

REMARKS ON THE LINEAR FRACTIONAL INTEGRO–DIFFERENTIAL EQUATION WITH VARIABLE COEFFICIENTS IN DISTRIBUTION

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Abstract. The goal of this paper is to study the following linear fractional integro-differential equation with variable coefficients, for the first time, in the distributional space $\mathcal{D}'(R^+)$ by Babenko's approach

$$u^{(\beta_n)}(x) + a_{n-1}(x)u^{(\beta_{n-1})}(x) + \cdots + a_1(x)u^{(\beta_1)}(x) + a_0(x)u^{(\beta_0)}(x) = g(x),$$

where $\beta_n > \beta_{n-1} > \cdots > \beta_0$ with $\beta_n > 0$. We obtain the solution as an infinite series and show its convergence. Furthermore, we investigate this equation with the Riemann-Liouville and Caputo derivatives (non-sequential) instead of distributional ones, and the initial conditions in the classical sense by a new and simpler method. Several interesting applications to solving the fractional differential and integral equations are presented using gamma functions, some of which cannot be achieved by ordinary integral transforms or numerical analysis.

1. Introduction

Generally speaking, a Green's function $u(t, x)$, of a linear differential operator $L = L(x)$ acting on functions (or distributions) over a subset of R^n , at a point t , is any fundamental solution of

$$Lu(t, x) = \delta(t - x),$$

where $\delta(x)$ is the Dirac delta function. It is well known that Green's function plays an important role in the study of fractional differential and integral equations appearing in mathematical and physical fields [1, 2, 3]. Many researchers have obtained explicit formulas for Green's functions of linear fractional differential equations with constant coefficients [4, 5, 6, 7, 8, 9, 10]. In 1991, Miller and Ross [4] derived Green's function from Laplace transform in the study of the following n -th order linear differential equations on the interval $I = [0, \infty)$

$$u^{(n)}(x) + a_{n-1}u^{(n-1)}(x) + \cdots + a_0u(x) = g(x)$$

with all zero initial conditions

$$u^{(k)}(0) = 0, \quad k = 0, 1, \dots, n - 1.$$

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Podlubny [7] obtained Green's function for the general linear fractional differential (sequential derivatives) equation with constant coefficients by Laplace transform and Mittag-Leffler function. Hilter [9] et al. constructed an operational calculus of Mikusiński type for the initial value problem of the fractional linear differential equation with the generalized Riemann-Liouville derivatives and constant coefficients. The Mikusiński operational calculus is an algebraic approach based on the interpretation of the Laplace convolution as a multiplication over a function space, which is widely used to solve differential and integral equations. Applying the fractional B-Splines wavelets and Mittag-Leffler function, Huang and Lu [11] discussed the existence and uniqueness of solutions of the following nonhomogeneous linear differential equation with all zero initial conditions (fractional order)

$$a_n D_{0,x}^{\beta_n} u(x) + \cdots + a_1 D_{0,x}^{\beta_1} u(x) = g(x), \quad a_n \neq 0$$

where all derivatives are in the Riemann-Liouville sense.

Kilbas [12] et al. studied solution by a power series method, near an ordinary point $x_0 \in [a, b]$, for the following fractional linear differential equation with variable coefficients

$$u^{(n\alpha)}(x) + \sum_{i=0}^{n-1} a_i(x) u^{(i\alpha)}(x) = g(x, \alpha),$$

where $\alpha \in (0, 1]$, $n \in \mathbb{N}$ and $u^{(i\alpha)}(x)$ denotes sequential fractional derivatives of order $i\alpha$ of the function $u(x)$ for $i = 0, 1, \dots, n$.

Using the integral representation and method of successive approximations, Kim and O [13] investigated the following fractional differential equation with continuous variable coefficients (as well as the nonhomogeneous equations with all zero initial conditions)

$$D_{0,x}^{\beta_n} u(x) + a_{n-1}(x) D_{0,x}^{\beta_{n-1}} u(x) + \cdots + a_0(x) D_{0,x}^{\beta_0} u(x) = 0, \quad x > 0 \quad (1.1)$$

with the initial conditions

$$D_{0,x}^{\beta_n - j} u(0) = \begin{cases} 1 & \text{if } j = 1, \\ 0 & \text{if } j = 2, 3, \dots, n_0, \end{cases} \quad (1.2)$$

where $\beta_n > \beta_{n-1} > \cdots > \beta_0 \geq 0$ and $n_0 - 1 < \beta_n \leq n_0 \in \mathbb{Z}^+$. Pak [14] et al. recently studied solutions of the following linear nonhomogeneous Caputo fractional differential equation with continuous variable coefficients for $x \in [0, T]$

$${}_C D_{0,x}^{\beta_n} u(x) + a_{n-1}(x) {}_C D_{0,x}^{\beta_{n-1}} u(x) + \cdots + a_0(x) {}_C D_{0,x}^{\beta_0} u(x) = g(x),$$

with all zero initial conditions

$$D^j u(0^+) = 0, \quad j = 0, 1, \dots, n_0 - 1.$$

As mentioned in the abstract, the aim of this paper is to solve the following linear fractional integro-differential equation with variable coefficients, for the first time, in the distributional space $\mathcal{D}'(\mathbb{R}^+)$

$$u^{(\beta_n)}(x) + a_{n-1}(x)u^{(\beta_{n-1})}(x) + \cdots + a_1(x)u^{(\beta_1)}(x) + a_0(x)u^{(\beta_0)}(x) = g(x), \quad (1.3)$$

where all derivatives or integrals are in the distributional sense (sequential law holds), and $g(x)$ is a distribution in $\mathcal{D}'(\mathbb{R}^+)$, such as

$$g(x) = \begin{cases} x^{-1.5} & \text{if } x > 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Note that this cannot be done in the classical sense as $g(x) = x_+^{-1.5}$ is not locally integrable, and furthermore its Laplace transform does not exist. Moreover, we apply Babenko's approach to solve the initial value problem of (1.1) and (1.2), as well as the nonhomogeneous case with all zero initial conditions in a much simpler way. Several applicable examples to solving linear fractional and integral equations with variable coefficients are presented utilizing gamma functions, some of which cannot be achieved in the normal sense since we deal with distribution $g(x)$ on the right-hand side of equation (1.3).

2. Fractional calculus of distribution

In order to study equation (1.3) in the generalized sense, we briefly introduce the following basic concepts with examples of finding distributional derivatives and Babenko's approach for an integro-differential equation in distribution with constant coefficients. Let $\mathcal{D}(R)$ be the Schwartz space (testing function space) [15] of infinitely differentiable functions with compact support in R , and $\mathcal{D}'(R)$ the (dual) space of distributions defined on $\mathcal{D}(R)$. A sequence $\phi_1, \phi_2, \dots, \phi_n, \dots$ goes to zero in $\mathcal{D}(R)$ if and only if these functions vanish outside a certain fixed and bounded set, and converge to zero uniformly together with their derivatives of any order. Clearly, $\mathcal{D}(R)$ is not empty since it contains zero as well as the following function

$$\phi(x) = \begin{cases} e^{\frac{1}{x^2-1}} & \text{if } |x| < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Evidently, any locally integrable function $f(x)$ on R is a (regular) distribution in $\mathcal{D}'(R)$ as

$$(f, \phi) = \int_{-\infty}^{\infty} f(x)\phi(x)dx$$

is well defined. Furthermore, f is linear and continuous on $\mathcal{D}(R)$. Indeed, we have for $\phi, \psi \in \mathcal{D}(R)$, and constants c_1 and c_2

$$(f, c_1\phi + c_2\psi) = c_1(f, \phi) + c_2(f, \psi).$$

If a sequence $\{\phi_n\}_{n=1}^{\infty}$ converges to zero in $\mathcal{D}(R)$ then it has compact support and converges to zero uniformly. This implies that

$$\lim_{n \rightarrow \infty} (f, \phi_n) = \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} f(x) \phi_n(x) dx = 0.$$

In particular, the unit step function $\theta(x)$ defined as

$$\theta(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x < 0, \end{cases}$$

is a member of $\mathcal{D}'(R)$ as it is locally integrable. In addition, the functional $\delta(x - x_0)$ on $\mathcal{D}(R)$, given by

$$(\delta(x - x_0), \phi(x)) = \phi(x_0),$$

is clearly linear and continuous on $\mathcal{D}(R)$. Therefore, $\delta(x - x_0) \in \mathcal{D}'(R)$.

Let $f \in \mathcal{D}'(R)$. The distributional derivative f' (or df/dx), is defined as

$$(f', \phi) = -(f, \phi')$$

for $\phi \in \mathcal{D}(R)$. Hence,

$$(\delta^{(n)}(x - x_0), \phi(x)) = (-1)^n (\delta(x - x_0), \phi^{(n)}(x)) = (-1)^n \phi^{(n)}(x_0).$$

The distributional derivative (global sense) is certainly an extension of the classical one (local sense). Any locally integrable function must have the distributional derivative, although its classical derivative may not exist.

Assume f is a distribution in $\mathcal{D}'(R)$ and ψ is a C^∞ function (infinitely differentiable). Then the product ψf is well defined by

$$(\psi f, \phi) = (f, \psi \phi)$$

for all functions $\phi \in \mathcal{D}(R)$, since $\psi \phi \in \mathcal{D}(R)$. The problem of defining products of two arbitrary distributions has been open and an active research area since distribution theory was introduced around 1950 [16, 17, 18]. For example, it seems hard to define the distribution $\delta^2(x)$, as

$$(\delta^2(x), \phi(x)) = (\delta(x)\delta(x), \phi(x)) = (\delta(x), \delta(x)\phi(x)) = \delta(0)\phi(0)$$

is undefined.

As an example of finding distributional derivatives, we show that

$$\frac{d^m}{dx^m} \left(\frac{\theta(x)}{1+x} \right) = \delta^{(m-1)}(x) + \dots + (-1)^{m-1} (m-1)! \delta(x) + \frac{(-1)^m m! \theta(x)}{(1+x)^{m+1}},$$

where $m = 0, 1, \dots$.

Clearly,

$$\begin{aligned} \frac{d^m}{dx^m} \left(\frac{1}{1+x} \right) &= \frac{(-1)^m m!}{(1+x)^{m+1}}, \quad x \geq 0, \\ \frac{d^m}{dx^m} \left(\frac{1}{1+x} \right) \Big|_{x=0} &= (-1)^m m!. \end{aligned}$$

Let $\phi \in \mathcal{D}(R)$. Using integration by parts,

$$\begin{aligned} \left(\frac{d^m}{dx^m} \left(\frac{\theta(x)}{1+x} \right), \phi(x) \right) &= (-1)^m \int_0^\infty \frac{\phi^{(m)}(x)}{1+x} dx \\ &= (-1)^m \frac{\phi^{(m-1)}(x)}{1+x} \Big|_0^\infty + (-1)^m \int_0^\infty \frac{\phi^{(m-1)}(x)}{(1+x)^2} dx \\ &= (-1)^{m-1} \phi^{(m-1)}(0) + (-1)^m \int_0^\infty \frac{\phi^{(m-1)}(x)}{(1+x)^2} dx \\ &= \left(\delta^{(m-1)}(x), \phi(x) \right) + (-1)^m \frac{\phi^{(m-2)}(x)}{(1+x)^2} \Big|_0^\infty + (-1)^m 2! \int_0^\infty \frac{\phi^{(m-2)}(x)}{(1+x)^3} dx \\ &= \left(\delta^{(m-1)}(x), \phi(x) \right) + (-1)^{m-1} \phi^{(m-2)}(0) + (-1)^m 2! \int_0^\infty \frac{\phi^{(m-2)}(x)}{(1+x)^3} dx \\ &= \dots \\ &= \left(\delta^{(m-1)}(x) + \dots + (-1)^{m-1} (m-1)! \delta(x), \phi(x) \right) \\ &\quad + (-1)^m m! \int_0^\infty \frac{\phi(x)}{(1+x)^{m+1}} dx. \end{aligned}$$

This infers that

$$\frac{d^m}{dx^m} \left(\frac{\theta(x)}{1+x} \right) = \delta^{(m-1)}(x) + \dots + (-1)^{m-1} (m-1)! \delta(x) + \frac{(-1)^m m! \theta(x)}{(1+x)^{m+1}}.$$

We should note that this function is not differentiable at $x = 0$ in the classical sense.

Assume that f and g are distributions in $\mathcal{D}'(R^+)$ (the set of all distributions concentrated on R^+ , which is a subspace of $\mathcal{D}'(R)$). Then the convolution $f * g$ is well defined by the equation [15]

$$((f * g)(x), \phi(x)) = (g(x), (f(y), \phi(x+y)))$$

for $\phi \in \mathcal{D}(R)$. This also implies that

$$f * g = g * f \quad \text{and} \quad (f * g)' = g' * f = g * f'.$$

Furthermore, the distribution $\Phi_\lambda = \frac{x_+^{\lambda-1}}{\Gamma(\lambda)} \in \mathcal{D}'(R^+)$ is an entire analytic function of λ on the complex plane [19, 20], and

$$\frac{x_+^{\lambda-1}}{\Gamma(\lambda)} \Big|_{\lambda=-n} = \delta^{(n)}(x), \quad \text{for } n = 0, 1, 2, \dots \quad (2.1)$$

which plays an important role in fractional calculus of distributions. Let λ and μ be arbitrary numbers, then the following identities

$$\Phi_\lambda * \Phi_\mu = \Phi_{\lambda+\mu} \quad (2.2)$$

$$\frac{d}{dx}\Phi_\lambda = \Phi_{\lambda-1} \quad (2.3)$$

are satisfied [21].

Let λ be an arbitrary complex number and $g(x)$ be a distribution in $\mathcal{D}'(R^+)$. We define the primitive of order λ of g as the distributional convolution

$$g_\lambda(x) = g(x) * \frac{x_+^{\lambda-1}}{\Gamma(\lambda)} = g(x) * \Phi_\lambda. \quad (2.4)$$

Note that this is well defined since the distributions g and Φ_λ are in $\mathcal{D}'(R^+)$. We shall write the convolution

$$g_{-\lambda} = \frac{d^\lambda}{dx^\lambda}g = g^{(\lambda)}(x) = g(x) * \Phi_{-\lambda}$$

as the fractional derivative of the distribution g of order λ if $\operatorname{Re} \lambda \geq 0$, and $\frac{d^\lambda}{dx^\lambda}g$ is interpreted as the fractional integral if $\operatorname{Re} \lambda < 0$.

It follows from equation (2.2) that differentiation and integration of the same order are mutually inverse processes, and the following sequential fractional derivative law holds

$$\frac{d^\lambda}{dx^\lambda} \left(\frac{d^\mu g}{dx^\mu} \right) = \frac{d^{\lambda+\mu} g}{dx^{\lambda+\mu}} = \frac{d^\mu}{dx^\mu} \left(\frac{d^\lambda g}{dx^\lambda} \right)$$

for any complex numbers λ and μ .

Note that there is no difference between the Riemann-Liouville derivative and the Caputo derivative in $\mathcal{D}'(R^+)$ [22]. Indeed, let $g(x)$ be any distribution concentrated on R^+ . Then

$$\begin{aligned} g_{-\lambda}(x) &= g(x) * \frac{x_+^{-\lambda-1}}{\Gamma(-\lambda)} = g(x) * \frac{d^m}{dx^m} \frac{x_+^{m-\lambda-1}}{\Gamma(m-\lambda)} \\ &= \frac{d^m}{dx^m} \left(g(x) * \frac{x_+^{m-\lambda-1}}{\Gamma(m-\lambda)} \right) = g^{(m)}(x) * \frac{x_+^{m-\lambda-1}}{\Gamma(m-\lambda)}, \end{aligned}$$

where $m-1 < \lambda < m \in Z^+$.

In this paper, we extend Babenko's approach [23] to distribution theory and solve equation (1.3), which is an integro-differential equation with variable coefficients. Generally speaking, Babenko's method itself is similar to Laplace transform method in the ordinary sense, but it can be used in more cases [8] such as solving integral or fractional differential equations with distributions whose Laplace transforms do not exist, as indicated below.

To illustrate Babenko's approach in detail, we are going to solve the following integro-differential equation with constant coefficients in the space $\mathcal{D}'(R^+)$

$$u^{(1.5)}(x) + u^{(0.5)}(x) - 2u^{(-0.5)}(x) = x_+^{-1.1} \quad (2.5)$$

where

$$u^{(-0.5)}(x) = \Phi_{0.5}(x) * u(x).$$

Note that this cannot be done in the classical sense as the distribution $x_+^{-1.1}$ is not locally integrable and its Laplace transform does not exist.

Clearly, equation (2.5) can be converted into

$$\Phi_{-1.5} * u + \Phi_{-0.5} * u - 2\Phi_{0.5} * u = \Gamma(-0.1)\Phi_{-0.1}$$

in terms of the distributional convolution. This implies

$$u + \Phi_1 * u - 2\Phi_2 * u = \Gamma(-0.1)\Phi_{1.4}.$$

By Babenko's approach (treating differential or integral operators as variables), we arrive at

$$\begin{aligned} u(x) &= \Gamma(-0.1) \frac{1}{\delta + \Phi_1 - 2\Phi_2} * \Phi_{1.4} = \Gamma(-0.1) \sum_{k=0}^{\infty} (-1)^k (\Phi_1 - 2\Phi_2)^k * \Phi_{1.4} \\ &= \Gamma(-0.1) \sum_{k=0}^{\infty} (-1)^k \sum_{i=0}^k \binom{k}{i} (-2\Phi_2)^i \Phi_1^{k-i} * \Phi_{1.4}. \end{aligned}$$

Using identities

$$\Phi_1^{k-i} = \Phi_{k-i}, \quad (-2\Phi_2)^i = (-2)^i \Phi_{2i},$$

we derive that

$$\begin{aligned} u(x) &= \Gamma(-0.1) \sum_{k=0}^{\infty} (-1)^k \sum_{i=0}^k \binom{k}{i} (-2)^i \Phi_{k+i+1.4} \\ &= \Gamma(-0.1) \sum_{k=0}^{\infty} \sum_{i=0}^k (-1)^k \binom{k}{i} (-2)^i \frac{x_+^{k+i+0.4}}{\Gamma(k+i+1.4)}. \end{aligned}$$

Using the following formula

$$\sum_{k=0}^{\infty} \sum_{i=0}^k a_{i,k-i} = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} a_{i,j},$$

we also imply that

$$u(x) = \Gamma(-0.1) \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} (-1)^{j+i} \binom{j+i}{i} (-2)^i \frac{x_+^{j+2i+0.4}}{\Gamma(j+2i+1.4)}.$$

We will show in the next section that this double series is absolutely and uniformly convergent on any interval $[0, T]$ for all $T > 0$. Therefore, $u(x)$ is continuous on R^+ .

3. The fractional integro-differential equation with variable coefficients

Let $a_i(x) \in C[0, \infty)$ for $i = 0, 1, 2, \dots, n - 1$ and $g \in \mathcal{D}'(R^+)$. In this section, we mainly focus on solving equation (1.3) distributionally with the conditions $\beta_n > \beta_{n-1} > \dots > \beta_0$ and $\beta_n > 0$. Note that this is an integro-differential equation if β_h is negative for some $0 \leq h \leq n - 1$. For example, we have

$$u^{(1.5)}(x) + 2x^2 u^{(0.2)}(x) + \frac{\sin x}{\sqrt{\pi}} \int_0^x (x-t)^{-0.5} u(t) dt = x_+^{-1.2} + \delta(x)$$

by choosing the values $\beta_2 = 1.5$, $\beta_1 = 0.2$, $\beta_0 = -0.5$, $a_0(x) = \sin x$, $a_1(x) = 2x^2$ and $g(x) = x_+^{-1.2} + \delta(x)$.

First, we consider the fundamental solution (Green's function) for the corresponding equation

$$u^{(\beta_n)}(x) + a_{n-1}(x)u^{(\beta_{n-1})}(x) + \dots + a_1(x)u^{(\beta_1)}(x) + a_0(x)u^{(\beta_0)}(x) = \delta(x). \tag{3.1}$$

Equation (3.1) can evidently be converted into

$$u * \Phi_{-\beta_n} + a_{n-1}(x) \cdot (u * \Phi_{-\beta_{n-1}}) + \dots + a_0(x) \cdot (u * \Phi_{-\beta_0}) = \delta(x).$$

Applying $*\Phi_{\beta_n}$ to the above equation from the right-hand side and considering the convolutional operator $*$ has a higher precedence than the product \cdot , we get

$$u + a_{n-1}(x) \cdot u * \Phi_{\beta_n - \beta_{n-1}} + \dots + a_0(x) \cdot u * \Phi_{\beta_n - \beta_0} = \delta(x) * \Phi_{\beta_n} = \Phi_{\beta_n}.$$

By Babenko's approach,

$$\begin{aligned} u(x) &= \frac{1}{\delta + a_{n-1}(x) \cdot \Phi_{\beta_n - \beta_{n-1}} + \dots + a_0(x) \cdot \Phi_{\beta_n - \beta_0}} * \Phi_{\beta_n} \\ &= \sum_{k=0}^{\infty} (-1)^k \left(\sum_{i=0}^{n-1} a_i(x) \cdot \Phi_{\beta_n - \beta_i} \right)^k * \Phi_{\beta_n}, \end{aligned}$$

where

$$\left(\sum_{i=0}^{n-1} a_i(x) \cdot \Phi_{\beta_n - \beta_i} \right)^k$$

is defined as k -time convolutions, rather than a power.

THEOREM 1. *Let $a_i(x) \in C[0, \infty)$ for $i = 0, 1, \dots, n - 1$. Then the linear fractional integro-differential equation (3.1) has the solution*

$$u(x) = \sum_{k=0}^{\infty} (-1)^k \left(\sum_{i=0}^{n-1} a_i(x) \cdot \Phi_{\beta_n - \beta_i} \right)^k * \Phi_{\beta_n},$$

which is continuous on the interval $[0, \infty)$ if $\beta_n \geq 1$, and $u(x) \in L(0, T)$ for all $T > 0$ if $0 < \beta_n < 1$.

Proof. Clearly,

$$\left| \sum_{k=0}^{\infty} (-1)^k \left(\sum_{i=0}^{n-1} a_i(x) \cdot \Phi_{\beta_n - \beta_i} \right)^k * \frac{x_+^{\beta_n - 1}}{\Gamma(\beta_n)} \right| \leq \sum_{k=0}^{\infty} \left(\sum_{i=0}^{n-1} |a_i(x)| \cdot \Phi_{\beta_n - \beta_i} \right)^k * \frac{x_+^{\beta_n - 1}}{\Gamma(\beta_n)}$$

by noting that

$$\Phi_{\beta_n - \beta_i}^j * \frac{x_+^{\beta_n - 1}}{\Gamma(\beta_n)} = \Phi_{\beta_n j - \beta_i j} * \frac{x_+^{\beta_n - 1}}{\Gamma(\beta_n)} = \Phi_{\beta_n j + \beta_n - \beta_i j} \geq 0$$

for all $j, i = 0, 1, \dots, n-1$.

Since $a_i(x) \in C[0, \infty)$, there exist constants $M_i > 0$ such that $|a_i(x)| \leq M_i$ for all $i = 0, 1, \dots, n-1$ and $x \in [0, T]$, where T is positive. Therefore,

$$\begin{aligned} |u(x)| &\leq \sum_{k=0}^{\infty} \left(\sum_{i=0}^{n-1} M_i \Phi_{\beta_n - \beta_i} \right)^k * \frac{x_+^{\beta_n - 1}}{\Gamma(\beta_n)} \\ &= \sum_{k=0}^{\infty} \sum_{k_0 + k_1 + \dots + k_{n-1} = k} \frac{k!}{k_0! k_1! \dots k_{n-1}!} M_0^{k_0} \dots M_{n-1}^{k_{n-1}} \\ &\quad \Phi_{\beta_n - \beta_0}^{k_0} * \dots * \Phi_{\beta_n - \beta_{n-1}}^{k_{n-1}} * \frac{x_+^{\beta_n - 1}}{\Gamma(\beta_n)} \\ &= \sum_{k=0}^{\infty} \sum_{k_0 + k_1 + \dots + k_{n-1} = k} \frac{k!}{k_0! k_1! \dots k_{n-1}!} M_0^{k_0} \dots M_{n-1}^{k_{n-1}} \\ &\quad \cdot \frac{x_+^{\beta_n - 1 + k_0(\beta_n - \beta_0) + \dots + k_{n-1}(\beta_n - \beta_{n-1})}}{\Gamma(\beta_n + k_0(\beta_n - \beta_0) + \dots + k_{n-1}(\beta_n - \beta_{n-1}))}. \end{aligned}$$

If $\beta_n \geq 1$ and $x \in [0, T]$, then

$$\begin{aligned} |u(x)| &\leq T^{\beta_n - 1} \sum_{k=0}^{\infty} \sum_{k_0 + k_1 + \dots + k_{n-1} = k} \frac{k!}{k_0! k_1! \dots k_{n-1}!} M_0^{k_0} \dots M_{n-1}^{k_{n-1}} \\ &\quad \cdot \frac{T^{k_0(\beta_n - \beta_0) + \dots + k_{n-1}(\beta_n - \beta_{n-1})}}{\Gamma(\beta_n + k_0(\beta_n - \beta_0) + \dots + k_{n-1}(\beta_n - \beta_{n-1}))} \\ &= T^{\beta_n - 1} E_{(\beta_n - \beta_0, \dots, \beta_n - \beta_{n-1}), \beta_n} \left(M_0 T^{\beta_n - \beta_0}, \dots, M_{n-1} T^{\beta_n - \beta_{n-1}} \right) \end{aligned}$$

where

$$E_{(\beta_n - \beta_0, \dots, \beta_n - \beta_{n-1}), \beta_n} \left(M_0 T^{\beta_n - \beta_0}, \dots, M_{n-1} T^{\beta_n - \beta_{n-1}} \right)$$

is the value at $z_0 = M_0 T^{\beta_n - \beta_0}, \dots, z_{n-1} = M_{n-1} T^{\beta_n - \beta_{n-1}}$ of the multivariate Mittag-Leffler function $E_{(\beta_n - \beta_0, \dots, \beta_n - \beta_{n-1}), \beta_n}(z_0, \dots, z_{n-1})$ given in [24]. This implies that

$$\sum_{k=0}^{\infty} (-1)^k \left(\sum_{i=0}^{n-1} a_i(x) \cdot \Phi_{\beta_n - \beta_i} \right)^k * \Phi_{\beta_n}$$

is absolutely and uniformly convergent on the interval $[0, T]$, and hence $u(x)$ is continuous on $[0, T]$. Since T is arbitrary, $u(x)$ is continuous on $[0, \infty)$. Assume $0 < \beta_n < 1$. Then for all $T > 0$,

$$\begin{aligned} \int_0^T |u(x)| dx &\leq \sum_{k=0}^{\infty} \sum_{k_0+k_1+\dots+k_{n-1}=k} \frac{k!}{k_0!k_1!\dots k_{n-1}!} M_0^{k_0} \dots M_{n-1}^{k_{n-1}} \\ &\quad \cdot \int_0^T \frac{x_+^{\beta_{n-1}+k_0(\beta_n-\beta_0)+\dots+k_{n-1}(\beta_n-\beta_{n-1})}}{\Gamma(\beta_n+k_0(\beta_n-\beta_0)+\dots+k_{n-1}(\beta_n-\beta_{n-1}))} dx \\ &= \sum_{k=0}^{\infty} \sum_{k_0+k_1+\dots+k_{n-1}=k} \frac{k!}{k_0!k_1!\dots k_{n-1}!} M_0^{k_0} \dots M_{n-1}^{k_{n-1}} \\ &\quad \cdot \frac{T^{\beta_n+k_0(\beta_n-\beta_0)+\dots+k_{n-1}(\beta_n-\beta_{n-1})}}{\Gamma(\beta_n+1+k_0(\beta_n-\beta_0)+\dots+k_{n-1}(\beta_n-\beta_{n-1}))} \\ &= T^{\beta_n} E_{(\beta_n-\beta_0, \dots, \beta_n-\beta_{n-1}), \beta_{n+1}} \left(M_0 T^{\beta_n-\beta_0}, \dots, M_{n-1} T^{\beta_n-\beta_{n-1}} \right), \end{aligned}$$

which is a finite value. This completes the proof of Theorem 1. \square

As an application of Theorem 1, we present the following example.

EXAMPLE 1. Assume $\alpha > 0$. The linear fractional integro-differential equation with a variable coefficient

$$u^{(2)}(x) + u(x) + x^\alpha u^{(-0.5)}(x) = \delta(x)$$

has the solution

$$u(x) = \sum_{k=0}^{\infty} (-1)^k \sum_{i=0}^k \binom{k}{i} C_{i,\alpha} \frac{x_+^{(\alpha+0.5)i+2k+1}}{\Gamma((\alpha+0.5)i+2k+2)}$$

in the distributional space $\mathcal{D}'(R^+)$, where

$$C_{i,\alpha} = \begin{cases} 1 & \text{if } i = 0, \\ \frac{\Gamma(\alpha+4.5)\Gamma(2\alpha+7)\dots\Gamma(i\alpha+4.5+(i-1)2.5)}{\Gamma(4.5)\Gamma(\alpha+7)\dots\Gamma((i-1)\alpha+4.5+(i-1)2.5)} & \text{if } i \geq 1. \end{cases}$$

By Theorem 1,

$$\begin{aligned} u(x) &= \sum_{k=0}^{\infty} (-1)^k (\Phi_2 + x^\alpha \cdot \Phi_{2.5})^k * \Phi_2 \\ &= \sum_{k=0}^{\infty} (-1)^k \sum_{i=0}^k \binom{k}{i} \Phi_2^{k-i} * (x^\alpha \cdot \Phi_{2.5})^i * \Phi_2. \end{aligned}$$

We need find the term $(x^\alpha \Phi_{2.5})^i * \Phi_2$. Clearly,

$$\begin{aligned} (x^\alpha \cdot \Phi_{2.5}) * \Phi_2 &= \frac{x_+^{\alpha+3.5}}{\Gamma(4.5)} = \frac{\Gamma(\alpha+4.5)}{\Gamma(4.5)} \Phi_{\alpha+4.5}, \\ (x^\alpha \cdot \Phi_{2.5}) * \frac{\Gamma(\alpha+4.5)}{\Gamma(4.5)} \Phi_{\alpha+4.5} &= \frac{\Gamma(\alpha+4.5)}{\Gamma(4.5)} \cdot \frac{\Gamma(2\alpha+7)}{\Gamma(\alpha+7)} \Phi_{2\alpha+7}, \\ \dots, \\ (x^\alpha \cdot \Phi_{2.5})^i * \Phi_2 &= \frac{\Gamma(\alpha+4.5)\Gamma(2\alpha+7)\cdots\Gamma(i\alpha+4.5+(i-1)2.5)}{\Gamma(4.5)\Gamma(\alpha+7)\cdots\Gamma((i-1)\alpha+4.5+(i-1)2.5)} \Phi_{i\alpha+4.5+(i-1)2.5} \\ &= C_{i,\alpha} \Phi_{i\alpha+4.5+(i-1)2.5}, \end{aligned}$$

where

$$C_{i,\alpha} = \begin{cases} 1 & \text{if } i = 0, \\ \frac{\Gamma(\alpha+4.5)\Gamma(2\alpha+7)\cdots\Gamma(i\alpha+4.5+(i-1)2.5)}{\Gamma(4.5)\Gamma(\alpha+7)\cdots\Gamma((i-1)\alpha+4.5+(i-1)2.5)} & \text{if } i \geq 1. \end{cases}$$

Hence,

$$\begin{aligned} u(x) &= \sum_{k=0}^{\infty} (-1)^k \sum_{i=0}^k \binom{k}{i} C_{i,\alpha} \Phi_{i\alpha+4.5+(i-1)2.5+2(k-i)} \\ &= \sum_{k=0}^{\infty} (-1)^k \sum_{i=0}^k \binom{k}{i} C_{i,\alpha} \frac{x_+^{(\alpha+0.5)i+2k+1}}{\Gamma((\alpha+0.5)i+2k+2)} \\ &= x_+ - \frac{x_+^2}{2!} - \frac{\Gamma(\alpha+4.5)}{\Gamma(4.5)} \frac{x_+^{\alpha+2.5}}{\Gamma(\alpha+3.5)} + \dots, \end{aligned}$$

which is continuous on the interval $[0, \infty)$ by Theorem 1 as $\beta_2 = 2$.

Clearly, the fractional integro-differential equation with a variable coefficient

$$u^{(2)}(x) + u(x) + x^m u^{(-0.5)}(x) = x_+^{-1.5}, \quad m \in \mathbb{Z}^+$$

has the solution

$$u(x) = 2\sqrt{\pi} \sum_{k=0}^{\infty} (-1)^{k+1} \sum_{i=0}^k \binom{k}{i} C_{i,m} \frac{x_+^{(m+0.5)i+2k+0.5}}{\Gamma((m+0.5)i+2k+1.5)}$$

in the distributional space $\mathcal{D}'(\mathbb{R}^+)$, although it cannot be solved in the classical sense.

COROLLARY 1. Let $a_i(x) = a_i$ for $i = 0, 1, \dots, n-1$. Then the linear fractional integro-differential equation (3.1) with all constant coefficients has the solution

$$\begin{aligned} u(x) &= \sum_{k=0}^{\infty} (-1)^k \sum_{k_0+k_1+\dots+k_{n-1}=k} \frac{k!}{k_0!k_1!\cdots k_{n-1}!} a_0^{k_0} \cdots a_{n-1}^{k_{n-1}} \\ &\quad \cdot \frac{x_+^{\beta_n-1+k_0(\beta_n-\beta_0)+\dots+k_{n-1}(\beta_n-\beta_{n-1})}}{\Gamma(\beta_n+k_0(\beta_n-\beta_0)+\dots+k_{n-1}(\beta_n-\beta_{n-1}))}, \end{aligned}$$

which is continuous on the interval $[0, \infty)$ if $\beta_n \geq 1$, and $u(x) \in L(0, T)$ for all $T > 0$ if $0 < \beta_n < 1$.

Proof. It follows immediately from the proof of Theorem 1. \square

We would like to mention that if $0 \geq \beta_n > \dots > \beta_0$, then equation (3.1) with constant coefficients is an integral equation and has the solution in the form of

$$u(x) = \text{a singular distribution} + \text{a locally integrable function}$$

in general. Indeed, we choose the minimum k such that

$$\min_{k \geq 1} \{ \beta_n - 1 + k_0(\beta_n - \beta_0) + \dots + k_{n-1}(\beta_n - \beta_{n-1}) \} > -1,$$

and denote it as k_0 . Then, we have from Corollary 1 that

$$\begin{aligned} u(x) &= \sum_{k=0}^{k_0-1} (-1)^k \sum_{k_0+k_1+\dots+k_{n-1}=k} \frac{k!}{k_0!k_1! \dots k_{n-1}!} a_0^{k_0} \dots a_{n-1}^{k_{n-1}} \\ &\quad \frac{x_+^{\beta_n-1+k_0(\beta_n-\beta_0)+\dots+k_{n-1}(\beta_n-\beta_{n-1})}}{\Gamma(\beta_n+k_0(\beta_n-\beta_0)+\dots+k_{n-1}(\beta_n-\beta_{n-1}))} \\ &+ \sum_{k=k_0}^{\infty} (-1)^k \sum_{k_0+k_1+\dots+k_{n-1}=k} \frac{k!}{k_0!k_1! \dots k_{n-1}!} a_0^{k_0} \dots a_{n-1}^{k_{n-1}} \\ &\quad \frac{x_+^{\beta_n-1+k_0(\beta_n-\beta_0)+\dots+k_{n-1}(\beta_n-\beta_{n-1})}}{\Gamma(\beta_n+k_0(\beta_n-\beta_0)+\dots+k_{n-1}(\beta_n-\beta_{n-1}))}. \end{aligned}$$

The first part is a distribution as the minimum power is less than or equal to -1 , and the second is a locally integrable function.

To end off this section, we would like to point out that we need the conditions $a_i(x) \in C^\infty[0, \infty)$ for $i = 0, 1, \dots, n-1$, in order to solve the general fractional integro-differential equation

$$u^{(\beta_n)}(x) + a_{n-1}(x)u^{(\beta_{n-1})}(x) + \dots + a_1(x)u^{(\beta_1)}(x) + a_0(x)u^{(\beta_0)}(x) = g(x),$$

where $g(x) \in \mathcal{D}'(R^+)$, as we deal with the distributional products $a_i(x)u^{(\beta_i)}(x)$, according to Section 2.

4. The initial value problems

In this section, we are going to apply Babenko’s approach to re-study equation (1.1) under the Riemann-Liouville derivatives (non-sequential law holds) with the initial conditions (1.2), which is mentioned in the introduction. Our approach will simplify the work due to Kim and O in [13]. Let us start with several soon-to-be used definitions.

DEFINITION 4.1. The fractional integral (or, the Riemann- Liouville) $I_{0,x}^\alpha$ (or I^α for short) of fractional order $\alpha \in R^+$ of function $f(x)$ is defined by

$$(I^\alpha f)(x) = (\Phi_\alpha * f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt.$$

Note that this definition is in the classical sense.

DEFINITION 4.2. The Riemann-Liouville derivative of fractional order $\alpha \in R^+$ of function $f(x)$ is defined as

$$D_{0,x}^\alpha f(x) = \frac{d^m}{dx^m} I^{m-\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_0^x (x-t)^{m-\alpha-1} f(t) dt,$$

where $m-1 \leq \alpha \leq m \in Z^+$.

Note that from [25]

$$I^\alpha D_{0,x}^\alpha f(x) = f(x) - \sum_{k=1}^m D_{0,x}^{\alpha-k} f(0) \frac{x^{\alpha-k}}{\Gamma(\alpha-k+1)}. \quad (4.1)$$

THEOREM 2. Equation (1.1) with the initial conditions (1.2) has the solution

$$u(x) = \frac{x^{\beta_n-1}}{\Gamma(\beta_n)} + \sum_{k=0}^{\infty} (-1)^{k+1} I^{\beta_n} \left(\sum_{i=0}^{n-1} a_i(x) I^{\beta_n-\beta_i} \right)^k \sum_{i=0}^{n-1} a_i(x) \cdot \frac{x^{\beta_n-\beta_i-1}}{\Gamma(\beta_n-\beta_i)}. \quad (4.2)$$

Proof. Clearly, the equivalent integral representation of equation (1.1) with the initial conditions (1.2) is

$$u(x) = \frac{x^{\beta_n-1}}{\Gamma(\beta_n)} - I^{\beta_n} a_{n-1}(x) D_{0,x}^{\beta_n-1} u(x) - \dots - I^{\beta_n} a_0(x) D_{0,x}^{\beta_0} u(x) \quad (4.3)$$

by equation (4.1).

Equation (4.3) can be converted into

$$\left(1 + I^{\beta_n} a_{n-1}(x) D_{0,x}^{\beta_n-1} + \dots + I^{\beta_n} a_0(x) D_{0,x}^{\beta_0} \right) \cdot u(x) = \frac{x^{\beta_n-1}}{\Gamma(\beta_n)},$$

which implies by Babenko's approach,

$$\begin{aligned} u(x) &= \frac{1}{1 + I^{\beta_n} a_{n-1}(x) D_{0,x}^{\beta_n-1} + \dots + I^{\beta_n} a_0(x) D_{0,x}^{\beta_0}} \cdot \frac{x^{\beta_n-1}}{\Gamma(\beta_n)} \\ &= \sum_{k=0}^{\infty} (-1)^k \left(\sum_{i=0}^{n-1} I^{\beta_n} a_i(x) D_{0,x}^{\beta_i} \right)^k \cdot \frac{x^{\beta_n-1}}{\Gamma(\beta_n)} \\ &= \frac{x^{\beta_n-1}}{\Gamma(\beta_n)} + \sum_{k=0}^{\infty} (-1)^{k+1} \left(\sum_{i=0}^{n-1} I^{\beta_n} a_i(x) D_{0,x}^{\beta_i} \right)^k \cdot \sum_{i=0}^{n-1} I^{\beta_n} a_i(x) D_{0,x}^{\beta_i} \cdot \frac{x^{\beta_n-1}}{\Gamma(\beta_n)}. \end{aligned}$$

Clearly,

$$\sum_{i=0}^{n-1} I^{\beta_n} a_i(x) D_{0,x}^{\beta_i} \cdot \frac{x^{\beta_n-1}}{\Gamma(\beta_n)} = I^{\beta_n} \sum_{i=0}^{n-1} a_i(x) \cdot \frac{x^{\beta_n-\beta_i-1}}{\Gamma(\beta_n-\beta_i)}, \quad \text{and}$$

$$\left(\sum_{i=0}^{n-1} I^{\beta_n} a_i(x) D_{0,x}^{\beta_i} \right)^k I^{\beta_n} = I^{\beta_n} \left(\sum_{i=0}^{n-1} a_i(x) D_{0,x}^{\beta_i} \cdot I^{\beta_n} \right)^k = I^{\beta_n} \left(\sum_{i=0}^{n-1} a_i(x) I^{\beta_n-\beta_i} \right)^k$$

by using

$$D_{0,x}^{\beta_i} \cdot I^{\beta_n} = I^{\beta_n-\beta_i}.$$

Therefore,

$$u(x) = \frac{x^{\beta_n-1}}{\Gamma(\beta_n)} + \sum_{k=0}^{\infty} (-1)^{k+1} I^{\beta_n} \left(\sum_{i=0}^{n-1} a_i(x) I^{\beta_n-\beta_i} \right)^k \sum_{i=0}^{n-1} a_i(x) \cdot \frac{x^{\beta_n-\beta_i-1}}{\Gamma(\beta_n-\beta_i)}.$$

This completes the proof of Theorem 2. \square

REMARK 1. Let

$$L_{loc}^{\alpha}(0, \infty) = \{f(x) \in L(0, T) \mid D_{0,x}^{\alpha} f \in L(0, T), \forall T > 0\}.$$

Kim and O [13] derived equation (4.2) by successive approximations (recursive technique) and showed that equation (1.1) with the initial conditions (1.2) has a unique solution $u(x)$ in the space $L_{loc}^{\beta_n}(0, \infty)$ by assuming $a_i(x) \in C[0, \infty)$ for all $i = 0, 1, \dots, n-1$. But their method is more complicated than ours.

In particular, if $a_i(x) = a_i = \text{const}$ for all $i = 0, 1, \dots, n-1$, then for $x > 0$ and $\beta_0 \geq 0$,

$$\begin{aligned} u(x) &= \sum_{k=0}^{\infty} (-1)^k \left(\sum_{i=0}^{n-1} I^{\beta_n} a_i D_{0,x}^{\beta_i} \right)^k \cdot \frac{x^{\beta_n-1}}{\Gamma(\beta_n)} \\ &= \sum_{k=0}^{\infty} (-1)^k \left(\sum_{i=0}^{n-1} a_i I^{\beta_n-\beta_i} \right)^k \cdot \frac{x^{\beta_n-1}}{\Gamma(\beta_n)} \\ &= \sum_{k=0}^{\infty} (-1)^k \sum_{k_0+k_1+\dots+k_{n-1}=k} \frac{k!}{k_0!k_1!\dots k_{n-1}!} a_0^{k_0} \dots a_{n-1}^{k_{n-1}} \\ &\quad \cdot \frac{1}{\Gamma(\beta_n + k_0(\beta_n - \beta_0) + \dots + k_{n-1}(\beta_n - \beta_{n-1}))}, \end{aligned}$$

which is in the classical sense, and clearly does not deal with the distributional equation like

$$a_n u^{(\beta_n)}(x) + a_{n-1} u^{(\beta_{n-1})}(x) + \dots + a_1 u^{(\beta_1)}(x) + a_0 u^{(\beta_0)}(x) = x_+^{-1.1}.$$

We are going to present the following example by Theorem 2.

EXAMPLE 2. The fractional differential equation

$$D_{0,x}^2 u(x) + x D_{0,x}^{1.5} u(x) + u(x) = 0,$$

with the initial conditions

$$D_{0,x} u(0) = 1, \quad u(0) = 0$$

has the solution

$$u(x) = x + \sum_{k=0}^{\infty} (-1)^{k+1} \sum_{i=0}^k \binom{k}{i} \left(A_i \frac{x^{1.5i+2(k-i)+3}}{\Gamma(1.5i+2(k-i)+4)} + B_i \frac{x^{1.5i+2(k-i)+2.5}}{\Gamma(1.5i+2(k-i)+3.5)} \right),$$

in the space $L_{loc}^2(0, \infty)$, where coefficients A_i and B_i are defined below.

By Theorem 2,

$$\begin{aligned} u(x) &= x + \sum_{k=0}^{\infty} (-1)^{k+1} I^2 \left(I^2 + x I^{0.5} \right)^k \left(x + x \frac{x^{-0.5}}{\Gamma(0.5)} \right) \\ &= x + \sum_{k=0}^{\infty} (-1)^{k+1} I^2 \sum_{i=0}^k \binom{k}{i} I^{2(k-i)} \left(x I^{0.5} \right)^i \left(x + \frac{x^{0.5}}{\Gamma(0.5)} \right). \end{aligned}$$

Note that

$$\begin{aligned} \left(x I^{0.5} \right) x &= \left(x I^{0.5} \right) \frac{x}{\Gamma(2)} = x \frac{x^{1.5}}{\Gamma(2.5)} = \frac{x^{2.5}}{\Gamma(2.5)} = \frac{\Gamma(3.5)}{\Gamma(2.5)} \frac{x^{2.5}}{\Gamma(3.5)}, \\ \left(x I^{0.5} \right) \frac{\Gamma(3.5)}{\Gamma(2.5)} \frac{x^{2.5}}{\Gamma(3.5)} &= \frac{\Gamma(3.5)\Gamma(5)}{\Gamma(2.5)\Gamma(4)} \frac{x^4}{\Gamma(5)}, \\ \dots, \\ \left(x I^{0.5} \right)^i x &= \frac{\Gamma(3.5)\Gamma(5) \cdots \Gamma(1.5i+2)}{\Gamma(2.5)\Gamma(4) \cdots \Gamma(1.5i+1)} \cdot \frac{x^{1.5i+1}}{\Gamma(1.5i+2)} = A_i \frac{x^{1.5i+1}}{\Gamma(1.5i+2)}, \end{aligned}$$

where

$$A_i = \begin{cases} 1 & \text{if } i = 0, \\ \frac{\Gamma(3.5)\Gamma(5) \cdots \Gamma(1.5i+2)}{\Gamma(2.5)\Gamma(4) \cdots \Gamma(1.5i+1)} & \text{if } i \geq 1. \end{cases}$$

On the other hand,

$$\begin{aligned} \left(x I^{0.5} \right) \frac{x^{0.5}}{\Gamma(0.5)} &= \frac{\Gamma(1.5)}{\Gamma(0.5)} \frac{x^2}{\Gamma(2)} = \frac{\Gamma(1.5)\Gamma(3)}{\Gamma(0.5)\Gamma(2)} \frac{x^2}{\Gamma(3)}, \\ \left(x I^{0.5} \right) \frac{\Gamma(1.5)\Gamma(3)}{\Gamma(0.5)\Gamma(2)} \frac{x^2}{\Gamma(3)} &= \frac{\Gamma(1.5)\Gamma(3)\Gamma(4.5)}{\Gamma(0.5)\Gamma(2)\Gamma(3.5)} \frac{x^{3.5}}{\Gamma(4.5)}, \\ \dots, \\ \left(x I^{0.5} \right)^i \frac{x^{0.5}}{\Gamma(0.5)} &= \frac{\Gamma(1.5)\Gamma(3) \cdots \Gamma(1.5(i+1))}{\Gamma(0.5)\Gamma(2) \cdots \Gamma(1.5i+0.5)} \frac{x^{1.5i+0.5}}{\Gamma(1.5(i+1))} = B_i \frac{x^{1.5i+0.5}}{\Gamma(1.5(i+1))}, \end{aligned}$$

where

$$B_i = \frac{\Gamma(1.5)\Gamma(3)\cdots\Gamma(1.5(i+1))}{\Gamma(0.5)\Gamma(2)\cdots\Gamma(1.5i+0.5)}.$$

Therefore,

$$\begin{aligned} & u(x) \\ = & x + \sum_{k=0}^{\infty} (-1)^{k+1} \sum_{i=0}^k \binom{k}{i} I^{2(k-i)+2} \left(A_i \frac{x^{1.5i+1}}{\Gamma(1.5i+2)} + B_i \frac{x^{1.5i+0.5}}{\Gamma(1.5(i+1))} \right) \\ & x + \sum_{k=0}^{\infty} (-1)^{k+1} \sum_{i=0}^k \binom{k}{i} \cdot \left(A_i \frac{x^{1.5i+2(k-i)+3}}{\Gamma(1.5i+2(k-i)+4)} + B_i \frac{x^{1.5i+2(k-i)+2.5}}{\Gamma(1.5i+2(k-i)+3.5)} \right). \end{aligned}$$

We would like to mention that we can follow the previous procedure to study equation (1.1) with different types of the initial conditions, such as

$$D_{0,x}^{\beta_n-j} u(0) = \begin{cases} 1 & \text{if } j = n_0, \\ 0 & \text{if } j = 2, 3, \dots, n_0 - 1, \end{cases}$$

where $n_0 - 1 < \beta_n \leq n_0 \in Z^+$, or

$$D_{0,x}^{\beta_n-j} u(0) = \begin{cases} 1 & \text{if } j = 1, 2, \\ 0 & \text{if } j = 3, \dots, n_0. \end{cases}$$

Clearly, the latter case will include more terms in its solution.

In addition, we can investigate the nonhomogeneous fractional differential equation for the continuous function $g(x)$

$$D_{0,x}^{\beta_n} u(x) + a_{n-1}(x) D_{0,x}^{\beta_{n-1}} u(x) + \cdots + a_0(x) D_{0,x}^{\beta_0} u(x) = g(x), \quad x > 0 \quad (4.4)$$

with the initial conditions (1.2) in the classical sense by Babenko's approach. In particular, equation (4.4) with the initial conditions

$$D_{0,x}^{\beta_n-j} u(0) = 0$$

for all $j = 1, 2, \dots, n_0$, has the solution

$$u(x) = \sum_{k=0}^{\infty} (-1)^k I^{\beta_n} \left(\sum_{i=0}^{n-1} a_i(x) I^{\beta_n - \beta_i} \right)^k g(x). \quad (4.5)$$

Indeed, we get

$$u(x) + I^{\beta_n} a_{n-1}(x) D_{0,x}^{\beta_{n-1}} u(x) + \cdots + I^{\beta_n} a_0(x) D_{0,x}^{\beta_0} u(x) = I^{\beta_n} g(x)$$

by applying I^{β_n} to equation (4.4) and noting that $D_{0,x}^{\beta_n-j} u(0) = 0$ for all $j = 1, 2, \dots, n_0$. This infers that

$$\begin{aligned} u &= \frac{1}{1 + I^{\beta_n} a_{n-1}(x) D_{0,x}^{\beta_{n-1}} + \dots + I^{\beta_n} a_0(x) D_{0,x}^{\beta_0}} I^{\beta_n} g(x) \\ &= \sum_{k=0}^{\infty} (-1)^k \left(\sum_{i=0}^{n-1} I^{\beta_n} a_i(x) D_{0,x}^{\beta_i} \right)^k I^{\beta_n} g(x) \\ &= \sum_{k=0}^{\infty} (-1)^k I^{\beta_n} \left(\sum_{i=0}^{n-1} a_i(x) I^{\beta_n - \beta_i} \right)^k g(x). \end{aligned}$$

We should point out that this solution is well defined. In fact,

$$\begin{aligned} u(x) &= \sum_{k=0}^{\infty} (-1)^k I^{\beta_n} \left(\sum_{i=0}^{n-1} a_i(x) I^{\beta_n - \beta_i} \right)^k g(x) \\ &= \sum_{k=0}^{\infty} (-1)^k \Phi_{\beta_n} * \left(\sum_{i=0}^{n-1} a_i(x) \cdot \Phi_{\beta_n - \beta_i} \right)^k * g(x) \\ &= \Phi_{\beta_n} * \sum_{k=0}^{\infty} (-1)^k \left(\sum_{i=0}^{n-1} a_i(x) \cdot \Phi_{\beta_n - \beta_i} \right)^k * \delta(x) * g(x). \end{aligned}$$

From the proof of Theorem 1,

$$w(x) = \sum_{k=0}^{\infty} (-1)^k \left(\sum_{i=0}^{n-1} a_i(x) \cdot \Phi_{\beta_n - \beta_i} \right)^k * \frac{x_{\pm}^{-1}}{\Gamma(0)} \in L(0, T)$$

for all $T > 0$. Furthermore, it is equal zero if $x < 0$. Hence, the convolution $\Phi_{\beta_n} * w(x) * g(x)$ is well defined in the classical sense if $g(x)$ is continuous if $x > 0$ and is zero if $x < 0$.

EXAMPLE 3. The non-homogeneous fractional differential equation with variable coefficients

$$D_{0,x}^{1.5} u(x) + x D_{0,x}^{0.5} u(x) + x^{1.5} u(x) = \frac{1}{2} x^2, \quad x > 0$$

with the initial conditions

$$D_{0,x}^{-0.5} u(0) = D_{0,x}^{0.5} u(0) = 0$$

has the solution

$$u(x) = \sum_{k=0}^{\infty} (-1)^k \sum_{i=0}^k \binom{k}{i} C_i B_{k-i} \frac{x^{2k+i+3.5}}{\Gamma(2k+i+4.5)},$$

where C_i and B_{k-i} are given below.

By equation (4.5),

$$\begin{aligned} u(x) &= \sum_{k=0}^{\infty} (-1)^k I^{1.5} \left(xI + x^{1.5} I^{1.5} \right)^k \frac{x^2}{2} \\ &= \sum_{k=0}^{\infty} (-1)^k I^{1.5} \sum_{i=0}^k \binom{k}{i} (xI)^{k-i} \left(x^{1.5} I^{1.5} \right)^i \frac{x^2}{2}. \end{aligned}$$

Clearly,

$$\begin{aligned} \left(x^{1.5} I^{1.5} \right) \frac{x^2}{\Gamma(3)} &= x^{1.5} \frac{x^{3.5}}{\Gamma(4.5)} = \frac{x^5}{\Gamma(4.5)} = \frac{\Gamma(6)}{\Gamma(4.5)} \frac{x^5}{\Gamma(6)}, \\ \left(x^{1.5} I^{1.5} \right) \frac{\Gamma(6)}{\Gamma(4.5)} \frac{x^5}{\Gamma(6)} &= \frac{\Gamma(6)}{\Gamma(4.5)} \frac{\Gamma(9)}{\Gamma(7.5)} \frac{x^8}{\Gamma(9)}, \\ \dots, \\ \left(x^{1.5} I^{1.5} \right)^i \frac{x^2}{\Gamma(3)} &= \frac{\Gamma(6) \cdots \Gamma(3(i+1))}{\Gamma(4.5) \cdots \Gamma(3i+1.5)} \frac{x^{3i+2}}{\Gamma(3i+3)} = C_i \frac{x^{3i+2}}{\Gamma(3i+3)}, \end{aligned}$$

where

$$C_i = \begin{cases} 1 & \text{if } i = 0, \\ \frac{\Gamma(6) \cdots \Gamma(3(i+1))}{\Gamma(4.5) \cdots \Gamma(3i+1.5)} & \text{if } i \geq 1. \end{cases}$$

To complete our calculation, we see that

$$\begin{aligned} (xI) \frac{x^{3i+2}}{\Gamma(3i+3)} &= \frac{\Gamma(3(i+1)+2)}{\Gamma(3(i+1)+1)} \frac{x^{3(i+1)+1}}{\Gamma(3(i+1)+2)} = (3(i+1)+1) \frac{x^{3(i+1)+1}}{\Gamma(3(i+1)+2)} \\ (xI)(3(i+1)+1) \frac{x^{3(i+1)+1}}{\Gamma(3(i+1)+2)} &= (3(i+1)+1)(3(i+1)+3) \frac{x^{3(i+1)+3}}{\Gamma(3(i+1)+4)} \\ \dots, \\ (xI)^{k-i} \frac{x^{3i+2}}{\Gamma(3i+3)} &= (3(i+1)+1)(3(i+1)+3) \cdots (2k+i+2) \frac{x^{2k+i+2}}{\Gamma(2k+i+3)} = B_{k-i} \frac{x^{2k+i+2}}{\Gamma(2k+i+3)}, \end{aligned}$$

where

$$B_{k-i} = \begin{cases} 1 & \text{if } i = k, \\ (3(i+1)+1)(3(i+1)+3) \cdots (2k+i+2) & \text{if } i < k. \end{cases}$$

The solution follows immediately from

$$I^{1.5} \frac{x^{2k+i+2}}{\Gamma(2k+i+3)} = \frac{x^{2k+i+3.5}}{\Gamma(2k+i+4.5)}.$$

DEFINITION 4.3. The Caputo derivative of fractional order $\alpha \in R^+$ of function $u(x)$ is defined as

$${}_CD_{0,x}^\alpha u(x) = I^{m-\alpha} \frac{d^m}{dx^m} u(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} u^{(m)}(t) dt,$$

where $m-1 < \alpha \leq m \in Z^+$.

Finally, we present solution for the following linear nonhomogeneous Caputo fractional differential equation with continuous variable coefficients (which was mentioned in the introduction),

$${}_CD_{0,x}^{\beta_n} u(x) + a_{n-1}(x) {}_CD_{0,x}^{\beta_{n-1}} u(x) + \dots + a_0(x) {}_CD_{0,x}^{\beta_0} u(x) = g(x), \quad x \in [0, T] \quad (4.6)$$

with all zero initial conditions

$$D^j u(0^+) = 0, \quad j = 0, 1, \dots, n_0 - 1$$

by Babenko’s approach.

Clearly, integration by parts and differentiation show that

$${}_CD_{0,x}^\alpha u(x) = D_{0,x}^\alpha \left(u(x) - \sum_{j=0}^{m-1} \frac{x^j}{j!} u^{(j)}(0) \right),$$

if $u(x) \in C^m[0, \infty)$ and $m-1 < \alpha \leq m \in Z^+$. This implies that for $k = 0, \dots, n-1$

$${}_CD_{0,x}^{\beta_k} u(x) = D_{0,x}^{\beta_k} u(x)$$

from all zero initial conditions above, by noting that $\beta_n > \beta_{n-1} > \dots > \beta_0 \geq 0$ and $n_0 - 1 < \beta_n \leq n_0 \in Z^+$. Integration by parts and the initial conditions also infer that,

$$\begin{aligned} I^{\beta_n} {}_CD_{0,x}^{\beta_n} u(x) &= I^{\beta_n} I^{n_0-\beta_n} \frac{d^{n_0}}{dx^{n_0}} u(x) = I^{n_0} \frac{d^{n_0}}{dx^{n_0}} u(x) = \frac{1}{(n_0-1)!} \int_0^x (x-t)^{n_0-1} u^{(n_0)}(t) dt \\ &= \frac{1}{(n_0-1)!} (x-t)^{n_0-1} u^{(n_0-1)}(t) \Big|_{t=0}^x + \frac{1}{(n_0-2)!} \int_0^x (x-t)^{n_0-2} u^{(n_0-1)}(t) dt \\ &= \frac{1}{(n_0-2)!} \int_0^x (x-t)^{n_0-2} u^{(n_0-1)}(t) dt \\ &\quad \dots, \\ &= \frac{1}{0!} \int_0^x (x-t)^{-1} u(t) dt = \delta(x) * u(x) = u(x). \end{aligned}$$

Hence, equation (4.6) is equivalent to the integral equation

$$u(x) + I^{\beta_n} a_{n-1}(x) D_{0,x}^{\beta_{n-1}} u(x) + \dots + I^{\beta_n} a_0(x) D_{0,x}^{\beta_0} u(x) = I^{\beta_n} g(x)$$

by applying I^{β_n} to equation (4.6), which has the well-defined solution

$$u(x) = \sum_{k=0}^{\infty} (-1)^k I^{\beta_n} \left(\sum_{i=0}^{n-1} a_i(x) I^{\beta_n - \beta_i} \right)^k g(x),$$

according the previous work for equation (4.5).

To end off this section, we would like to point out that solution for equation (4.6) with all zero initial conditions has been investigated by several researchers using different approaches, including the classical Green function, generalized Green function and modified Green function [14, 26].

5. Conclusion

Applying Babenko's approach, we have investigated the linear fractional integro-differential equation (1.3) with variable coefficients in the distributional space $\mathcal{D}'(R^+)$ for the first time, and obtained its solution as the convergent series. Furthermore, we studied this equation with the Riemann-Liouville and Caputo derivatives and the initial conditions by a new technique in the classical sense. Several applicable examples of solving fractional differential equations were presented using gamma functions.

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