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Article

On the Generalized Riesz Derivative

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Abstract: The goal of this paper is to construct an integral representation for the generalized Riesz derivative ${}_{RZ}D_x^{2s}u(x)$ for $k < s < k+1$ with $k = 0, 1, \cdots$, which is proved to be a one-to-one and linearly continuous mapping from the normed space $W_{k+1}(R)$ to the Banach space $C(R)$. In addition, we show that ${}_{RZ}D_x^{2s}u(x)$ is continuous at the end points and well defined for $s = \frac{1}{2} + k$. Furthermore, we extend the generalized Riesz derivative ${}_{RZ}D_x^{2s}u(x)$ to the space $C_k(R^n)$, where k is an n-tuple of nonnegative integers, based on the normalization of distribution and surface integrals over the unit sphere. Finally, several examples are presented to demonstrate computations for obtaining the generalized Riesz derivatives.

Keywords: Riesz derivative; fractional Laplacian; normalization; distribution; Gamma function

1. Introduction

During the past few decades, fractional calculus [1–4] has been an emergent tool which uses fractional differential and integral equations to develop more sophisticated mathematical models that can accurately describe complex systems. Fractional powers of the Laplacian operator arise naturally in the study of partial differential equations related to anomalous diffusion, where the fractional operator plays a similar role to that of the integer-order Laplacian for ordinary diffusion [5,6]. By replacing Brownian motion of particles with Lévy flights [7], one obtains a fractional diffusion equation in terms of the fractional Laplacian operator [8] of order $s \in (0, 1)$ via the Cauchy principal value (P.V. for short) integral [9], given as

$$(-\triangle)^s u(x) = C_{n,s}\text{P.V.} \int_{R^n} \frac{u(x) - u(\zeta)}{|x-\zeta|^{n+2s}} d\zeta, \tag{1}$$

where $\triangle = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2$, and the constant $C_{n,s}$ is given by

$$C_{n,s} = \left(\int_{R^n} \frac{1 - \cos y_1}{|y|^{n+2s}} dy \right)^{-1} = \pi^{-n/2} 2^{2s} \frac{\Gamma(\frac{n+2s}{2})}{\Gamma(1-s)} s. \tag{2}$$

Let $x = (x_1, x_2, \cdots, x_n) \in R^n$. For a given n-tuple $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ of nonnegative integers (or called a multi-index), we define

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n, \quad \alpha! = \alpha_1!\alpha_2! \cdots \alpha_n!$$
$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n},$$
$$\partial^\alpha u = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n} u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}.$$

The Schwartz space $S(R^n)$ (space of rapidly decreasing functions on R^n) is the function space [10] defined as

$$S(R^n) = \left\{ u(x) \in C^\infty(R^n) : \ \|u(x)\|_{\alpha,k} \le C_{\alpha,k}(\text{const}) \ \ \forall \alpha, k \in N_0^n \right\},$$

where $N_0 = \{0\} \cup N$ is the set of nonnegative integers and

$$\|u(x)\|_{\alpha,k} = \sup_{x\in R^n} \left| x^\alpha \partial^k u(x) \right|.$$

Let $r^2 = x_1^2 + x_2^2 + \cdots + x_n^2$. The function space $C_k(R^n)$ is defined in Reference [11] as follows.

$$\begin{aligned} C_k(R^n) \quad = \quad & \Big\{ u(x) \text{ is bounded and } \partial^{2k} u(x) \text{ is continuous on } R^n : \\ & \exists\, M_k(\text{const}) > 0, \text{ such that } \left| \partial^{2k} u(x) \right| \le \frac{M_k}{r^2} \text{ as } r \to \infty \Big\} \end{aligned}$$

where $k = (k_1, k_2, \cdots, k_n)$ is an n-tuple of nonnegative integers.

Applying the normalization in distribution theory, Pizzetti's formula, and surface integrals on R^n, Li [11] very recently extended the fractional Laplacian $(-\triangle)^s u(x)$ over the space $C_k(R^n)$ (which contains $S(R^n)$ as a proper subspace) for all $s > 0$ and $s \neq 1, 2, \cdots$, and obtained Theorem 1 below.

Theorem 1. *Let $i = 0, 1, \cdots$ and $i < s < i + 1$. Then the generalized fractional Laplacian $(-\triangle)^s$ is normalized over the space $C_k(R^n)$ as*

$$\begin{aligned} (-\triangle)^s u(x) \quad = \quad & -\frac{1}{2} C_{n,s} \int_0^\infty r^{-1-2s} \cdot \\ & \left[S(r) - \frac{r^2 \Omega_n \triangle u(x)}{n} - \cdots - \frac{2 r^{2i} \Omega_n \triangle^i u(x)}{2^i\, i!\, n(n+2)\cdots(n+2i-2)} \right] dr, \end{aligned} \tag{3}$$

where $\Omega_n = 2\pi^{n/2}/\Gamma(\frac{n}{2})$ is the area of the unit sphere $\Omega \subset R^n$, $k = (k_1, k_2, \cdots, k_n)$ with $k_1 + \cdots + k_n = i + 1$, and

$$S(r) = \int_\Omega [u(x + r\sigma) - 2u(x) + u(x - r\sigma)] d\sigma.$$

In particular for $n = 1$, we have the following.

Theorem 2. *Let $k < s < k + 1$ and $k = 0, 1, 2, \cdots$. Then the fractional Laplacian operator $(-\triangle)^s$ is normalized over $C_{k+1}(R)$ as*

$$(-\triangle)^s u(x) = -C_{1,s} \int_0^\infty y^{-1-2s} \left[S(y) - u^{(2)}(x) y^2 - \cdots - \frac{2 y^{2k}}{(2k)!} u^{(2k)}(x) \right] dy, \tag{4}$$

where

$$S(y) = u(x + y) - 2u(x) + u(x - y).$$

Definition 1. *For a sufficiently nice function $u(x)$ defined on R the left- and right- sided Riemann-Liouville derivatives of order α, with $m - 1 < \alpha < m \in N$, given by*

$${}_{RL}D^\alpha_{-\infty,x} u(x) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_{-\infty}^x \frac{u(t)}{(t-x)^{\alpha-m+1}} dt,$$

and

$${}_{RL}D^\alpha_{x,\infty} u(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_x^\infty \frac{u(t)}{(t-x)^{\alpha-m+1}} dt$$

respectively.

From integration by parts we have

$$\lim_{\alpha \to m^-} {}_{RL}D^{\alpha}_{-\infty,x} u(x) = u^{(m)}(x), \quad \text{and}$$
$$\lim_{\alpha \to (m-1)^+} {}_{RL}D^{\alpha}_{-\infty,x} u(x) = u^{(m-1)}(x).$$

Definition 2. *The α-order Riesz derivative of a function $u(x)$ ($x \in R$) is defined as*

$$ {}_{RZ}D^{\alpha}_{x} u(x) = -\Psi_{\alpha} \left({}_{RL}D^{\alpha}_{-\infty,x} + {}_{RL} D^{\alpha}_{x,\infty} \right) u(x)$$

where

$$\Psi_{\alpha} = \frac{1}{2\cos\dfrac{\alpha\pi}{2}}$$

for $\alpha \neq 1, 3, \cdots$.

In general, the following definition regarding the Riesz derivative on R^n can be given.

Definition 3. *The Riesz fractional derivative is defined for suitably smooth function $u(x)$ ($x \in R^n$) in arbitrary dimensions by [1,12]*

$$ {}_{RZ}D^{\alpha}_{x} u(x) = \frac{1}{d_{n,l}(\alpha)} \int_{R^n} \frac{(\triangle^l_y u)(x)}{|y|^{n+\alpha}} dy, \;\; 0 < \alpha < l$$

where l is an arbitrary integer bigger than α, and $(\triangle^l_y u)(x)$ denotes either the centred difference

$$(\triangle^l_y u)(x) = \sum_{k=0}^{l} (-1)^k \binom{l}{k} u(x + (l/2 - k)y),$$

or non-centred differences

$$(\triangle^l_y u)(x) = \sum_{k=0}^{l} (-1)^k \binom{l}{k} u(x - ky).$$

The $d_{n,l}(\alpha)$ are normalizing constants which are independent of the choice of $l > \alpha$, and are analytic functions with respect to the parameter α by

$$d_{n,l}(\alpha) = \frac{2^{-\alpha}\pi^{1+n/2}}{\Gamma\left(1+\frac{\alpha}{2}\right)\Gamma\left(\frac{n+\alpha}{2}\right)} \frac{A_l(\alpha)}{\sin\left(\frac{\pi\alpha}{2}\right)},$$

and

$$A_l(\alpha) = \begin{cases} \displaystyle\sum_{k=0}^{l} (-1)^{k-1} \binom{l}{k} k^{\alpha}, & \text{in the case of non-centred difference,} \\ \displaystyle 2\sum_{k=0}^{\frac{l}{2}} (-1)^{k-1} \binom{l}{k} \left(\frac{l}{2} - k\right)^{\alpha}, & \text{in the case of centred difference,} \end{cases}$$

for an even number $l > \alpha$.

It is well known that the Riesz derivative plays an important role in anomalous diffusion [13–15] and space of fractional quantum mechanics. For example, the Riesz derivative satisfies the fractional diffusion equation, which has lots of physical applications [13]:

$$\frac{\partial P_L(x,t;\alpha)}{\partial t} - \sigma_{\alpha} \, {}_{RZ}D^{\alpha}_{x} P_L(x,t;\alpha) = 0,$$

where $P_L(x, t; \alpha)$ is the α-stable Lévy distribution and α, $0 < \alpha \leq 2$, is called the Lévy index. There are also many studies, including numerical analysis [16–19], scientific computing and Fourier transform methods [20,21], on differential equations involving the Riesz derivative with applications in several fields, including mathematical physics and engineering.

It is widely considered that the Riesz derivative is equivalent to the fractional Laplacian in arbitrary dimensions [22–24]. Cai and Li [25] showed that for $s \in (0, 1)$

$$-(-\triangle)^s u(x) = {}_{RZ}D_x^{2s} u(x), \quad u(x) \in S(R) \text{ and } s \neq 1/2,$$
$$(-\triangle)^s u(x) = {}_{RZ}D_x^{2s} u(x), \quad u(x) \in S(R^n) \text{ with } n > 1.$$

Furthermore, on page 205 and 206 in the same reference they stated

(i) for the case with $\alpha = 3, 5, \cdots$, the Riesz derivative of the given function $u(x)$ ($x \in R$) can be defined in the form

$$_{RZ}D_x^{\alpha} u(x) = \frac{2^{\alpha-1}\alpha\Gamma\left(\frac{1+\alpha}{2}\right)}{\pi^{\frac{1}{2}}\Gamma(1-\frac{\alpha}{2})} \int_0^\infty \frac{u(x+y) - 2u(x) + u(x-y)}{y^{1+\alpha}} dy,$$

which is suitable for positive values of $\alpha \neq 2, 4, 6, \cdots$.

(ii) For $k = 1, 2, \cdots$. Then,

$$\lim_{\alpha \to 4k+1} {}_{RZ}D_x^{\alpha} u(x) = \frac{2^{4k}(4k+1)\Gamma(2k+1)}{\pi^{1/2}\Gamma\left(\frac{1-4k}{2}\right)} \int_0^\infty \frac{u(x+y) - 2u(x) + u(x-y)}{y^{4k+2}} dy,$$

and for $k = 0, 1, 2, \cdots$,

$$\lim_{\alpha \to 4k+3} {}_{RZ}D_x^{\alpha} u(x) = \frac{2^{4k+2}(4k+3)\Gamma(2k+2)}{\pi^{1/2}\Gamma\left(-\frac{1+4k}{2}\right)} \int_0^\infty \frac{u(x+y) - 2u(x) + u(x-y)}{y^{4k+4}} dy.$$

We would like to reconsider cases (i) and (ii) in this paper as the integrals on the right-hand side do not exist even for a sufficiently good function $u(x) \in S(R)$. Indeed, by Taylor's expansion

$$u(x+y) - 2u(x) + u(x-y) = \frac{u''(x+\theta y) + u''(x-\theta y)}{2!} y^2 \sim u''(x) y^2 \quad \text{as } y \to 0^+$$

where $\theta \in (0, 1)$. This clearly makes all the integrals on the right-hand side divergent near the origin.

As outlined in the abstract, we establish an integral representation for the generalized Riesz derivative ${}_{RZ}D_x^{2s} u(x)$ for $k < s < k+1$ with $k = 0, 1, \cdots$, as a linearly continuous mapping from the normed space $W_{k+1}(R)$ to the Banach space $C(R)$. Then we study the generalized Riesz derivative in arbitrary dimensions and further show that ${}_{RZ}D_x^{2s} u(x)$ is continuous at the end points based on the normalization of distribution and the surface integrals. In particular, the derivative ${}_{RZ}D_x^{2s} u(x)$ is well defined for all $s = k + \frac{1}{2}$, which extends Definition 2.

2. The Generalized Riesz Derivative on R

Let $C(R)$ be the space of continuous functions on R given as

$$C(R) = \{u(x) : \ u(x) \text{ is continuous on } R \text{ and } \|u\|_\infty < \infty\}$$

where

$$\|u\|_\infty = \sup_{x \in R} |u(x)|.$$

Clearly, $C(R)$ is a Banach space. The following space will play an important role in defining the generalized Riesz derivative on R.

Let $k = 1, 2, \cdots$. We define the normed space $W_k(R)$ as

$$W_k(R) = \left\{ u(x) : u^{(2k)}(x) \text{ is continuous on } R \text{ and } \|u\|_k < \infty \right\}$$

where

$$\|u\|_k = \max \left\{ \sup_{x \in R} |xu(x)|,\ \sup_{x \in R} |xu'(x)|,\ \sup_{x \in R} \left| (x^2+1) u^{(2k)}(x) \right| \right\}.$$

Clearly, $u(x) = \dfrac{x}{x^2+1} \in W_k(R)$ but $u(x) = \dfrac{x}{x^2+1} \notin S(R)$, and

$$S(R) \subset W_k(R) \subset C_k(R) \subset C(R)$$

for all $k = 1, 2, \cdots$.

We are ready to prove the following theorem which establishes an initial equivalence between the Riesz derivative and the fractional Laplacian on the space $W_1(R)$.

Theorem 3. *Let $u \in W_1(R)$. Then both ${}_{RZ}D_x^{2s} u(x)$ and $(-\triangle)^s u(x)$ exist and*

$${}_{RZ}D_x^{2s} u(x) = -(-\triangle)^s u(x) = C_{1,s} \int_0^\infty \frac{u(x+y) - 2u(x) + u(x-y)}{y^{1+2s}} dy$$

for $0 < s < 1$.

Proof of Theorem 3. Making the variable change $z = x - \zeta$, we derive from Equation (1) that (for $n = 1$)

$$(-\triangle)^s u(x) = C_{1,s} \text{P.V.} \int_R \frac{u(x) - u(x-z)}{|z|^{1+2s}} dz.$$

Setting $w = -z$ on the right-hand side of the above equality, we come to

$$\text{P.V.} \int_R \frac{u(x) - u(x-z)}{|z|^{1+2s}} dz = \text{P.V.} \int_R \frac{u(x) - u(x+w)}{|w|^{1+2s}} dw.$$

Therefore,

$$\begin{aligned} &2\text{P.V.} \int_R \frac{u(x) - u(x-z)}{|z|^{1+2s}} dz \\ &= \text{P.V.} \int_R \frac{u(x) - u(x-z)}{|z|^{1+2s}} dz + \text{P.V.} \int_R \frac{u(x) - u(x+w)}{|w|^{1+2s}} dw \\ &= -\text{P.V.} \int_R \frac{u(x+y) - 2u(x) + u(x-y)}{|y|^{1+2s}} dy \end{aligned}$$

after relabeling $y = z$ and $y = w$. This implies that

$$(-\triangle)^s u(x) = -\frac{C_{1,s}}{2} \text{P.V.} \int_R \frac{u(x+y) - 2u(x) + u(x-y)}{|y|^{1+2s}} dy.$$

Note that the above integral is well defined for $u(x) \in W_1(R)$. Indeed, a second order Taylor expansion infers

$$\frac{|u(x+y) - 2u(x) + u(x-y)|}{|y|^{1+2s}} \leq \frac{\sup_{y \in R} |u''(y)|}{|y|^{2s-1}}.$$

Hence, it is absolutely integrable near the origin. Furthermore, $u(x) \in W_1(R)$ implies that there exists a constant $C > 0$ such that

$$\left|(y^2+1)u''(y)\right| \le C \quad \text{as } |y| \to \infty.$$

This indicates that the integral is absolutely integrable at infinity. In summary,

$$\begin{aligned}(-\triangle)^s u(x) &= -\frac{C_{1,s}}{2}\int_R \frac{u(x+y)-2u(x)+u(x-y)}{|y|^{1+2s}}dy \\ &= -C_{1,s}\int_0^\infty \frac{u(x+y)-2u(x)+u(x-y)}{y^{1+2s}}dy,\end{aligned}$$

as

$$\frac{u(x+y)-2u(x)+u(x-y)}{|y|^{1+2s}}$$

is an even function with respect to y.

Assume $0 < s < 1/2$. Integration by parts yields

$$\begin{aligned}&\int_0^\infty \frac{u(x+y)-2u(x)+u(x-y)}{y^{1+2s}}dy \\ &= -\frac{1}{2s}\frac{u(x+y)-2u(x)+u(x-y)}{y^{2s}}\Bigg|_{y=0}^{\infty} + \frac{1}{2s}\int_0^\infty \frac{u'(x+y)-u'(x-y)}{y^{2s}}dy \\ &= \frac{1}{2s}\frac{d}{dx}\int_0^\infty \frac{u(x+y)}{y^{2s}}dy - \frac{1}{2s}\frac{d}{dx}\int_0^\infty \frac{u(x-y)}{y^{2s}}dy\end{aligned}$$

by applying the facts that all four integrals

$$\int_0^\infty \frac{u(x+y)}{y^{2s}}dy, \quad \int_0^\infty \frac{u'(x+y)}{y^{2s}}dy$$
$$\int_0^\infty \frac{u(x-y)}{y^{2s}}dy, \quad \text{and} \quad \int_0^\infty \frac{u'(x-y)}{y^{2s}}dy$$

are uniformly convergent with respect to x using the conditions

$$\sup_{x\in R}|xu(x)| \quad \text{and} \quad \sup_{x\in R}|xu'(x)|$$

are bounded. Since

$$\begin{aligned}\frac{d}{dx}\int_0^\infty \frac{u(x+y)}{y^{2s}}dy &= -\Gamma(1-2s)\ {}_{RL}D^{2s}_{x,\infty}u(x), \text{ and} \\ \frac{d}{dx}\int_0^\infty \frac{u(x-y)}{y^{2s}}dy &= \Gamma(1-2s)\ {}_{RL}D^{2s}_{-\infty,x}u(x)\end{aligned}$$

we come to

$$(-\triangle)^s u(x) = \frac{C_{1,s}\Gamma(1-2s)}{2s}\left({}_{RL}D^{2s}_{x,\infty}u(x) + {}_{RL}D^{2s}_{-\infty,x}u(x)\right).$$

From the formula [26]

$$C_{1,s}^{-1} = \int_{-\infty}^{\infty} \frac{1-\cos y}{|y|^{1+2s}}dy = \frac{1}{s}\Gamma(1-2s)\cos(\pi s),$$

we have

$$(-\triangle)^s u(x) = \Psi_{2s}\left({}_{RL}D^{2s}_{-\infty,x} + {}_{RL}D^{2s}_{x,\infty}\right)u(x) = -{}_{RZ}D^{2s}_x u(x).$$

Finally we assume $1/2 < s < 1$. Applying

$$\frac{d^2}{dx^2}\int_0^\infty \frac{u(x+y)}{y^{2s-1}}dy = \Gamma(2-2s)\ {}_{RL}D^{2s}_{x,\infty}u(x) \text{ and}$$
$$\frac{d^2}{dx^2}\int_0^\infty \frac{u(x-y)}{y^{2s-1}}dy = \Gamma(2-2s)\ {}_{RL}D^{2s}_{-\infty,x}u(x),$$

we deduce that

$$\begin{aligned}(-\triangle)^s u(x) &= \frac{-C_{1,s}}{2s(2s-1)}\int_0^\infty \frac{u''(x+y)+u''(x-y)}{y^{2s-1}}dy \\ &= \frac{-C_{1,s}}{2s(2s-1)}\left[\frac{d^2}{dx^2}\int_0^\infty \frac{u(x+y)}{y^{2s-1}}dy + \frac{d^2}{dx^2}\int_0^\infty \frac{u(x-y)}{y^{2s-1}}dy\right] \\ &= \frac{-C_{1,s}}{2s(2s-1)}\Gamma(2-2s)\left[{}_{RL}D^{2s}_{-\infty,x} + {}_{RL}D^{2s}_{x,\infty}\right]u(x) \\ &= \frac{C_{1,s}}{2s}\Gamma(1-2s)\left[{}_{RL}D^{2s}_{-\infty,x} + {}_{RL}D^{2s}_{x,\infty}\right]u(x) = -\ {}_{RZ}D^{2s}_{x}u(x).\end{aligned}$$

In particular for $s = 1/2$, we have

$$\begin{aligned}{}_{RZ}D^{1}_{x}u(x) = -(-\triangle)^{1/2}u(x) &= C_{1,1/2}\int_0^\infty \frac{u(x+y)-2u(x)+u(x-y)}{y^2}dy \\ &= \frac{1}{\pi}\int_0^\infty \frac{u(x+y)-2u(x)+u(x-y)}{y^2}dy,\end{aligned}$$

which is well defined and extends Definition 2 to the value $\alpha = 1$. □

Remark 1.

(a) *Using the formula*

$$\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = \sqrt{\pi}2^{1-2z}\Gamma(2z)$$

for $2z \neq 0, -1, -2, \cdots$, *we have for* $u \in W_1(R)$ *that*

$${}_{RZ}D^{2s}_{x}u(x) = -(-\triangle)^s u(x) = \Gamma(1+2s)\frac{\sin \pi s}{\pi}\int_0^\infty \frac{u(x+y)-2u(x)+u(x-y)}{y^{1+2s}}dy$$

for $0 < s < 1$. *This expression has symbolically appeared in several existing literatures, such as References [13,21,24], for a suitable smooth function* $u(x)$.

(b) *Cai and Li presented Theorem 3 in Reference [25] under the conditions that* $u(x) \in S(R)$ *which is a proper subspace of* $W_1(R)$, *and* $s \in (0,1)$ *with* $s \neq 1/2$.

In order to study the generalized Riesz derivative, we briefly introduce the following basic concepts in distribution and the normalization of x_+^λ. Let $\mathcal{D}(R)$ be the Schwartz space [27] of infinitely differentiable functions (or so-called the Schwartz space of testing functions) with compact support in R, and $\mathcal{D}'(R)$ be the space of distributions (linearly continuous functionals) defined on $\mathcal{D}(R)$. Furthermore, we shall define a sequence $\phi_1(x)$, $\phi_2(x)$, $\cdots$, $\phi_m(x)$, $\cdots$ which converges to zero in $\mathcal{D}(R)$ if all these functions vanish outside a certain fixed and bounded smooth set in R^n and converge uniformly to zero (in the usual sense) together with their derivatives of any order. We further assume that $\mathcal{D}'(R^+)$ is the subspace of $\mathcal{D}'(R)$ with support contained in R^+. The functional δ is defined as

$$(\delta, \phi) = \phi(0),$$

where $\phi \in \mathcal{D}(R)$. Clearly, δ is a linear and continuous functional on $\mathcal{D}(R)$, and hence $\delta \in \mathcal{D}'(R)$.

Let $f \in \mathcal{D}'(R)$. Then the distributional derivative f' on $\mathcal{D}'(R)$ is defined as:

$$(f', \phi) = -(f, \phi')$$

for $\phi \in \mathcal{D}(R)$. In particular,

$$(\delta^{(m)}(x), \phi(x)) = (-1)^m \phi^{(m)}(0),$$

where m is a nonnegative integer.

The distribution x_+^λ on $\mathcal{D}(R)$ is normalized in Reference [27] as:

$$(x_+^\lambda, \phi(x)) = \int_0^\infty x^\lambda \left[\phi(x) - \phi(0) - x\phi'(0) - \cdots - \frac{x^{m-1}}{(m-1)!}\phi^{(m-1)}(0) \right] dx, \tag{5}$$

where $-m-1 < \lambda < -m$ ($m \in N$) and $\phi \in \mathcal{D}(R)$.

Let $\tau(x)$ be an infinitely differentiable function on $[0, +\infty) \subset R$ satisfying the following conditions:

(i) $0 \le \tau(x) \le 1$,
(ii) $\tau(x) = 1$ if $0 \le x \le 1/2$,
(iii) $\tau(x) = 0$ if $x \ge 1$.

Let $r = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$. We construct the sequence $I_m(r)$ for $m = 1, 2, \cdots$ as:

$$I_m(r) = \begin{cases} 1 & \text{if } r \le m, \\ \tau\left(\dfrac{m^{2m}}{1+2m^{1+m}} r^2 - \dfrac{m^{2m+2}}{1+2m^{1+m}} \right) & \text{if } r > m. \end{cases}$$

Clearly, $I_m(r)$ is infinitely differentiable with respect to $x_1, x_2, \cdots, x_n$ and r, and $I_m(r) = 0$ if $r \ge m + m^{-m}$, as

$$\frac{m^{2m}}{1+2m^{1+m}}(m + m^{-m})^2 - \frac{m^{2m+2}}{1+2m^{1+m}} = 1.$$

Furthermore,

$$0 \le I_m(r) \le 1.$$

Applying Equation (5) and the identity sequence $I_m(r)$ for $m = 1, 2, \cdots$, Li [11] established Theorems 1 and 2 outlined in the introduction. Based on Theorems 2 and 3, the generalized Riesz derivative on R is well defined, for $k < s < k+1$ with $k = 0, 1, 2, \cdots$, as

$${}_{RZ}D_x^{2s} u(x) = C_{1,s} \int_0^\infty y^{-1-2s} \left[S(y) - u^{(2)}(x) y^2 - \cdots - \frac{2y^{2k}}{(2k)!} u^{(2k)}(x) \right] dy,$$

where $u(x) \in W_{k+1}(R)$, and

$$S(y) = u(x+y) - 2u(x) + u(x-y).$$

The following theorem is to construct a relationship between the normed space $W_{k+1}(R)$ and the Banach space $C(R)$ by the generalized Riesz derivative.

Theorem 4. *Let $k < s < k+1$ with $k = 0, 1, 2, \cdots$. Then the generalized Riesz derivative ${}_{RZ}D_x^{2s}$ given by*

$${}_{RZ}D_x^{2s} u(x) = C_{1,s} \int_0^\infty y^{-1-2s} \left[S(y) - u^{(2)}(x) y^2 - \cdots - \frac{2y^{2k}}{(2k)!} u^{(2k)}(x) \right] dy$$

is a one-to-one and linearly continuous mapping from $W_{k+1}(R)$ to $C(R)$.

Proof of Theorem 4. From the above integral expression, the generalized Riesz derivative ${}_{RZ}D_x^{2s}$ is a linear mapping on the space $W_{k+1}(R)$. Let $u_m(x) \in W_{k+1}(R)$ and $u_m(x) \to 0$ in $W_{k+1}(R)$. It follows from Taylor's expansion that

$$\begin{aligned} S_m(y) &= u_m(x+y) - 2u_m(x) + u_m(x-y) \\ &= u_m^{(2)}(x)y^2 + \cdots + \frac{2y^{2k}}{(2k)!}u_m^{(2k)}(x) + \frac{y^{2k+2}}{(2k+2)!}\left(u_m^{(2k+2)}(x+\theta y) + u_m^{(2k+2)}(x-\theta y)\right), \end{aligned}$$

where $\theta \in (0,1)$. Clearly,

$$\begin{aligned} {}_{RZ}D_x^{2s}u_m(x) &= \frac{C_{1,s}}{(2k+2)!}\int_0^1 y^{-1-2s+2k+2}\left(u_m^{(2k+2)}(x+\theta y) + u_m^{(2k+2)}(x-\theta y)\right)dy \\ &= \frac{C_{1,s}}{(2k+2)!}\int_1^\infty y^{-1-2s+2k}\left(y^2u_m^{(2k+2)}(x+\theta y) + y^2u_m^{(2k+2)}(x-\theta y)\right)dy. \end{aligned}$$

Therefore,

$$\begin{aligned} \left\|{}_{RZ}D_x^{2s}u_m(x)\right\|_\infty &\le \frac{2|C_{1,s}|}{(2k+2)!(2k+2-2s)}\sup_{y\in R}\left|u_m^{(2k+2)}(y)\right| \\ &\quad + \frac{2|C_{1,s}|}{(2k+2)!(2s-2k)}\sup_{y\in R}\left|y^2\, u_m^{(2k+2)}(y)\right|, \end{aligned}$$

which converges to zero, as $\|u_m(x)\|_{k+1} \to 0$ implies that both $\sup_{y\in R}\left|u_m^{(2k+2)}(y)\right|$ and $\sup_{y\in R}\left|y^2\, u_m^{(2k+2)}(y)\right|$ go to zero as $m \to \infty$.

It remains to show that ${}_{RZ}D_x^{2s}$ is one-to-one from $W_{k+1}(R)$ to $C(R)$. Assume $u_1(x), u_2(x) \in W_{k+1}(R)$ such that

$${}_{RZ}D_x^{2s}u_1(x) = {}_{RZ}D_x^{2s}u_2(x).$$

This infers that

$$\begin{aligned} &\int_0^\infty y^{-1-2s}\left[S_1(y) - u_1^{(2)}(x)y^2 - \cdots - \frac{2y^{2k}}{(2k)!}u_1^{(2k)}(x)\right]dy \\ &= \int_0^\infty y^{-1-2s}\left[S_2(y) - u_2^{(2)}(x)y^2 - \cdots - \frac{2y^{2k}}{(2k)!}u_2^{(2k)}(x)\right]dy. \end{aligned}$$

Using the formula [28]

$$\lim_{s\to(k+1)^-}\frac{y^{-1-2s}}{\Gamma(-2s)} = \delta^{(2k+2)}(y),$$

we arrive at

$$\begin{aligned} &\int_0^\infty \delta^{(2k+2)}(y)\left[S_1(y) - u_1^{(2)}(x)y^2 - \cdots - \frac{2y^{2k}}{(2k)!}u_1^{(2k)}(x)\right]dy \\ &= \int_0^\infty \delta^{(2k+2)}(y)\left[S_2(y) - u_2^{(2)}(x)y^2 - \cdots - \frac{2y^{2k}}{(2k)!}u_2^{(2k)}(x)\right]dy. \end{aligned}$$

Hence,

$$S_1^{(2k+2)}(0) = S_2^{(2k+2)}(0),$$

by noting that

$$\int_0^\infty \delta^{(2k+2)}(y)\left[S_1(y) - u_1^{(2)}(x)y^2 - \cdots - \frac{2y^{2k}}{(2k)!}u_1^{(2k)}(x)\right]dy = S_1^{(2k+2)}(0).$$

Evidently,

$$S_1^{(2k+2)}(0) = 2u_1^{(2k+2)}(x) = 2u_2^{(2k+2)}(x) = S_2^{(2k+2)}(0),$$

which further claims that

$$u_1(x) = u_2(x) + P_{2k+1}(x),$$

where $P_{k+1}(x)$ is a polynomial of degree $2k+1$ in the space $W_{k+1}(R)$, which must be zero due to the condition

$$\sup_{x\in R} |xP_{k+1}(x)| < \infty.$$

□

Remark 2. *At this moment, we are unable to describe a subspace (say $C_s(R)$) of $C(R)$ such that the generalized Riesz derivative ${}_{RZ}D_x^{2s}$ is bijective and linearly continuous mapping from $W_{k+1}(R)$ to $C_s(R)$. This further study is of interest since we can define an inverse operation of the Riesz derivative on $C_s(R)$ if it exists.*

In addition, we have the following theorem regarding the limits at the end points for the generalized Riesz derivative ${}_{RZ}D_x^{2s}u(x)$ over the space $W_{k+1}(R)$.

Theorem 5. *Let $u(x) \in W_{k+1}(R)$ and $k < s < k+1$ with $k = 0, 1, 2, \cdots$. Then,*

$$\lim_{s\to(k+1)^-} {}_{RZ}D_x^{2s}u(x) = (-1)^k u^{(2k+2)}(x), \ \ \text{and}$$
$$\lim_{s\to k^+} {}_{RZ}D_x^{2s}u(x) = (-1)^{k+1} u^{(2k)}(x)$$

in the space $C(R)$.

In particular,

$$\lim_{s\to k} {}_{RZ}D_x^{2s}u(x) = (-1)^{k+1} u^{(2k)}(x)$$

for all $k = 1, 2, \cdots$.

Proof of Theorem 5. Let $k < s < k+1$ with $k = 0, 1, 2, \cdots$. Then,

$$\lim_{s\to(k+1)^-} \left\| {}_{RZ}D_x^{2s}u(x) - (-1)^k u^{(2k+2)}(x) \right\|_\infty$$
$$= \lim_{s\to(k+1)^-} \sup_{x\in R} \left| C_{1,s}\int_0^\infty y^{-1-2s}\left[S(y) - u^{(2)}(x)y^2 - \cdots - \frac{2y^{2k}}{(2k)!}u^{(2k)}(x)\right]dy \right.$$
$$\left. -(-1)^k u^{(2k+2)}(x) \right|.$$

Using

$$\lim_{s\to(k+1)^-} \frac{\Gamma(-2s)}{\Gamma(1-s)} = -\frac{k!}{(2k+2)!}\frac{(-1)^{k+1}}{2}, \ \text{and}$$
$$\Gamma\left(\frac{1}{2}+k+1\right) = \frac{(2k+2)!}{2^{2k+2}(k+1)!}\sqrt{\pi},$$

we derive that

$$\lim_{s\to(k+1)^-} C_{1,s}\Gamma(-2s) = -\pi^{-1/2}2^{2(k+1)}(k+1)\Gamma\left(\frac{1}{2}+k+1\right)\frac{k!}{(2k+2)!}\frac{(-1)^{k+1}}{2}$$
$$= \frac{(-1)^k}{2}.$$

Furthermore, the integral

$$\int_0^\infty \frac{y^{-1-2s}}{\Gamma(-2s)}\left[S(y) - u^{(2)}(x)y^2 - \cdots - \frac{2y^{2k}}{(2k)!}u^{(2k)}(x)\right]dy$$

converges uniformly with respect to s. Hence,

$$\lim_{s\to(k+1)^-} \int_0^\infty \frac{y^{-1-2s}}{\Gamma(-2s)}\left[S(y) - u^{(2)}(x)y^2 - \cdots - \frac{2y^{2k}}{(2k)!}u^{(2k)}(x)\right]dy$$
$$= \int_0^\infty \delta^{(2k+2)}(y)\left[S(y) - u^{(2)}(x)y^2 - \cdots - \frac{2y^{2k}}{(2k)!}u^{(2k)}(x)\right]dy$$
$$= S^{(2k+2)}(0) = 2u^{(2k+2)}(x).$$

In summary, we get

$$\lim_{s\to(k+1)^-} \left\|{}_{RZ}D_x^{2s}u(x) - (-1)^k u^{(2k+2)}(x)\right\|_\infty = 0,$$

which implies that

$$\lim_{s\to(k+1)^-} {}_{RZ}D_x^{2s}u(x) = (-1)^k u^{(2k+2)}(x)$$

in the space $C(R)$.

On the other hand,

$$\lim_{s\to k^+} \left\|{}_{RZ}D_x^{2s}u(x) - (-1)^{k+1}u^{(2k)}(x)\right\|_\infty$$
$$= \lim_{s\to k^+}\sup_{x\in R}\left|C_{1,s}\int_0^\infty y^{-1-2s}\left[S(y) - u^{(2)}(x)y^2 - \cdots - \frac{2y^{2k}}{(2k)!}u^{(2k)}(x)\right]dy \right.$$
$$\left. -(-1)^{k+1}u^{(2k)}(x)\right|.$$

Using

$$\lim_{s\to k^+}\frac{\Gamma(-2s)}{\Gamma(1-s)} = -\frac{(k-1)!}{(2k)!}\frac{(-1)^k}{2}, \text{ and}$$
$$\Gamma\left(\frac{1}{2}+k\right) = \frac{(2k)!}{2^{2k}k!}\sqrt{\pi}$$

we derive that

$$\lim_{s\to k^+} C_{1,s}\Gamma(-2s) = -\pi^{-1/2}2^{2k}k\Gamma\left(\frac{1}{2}+k\right)\frac{(k-1)!}{(2k)!}\frac{(-1)^k}{2} = \frac{(-1)^{k+1}}{2}.$$

Thus, from

$$\frac{y_+^{-1-2s}}{\Gamma(-2s)} = \delta^{(2s)}(y),$$
$$\frac{\partial^{2s}}{\partial y^{2s}}\left[u^{(2)}(x)y^2 + \cdots + \frac{2y^{2k}}{(2k)!}u^{(2k)}(x)\right] dy = 0 \quad \text{for } s > k, \text{ and}$$
$$\lim_{s\to k^+}\int_0^\infty \delta^{(2s)}(y)\left[S(y) - u^{(2)}(x)y^2 + \cdots + \frac{2y^{2k}}{(2k)!}u^{(2k)}(x)\right] dy$$
$$= \lim_{s\to k^+} S^{(2s)}(0) = S^{(2k)}(0) = 2u^{(2k)}(x)$$

it follows that

$$\lim_{s\to k^+}\left\|{}_{RZ}D_x^{2s}u(x) - (-1)^{k+1}u^{(2k)}(x)\right\|_\infty = 0.$$

Therefore,

$$\lim_{s\to k^+} {}_{RZ}D_x^{2s}u(x) = (-1)^{k+1}u^{(2k)}(x)$$

in the space $C(R)$. □

Remark 3.

(a) *From Theorem 5, we have*

$$\lim_{s\to 2k+1} {}_{RZ}D_x^{2s}u(x) = \lim_{s\to(2k+1)^+} {}_{RZ}D_x^{2s}u(x) = \lim_{s\to(2k+1)^-} {}_{RZ}D_x^{2s}u(x) = u^{(4k+2)}(x)$$

for all $k = 0, 1, 2, \cdots$, and

$$\lim_{s\to 2k} {}_{RZ}D_x^{2s}u(x) = \lim_{s\to(2k)^+} {}_{RZ}D_x^{2s}u(x) = \lim_{s\to(2k)^-} {}_{RZ}D_x^{2s}u(x) = -u^{(4k)}(x)$$

for all $k = 1, 2, \cdots$.

(b) *Clearly for $k = 0, 1, \cdots$,*

$$\begin{aligned} {}_{RZ}D_x^{2k+1}u(x) &= \frac{2^{2k+1}(k+1/2)(2k)!}{(-4)^k\pi}\int_0^\infty y^{-2-2k} \\ &\quad \left[u(x+y) - 2u(x) + u(x-y) - u^{(2)}(x)y^2 - \cdots - \frac{2y^{2k}}{(2k)!}u^{(2k)}(x)\right] dy \end{aligned}$$

using the identity

$$\Gamma\left(-k + \frac{1}{2}\right) = \frac{(-4)^k k!}{(2k)!}\sqrt{\pi}.$$

In particular,

$${}_{RZ}D_x^1u(x) = \frac{1}{\pi}\int_0^\infty \frac{u(x+y) - 2u(x) + u(x-y)}{y^2}dy, \quad \text{and}$$
$${}_{RZ}D_x^3u(x) = -\frac{6}{\pi}\int_0^\infty y^{-4}\left[u(x+y) - 2u(x) + u(x-y) - u^{(2)}(x)y^2\right] dy.$$

To end off this section, we use the following example to demonstrate computations of the generalized Riesz derivative.

Theorem 6. *Let $s > 0$ and $s \neq 1, 2, \cdots$. Then,*

$$_{RZ}D_x^{2s} e^{-x^2} = 2^{2s} \pi^{-\frac{1}{2}} \frac{\Gamma\left(\frac{1}{2}+s\right)}{\Gamma(1-s)} s e^{-x^2} \sum_{j=0}^{\infty} \frac{(2x)^{2j}}{(2j)!} \Gamma(j-s).$$

Furthermore,

$$\lim_{s \to 0^+} {}_{RZ}D_x^{2s} e^{-x^2} = -e^{-x^2} \quad \text{and} \quad \lim_{s \to k} {}_{RZ}D_x^{2s} e^{-x^2} = (-1)^{k+1} \frac{d^{2k}}{dx^{2k}} e^{-x^2},$$

where $k = 1, 2, \cdots$.

Proof of Theorem 6. We first assume $2 < s < 3$. Letting $u(x) = e^{-x^2}$ we come to

$$u^{(2)}(x) = e^{-x^2} 4x^2 - 2e^{-x^2}, \quad \text{and}$$
$$u^{(4)}(x) = 12e^{-x^2} - 48x^2 e^{-x^2} + 16x^4 e^{-x^2}.$$

By Theorem 4 (as $e^{-x^2} \in W_3(R)$),

$$\begin{aligned} {}_{RZ}D_x^{2s} e^{-x^2} &= C_{1,s} \int_0^{\infty} y^{-1-2s} \left[S(y) - u^{(2)}(x) y^2 - \frac{2y^4}{4!} u^{(4)}(x) \right] dy \\ &= C_{1,s} \int_0^{\infty} y^{-1-2s} \left\{ \left[e^{-(x+y)^2} - 2e^{-x^2} + e^{-(x-y)^2} \right] - u^{(2)}(x) y^2 - \frac{2y^4}{4!} u^{(4)}(x) \right\} dy. \end{aligned}$$

Clearly,

$$\begin{aligned} &\left[e^{-(x+y)^2} - 2e^{-x^2} + e^{-(x-y)^2} \right] - u^{(2)}(x) y^2 - \frac{2y^4}{4!} u^{(4)}(x) \\ &= \left[e^{-(x+y)^2} - 2e^{-x^2} + e^{-(x-y)^2} \right] - e^{-x^2} 4x^2 y^2 + 2e^{-x^2} y^2 \\ &\quad - y^4 e^{-x^2} + 4y^4 x^2 e^{-x^2} - \frac{4}{3} y^4 x^4 e^{-x^2} \\ &= 2e^{-x^2} (e^{-y^2} - 1 + y^2 - \frac{1}{2} y^4) + 4x^2 y^2 e^{-x^2} (e^{-y^2} - 1 + y^2) + \frac{4}{3} x^4 y^4 e^{-x^2} (e^{-y^2} - 1) \\ &\quad + e^{-x^2} e^{-y^2} \sum_{j=3}^{\infty} \frac{2(2xy)^{2j}}{(2j)!} \end{aligned}$$

using

$$e^{2xy} + e^{-2xy} = \sum_{j=0}^{\infty} \frac{2(2xy)^{2j}}{(2j)!} = 2 + 4x^2 y^2 + \frac{4}{3} x^4 y^4 + \sum_{j=3}^{\infty} \frac{2(2xy)^{2j}}{(2j)!}.$$

Making the variable change $u = y^2$,

$$\int_0^{\infty} y^{-1-2s} \left(e^{-y^2} - 1 + y^2 - \frac{1}{2} y^4 \right) dy = \frac{1}{2} \int_0^{\infty} u^{-1-s} \left(e^{-u} - 1 + u - \frac{1}{2} u^2 \right) du.$$

Using integration by parts, we get

$$
\begin{aligned}
&\frac{1}{2}\int_0^\infty u^{-1-s}\left(e^{-u}-1+u-\frac{1}{2}u^2\right)du \\
&= -\frac{1}{2s}\frac{e^{-u}-1+u-\frac{1}{2}u^2}{u^s}\bigg|_{u=0}^{\infty} + \frac{1}{2s}\int_0^\infty u^{-s}[-e^{-u}+1-u]du \\
&= -\frac{1}{2s}\int_0^\infty u^{-s}[e^{-u}-1+u]du \\
&= \frac{1}{2s(-s+1)}\ \frac{e^{-u}-1+u}{u^{s-1}}\bigg|_0^\infty + \frac{1}{2s(-s+1)}\int_0^\infty u^{-s+1}[1-e^{-u}]du \\
&= \frac{1}{2s(-s+1)(-s+2)}\ \frac{1-e^{-u}}{u^{-s+2}}\bigg|_0^\infty + \frac{1}{2s(s-1)(-s+2)}\int_0^\infty u^{-s+3}e^{-u}du \\
&= \frac{1}{2s(s-1)(-s+2)}\Gamma(-s+3) \\
&= \frac{\Gamma(2-s)}{2s(s-1)},
\end{aligned}
$$

by noting that

$$
\begin{aligned}
&\lim_{u\to\infty}\frac{e^{-u}-1+u-\frac{1}{2}u^2}{u^s} = \lim_{u\to 0^+}\frac{e^{-u}-1+u-\frac{1}{2}u^2}{u^s} = 0, \\
&\lim_{u\to\infty}\frac{e^{-u}-1+u}{u^{s-1}} = \lim_{u\to 0^+}\frac{e^{-u}-1+u}{u^{s-1}} = 0, \text{ and} \\
&\lim_{u\to\infty}\frac{1-e^{-u}}{u^{-s+2}} = \lim_{u\to 0^+}\frac{1-e^{-u}}{u^{-s+2}} = 0
\end{aligned}
$$

if $2 < s < 3$. Similarly, we obtain

$$
\begin{aligned}
&\int_0^\infty y^{1-2s}\left(e^{-y^2}-1+y^2\right)dy = -\frac{\Gamma(2-s)}{2(s-1)}, \\
&\int_0^\infty y^{3-2s}\left(e^{-y^2}-1\right)dy = \frac{\Gamma(2-s)}{2}, \\
&\int_0^\infty y^{2j-1-2s}e^{-y^2}dy = \frac{\Gamma(j-s)}{2}, \quad \text{for } j = 3, 4, \cdots.
\end{aligned}
$$

Hence,

$$\begin{aligned}
{}_{RZ}D_x^{2s} e^{-x^2} &= 2C_{1,s} e^{-x^2} \int_0^\infty y^{-1-2s} \left(e^{-y^2} - 1 + y^2 - \frac{1}{2} y^4 \right) dy \\
&\quad + 4C_{1,s} x^2 e^{-x^2} \int_0^\infty y^{1-2s} \left(e^{-y^2} - 1 + y^2 \right) dy \\
&\quad + \frac{4}{3} C_{1,s} x^4 e^{-x^2} \int_0^\infty y^{3-2s} (e^{-y^2} - 1) dy \\
&\quad + 2C_{1,s} e^{-x^2} \sum_{j=3}^{\infty} \frac{(2x)^{2j}}{(2j)!} \int_0^\infty y^{2j-1-2s} e^{-y^2} dy \\
&= -2^{2s} \pi^{-\frac{1}{2}} \Gamma\left(\frac{1}{2} + s\right) e^{-x^2} + 2^{2s+1}\, s\, \pi^{-\frac{1}{2}} \Gamma\left(\frac{1}{2} + s\right) x^2 e^{-x^2} \\
&\quad + \frac{2^{2s+1}}{3} \pi^{-\frac{1}{2}} s \frac{\Gamma(2-s)}{\Gamma(1-s)} \Gamma\left(\frac{1}{2} + s\right) x^4 e^{-x^2} \\
&\quad + 2^{2s} \pi^{-\frac{1}{2}} \frac{\Gamma\left(\frac{1}{2} + s\right)}{\Gamma(1-s)} s e^{-x^2} \sum_{j=3}^{\infty} \frac{(2x)^{2j}}{(2j)!} \Gamma(j-s) \\
&= 2^{2s} \pi^{-\frac{1}{2}} \frac{\Gamma\left(\frac{1}{2} + s\right)}{\Gamma(1-s)} s e^{-x^2} \sum_{j=0}^{\infty} \frac{(2x)^{2j}}{(2j)!} \Gamma(j-s).
\end{aligned}$$

Clearly, the series

$$2^{2s} \pi^{-\frac{1}{2}} \frac{\Gamma\left(\frac{1}{2} + s\right)}{\Gamma(1-s)} s e^{-x^2} \sum_{j=0}^{\infty} \frac{(2x)^{2j}}{(2j)!} \Gamma(j-s)$$

can be extended to all values of $s > 0$ and $s \neq 1, 2, \cdots$. For example, a similar calculation leads to

$$\begin{aligned}
{}_{RZ}D_x^{2s} e^{-x^2} &= -\frac{2^{2s} \Gamma(s + \frac{1}{2}) e^{-x^2}}{\sqrt{\pi}} + \frac{2^{2s+1} s \Gamma(s + \frac{1}{2})}{\sqrt{\pi}}\, x^2 e^{-x^2} \\
&\quad + \frac{2^{2s} s \Gamma(s + \frac{1}{2})}{\Gamma(1-s)\, \sqrt{\pi}}\, e^{-x^2} \sum_{j=2}^{\infty} \frac{(2x)^{2j}}{(2j)!} \Gamma(j-s) \\
&= 2^{2s} \pi^{-\frac{1}{2}} \frac{\Gamma\left(\frac{1}{2} + s\right)}{\Gamma(1-s)} s e^{-x^2} \sum_{j=0}^{\infty} \frac{(2x)^{2j}}{(2j)!} \Gamma(j-s)
\end{aligned}$$

if $1 < s < 2$. In addition,

$$\begin{aligned}
&\lim_{s \to 0^+} {}_{RZ}D_x^{2s} e^{-x^2} = \lim_{s \to 0^+} 2^{2s} \pi^{-\frac{1}{2}} \frac{\Gamma\left(\frac{1}{2} + s\right)}{\Gamma(1-s)} s e^{-x^2} \left[\Gamma(-s) + \sum_{j=1}^{\infty} \frac{(2x)^{2j}}{(2j)!} \Gamma(j-s) \right] \\
&= \lim_{s \to 0^+} 2^{2s} \pi^{-\frac{1}{2}} \frac{\Gamma\left(\frac{1}{2} + s\right)}{\Gamma(1-s)} s e^{-x^2} \Gamma(-s) \\
&\quad + \lim_{s \to 0^+} 2^{2s} \pi^{-\frac{1}{2}} \frac{\Gamma\left(\frac{1}{2} + s\right)}{\Gamma(1-s)} s e^{-x^2} \sum_{j=1}^{\infty} \frac{(2x)^{2j}}{(2j)!} \Gamma(j-s) = -e^{-x^2},
\end{aligned}$$

by applying the formula

$$-s\Gamma(-s) = \Gamma(1-s).$$

Clearly for $j = 2, 3, \cdots, k$,

$$\begin{aligned}\lim_{s\to k}\frac{\Gamma(j-s)}{\Gamma(1-s)} &= \lim_{s\to k}\frac{(j-1-s)(j-2-s)\cdots(1-s)\Gamma(1-s)}{\Gamma(1-s)} \\ &= (-1)^{j-1}(k-1)(k-2)\cdots(k-j+1).\end{aligned}$$

Hence for $k = 1, 2, \cdots$,

$$\begin{aligned}&\lim_{s\to k} {}_{RZ}D_x^{2s}e^{-x^2} = -2^{2k}\pi^{-\frac{1}{2}}\Gamma\left(\frac{1}{2}+k\right)e^{-x^2} + 2^{2k+1}\,k\,\pi^{-\frac{1}{2}}\Gamma\left(\frac{1}{2}+k\right)x^2e^{-x^2} \\ &-2^{2k}\pi^{-\frac{1}{2}}\Gamma\left(\frac{1}{2}+k\right)e^{-x^2}\sum_{j=2}^{k}\frac{(2x)^{2j}}{(2j)!}(-1)^j k(k-1)\cdots(k-j+1) \\ &= (-1)^{k+1}\frac{d^{2k}}{dx^{2k}}e^{-x^2}\end{aligned}$$

by Theorem 5, which can be verified directly by mathematical induction. □

Remark 4. *From the physicists' Hermite polynomials given by*

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2},$$

we derive

$$\lim_{s\to k} {}_{RZ}D_x^{2s}e^{-x^2} = (-1)^{k+1}e^{-x^2}H_{2k}(x).$$

3. The Generalized Riesz Derivative on R^n with $n \geq 2$

In this section, we begin to study the generalized Riesz derivative ${}_{RZ}D_x^{2s}u(x)$ for $s > 0$ on R^n, and obtain its integral representation using Theorem 1 mentioned in the introduction. In particular, we derive explicit integral expressions for ${}_{RZ}D_x^{2k+1}u(x)$ when $k = 0, 1, 2, \cdots$.

Theorem 7. *Let $0 < s < 1$ and $k = (k_1, k_2, \cdots, k_n)$ be an n-tuple of nonnegative integers with $k_1 + \cdots + k_n = 1$. Then for $u(x) \in C_k(R^n)$ (defined in the introduction),*

$$ {}_{RZ}D_x^{2s}u(x) = (-\triangle)^s u(x) = -\frac{1}{2}C_{n,s}\int_0^\infty r^{-1-2s}S(r)dr \tag{6}$$

where $S(r)$ is the surface integral on the unit sphere $\Omega \subset R^n$, given by

$$S(r) = \int_\Omega [u(x+r\sigma) - 2u(x) + u(x-r\sigma)]d\sigma.$$

Proof of Theorem 7. We let $l = 2$ in the case of centred difference from Definition 3 and derive that

$$(\triangle_y^2 u)(x) = \sum_{k=0}^{2}(-1)^k\binom{2}{k}u(x+(1-k)y) = u(x+y) - 2u(x) + u(x-y)$$

and direct computation implies that

$$\frac{1}{d_{n,l}(2s)} = -\frac{2^{2s-1}s\Gamma\left(\frac{n}{2}+s\right)}{\pi^{\frac{n}{2}}\Gamma(1-s)} = -\frac{C_{n,s}}{2}$$

by making use of the identity

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin \pi z}$$

for any non-integer z. Hence,

$$ {}_{RZ}D_x^{2s}u(x) = -\frac{C_{n,s}}{2}\int_R \frac{u(x+y) - 2u(x) + u(x-y)}{|y|^{n+2s}}dy, \tag{7} $$

which is well defined for $u(x) \in C_k(R^n)$. Indeed, a second order Taylor expansion derives

$$ \frac{|[u(x+y) - 2u(x) + u(x-y)]|}{|y|^{n+2s}} \leq \frac{\|D^2u\|_{L^\infty}}{|y|^{n+2s-2}}, \quad 0 < s < 1, $$

which is integrable near zero. Furthermore, $u(x) \in C_k(R^n)$ implies that

$$ \sup_{y \in R^n} ||y|^2 D^2 u(y)| $$

is bounded as $|y| \to \infty$. This deduces that the integral converges at infinity.

Using the spherical coordinates below

$$ \begin{aligned} y_1 &= r\cos\theta_1 \\ y_2 &= r\sin\theta_1\cos\theta_2 \\ y_3 &= r\sin\theta_1\sin\theta_2\cos\theta_3 \\ &\cdots \\ y_{n-1} &= r\sin\theta_1\cdots\sin\theta_{n-2}\cos\theta_{n-1} \\ y_n &= r\sin\theta_1\cdots\sin\theta_{n-2}\sin\theta_{n-1}, \end{aligned} $$

where the angles $\theta_1, \theta_2, \cdots, \theta_{n-2}$ range over $[0, \pi]$ and θ_{n-1} ranges over $[0, 2\pi]$. Then Equation (7) turns out to be

$$ {}_{RZ}D_x^{2s}u(x) = -\frac{1}{2}C_{n,s}\int_0^\infty \frac{r^{n-1}S(r)}{r^{n+2s}}dr = -\frac{1}{2}C_{n,s}\int_0^\infty \frac{S(r)}{r^{1+2s}}dr, $$

where

$$ S(r) = \int_\Omega [u(x+r\sigma) - 2u(x) + u(x-r\sigma)]d\sigma. $$

Clearly, the integral

$$ \int_0^\infty \frac{S(r)}{r^{1+2s}}dr = \int_0^\infty \frac{S(r) - S(0)}{r^{1+2s}}dr $$

converges as $S(0) = 0$ and $S(r)$ is an even function with respect to r. It follows from Theorem 1 for $0 < s < 1$ that

$$ {}_{RZ}D_x^{2s}u(x) = (-\triangle)^s u(x) = -\frac{1}{2}C_{n,s}\int_0^\infty r^{-1-2s}S(r)dr. $$

□

Remark 5. *There is a sign difference between Definition 2 and Definition 3 for $n = 1$. Indeed for $u \in W_1(R)$ and $0 < s < 1$,*

$$ {}_{RZ}D_x^{2s}u(x) = -(-\triangle)^s u(x) = C_{1,s}\int_0^\infty \frac{u(x+y) - 2u(x) + u(x-y)}{y^{1+2s}}dy $$

from Definition 2, and

$$ {}_{RZ}D_x^{2s}u(x) = (-\triangle)^s u(x) = -C_{1,s}\int_0^\infty \frac{u(x+y) - 2u(x) + u(x-y)}{y^{1+2s}}dy $$

by Equation (7), which is directly from Definition 3.

Let $i = 0, 1, \cdots$ and $i < s < i + 1$. Applying Theorem 7 and Theorem 1, we can extend the generalized Riesz derivative ${}_{RZ}D_x^{2s}$ over the space $C_k(R^n)$ as

$$\begin{aligned} {}_{RZ}D_x^{2s}u(x) &= (-\triangle)^s u(x) = -\frac{1}{2}C_{n,s}\int_0^\infty r^{-1-2s} \\ &\left[S(r) - \frac{r^2\Omega_n \triangle u(x)}{n} - \cdots - \frac{2r^{2i}\Omega_n \triangle^i u(x)}{2^i\, i!\, n(n+2)\cdots(n+2i-2)}\right] dr, \end{aligned} \tag{8}$$

where $k = (k_1, k_2, \cdots, k_n)$ is an n-tuple of nonnegative integers with $k_1 + \cdots + k_n = i + 1$.

In particular,

$$\begin{aligned} {}_{RZ}D_x^1 u(x) &= -\frac{\Gamma\left(\frac{n+1}{2}\right)}{2\pi^{\frac{n+1}{2}}}\int_0^\infty \frac{S(r)}{r^2} dr, \\ {}_{RZ}D_x^3 u(x) &= \frac{3\Gamma\left(\frac{n+3}{2}\right)}{\pi^{\frac{n+1}{2}}}\int_0^\infty r^{-4}\left[S(r) - \frac{r^2\Omega_n\triangle u(x)}{n}\right] dr, \\ &\cdots \\ {}_{RZ}D_x^{2k+1}u(x) &= -\frac{2^{2k}(k+\frac{1}{2})(2k)!\Gamma\left(\frac{n+1}{2}+k\right)}{\pi^{\frac{n+1}{2}}(-4)^k k!}\int_0^\infty r^{-1-2s} \\ &\left[S(r) - \frac{r^2\Omega_n\triangle u(x)}{n} - \cdots - \frac{2r^{2k}\Omega_n\triangle^k u(x)}{2^k\, k!\, n(n+2)\cdots(n+2k-2)}\right] dr. \end{aligned}$$

The following theorem can be found in Reference [11].

Theorem 8. *Let $u(x) \in C_k(R^n)$ with $n > 1$ and $i < s < i + 1$ for $i = 0, 1, \cdots$. Then,*

$$\begin{aligned} &\lim_{s\to(i+1)^-}(-\triangle)^s u(x) = (-1)^{i+1}\triangle^{i+1}u(x), \quad \textit{and} \\ &\lim_{s\to i^+}(-\triangle)^s u(x) = (-1)^i\triangle^i u(x) \end{aligned}$$

where $k = (k_1, k_2, \cdots, k_n)$ is an n-tuple of nonnegative integers and $k_1 + k_2 + \cdots + k_n = i + 1$.

From Theorem 8 we have

$$\begin{aligned} &\lim_{s\to(i+1)^-} {}_{RZ}D_x^{2s}u(x) = (-1)^{i+1}\triangle^{i+1}u(x), \quad \text{and} \\ &\lim_{s\to i^+} {}_{RZ}D_x^{2s}u(x) = (-1)^i\triangle^i u(x). \end{aligned}$$

Hence,

$$\lim_{s\to i} {}_{RZ}D_x^{2s}u(x) = (-1)^i\triangle^i u(x).$$

for $i = 1, 2, \cdots$.

An an example, we are going to compute ${}_{RZ}D_x^1 u(x)$, where $u(x) = e^{-x_1^2-x_2^2}$. It follows from Reference [11] that

$$S(r) = \int_\Omega [u(x+r\sigma) - 2u(x) + u(x-r\sigma)]d\sigma = 4\pi e^{-x_1^2-x_2^2}\left[e^{-r^2}\sum_{k=0}^\infty \frac{r^{2k}(x_1^2+x_2^2)^k}{(k!)^2} - 1\right].$$

Then,

$$
\begin{aligned}
{}_{RZ}D_x^1 u(x) &= -\frac{1}{4\pi}\int_0^\infty \frac{S(r)}{r^2}dr \\
&= -e^{-x_1^2-x_2^2}\int_0^\infty \frac{1}{r^2}\left[e^{-r^2}-1+e^{-r^2}\sum_{k=1}^{\infty}\frac{r^{2k}(x_1^2+x_2^2)^k}{(k!)^2}\right] \\
&= -e^{-x_1^2-x_2^2}\int_0^\infty \frac{e^{-r^2}-1}{r^2}dr \\
&\quad -e^{-x_1^2-x_2^2}\sum_{k=1}^{\infty}\frac{(x_1^2+x_2^2)^k}{(k!)^2}\int_0^\infty e^{-r^2}r^{2k-2}dr \\
&= \sqrt{\pi}e^{-x_1^2-x_2^2}-\frac{1}{2}e^{-x_1^2-x_2^2}\sum_{k=1}^{\infty}\frac{(x_1^2+x_2^2)^k}{(k!)^2}\Gamma\left(k-\frac{1}{2}\right) \\
&= \sqrt{\pi}e^{-x_1^2-x_2^2}-\frac{\sqrt{\pi}}{2}e^{-x_1^2-x_2^2}\sum_{k=1}^{\infty}\frac{(x_1^2+x_2^2)^k}{(k!)^2}\frac{(2k-2)!}{4^{k-1}(k-1)!}
\end{aligned}
$$

using

$$
\begin{aligned}
&\int_0^\infty \frac{e^{-r^2}-1}{r^2}dr = \frac{\Gamma(-1/2)}{2} = -\sqrt{\pi}, \\
&\int_0^\infty e^{-r^2}r^{2k-2}dr = \frac{\Gamma\left(k-\frac{1}{2}\right)}{2}, \\
&\Gamma\left(k-\frac{1}{2}\right) = \frac{(2k-2)!}{4^{k-1}(k-1)!}\sqrt{\pi}.
\end{aligned}
$$

Let $u(x) \in C^\infty(R^n)$. Then $u(x)I_m(r)$ has a compact support and belongs to the space $C_k(R^n)$ for all n-tuple of nonnegative integers k where the identity sequence $I_m(r)$ is given in Section 2.

Let $i < s < i+1$ with $i = 0, 1, \cdots$, and set

$$S_m(r) = S(r)I_m(r).$$

Applying Equation (8) we can define the generalized Riesz derivative ${}_{RZ}D_x^{2s}$ over the space $C^\infty(R^n)$ as

$$
\begin{aligned}
{}_{RZ}D_x^{2s}u(x) \quad = \quad & (-\triangle)^s u(x) = -\frac{1}{2}C_{n,s}\lim_{m\to}\int_0^{m+m^{-m}} r^{-1-2s} \\
& \left[S_m(r)-\frac{r^2\Omega_n\triangle u(x)}{n}-\cdots-\frac{2r^{2i}\Omega_n\triangle^i u(x)}{2^i\, i!\, n(n+2)\cdots(n+2i-2)}\right]dr, \qquad (9)
\end{aligned}
$$

if the limit exists.

To complete this section, we present the following theorem.

Theorem 9. *Let $s > 1$ and $n > 1$. Then ${}_{RZ}D_x^{2s}(x_1^2x_2) = 0$ on R^n.*

Proof of Theorem 9. We first note that the function $x_1^2x_2 \in C^\infty(R^n)$, but not bounded. Clearly,

$$
\begin{aligned}
&\triangle(x_1^2x_2) = (\partial^2/\partial x_1^2+\cdots+\partial^2/\partial x_n^2)(x_1^2x_2) = 2x_2, \\
&(\triangle)^2(x_1^2x_2) = 0.
\end{aligned}
$$

Assume $1 < s < 2$ first. Then from Equation (9),

$$\begin{aligned} {}_{RZ}D_x^{2s}(x_1^2 x_2) &= -\frac{1}{2}C_{n,s}\lim_{m\to\infty}\int_0^{m+m^{-m}} r^{-1-2s}\left[S_m(r) - \frac{r^2\Omega_n \triangle u(x)}{n}\right]dr \\ &= -\frac{1}{2}C_{n,s}\lim_{m\to\infty}\int_0^{m+m^{-m}} r^{-1-2s}\left[S_m(r) - \frac{2x_2 r^2\Omega_n}{n}\right]dr. \end{aligned}$$

To compute $S_m(r)$ we come to

$$S_m(r) = I_m(r)\int_\Omega [u(x+r\sigma) - 2u(x) + u(x-r\sigma)]d\sigma$$

and

$$\begin{aligned} &u(x+r\sigma) - 2u(x) + u(x-r\sigma) \\ &= (x_1 + r\sigma_1)^2(x_2 + r\sigma_2) - 2x_1^2 x_2 + (x_1 - r\sigma_1)^2(x_2 - r\sigma_2) \\ &= 4x_1 r^2\sigma_1\sigma_2 + 2x_2 r^2\sigma_1^2. \end{aligned}$$

Clearly,

$$\int_\Omega 2x_2 r^2\sigma_1^2 d\sigma = 2x_2 r^2\int_\Omega \sigma_1^2 d\sigma = 2x_2 r^2 V_n = \frac{2x_2 r^2\Omega_n}{n},$$

where V_n is the volume of the unit ball in R^n. Furthermore,

$$\int_\Omega \sigma_1\sigma_2 d\sigma = \int_\Omega \sigma_1\sigma_2 d\sigma_1\cdots d\sigma_n = 0$$

due to the integral cancellation over the unit sphere. Hence,

$$S(r) = \frac{2x_2 r^2\Omega_n}{n},$$

and

$$\begin{aligned} {}_{RZ}D_x^{2s}(x_1^2 x_2) &= -\frac{1}{2}C_{n,s}\lim_{m\to\infty}\int_0^{m} r^{-1-2s}\left[S(r) - \frac{2x_2 r^2\Omega_n}{n}\right]dr \\ &\quad -\frac{1}{2}C_{n,s}\lim_{m\to\infty}\int_m^{m+m^{-m}} r^{-1-2s}\left[I_m(r)S(r) - \frac{2x_2 r^2\Omega_n}{n}\right]dr \\ &= \frac{x_2\Omega_n}{n}C_{n,s}\lim_{m\to\infty}\int_m^{m+m^{-m}} r^{1-2s}[1 - I_m(r)]dr = 0. \end{aligned}$$

It follows from

$$(\triangle)^2(x_1^2 x_2) = (\triangle)^3(x_1^2 x_2) = \cdots = 0$$

that the result still holds for $s > 2$. $\square$

4. Conclusions

An integral representation is constructed for the generalized Riesz derivative ${}_{RZ}D_x^{2s}u(x)$ for $k < s < k+1$ with $k = 0, 1, \cdots$ in arbitrary dimensions by applying the normalization of distribution and the surface integrals. We further show that ${}_{RZ}D_x^{2s}u(x)$ is continuous at the end points and well defined for $s = \frac{1}{2} + k$. In addition, several examples are presented to demonstrate computations for obtaining the generalized Riesz derivatives.

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