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An Example of the Generalized Fractional Laplacian on R^n

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Abstract: Using the normalization of the fractional Laplacian $(-\triangle)^s u(x)$ over the space $C_k(R^n)$ for all $s > 0$ and $s \neq 1, 2, \cdots$, we evaluate $(-\triangle)^s e^{-\|x\|^2}$ based on the multidimensional surface integrals over the unit sphere and special functions.
Keywords: fractional Laplacian, normalization; distribution, surface integral, gamma function

1. Introduction

Let $\triangle = \partial^2 / \partial x_1^2 + \cdots + \partial^2 / \partial x_n^2$. The fractional Laplacian operator $(-\triangle)^s$ of order s $\in$ (0,1) is defined, via the Cauchy principal value integral[1-5], as

$$(-\triangle)^s u(x) = C_{n,s} \text{P.V.} \int_{R^n} \frac{u(x) - u(\zeta)}{|x - \zeta|^{n+2s}} d\zeta,$$

where $u(x)$ is a function from R^n to R, and the constant $C_{n,s}$ is given by

$$C_{n,s} = \pi^{-n/2} 2^{2s} \frac{\Gamma(\frac{n+2s}{2})}{\Gamma(1-s)} s.$$

One typically needs two main requisites on the function u in order to make the above integral exist:
(i) u has to be sufficiently smooth near x,
(ii) u needs to have a controlled growth at infinity, for instance

$$\int_{R^n} \frac{|u(x)|}{1 + |x|^{n+2s}} dx < \infty.$$

In [6], Dipierro et al. defined the fractional Laplacian for functions which grow more than linearly at infinity. The basic idea for this is that, if the function grows too much at infinity, its fractional Laplacian diverges, but it can be written as a given function plus a diverging sequence of polynomials of a given degree.

The fractional Laplacian, widely considered as the Riesz derivative[7-11] or a nonlocal pseudo-differential operator, has ten equivalent definitions over certain function spaces[12]. For example, it can be defined

(i) as a Fourier multiplier given by the formula

$$\mathcal{F}((-\triangle)^s u)(\zeta) = |\zeta|^{2s} \mathcal{F}(u)(\zeta)$$

where the Fourier transform $\mathcal{F}(u)$ of a function u is given by

$$\mathcal{F}(u)(\zeta) = \int_{R^n} u(x) e^{-ix\zeta} dx.$$

(ii) by singular integral definition

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$$(-\triangle)^s u(\zeta) = -\lim_{r\to 0^+} \frac{2^{2s}\Gamma\left(\frac{n}{2}+s\right)}{\pi^{n/2}\,|\Gamma(-s)|}\int_{R^n\setminus B(x,r)} \frac{u(\zeta+z)-u(\zeta)}{|z|^{n+2s}}dz,$$

with the limit in Lebesgue spaces.

The fractional Laplacian with Fourier transform has recently gained a great deal of attention from the research area of differential equations[13-18], including fractional diffusion[19-21] and porous medium equations[22-24].

Let $x = (x_1, x_2, \cdots, x_n) \in R^n$. For a given n-tuple $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ of nonnegative integers (or called a multi-index), we define

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n, \alpha! = \alpha_1!\alpha_2!\cdots\alpha_n!$$

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n},$$

$$\partial^\alpha u = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_n^{\alpha_n} u = \frac{\partial^{|\alpha|}u}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\cdots\partial x_n^{\alpha_n}}.$$

The Schwartz space $S(R^n)$ (space of rapidly decreasing functions on R^n) is the function space[25] defined as

$$S(R^n) = \left\{u(x) \in C^\infty(R^n) : \|u(x)\|_{\alpha,k} \le C_{\alpha,k}(\text{const})\ \forall \alpha, k \in N_0^n\right\}$$

where $N_0 = \{0\} \cup N$ is the set of nonnegative integers and

$$\|u(x)\|_{\alpha,k} = \sup_{x\in R^n}\left|x^\alpha \partial^k u(x)\right|.$$

Let $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$. The function space $C_k(R^n)$ is defined in [27] as the following.

$C_k(R^n) = \{u(x)$ is bounded and $\partial^{2k}u(x)$ is continuous on R^n:

$$\exists M_k(\text{const}) > 0, \text{ such that } \left|\partial^{2k}u(x)\right| \le \frac{M_k}{\|x\|^2} \text{ as } \|x\| \to \infty\}$$

where $k = (k_1, k_2, \cdots, k_n)$ is an n-tuple of nonnegative integers.

Applying the normalization of the distribution x_+^λ, Pizzetti's formula[26], and surface integrals on R^n, Li[27] very recently extended the fractional Laplacian $(-\triangle)^s u(x)$ over the space $C_k(R^n)$ (which contains $S(R^n)$ as a proper subspace) for all $s > 0$ and $s \neq 1, 2, \cdots$, and obtained Theorem 1 below.

Theorem 1. Let $i = 0, 1, \cdots$ and $i < s < i + 1$. Then the generalized fractional Laplacian $(-\triangle)^s$ is normalized over the space $C_k(R^n)$ as

$$(-\triangle)^s u(x) = -\frac{1}{2}C_{n,s}\int_0^\infty r^{-1-2s}\cdot$$

$$\left[S(r) - \frac{r^2\Omega_n \triangle u(x)}{n} - \cdots - \frac{2r^{2i}\Omega_n \triangle^i u(x)}{2^i i! n(n+2)\cdots(n+2i-2)}\right]dr, \tag{1}$$

where $\Omega_n = 2\pi^{n/2}/\Gamma(\frac{n}{2})$ is the area of the unit sphere in R^n, $k = (k_1, k_2, \cdots, k_n)$ with $k_1 + \cdots + k_n = i + 1$, and

$$S(r) = \int_{\|\sigma\|^2=1}\left[u(x+r\sigma) - 2u(x) + u(x-r\sigma)\right]d\sigma.$$

Furthermore,

$$\lim_{s\to m}(-\triangle)^s u(x) = (-1)^m \triangle^m u(x)$$

for all m = 1, 2, $\cdots$, and

$$\lim_{s\to 0^+}(-\triangle)^s u(x) = u(x).$$

Note that for i = 0, we define

$$\frac{r^2\Omega_n \triangle u(x)}{n} + \cdots + \frac{2r^{2i}\Omega_n \triangle^i u(x)}{2^i\, i!n(n+2)\cdots(n+2i-2)} = 0.$$

The goal of this paper is to directly compute the fractional Laplacian $(-\triangle)^s e^{-\|x\|^2}$ in an n-dimensional space by making use of Theorem 1, value of the surface integral

$$\int_{\|\sigma\|^2=1} \sigma_1^{2j_1}\cdots\sigma_n^{2j_n} d\sigma,$$

as well as special functions.

2. Main result

It follows from integration by parts that

$$\int_0^\infty r^{-1-2s}(e^{-r^2} - 1 + r^2)dr = -\frac{\Gamma(1-s)}{2s} \tag{2}$$

for 1 < s < 2.

In general, we have

$$\int_0^\infty r^{-1-2s}\left(e^{-r^2} - \sum_{j=0}^{k}\frac{(-r^2)^j}{j!}\right)dr = \frac{1}{2}\Gamma(-s) \tag{3}$$

for $k < s < k + 1$ and k = 1, 2, $\cdots$. Indeed, by the variable change $u = r^2$

$$\int_0^\infty r^{-1-2s}\left(e^{-r^2} - \sum_{j=0}^{k}\frac{(-r^2)^j}{j!}\right)dr = \frac{1}{2}\int_0^\infty u^{-1-s}\left(e^{-u} - \sum_{j=0}^{k}\frac{(-u)^j}{j!}\right)du = \frac{1}{2}\Gamma(-s),$$

which is the normalization of gamma function on page 53 in [28]. Clearly for k = 1, equations (2) and (3) agree since

$$-\frac{\Gamma(1-s)}{2s} = \frac{1}{2}\Gamma(-s).$$

The multinomial theorem can be written as

$$(x_1 + \cdots + x_n)^m = \sum_{k_1+\cdots+k_n=m}\binom{m}{k_1, k_2, \cdots, k_n} x_1^{k_1}\cdots x_n^{k_n}$$

where m = 0, 1, 2, $\cdots$ and

$$\binom{m}{k_1, k_2, \cdots, k_n} = \frac{m!}{k_1! \cdots k_n!}.$$

In particular,

$$\sum_{k_1+\cdots+k_n=m} \binom{m}{k_1, k_2, \cdots, k_n} = n^m$$

by setting $x_1 = \cdots = x_n = 1$.

We are ready to present the main result as follows.

Theorem 2. Let $s > 0$ and $s \neq 1, 2, \cdots$. Then,

$$(-\triangle)^s e^{-\|x\|^2} = -\pi^{-n/2} 2^{2s} \frac{\Gamma\left(\frac{n}{2}+s\right)}{\Gamma(1-s)} s\, e^{-\|x\|^2} \sum_{j=0}^{\infty} \frac{2^{2j}\Gamma(j-s)}{(2j)!\,\Gamma\left(j+\frac{n}{2}\right)} \cdot$$

$$\sum_{j_1+\cdots+j_n=j} \binom{2j}{2j_1, \cdots, 2j_n} x_1^{2j_1} \cdots x_n^{2j_n} \prod_{i=1}^{n} \Gamma\left(\frac{1}{2}+j_i\right).$$

In particular,

$$\lim_{s\to 0^+} (-\triangle)^s e^{-\|x\|^2} = e^{-\|x\|^2},$$

and

$$\lim_{s\to k} (-\triangle)^s e^{-\|x\|^2} = 2^{2k} \frac{\Gamma\left(\frac{n}{2}+k\right)}{\Gamma(n/2)} e^{-\|x\|^2} - \frac{1}{n} 2^{2k+1} k \frac{\Gamma\left(\frac{n}{2}+k\right)}{\Gamma(n/2)} e^{-\|x\|^2} \|x\|^2$$

$$-\pi^{-n/2} 2^{2k} \Gamma\left(\frac{n}{2}+k\right) e^{-\|x\|^2} \sum_{j=2}^{k} \frac{2^{2j}}{(2j)!\,\Gamma\left(j+\frac{n}{2}\right)} (-1)^{j-1} k(k-1)\cdots(k-j+1)$$

$$\sum_{j_1+\cdots+j_n=j} \binom{2j}{2j_1, \cdots, 2j_n} x_1^{2j_1} \cdots x_n^{2j_n} \prod_{i=1}^{n} \Gamma\left(\frac{1}{2}+j_i\right) = (-1)^k \triangle^k e^{-\|x\|^2}$$

for $k = 1, 2, \cdots$.

Proof. Clearly,

$$\triangle e^{-\|x\|^2} = e^{-\|x\|^2} (4\|x\|^2 - 2n).$$

Let us assume $1 < s < 2$ first and $u(x) = e^{-\|x\|^2} \in S(R^n) \subset C_k(R^n)$. From equation (1), we need to evaluate $S(r)$ for $r > 0$.

$$S(r) = \int_{\|\sigma\|^2=1} \left[u(x + r\sigma) - 2u(x) + u(x - r\sigma) \right] d\sigma$$

$$= \int_{\|\sigma\|^2=1} \left[e^{-(x_1+r\sigma_1)^2-...-(x_n+r\sigma_n)^2} - 2u(x) + e^{-(x_1-r\sigma_1)^2-...-(x_n-r\sigma_n)^2} \right] d\sigma$$

$$= e^{-\|x\|^2-r^2} \int_{\|\sigma\|^2=1} \left[e^{-2x_1r\sigma_1-...-2x_nr\sigma_n} + e^{2x_1r\sigma_1+...+2x_nr\sigma_n} \right] d\sigma - 2u(x)\Omega_n.$$

Using

$$e^{-x} + e^x = \sum_{j=0}^{\infty} \frac{2x^{2j}}{(2j)!}, \; x \in R$$

we come to

$$e^{-2x_1r\sigma_1-\cdots-2x_nr\sigma_n} + e^{2x_1r\sigma_1+\cdots+2x_nr\sigma_n} = 2\sum_{j=0}^{\infty} \frac{2^{2j} r^{2j} (x_1\sigma_1 + \cdots + x_n\sigma_n)^{2j}}{(2j)!}$$

$$= 2\sum_{j=0}^{\infty} \frac{2^{2j} r^{2j}}{(2j)!} \sum_{k_1+\cdots+k_n=2j} \binom{2j}{k_1,\cdots,k_n} (x_1\sigma_1)^{k_1} \cdots (x_n\sigma_n)^{k_n}$$

$$= 2\sum_{j=0}^{\infty} \frac{2^{2j} r^{2j}}{(2j)!} \sum_{k_1+\cdots+k_n=2j} \binom{2j}{k_1,\cdots,k_n} x_1^{k_1} \cdots x_n^{k_n} \, \sigma_1^{k_1} \cdots \sigma_n^{k_n}.$$

From the cancellation over the unit sphere, we obtain

$$\int_{\|\sigma\|^2=1} \sigma_1^{k_1} \cdots \sigma_n^{k_n} d\sigma = 0$$

if one of k_i for $i = 1, 2, \cdots, n$ is odd. Hence,

$$S(r) = e^{-\|x\|^2-r^2} \sum_{j=0}^{\infty} \frac{2^{2j+1} r^{2j}}{(2j)!} \sum_{k_1+\cdots+k_n=2j} \binom{2j}{k_1,\cdots,k_n} x_1^{k_1} \cdots x_n^{k_n} \cdot \int_{\|\sigma\|^2=1} \sigma_1^{k_1} \cdots \sigma_n^{k_n} d\sigma - 2u(x)\Omega_n$$

$$= e^{-\|x\|^2-r^2} \sum_{j=0}^{\infty} \frac{2^{2j+1} r^{2j}}{(2j)!} \sum_{j_1+\cdots+j_n=j} \binom{2j}{2j_1,\cdots,2j_n} x_1^{2j_1} \cdots x_n^{2j_n} \cdot \int_{\|\sigma\|^2=1} \sigma_1^{2j_1} \cdots \sigma_n^{2j_n} d\sigma - 2u(x)\Omega_n.$$

In order to evaluate the surface integral over the unit sphere in R^n

$$S_n(1, j_1, \cdots, j_n) = \int_{\|\sigma\|^2=1} \sigma_1^{2j_1} \cdots \sigma_n^{2j_n} d\sigma,$$

we need to consider the following integral over the unit ball

$$I_n(1, j_1, \cdots, j_n) = \int_{\|\sigma\|^2 \le 1} \sigma_1^{2j_1} \cdots \sigma_n^{2j_n} d\sigma.$$

Making the variable change

$\sigma_n = \cos\theta,\ 0 \le \theta \le \pi,$

we arrive at

$$\int_{\|\sigma\|^2\le 1}\sigma_1^{2j_1}\cdots\sigma_n^{2j_n}d\sigma=\int_0^{\pi}\left(\int_{\sigma_1^2+\cdots+\sigma_{n-1}^2\le\sin^2\theta}\sigma_1^{2j_1}\cdots\sigma_{n-1}^{2j_{n-1}}d\sigma\right)(\cos^2\theta)^{j_n}\sin\theta d\theta.$$

Clearly,

$$\int_{\sigma_1^2+\cdots+\sigma_{n-1}^2\le\sin^2\theta}\sigma_1^{2j_1}\cdots\sigma_{n-1}^{2j_{n-1}}d\sigma=(\sin^2\theta)^{j_1}\cdots(\sin^2\theta)^{j_{n-1}}\sin^{n-1}\theta\int_{y_1^2+\cdots+y_{n-1}^2\le 1}y_1^{2j_1}\cdots y_{n-1}^{2j_{n-1}}dy$$

$$=(\sin\theta)^{2(j_1+\cdots+j_{n-1})+n-1}I_{n-1}(1,j_1,\cdots,j_{n-1})$$

by the substitution

$\sigma_i=y_i\sin\theta$

for $1\le i\le n-1$.

Therefore,

$$I_n(1,j_1,\cdots,j_n)=I_{n-1}(1,j_1,\cdots,j_{n-1})\int_0^{\pi}(\sin\theta)^{2(j_1+\cdots+j_{n-1})+n-1}(\cos^2\theta)^{j_n}\sin\theta\,d\theta$$

$$=I_{n-1}(1,j_1,\cdots,j_{n-1})\int_0^{\pi}(\sin\theta)^{2(j_1+\cdots+j_{n-1})+n}(\cos^2\theta)^{j_n}\,d\theta.$$

Since

$$\int_0^{\pi/2}(\sin\theta)^{2(j_1+\cdots+j_{n-1})+n}(\cos^2\theta)^{j_n}\,d\theta=\int_{\pi/2}^{\pi}(\sin\theta)^{2(j_1+\cdots+j_{n-1})+n}(\cos^2\theta)^{j_n}\,d\theta,$$

we derive that

$$I_n(1,j_1,\cdots,j_n)=I_{n-1}(1,j_1,\cdots,j_{n-1})\,2\int_0^{\pi/2}(\sin\theta)^{2(j_1+\cdots+j_{n-1})+n}(\cos^2\theta)^{j_n}\,d\theta$$

$$=I_{n-1}(1,j_1,\cdots,j_{n-1})\beta\left(\frac{n+1}{2}+\sum_{i=1}^{n-1}j_i,\frac{1}{2}+j_n\right)$$

$$=\frac{\Gamma\left(\frac{n+1}{2}+\sum_{i=1}^{n-1}j_i\right)\Gamma\left(\frac{1}{2}+j_n\right)}{\Gamma\left(j+\frac{n}{2}+1\right)}I_{n-1}(1,j_1,\cdots,j_{n-1}),$$

where β is the beta function.

By the above recursion,

$$I_{n-1}(1,j_1,\cdots,j_{n-1})=\frac{\Gamma\left(\frac{n}{2}+\sum_{i=1}^{n-2}j_i\right)\Gamma\left(\frac{1}{2}+j_{n-1}\right)}{\Gamma\left(\frac{n+1}{2}+\sum_{i=1}^{n-1}j_i\right)}I_{n-2}(1,j_1,\cdots,j_{n-2}).$$

Repeating this procedure, we get

$$I_n(1, j_1, \cdots, j_n) = \frac{\Gamma\left(\frac{1}{2}+j_n\right)\cdots\Gamma\left(\frac{1}{2}+j_1\right)}{\Gamma\left(j+\frac{n}{2}+1\right)} = \frac{2\Gamma\left(\frac{1}{2}+j_n\right)\cdots\Gamma\left(\frac{1}{2}+j_1\right)}{(2j+n)\Gamma\left(j+\frac{n}{2}\right)}$$

$$= \frac{2\prod_{i=1}^{n}\Gamma\left(\frac{1}{2}+j_i\right)}{(2j+n)\Gamma\left(j+\frac{n}{2}\right)},$$

by noting that $I_0(1) = 1$.

Let $t > 0$ and

$$S_n(t, j_1, \cdots, j_n) = \int_{\|\sigma\|^2=t^2} \sigma_1^{2j_1} \cdots \sigma_n^{2j_n} d\sigma.$$

Then,

$$S_n(t, j_1, \cdots, j_n) = t^{2j+n-1} S_n(1, j_1, \cdots, j_n)$$

and

$$I_n(1, j_1, \cdots, j_n) = \int_0^1 S_n(t, j_1, \cdots, j_n) dt = S_n(1, j_1, \cdots, j_n) \int_0^1 t^{2j+n-1} dt$$

$$= \frac{S_n(1, j_1, \cdots, j_n)}{2j+n}.$$

Thus,

$$S_n(1, j_1, \cdots, j_n) = (2j+n) I_n(1, j_1, \cdots, j_n) = \frac{2\prod_{i=1}^{n}\Gamma\left(\frac{1}{2}+j_i\right)}{\Gamma\left(j+\frac{n}{2}\right)}.$$

In particular,

$$\Omega_n = S_n(1, 0, \cdots, 0) = \int_{\|\sigma\|^2=1} d\sigma = \frac{2\pi^{n/2}}{\Gamma(n/2)},$$

$$S_n(1, 1, 0, \cdots, 0) = \int_{\|\sigma\|^2=1} \sigma_1^2 d\sigma = \frac{\Omega_n}{n} = \text{Volume of the unit ball in } R^n.$$

In summary,

$$S(r) = 4e^{-\|x\|^2 - r^2} \sum_{j=0}^{\infty} \frac{2^{2j} r^{2j}}{(2j)!} \sum_{j_1+\cdots+j_n=j} \binom{2j}{2j_1, \cdots, 2j_n} x_1^{2j_1} \cdots x_n^{2j_n} \cdot \frac{\prod_{i=1}^{n}\Gamma\left(\frac{1}{2}+j_i\right)}{\Gamma\left(j+\frac{n}{2}\right)} - 2e^{-\|x\|^2}\Omega_n.$$

This implies that

$$S(r)-\frac{r^2\Omega_n \triangle e^{-\|x\|^2}}{n}=2e^{-\|x\|^2-r^2}\Omega_n+4e^{-\|x\|^2-r^2}\frac{4r^2}{2!}\|x\|^2\frac{\Gamma\left(\frac{1}{2}+1\right)\pi^{\frac{n-1}{2}}}{\Gamma\left(1+\frac{n}{2}\right)}$$

$$+4e^{-\|x\|^2-r^2}\sum_{j=2}^{\infty}\frac{2^{2j}r^{2j}}{(2j)!}\sum_{j_1+\cdots+j_n=j}\binom{2j}{2j_1,\cdots,2j_n}x_1^{2j_1}\cdots x_n^{2j_n}\cdot$$

$$\frac{\prod_{i=1}^{n}\Gamma\left(\frac{1}{2}+j_i\right)}{\Gamma\left(j+\frac{n}{2}\right)}-2e^{-\|x\|^2}\Omega_n-\frac{r^2\Omega_n 4\|x\|^2 e^{-\|x\|^2}}{n}+2r^2\Omega_n e^{-\|x\|^2}=T_1+T_2+T_3,$$

where

$$T_1=2\Omega_n e^{-\|x\|^2}\left(e^{-r^2}+r^2-1\right),$$

$$T_2=4e^{-\|x\|^2}\|x\|^2 r^2\frac{\Omega_n}{n}\left[e^{-r^2}-1\right],$$

$$T_3=4e^{-\|x\|^2-r^2}\sum_{j=2}^{\infty}\frac{2^{2j}r^{2j}}{(2j)!}\sum_{j_1+\cdots+j_n=j}\binom{2j}{2j_1,\cdots,2j_n}x_1^{2j_1}\cdots x_n^{2j_n}\frac{\prod_{i=1}^{n}\Gamma\left(\frac{1}{2}+j_i\right)}{\Gamma\left(j+\frac{n}{2}\right)}.$$

From Theorem 1 for $1 < s < 2$, we get

$$(-\triangle)^s e^{-\|x\|^2}=-\frac{1}{2}C_{n,s}\int_0^{\infty}r^{-1-2s}\left[T_1+T_2+T_3\right]dr$$

$$=-C_{n,s}\Omega_n e^{-\|x\|^2}\int_0^{\infty}r^{-1-2s}\left(e^{-r^2}+r^2-1\right)dr-\frac{2}{n}C_{n,s}\Omega_n e^{-\|x\|^2}\|x\|^2\int_0^{\infty}r^{1-2s}\left(e^{-r^2}-1\right)dr$$

$$-2C_{n,s}e^{-\|x\|^2}\sum_{j=2}^{\infty}\frac{2^{2j}}{(2j)!\Gamma\left(j+\frac{n}{2}\right)}\int_0^{\infty}r^{2j-1-2s}e^{-r^2}dr\cdot$$

$$\sum_{j_1+\cdots+j_n=j}\binom{2j}{2j_1,\cdots,2j_n}x_1^{2j_1}\cdots x_n^{2j_n}\prod_{i=1}^{n}\Gamma\left(\frac{1}{2}+j_i\right).$$

From equation (2), we deduce that

$$-C_{n,s}\Omega_n e^{-\|x\|^2}\int_0^{\infty}r^{-1-2s}\left(e^{-r^2}+r^2-1\right)dr=2^{2s}\frac{\Gamma\left(\frac{n}{2}+s\right)}{\Gamma(n/2)}e^{-\|x\|^2}.$$

Using the identities

$$\int_0^\infty r^{1-2s}\left(e^{-r^2}-1\right)dr=\frac{\Gamma(1-s)}{2} \text{ and } \int_0^\infty r^{2j-1-2s}e^{-r^2}dr=\frac{\Gamma(j-s)}{2},$$

we imply that

$$-\frac{2}{n}C_{n,s}\Omega_n e^{-\|x\|^2}\|x\|^2\int_0^\infty r^{1-2s}\left(e^{-r^2}-1\right)dr=-\frac{1}{n}2^{2s+1}s\frac{\Gamma\left(\frac{n}{2}+s\right)}{\Gamma(n/2)}e^{-\|x\|^2}\|x\|^2,$$

and

$$-2C_{n,s}e^{-\|x\|^2}\sum_{j=2}^\infty\frac{2^{2j}}{(2j)!\Gamma\left(j+\frac{n}{2}\right)}\int_0^\infty r^{2j-1-2s}e^{-r^2}dr\cdot\sum_{j_1+\cdots+j_n=j}\binom{2j}{2j_1,\cdots,2j_n}x_1^{2j_1}\cdots x_n^{2j_n}\prod_{i=1}^n\Gamma\left(\frac{1}{2}+j_i\right)$$

$$=-\pi^{-n/2}2^{2s}\frac{\Gamma\left(\frac{n}{2}+s\right)}{\Gamma(1-s)}s\,e^{-\|x\|^2}\sum_{j=2}^\infty\frac{2^{2j}\Gamma(j-s)}{(2j)!\Gamma\left(j+\frac{n}{2}\right)}\cdot\sum_{j_1+\cdots+j_n=j}\binom{2j}{2j_1,\cdots,2j_n}x_1^{2j_1}\cdots x_n^{2j_n}\prod_{i=1}^n\Gamma\left(\frac{1}{2}+j_i\right),$$

where

$$C_{n,s}=\pi^{-n/2}2^{2s}\frac{\Gamma(\frac{n+2s}{2})}{\Gamma(1-s)}s$$

is given earlier. Therefore,

$$(-\triangle)^s e^{-\|x\|^2}=2^{2s}\frac{\Gamma\left(\frac{n}{2}+s\right)}{\Gamma(n/2)}e^{-\|x\|^2}-\frac{1}{n}2^{2s+1}s\frac{\Gamma\left(\frac{n}{2}+s\right)}{\Gamma(n/2)}e^{-\|x\|^2}\|x\|^2$$

$$-\pi^{-n/2}2^{2s}\frac{\Gamma\left(\frac{n}{2}+s\right)}{\Gamma(1-s)}s\,e^{-\|x\|^2}\sum_{j=2}^\infty\frac{2^{2j}\Gamma(j-s)}{(2j)!\Gamma\left(j+\frac{n}{2}\right)}\cdot\sum_{j_1+\cdots+j_n=j}\binom{2j}{2j_1,\cdots,2j_n}x_1^{2j_1}\cdots x_n^{2j_n}\prod_{i=1}^n\Gamma\left(\frac{1}{2}+j_i\right)$$

$$=-\pi^{-n/2}2^{2s}\frac{\Gamma\left(\frac{n}{2}+s\right)}{\Gamma(1-s)}s\,e^{-\|x\|^2}\sum_{j=0}^\infty\frac{2^{2j}\Gamma(j-s)}{(2j)!\Gamma\left(j+\frac{n}{2}\right)}\cdot\sum_{j_1+\cdots+j_n=j}\binom{2j}{2j_1,\cdots,2j_n}x_1^{2j_1}\cdots x_n^{2j_n}\prod_{i=1}^n\Gamma\left(\frac{1}{2}+j_i\right).$$

This formula can be extended to all values of $s > 0$ with $s \neq 1, 2, \cdots$ using equation (3), which deals with new terms, due to the increasing range of s, in equation (1). Indeed, we can deduce the same expression as the above for $(-\triangle)^s e^{-\|x\|^2}$ with $2 < s < 3$ by following the similar steps.

Clearly for $j = 2, 3, \cdots, k$,

$$\lim_{s\to k}\frac{\Gamma(j-s)}{\Gamma(1-s)}=(-1)^{j-1}(k-1)\cdots(k-j+1).$$

This infers that

$$\lim_{s\to 0^+}(-\triangle)^s e^{-\|x\|^2} = e^{-\|x\|^2},$$

and

$$\lim_{s\to k}(-\triangle)^s e^{-\|x\|^2} = 2^{2k}\frac{\Gamma\left(\frac{n}{2}+k\right)}{\Gamma(n/2)}e^{-\|x\|^2} - \frac{1}{n}2^{2k+1}k\frac{\Gamma\left(\frac{n}{2}+k\right)}{\Gamma(n/2)}e^{-\|x\|^2}\|x\|^2$$

$$-\pi^{-n/2}2^{2k}\Gamma\left(\frac{n}{2}+k\right)e^{-\|x\|^2}\sum_{j=2}^{k}\frac{2^{2j}}{(2j)!\,\Gamma\left(j+\frac{n}{2}\right)}(-1)^{j-1}k(k-1)\cdots(k-j+1)$$

$$\sum_{j_1+\cdots+j_n=j}\binom{2j}{2j_1,\cdots,2j_n}x_1^{2j_1}\cdots x_n^{2j_n}\prod_{i=1}^{n}\Gamma\left(\frac{1}{2}+j_i\right) = (-1)^k\triangle^k e^{-\|x\|^2}$$

by Theorem 1. This completes the proof of Theorem 2.

Remark. (a) Let $j = (j_1, \cdots, j_n)$ be an n-tuple of real numbers in R^n with $j_i > -1/2$ for $1 \le i \le n$. Then,

$$S_n(1, j_1, \cdots, j_n) = \frac{2\prod_{i=1}^{n}\Gamma\left(\frac{1}{2}+j_i\right)}{\Gamma\left(j+\frac{n}{2}\right)} \tag{4}$$

still holds. In reference [29], Michelitsch et al. established a regularized representation for the fractional Laplacian in n dimension and showed the following identity

$$U_{n,\alpha} = \int_{\|\sigma\|=1}|\sigma_i|^\alpha d\sigma = \frac{2\pi^{\frac{n-1}{2}}\Gamma\left(\frac{\alpha+1}{2}\right)}{\Gamma\left(\frac{\alpha+n}{2}\right)} \text{ for } 1 \le i \le n,$$

which is a special case of equation (4) by setting $j_i = \alpha / 2$ and the rest being zero.

(b) In particular for $n = 1$ and $s > 0$ with $s \ne 1, 2, \cdots$,

$$(-\triangle)^s e^{-x^2} = -2^{2s}\pi^{-\frac{1}{2}}\frac{\Gamma\left(\frac{1}{2}+s\right)}{\Gamma(1-s)}se^{-x^2}\sum_{j=0}^{\infty}\frac{(2x)^{2j}}{(2j)!}\Gamma(j-s)$$

for all $x \in R$.

(c) For $n = 2$ and $s > 0$ with $s \ne 1, 2, \cdots$,

$$(-\triangle)^s e^{-x_1^2-x_2^2} = -2^{2s}s\frac{\Gamma(1+s)}{\Gamma(1-s)}e^{-x_1^2-x_2^2}\sum_{k=0}^{\infty}\frac{\Gamma(k-s)(x_1^2+x_2^2)^k}{(k!)^2},$$

which is identical with the result derived by a totally different method in [27].

(d) Theorem 2 for $0 < s < 1$ was proved by Samko in 2001.

3. Conclusion

Utilizing the surface integrals in R^n and special functions, we obtained the formula for the generalized fractional

Laplacian $(-\triangle)^s e^{-\|x\|^2}$ for all $s > 0$ and $s \neq 1, 2, \cdots$, based on the normalization in distribution.

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