n-2)!!\sqrt{\pi}} (\Gamma(s+1)\Phi_{-n/2+s+1}(x) + \Phi_{-n/2-k}(x)).$$

Proof. Equation (9) is equivalent to

$$x_+^s + \delta^{(k)}(x) = \frac{\Gamma(n/2)}{\Gamma(n/2)} \int_0^x y(\tau) (x - \tau)^{n/2-1} d\tau$$

which gives

$$x_+^s + \delta^{(k)}(x) = \Gamma(n/2) (y(x) * \Phi_{n/2}(x)).$$

Theorem 1 implies

$$y(x) = \frac{1}{\Gamma(n/2)} (\Phi_{-n/2}(x) * (x_+^s + \delta^{(k)}(x)))$$

which simplifies to

$$\begin{aligned}
 y(x) &= \frac{1}{\Gamma(n/2)} ((\Phi_{-n/2}(x) * x_+^s) + (\Phi_{-n/2}(x) * \delta^{(k)}(x))) \\
 &= \frac{1}{\Gamma(n/2)} ((\Phi_{-n/2} * \frac{\Gamma(s+1)}{\Gamma(s+1)} x_+^s) + (\Phi_{-n/2}(x) * \Phi_{-k}(x))) \\
 &= \frac{1}{\Gamma(n/2)} ((\Phi_{-n/2} * \Gamma(s+1)\Phi_{s+1}(x)) + \Phi_{-n/2-k}(x)) \\
 &= \frac{1}{\Gamma(n/2)} ((\Gamma(s+1)\Phi_{-n/2+s+1}) + \Phi_{-n/2-k}(x)).
 \end{aligned}$$

Using the formula

$$\Gamma(n/2) = \frac{(n-2)!!\sqrt{\pi}}{2^{(n-1)/2}},$$

we infer that

$$y(x) = \frac{2^{(n-1)/2}}{(n-2)!!\sqrt{\pi}} (\Gamma(s+1)\Phi_{-n/2+s+1}(x) + \Phi_{-n/2-k}(x)).$$

This completes the proof of Example 1. \square

In particular, the integral equation

$$x_+^{-3/2} + \delta'(x) = \int_0^x y(\tau)(x-\tau)^{-1/2}d\tau \tag{10}$$

has the solution in the space $\mathcal{D}'(R^+)$

$$y(x) = -2\delta'(x) + \frac{1}{\sqrt{\pi}}\Phi_{-3/2}(x)$$

using

$$\Gamma(-1/2) = -2\sqrt{\pi}.$$

Remark 1. We must mention that Equation (10) cannot be solved by the Laplace transform since the distribution $x_+^{-3/2}$ is not locally integrable and its Laplace transform does not exist.

Similarly, the integral equation

$$x_+ + \delta(x) = \int_0^x y(\tau)(x-\tau)d\tau$$

has the solution in the space $\mathcal{D}'(R^+)$

$$y(x) = \delta(x) + \delta''(x)$$

by Equation (2).

Example 2. Let $s, \alpha \in R$, and $\alpha \neq 0, -1, -2 \dots$. Then, the integral equation

$$\sin x_+ + \Phi_s(x) = \int_0^x y(\tau)(x-\tau)^{\alpha-1}d\tau \tag{11}$$

has the solution in the space $\mathcal{D}'(R^+)$

$$y(x) = \frac{1}{\Gamma(\alpha)} \left(\sum_{k=0}^{\infty} (-1)^k \Phi_{2k-\alpha+2}(x) + \Phi_{s-\alpha}(x) \right),$$

where

$$\sin x_+ = \begin{cases} \sin x & \text{if } x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Equation (11) can be written as

$$\sin x_+ + \Phi_s(x) = \frac{\Gamma(\alpha)}{\Gamma(\alpha)} \int_0^x y(\tau)(x - \tau)^{\alpha-1} d\tau$$

which is equal to

$$\sin x_+ + \Phi_s(x) = \Gamma(\alpha)(\Phi_\alpha * y)(x).$$

Applying Theorem 1, we get

$$y(x) = \frac{1}{\Gamma(\alpha)} (\Phi_{-\alpha}(x) * (\sin x_+ + \Phi_s(x)))$$

which distributes to

$$y(x) = \frac{1}{\Gamma(\alpha)} (\Phi_{-\alpha}(x) * \sin x_+ + \Phi_{-\alpha}(x) * \Phi_s(x)).$$

The Taylor expansion

$$\sin x_+ = \sum_{k=0}^{\infty} \frac{(-1)^k x_+^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} (-1)^k \Phi_{2k+2}(x)$$

gives

$$y(x) = \frac{1}{\Gamma(\alpha)} \left(\sum_{k=0}^{\infty} (-1)^k \Phi_{2k-\alpha+2}(x) + \Phi_{s-\alpha}(x) \right).$$

We note that the series

$$\sum_{k=0}^{\infty} (-1)^k \Phi_{2k-\alpha+2}$$

is absolutely convergent by the ratio test. Indeed,

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{\Phi_{2(k+1)-\alpha+2}(x)}{\Phi_{2k-\alpha+2}(x)} &= \lim_{k \rightarrow \infty} \frac{\frac{x_+^{2k-\alpha+3}}{\Gamma(2k-\alpha+4)}}{\frac{x_+^{2k-\alpha+1}}{\Gamma(2k-\alpha+2)}} \\ &= \lim_{k \rightarrow \infty} \frac{x_+^2}{(2k-\alpha+3)(2k-\alpha+2)} = 0. \end{aligned}$$

This completes the proof of Example 2. \square

Remark 2. We should point out that the series $\sum_{k=0}^{\infty} (-1)^k \Phi_{2k-\alpha+2}(x)$ is the sum of singular and regular distributions. Indeed, let j be the largest non-negative integer, such that $2j - \alpha + 2 \leq 0$. Then

$$\sum_{k=0}^{\infty} (-1)^k \Phi_{2k-\alpha+2}(x) = \sum_{k=0}^j (-1)^k \Phi_{2k-\alpha+2}(x) + \sum_{k=j+1}^{\infty} (-1)^k \Phi_{2k-\alpha+2}(x)$$

where the term

$$\sum_{k=0}^j (-1)^k \Phi_{2k-\alpha+2}(x)$$

is a singular distribution, while

$$\sum_{k=j+1}^{\infty} (-1)^k \Phi_{2k-\alpha+2}(x)$$

is regular.

In the special case of $\alpha < 2$, we get

$$\begin{aligned} y(x) &= \frac{1}{\Gamma(\alpha)} \left(\sum_{k=0}^{\infty} \frac{(-1)^k x_+^{2k-\alpha+1}}{\Gamma(2k-\alpha+2)} + \Phi_{s-\alpha}(x) \right) \\ &= \frac{1}{\Gamma(\alpha)} \left(x_+^{-\alpha+1} \sum_{k=0}^{\infty} \frac{(-x_+^2)^k}{\Gamma(2k-\alpha+2)} + \Phi_{s-\alpha}(x) \right) \\ &= \frac{1}{\Gamma(\alpha)} \left(x_+^{-\alpha+1} E_{2,-\alpha+2}(-x_+^2) + \Phi_{s-\alpha}(x) \right). \end{aligned}$$

Note that the function $x_+^{-\alpha+1}$ is locally integrable on R .

In particular, we have for $\alpha = 1$ that

$$y(x) = \theta(x) \cosh(ix) + \Phi_{s-1}(x) = \theta(x) \cos x + \Phi_{s-1}(x)$$

using

$$E_{2,1}(z) = \cosh \sqrt{z}.$$

This also can be derived directly from Equation (11). In fact, it becomes for $\alpha = 1$

$$\sin x_+ + \Phi_s(x) = \int_0^x y(\tau) d\tau$$

which claims that

$$y(x) = \frac{d}{dx}(\theta(x) \sin x + \Phi_s(x)) = \theta(x) \cos x + \Phi_{s-1}(x)$$

by noting that

$$\frac{d}{dx} \theta(x) \sin x = \delta(x) \sin x + \theta(x) \cos x = \theta(x) \cos x.$$

We further note that Equation (11) becomes

$$\sin x_+ + \delta^{(-s)}(x) = \int_0^x y(\tau)(x-\tau)^{\alpha-1} d\tau.$$

for $s = 0, -1, -2, \dots$.

Similarly, the integral equation

$$\cos x_+ + e_+^x = \int_0^x y(\tau)(x-\tau)^{\alpha-1} d\tau \tag{12}$$

has the solution in the space $\mathcal{D}'(R^+)$

$$y(x) = \frac{1}{\Gamma(\alpha)} \left(\sum_{k=0}^{\infty} (-1)^k \Phi_{2k-\alpha+1}(x) + \sum_{k=0}^{\infty} \Phi_{k-\alpha+1}(x) \right),$$

where $\alpha \neq 0, -1, \dots$ and

$$e_+^x = \begin{cases} e^x & \text{if } x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, we apply Theorem 1 to get

$$y(x) = \frac{1}{\Gamma(\alpha)} (\Phi_{-\alpha}(x) * (\cos x_+ + e_+^x)).$$

Applying the following Taylor’s expansions

$$\begin{aligned} \cos x_+ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x_+^{2k} = \sum_{k=0}^{\infty} (-1)^k \Phi_{2k+1}(x), \\ e_+^x &= \sum_{k=0}^{\infty} \frac{x_+^k}{k!} = \sum_{k=0}^{\infty} \Phi_{k+1}(x) \end{aligned}$$

we arrive at

$$y(x) = \frac{1}{\Gamma(\alpha)} \left(\sum_{k=0}^{\infty} (-1)^k \Phi_{2k+1-\alpha}(x) + \sum_{k=0}^{\infty} \Phi_{k+1-\alpha}(x) \right).$$

This completes the proof by noting that both $\sum_{k=0}^{\infty} (-1)^k \Phi_{2i-\alpha+1}(x)$ and $\sum_{k=0}^{\infty} \Phi_{k-\alpha+1}(x)$ are absolutely convergent by the ratio test.

In the case of $\alpha < 1$ we get

$$y(x) = \frac{x_+^{-\alpha}}{\Gamma(\alpha)} (E_{2,-\alpha+1}(-x_+^2) + E_{1,-\alpha+1}(x_+)).$$

In particular when $\alpha = 1/2$

$$f(x) = \frac{x_+^{-1/2}}{\sqrt{\pi}} \left(E_{2,1/2}(-x_+^2) + E_{1,1/2}(x_+) \right).$$

We shall extend the techniques used by Yu. I. Babenko in his book [48], for solving various types of fractional differential and integral equations in the classical sense, to generalized functions. The method itself is close to the Laplace transform method in the ordinary sense, but it can be used in more cases [46], such as solving integral or fractional differential equations with distributions whose Laplace transforms do not exist in the classical sense as indicated below. Clearly, it is always necessary to show convergence of the series obtained as solutions. In [46], Podlubny also provided interesting applications to solving certain partial differential equations for heat and mass transfer by Babenko’s method.

To illustrate Babenko’s approach in detail, we solve the following Abel’s integral equation of the second kind in the space $\mathcal{D}'(R^+)$

$$\Phi_{-1/2} = y(x) + \int_0^x (x - \zeta)^\alpha y(\zeta) d\zeta$$

for $\alpha > 0$. Note that the Laplace transform does not work for this equation, since the Laplace transform of $\Phi_{-1/2}$ does not exist. However, this equation can be converted into

$$\Phi_{-1/2} = y(x) + \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1)} \int_0^x (x - \zeta)^\alpha y(\zeta) d\zeta = (\delta + \Gamma(\alpha + 1)\Phi_{\alpha+1}) * y(x)$$

This implies by Babenko’s method that

$$\begin{aligned}
 y(x) &= (\delta + \Gamma(\alpha + 1)\Phi_{\alpha+1})^{-1} * \Phi_{-1/2} \\
 &= \sum_{n=0}^{\infty} (-1)^n \Gamma^n(\alpha + 1)\Phi_{\alpha+1}^n * \Phi_{-1/2} \\
 &= \sum_{n=0}^{\infty} (-1)^n \Gamma^n(\alpha + 1)\Phi_{(\alpha+1)n} * \Phi_{-1/2} \\
 &= \sum_{n=0}^{\infty} (-1)^n \Gamma^n(\alpha + 1)\Phi_{(\alpha+1)n-1/2} \\
 &= \Phi_{-1/2} + \sum_{n=1}^{\infty} (-1)^n \Gamma^n(\alpha + 1)\Phi_{(\alpha+1)n-1/2} \\
 &= \Phi_{-1/2} + \sum_{n=0}^{\infty} (-1)^{n+1} \Gamma^{n+1}(\alpha + 1)\Phi_{(\alpha+1)n+\alpha+1/2} \\
 &= \Phi_{-1/2} - \Gamma(\alpha + 1)x_+^{\alpha-1} E_{\alpha+1, \alpha+1/2} \left(-\Gamma(\alpha + 1)x_+^{\alpha+1} \right)
 \end{aligned}$$

using

$$\Phi_{\alpha+1}^n = \Phi_{(\alpha+1)n}.$$

Example 3. Let $\alpha > \beta \geq 0$, and $\gamma \geq 0$. Then the fractional differential equation

$$ay^{(\alpha)}(x) + by^{(\beta)}(x) = c\Phi_{\gamma}(x) + x_+^m \tag{13}$$

has the solution in the space $\mathcal{D}'(R^+)$

$$y(x) = \frac{cx_+^{\alpha+\gamma-1}}{a} E_{\alpha-\beta, \alpha+\gamma} \left(-\frac{b}{a}x_+^{\alpha-\beta} \right) + \frac{m! x_+^{\alpha+m}}{a} E_{\alpha-\beta, \alpha+m+1} \left(-\frac{b}{a}x_+^{\alpha-\beta} \right)$$

where $m = 0, 1, 2, \dots$, and $a, b, c \in R$ with $a \neq 0$.

Proof. We see that Equation (13) is equivalent to

$$ay(x) * \Phi_{-\alpha}(x) + by(x) * \Phi_{-\beta}(x) = c\Phi_{\gamma}(x) + x_+^m.$$

Applying Φ_{α} to both sides, we get

$$\begin{aligned}
 ay(x) + by(x) * \Phi_{\alpha-\beta}(x) &= ay(x) * \left(\delta(x) + \frac{b}{a}\Phi_{\alpha-\beta}(x) \right) \\
 &= c\Phi_{\alpha+\gamma}(x) + x_+^m * \Phi_{\alpha}(x).
 \end{aligned}$$

This implies, by Babenko’s approach

$$\begin{aligned}
 y(x) &= \frac{1}{a} \left(\delta(x) + \frac{b}{a}\Phi_{\alpha-\beta}(x) \right)^{-1} (c\Phi_{\alpha+\gamma}(x) + x_+^m * \Phi_{\alpha}(x)) \\
 &= \frac{1}{a} \sum_{k=0}^{\infty} (-1)^k \left(\frac{b}{a}\Phi_{\alpha-\beta}(x) \right)^k * (c\Phi_{\alpha+\gamma}(x) + x_+^m * \Phi_{\alpha}(x)) \\
 &= \frac{c}{a} \sum_{k=0}^{\infty} \frac{(-b)^k}{a^k} \Phi_{k(\alpha-\beta)+\alpha+\gamma}(x) + \frac{m!}{a} \sum_{k=0}^{\infty} \frac{(-b)^k}{a^k} \Phi_{k(\alpha-\beta)+\alpha+m+1}(x) \\
 &= \frac{c}{a} \sum_{k=0}^{\infty} \frac{(-b)^k x_+^{k(\alpha-\beta)+\alpha+\gamma-1}}{a^k \Gamma(k(\alpha-\beta) + \alpha + \gamma)} + \frac{m!}{a} \sum_{k=0}^{\infty} \frac{(-b)^k x_+^{k(\alpha-\beta)+\alpha+m}}{a^k \Gamma(k(\alpha-\beta) + \alpha + m + 1)} \\
 &= \frac{cx_+^{\alpha+\gamma-1}}{a} E_{\alpha-\beta, \alpha+\gamma} \left(-\frac{b}{a}x_+^{\alpha-\beta} \right) + \frac{m! x_+^{\alpha+m}}{a} E_{\alpha-\beta, \alpha+m+1} \left(-\frac{b}{a}x_+^{\alpha-\beta} \right).
 \end{aligned}$$

This completes the proof of Example 3. \square

In particular, the ordinary differential equation

$$ay'(x) + by(x) = c\Phi_2(x) + x_+,$$

has the solution in the space $\mathcal{D}'(R^+)$

$$y(x) = \frac{(c+1)x_+^2}{a} E_{1,3} \left(-\frac{bx_+}{a} \right)$$

where

$$E_{1,3}(z) = \frac{e^z - z - 1}{z^2}.$$

On the other hand, we derive that the fractional differential equation

$$ay'(x) + by^{(1/2)}(x) = c\delta(x) + x_+^m, \quad a \neq 0$$

has the solution in the space $\mathcal{D}'(R^+)$

$$y(x) = \frac{c\theta(x)}{a} e^{\frac{b^2}{a^2}x_+} \operatorname{erfc} \left(\frac{b}{a} \sqrt{x_+} \right) + \frac{x_+^{1+m} m!}{a} E_{0.5,2+m} \left(-\frac{b}{a} \sqrt{x_+} \right).$$

by using

$$E_{0.5,1}(z) = e^{z^2} \operatorname{erfc}(-z),$$

where erfc is the complement to the error function (erf),

$$\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-u^2} du = 1 - \operatorname{erf}(z), \quad z \in \mathbb{C}.$$

Clearly, this example can also be solved using the Laplace transform. Applying the Laplace transform to the equation

$$ay^{(\alpha)}(x) + by^{(\beta)}(x) = c\Phi_\gamma(x) + x_+^m,$$

we come to

$$y^*(s) = \frac{cs^{m+1} + m!s^\gamma}{as^{\alpha+\gamma+m+1} + bs^{\beta+\gamma+m+1}} = \left(\frac{c}{a}\right) \left(\frac{s^{-\gamma-\beta}}{s^{\alpha-\beta} + \frac{b}{a}}\right) + \left(\frac{m!}{a}\right) \left(\frac{s^{-\beta-m-1}}{s^{\alpha-\beta} + \frac{b}{a}}\right).$$

Using the inverse transform, we have

$$y(x) = \frac{cx_+^{\alpha+\gamma-1}}{a} E_{\alpha-\beta, \alpha+\gamma} \left(-\frac{b}{a} x_+^{\alpha-\beta} \right) + \frac{m! x_+^{\alpha+m}}{a} E_{\alpha-\beta, \alpha+m+1} \left(-\frac{b}{a} x_+^{\alpha-\beta} \right)$$

by the formula [46]

$$\int_0^\infty e^{-sx} x^{\beta-1} E_{\alpha,\beta}(-ax^\alpha) dx = \frac{s^{\alpha-\beta}}{s^\alpha + a}, \quad \operatorname{Re}(s) > |a|^{1/\alpha}.$$

Remark 3. We must add that the following fractional differential equation

$$ay^{(\alpha)}(x) + by^{(\beta)}(x) = c\Phi_\gamma(x) + x_+^{-3/2}$$

can also be solved by the same technique used in Example 3. Though it fails to do so by the Laplace transform, as the distribution $x_+^{-3/2}$ is singular.

Many applied problems from physical, engineering and chemical processes lead to integral equations, which at first glance have nothing in common with Abel’s integral equations. Due to this perception, additional efforts are undertaken for the development of analytical or numerical procedure for solving these equations. However, their transformations to the form of Abel’s integral equations will speed up the solution process [20], or, more significantly, lead to distributional solutions in cases where classical ones do not exist [40,41].

Example 4. Let $0 \leq \theta < \pi/2$ and $\alpha < 1$. Then the following integral equation

$$\int_{\theta}^{\pi/2} \frac{\sin \varphi y(\varphi)}{(\cos \theta - \cos \varphi)^\alpha} d\varphi = f(\theta) \tag{14}$$

has the solution

$$y(\arccos x) = \frac{1}{\Gamma(1-\alpha)} \frac{d^{1-\alpha}}{dx^{1-\alpha}} f(\arccos x)$$

where f is a differential function in $\mathcal{D}'(R^+)$.

Proof. Making the variable changes $\tau = \cos \varphi$ and $x = \cos \theta$. Then Equation (14) becomes

$$\int_0^x \frac{1}{(x-\tau)^\alpha} y(\arccos \tau) d\tau = \frac{\Gamma(1-\alpha)}{\Gamma(1-\alpha)} \int_0^x \frac{1}{(x-\tau)^\alpha} y(\arccos \tau) d\tau = f(\arccos x)$$

which is Abel’s integral equation of the first kind. Therefore, we arrive at

$$\Phi_{1-\alpha}(\tau) * y(\arccos \tau) = \frac{1}{\Gamma(1-\alpha)} f(\arccos x)$$

which implies that

$$y(\arccos x) = \frac{1}{\Gamma(1-\alpha)} \Phi_{\alpha-1}(\tau) * f(\arccos \tau) = \frac{1}{\Gamma(1-\alpha)} \frac{d^{1-\alpha}}{dx^{1-\alpha}} f(\arccos x).$$

This completes the proof of Example 4. \square

In particular, we have that for $\alpha = 1/2$ and $f(\theta) = \theta$

$$y(\arccos x) = \frac{1}{\Gamma(1/2)} \frac{d^{1/2}}{dx^{1/2}} \arccos x = \frac{1}{\sqrt{\pi}} \frac{d^{1/2}}{dx^{1/2}} \arccos x.$$

Using the Taylor series

$$\arccos x = \frac{1}{2}\pi - \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2(2n+1)} x^{2n+1}$$

if $0 < x \leq 1$, and

$$\frac{d^{1/2}}{dx^{1/2}} x_+^{2n+1} = (2n+1)! \Phi_{-1/2}(x) * \frac{x_+^{2n+1}}{\Gamma(2n+2)} = (2n+1)! \Phi_{2n+3/2}(x)$$

we come to

$$y(\arccos x) = -\frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{[(2n)!]^2}{2^{2n}(n!)^2} \Phi_{2n+3/2}(x)$$

which is obviously convergent. Furthermore, setting $t = \arccos x$ we finally infer that

$$y(t) = -\frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{[(2n)!]^2}{2^{2n}(n!)^2} \Phi_{2n+3/2}(\cos t).$$

Remark 4. Clearly, Equation (14) can be converted into

$$\begin{aligned} \int_{\theta}^{\pi/2} \frac{\sin \varphi y(\varphi)}{(\cos \theta - \cos \varphi)^{\alpha}} d\varphi &= \int_{\pi/2}^{\theta} \frac{y(\varphi)}{(\cos \theta - \cos \varphi)^{\alpha}} d \cos \varphi \\ &= \int_0^{\cos \theta} \frac{y(\arccos \tau)}{(\cos \theta - \tau)^{\alpha}} d\tau \\ &= \int_0^x \frac{y(\arccos \tau)}{(x - \tau)^{\alpha}} d\tau \\ &= f(\arccos x). \end{aligned}$$

Setting

$$y(\varphi) = \frac{1}{\sin \varphi} Y(\varphi),$$

then the integral equation

$$\int_{\theta}^{\pi/2} \frac{Y(\varphi)}{(\cos \theta - \cos \varphi)^{\alpha}} d\varphi = f(\theta)$$

has the solution

$$Y(\arccos x) = \frac{\sqrt{1-x^2}}{\Gamma(1-\alpha)} \frac{d^{1-\alpha}}{dx^{1-\alpha}} f(\arccos x)$$

since

$$\sin(\arccos x) = \sqrt{1-x^2}.$$

Further, setting

$$y(\varphi) = \frac{1}{\sin \varphi \cos^{\beta} \varphi} Y(\varphi) \quad \text{for } \beta < 1,$$

then the integral equation

$$\int_{\theta}^{\pi/2} \frac{Y(\varphi)}{\cos^{\beta} \varphi (\cos \theta - \cos \varphi)^{\alpha}} d\varphi = f(\theta)$$

has the solution

$$Y(\arccos x) = \frac{x^{\beta} \sqrt{1-x^2}}{\Gamma(1-\alpha)} \frac{d^{1-\alpha}}{dx^{1-\alpha}} f(\arccos x).$$

Example 5. Assume that the functions g and f are given and g is a nonzero function satisfying the condition

$$g(x+t) = g(x)g(t)$$

for all $x, t \in \mathbb{R}$. Then the integral equation

$$\int_0^{\infty} x^{1/2} g(x) y(x+t) dx = f(t) \tag{15}$$

has the solution

$$y(1/s) = \frac{2s^{2.5}}{\sqrt{\pi} g\left(\frac{1}{s}\right)} \left(\frac{d^{1.5}}{ds^{1.5}} \frac{f(1/s)s^{0.5}}{g(-1/s)} \right).$$

Proof. Making the substitution

$$\tau = \frac{1}{x + t},$$

Equation (15) becomes

$$\int_0^{1/t} \left(\frac{1}{\tau} - t\right)^{1/2} g\left(\frac{1}{\tau} - t\right) \frac{y\left(\frac{1}{\tau}\right)}{\tau^2} d\tau = f(t)$$

which infers that

$$\int_0^{1/t} \left(\frac{1}{\tau} - t\right)^{1/2} g\left(\frac{1}{\tau}\right) \frac{y\left(\frac{1}{\tau}\right)}{\tau^2} d\tau = \frac{f(t)}{g(-t)}$$

since g is a nonzero function. Further, setting $s = 1/t$ we come to

$$\frac{\Gamma(1.5)}{\Gamma(1.5)} \int_0^s (s - \tau)^{1/2} \frac{g\left(\frac{1}{\tau}\right) y\left(\frac{1}{\tau}\right)}{\tau^{2.5}} d\tau = \frac{s^{0.5} f(1/s)}{g(-1/s)}$$

which is Abel’s integral equation. Hence, we get the solution

$$y(1/s) = \frac{2s^{2.5}}{\sqrt{\pi} g\left(\frac{1}{s}\right)} \left(\frac{d^{1.5} f(1/s)s^{0.5}}{ds^{1.5} g(-1/s)} \right)$$

using

$$\Gamma(1.5) = \sqrt{\pi}/2.$$

This completes the proof of Example 5. \square

A particular example can be derived from setting $g(x) = e^{-x}$. We leave this to interested readers. We should point out that the term

$$\frac{d^{1.5}}{ds^{1.5}} \left(\frac{f(1/s)s^{0.5}}{g(-1/s)} \right)$$

is in the distributional sense. Otherwise, it is undefined if we let $g(x) \equiv 1$ and f be chosen in $\mathcal{D}'(R^+)$ such that $f(1/s)s^{0.5} = s_+^{-1.4}$.

3. The Applications in Viscoelastic Systems

A modeling is a cognitive activity which we use to describe how devices, or objects of interest, behave.

Elasticity is the ability of a material to resist a distortion or a deforming force and return to its original form when the force is removed. According to the classical theory in the infinitesimal deformation, the most elastic materials, based on Hooke’s Law, can be described by a linear relation between the strain ϵ and stress σ and

$$\epsilon(t) = \frac{1}{E} \sigma(t)$$

where E is a constant, known as the elastic or Young’s modulus.

However, in a more complicated fractional viscoelastic model, one [49,50] constructs the following integral equation

$$\epsilon(t) = \sigma(t)J(0^+) + \frac{1}{E\tau^\alpha} \left[\frac{1}{\Gamma(\alpha)} \int_0^t (t - \zeta)^{\alpha-1} \sigma(\zeta) d\zeta \right] \tag{16}$$

where

$$J\alpha(t) = \frac{1}{E} \frac{(t/\tau)^\alpha}{\Gamma(1 + \alpha)},$$

and $\tau = E/\eta$, η being the shear modulus.

According to the kernel function $t^{\alpha-1}/\Gamma(\alpha)$ in Equation (16), $\alpha = 0$ and $\alpha = 1$ are equal with a memory-less system and full-memory system in creeping state respectively owing to $\Gamma(0) = \infty$. Clearly, we can derive that for $\alpha = 0$

$$\sigma(t) = \frac{\epsilon(t)}{J(0^+) + \frac{1}{E}} = \frac{E\epsilon(t)}{1 + EJ(0^+)}$$

using Equation (2) in distribution.

When $0 < \alpha \leq 1$, we convert Equation (16) into

$$\frac{\epsilon(t)}{J(0^+)} = \sigma(t) + \frac{1}{E\tau^\alpha J(0^+)} (\Phi_\alpha * \sigma)(t) = \left(\delta + \frac{1}{E\tau^\alpha J(0^+)} \Phi_\alpha \right) * \sigma(t).$$

By Babenko’s approach, we imply that

$$\begin{aligned} \sigma(t) &= \frac{1}{J(0^+)} \left(\delta + \frac{1}{E\tau^\alpha J(0^+)} \Phi_\alpha \right)^{-1} * \epsilon(t) \\ &= \frac{1}{J(0^+)} \sum_{n=0}^{\infty} \frac{(-1)^n}{E^n \tau^{\alpha n} J^n(0^+)} \Phi_{\alpha n} * \epsilon(t) \end{aligned}$$

which is the relation between the stress $\sigma(t)$ and strain $\epsilon(t)$.

In particular, we derive that

$$\sigma(t) = \frac{1}{J(0^+)} \sum_{n=0}^{\infty} \frac{(-1)^n}{E^n \tau^{\alpha n} J^n(0^+)} \Phi_{\alpha n+1}$$

if the strain $\epsilon(t) = \theta(t)$.

4. Conclusions

With Babenko’s approach, we have studied and solved several Abel’s integral and fractional differential equations based on fractional calculus and convolutions of distributions in the space $\mathcal{D}'(R^+)$. Some of the results obtained are not achievable in the classical sense, such as numerical analysis methods or the Laplace transform, since the equations involve generalized functions which are not locally integrable, and undefined at points in R . Generally speaking, these equations can be expressed in terms of series or the Mittag–Leffer functions using inverse convolution operators in distribution. At the end, we demonstrate applications of Abel’s integral equations in viscoelastic systems, and for solving other different types of integral equations with potential demands in physical problems.

Author Contributions: The order of the author list reflects contributions to the paper.

Funding: This research was funded by NSERC (Canada) under grant number 2017-00001.

Acknowledgments: The authors are grateful to the reviewers and academic editor for their careful reading of the paper with very productive suggestions and corrections, which certainly improved its quality.

Conflicts of Interest: The authors declare no conflict of interest.

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