The fractional Green's function by Babenko's approach

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Abstract

The goal of this paper is to derive the fractional Green's function for the first time in the distributional space for the fractional-order integro-differential equation with constant coefficients. Our new technique is based on Babenko's approach, without using any integral transforms such as the Laplace transform along with Mittag-Leffler function. The results obtained are not only much simpler, but also more generalized than the classical ones as they deal with distributions which are undefined in the ordinary sense in general. Furthermore, several interesting applications to solving the fractional differential and integral equations, as well as in the wave reaction-diffusion equation are provided, some of which cannot be achieved by integral transforms or numerical analysis.

2010 Mathematics Subject Classification. **46F10**. 45J05, 26A33.
Keywords. distribution, fractional calculus, Mittag-Leffler function, Babenko's approach, gamma function, multinomial theorem, Green's function.

1 Introduction

A Green's function $u(t,x)$, of a linear differential operator $L = L(x)$ acting on functions (or distributions) over a subset of R^n, at a point t, is any solution of

$$Lu(t,x) = \delta(t-x)$$

where $\delta(x)$ is the Dirac delta function. In particular, when $t = 0$ we have

$$Lu(0,x) = \delta(0-x) = \delta(x).$$

Green's function derived from Laplace transform is in the study of the following n-th order linear differential equations on the interval $I = [0,\infty)$

$$u^{(n)}(x) + a_{n-1}u^{(n-1)}(x) + \cdots + a_0 u(x) = g(x) \tag{1}$$

with the initial conditions

$$u^{(k)}(0) = 0, \quad \text{for } 0 \le k \le n-1.$$

Let $\mathcal{L}^{-1}$ denote the Laplace inverse transform. Then the solution of equation (1) is given in [1] by

$$u(x) = (H * g)(x) = \int_0^x H(x-\zeta)g(\zeta)d\zeta,$$

where

$$H(x) = \mathcal{L}^{-1}\left\{\frac{1}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}\right\}(x)$$

Tbilisi Mathematical Journal 13(3) (2020), pp. 19–42.
Tbilisi Centre for Mathematical Sciences.

Received by the editors: 10 September 2019.
Accepted for publication: 25 May 2020.

is the one-sided Green's function satisfying the fundamental differential equation

$$u^{(n)}(x) + a_{n-1}u^{(n-1)}(x) + \cdots + a_0 u(x) = \delta(x).$$

On the other hand, fractional calculus [2, 3, 4, 5, 6, 7, 8] that deals with operations of integration and differentiation of non-integer order is a generalization of classical calculus. The notion of fractional operators has been introduced almost simultaneously with the development of the classical theories. Fractional modeling is to use fractional differential and integral equations to describe numerous physical problems [9, 10] in the fields of chemistry, biology, electronics, noncommutative quantum field theories [11], and quantum mechanics [12]. Integral transforms, such as Laplace, Fourier, Hankel, and Mellin transforms, are vital tools of seeking solutions for linear and fractional ordinary or partial differential equations, especially with constant coefficients, by solving algebraic equations and inverse operations.

It is well known that Green's function plays an important role in solving fractional differential and integral equations which appear in mathematical and physical fields, particularly for the so called wave reaction-diffusion equation and its special cases [13]. Using Laplace transform, the Mittag-Leffler and Green's functions, Gorenflo et al. [14] studied the fractional relaxation-oscillation equation

$$\frac{d^\alpha}{dt^\alpha}u + \omega^\alpha u(t;\alpha) = 0, \quad 0 < \alpha \leq 2,$$

where ω is a positive constant with the dimensions T^{-1} as a frequency, and the field variable $u(t;\alpha)$ is assumed to be a causal function of time, as well as the fractional diffusion-wave equation

$$\frac{\partial^{2\beta}}{\partial t^{2\beta}}u = D\frac{\partial^2}{\partial x^2}u, \quad x \in R, \ \ 0 < \beta \leq 1,$$

where D denotes a positive constant with the dimensions $L^2T^{-2\beta}$, and the field variable $u(x,t;\beta)$ is assumed to be a causal function of time with $u(\mp, t; \beta) = 0$. Very recently, Fernandez et al.[15] investigated several different models of fractional calculus, based on the Prabhakar fractional integral transform, involving generalized multi-parameters Mittag-Leffler functions. They derived a new series expression for this transform, in terms of the classical Riemann-Lioville fractional integrals. Srivastava et al. [16] introduced and studied a fractional integral operator containing a certain generalized multi-index Mittag-Leffler function in its kernel. The authors are particularly interested in their Theorem 2 which states the fractional integral operator is bounded on $L(a,b)$. This property may be useful in seeking for solution of an integral equation involving such operators.

Ma studied Green's function associated the higher-order fractional boundary value problem [17]

$$\begin{aligned} &-D^\nu_{0+}u(x) = a(x)f(x,u(x)), \quad 0 < x < 1, \\ &u(0) = u'(0) = u''(0) = \cdots = u^{(n-2)}(0) = 0, \quad D^\alpha_{0+}u(x)|_{x=1} = 0, \end{aligned}$$

where ν and α are two given constants satisfying $n-1 < \nu \leq n$ with $n \geq 3, \ 0 \leq \alpha \leq n-2$, and established the existence of positive solutions of the equation.

As far as we know, there are several existing methods of finding Green's function in the classical sense for the fractional differential equation. The common and popular approach is based on the inverse of Laplace transform, and Mittag-Leffler function along with its derivatives. To demonstrate the ides in detail, we solve the following four-term fractional differential equation to avoid complicated operations of an n-term equation

$$a_3 u^{(\beta_3)}(x) + a_2 u^{(\beta_2)}(x) + a_1 u^{(\beta_1)}(x) + a_0 u(x) = \delta(x) \tag{2}$$

where $\beta_3 > \beta_2 > \beta_1$, and the derivatives are sequential. Applying the Laplace transform to both sides of the above equation, we arrive at

$$\begin{aligned}
\tilde{u}(s) &= \frac{1}{a_3 s^{\beta_3} + a_2 s^{\beta_2} + a_1 s^{\beta_1} + a_0} \\
&= \frac{1}{a_3 s^{\beta_3} + a_2 s^{\beta_2}} \frac{1}{1 + \dfrac{a_1 s^{\beta_1} + a_0}{a_3 s^{\beta_3} + a_2 s^{\beta_2}}} \\
&= \frac{a_3^{-1} s^{-\beta_2}}{s^{\beta_3-\beta_2} + a_2/a_3} \frac{1}{1 + \dfrac{a_1 s^{\beta_1-\beta_2}/a_3 + a_0 s^{-\beta_2}/a_3}{s^{\beta_3-\beta_2} + a_2/a_3}} \\
&= \sum_{m=0}^{\infty} (-1)^m \frac{a_3^{-1} s^{-\beta_2}}{s^{\beta_3-\beta_2} + a_2/a_3} \left(\frac{a_1 s^{\beta_1-\beta_2}/a_3 + a_0 s^{-\beta_2}/a_3}{s^{\beta_3-\beta_2} + a_2/a_3} \right)^m \\
&= \sum_{m=0}^{\infty} (-1)^m \frac{a_3^{-1} s^{-\beta_2}}{(s^{\beta_3-\beta_2} + a_2/a_3)^{m+1}} \sum_{k=0}^{m} \binom{m}{k} \frac{a_1^k a_0^{m-k}}{a_3^m} s^{k\beta_1 - m\beta_2} \\
&= \frac{1}{a_3} \sum_{m=0}^{\infty} (-1)^m \left(\frac{a_0}{a_3}\right)^m \sum_{k=0}^{m} \binom{m}{k} \left(\frac{a_1}{a_0}\right)^k \frac{s^{k\beta_1 - m\beta_2 - \beta_2}}{(s^{\beta_3-\beta_2} + a_2/a_3)^{m+1}}.
\end{aligned}$$

Using the formula [4]

$$\int_0^{\infty} e^{-xs} x^{\alpha m + \beta - 1} E_{\alpha,\beta}^{(m)}(\pm a x^{\alpha}) dx = \frac{m! s^{\alpha-\beta}}{(s^{\alpha} \mp a)^{m+1}} \tag{3}$$

where $\mathrm{Re}\, s > |a|^{1/\alpha}$, and term-by-term the Laplace inverse, we deduce that

$$\begin{aligned}
u(x) &= \frac{1}{a_3} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left(\frac{a_0}{a_3}\right)^m \sum_{k=0}^{m} \binom{m}{k} \left(\frac{a_1}{a_0}\right)^k x^{\beta_3(m+1) - \beta_1 k - 1} \\
&\quad \times E^{(m)}_{\beta_3-\beta_2,\, \beta_3+\beta_2 m - \beta_1 k}(-a_2 x^{\beta_3-\beta_2}/a_3)
\end{aligned}$$

where where $E_{\alpha,\,\beta}(z)$ is the Mittag-Leffler function of two parameters given by

$$E_{\alpha,\,\beta}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n + \beta)}, \quad (\alpha > 0, \ \ \beta > 0)$$

and

$$E^{(m)}_{\alpha,\,\beta}(x) = \frac{d^m}{dx^m} E_{\alpha,\,\beta}(x) = \sum_{j=0}^{\infty} \frac{(j+m)! x^j}{j!\, \Gamma(\alpha j + \alpha m + \beta)}, \quad m = 0, 1, 2, \cdots. \tag{4}$$

Therefore,

$$
\begin{aligned}
u(x) &= \frac{1}{a_3}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\left(\frac{a_0}{a_3}\right)^m\sum_{k=0}^{m}\binom{m}{k}\left(\frac{a_1}{a_0}\right)^k x^{\beta_3(m+1)-\beta_1 k-1}\\
&\quad\times\sum_{j=0}^{\infty}\frac{(j+m)!\,(-1)^j\left(\frac{a_2}{a_3}\right)^j x^{j\,(\beta_3-\beta_2)}}{j!\,\Gamma((j+m+1)\beta_3-j\beta_2-\beta_1 k)}
\end{aligned}
$$

which is the solution of equation (2).

In 2014, Kim and O [18] provided an explicit Greens function for the fractional differential equation (under Riemann-Liouville derivatives) in the classical sense, with continuous variable coefficients and the initial conditions. Their approach is based on representation of solution of the corresponding integral equation and the method of successive approximations (recursive technique). Pak et al. [19] recently studied solutions of the following linear nonhomogeneous Caputo fractional differential equation with continuous variable coefficients and $g(x)$

$$
{}_C D_{0,x}^{\beta_n}u(x)+a_{n-1}(x)\,{}_C D_{0,x}^{\beta_{n-1}}u(x)+\cdots+a_0(x)\,{}_C D_{0,x}^{\beta_0}u(x)=g(x),
$$

with all zero initial conditions

$$
D^j u(0^+)=0,\quad j=0,1,\cdots,n_0,\quad n_0-1<\beta_n\le n_0\in Z^+.
$$

Srivastava et al. [20, 21, 22] investigated various classes of the Mittag-Leffler type functions which are associated with several families of generalized Riemann-Liouville and other related fractional derivative operators, and presented solutions of many different classes of fractional differential equations with constant or variable coefficients and some general Volterra-type differential equations in the space of Lebesgue integrable functions. For example, the following general class of differential equations of the Volterra-type involving the generalized fractional derivative operators [22]

$$
(D_{0+}^{\alpha,\mu}u)(x)+\frac{a}{\Gamma(\nu)}\int_0^x (x-t)^{\nu-1}u(t)dt=g(x),\ \ 0<\alpha<1;\ 0\le\mu\le 1;\ Re(\nu)>0
$$

with the initial condition

$$
\left(I_{0+}^{(1-\mu)(1-\alpha)}u\right)(0+)=c
$$

has the solution in $L(0,\infty)$

$$
u(x)=cx^{\alpha-\mu(\alpha-1)-1}E_{\alpha+\nu,\alpha-\mu(\alpha-1)}(-ax^{\alpha+\nu})+(E^1_{\alpha+\nu,\alpha,-\alpha;0+}g)(x),
$$

where c is a constant and $E^1_{\alpha+\nu,\alpha,-\alpha;0+}$ is the Prabhaker fractional integral operator.

The aim of this paper is to imply Green's function with a new and simpler method in distribution, without using any integral transforms along with Mittag-Leffler functions. The obtained Green's function can further solve the fractional integro-differential equation

$$
a_n u^{(\beta_n)}(x)+a_{n-1}u^{(\beta_{n-1})}(x)+\cdots+a_1 u^{(\beta_1)}(x)+a_0 u^{(\beta_0)}(x)=g(x)
$$

with singular generalized functions on the right-hand side which do not have Laplace transforms, and it reduces to the classical Green's function when $g(x)$ has the well-defined Laplace transform. Some applicable examples of solving fractional differential and integral equations, as well as in the wave reaction-diffusion equation are presented by utilizing our derived results. We start with fundamental concepts of distribution theory.

2 Fractional calculus of distribution

In order to study the fractional Green's function in the generalized sense, we briefly introduce the following basic concepts with several interesting examples of solving Abel's integral equations in distribution. Let $\mathcal{D}(R)$ be the Schwartz space (testing function space) [23] of infinitely differentiable functions with compact support in R, and $\mathcal{D}'(R)$ the (dual) space of distributions defined on $\mathcal{D}(R)$. A sequence $\varphi_1, \varphi_2, \cdots, \varphi_n, \cdots$ goes to zero in $\mathcal{D}(R)$ if and only if these functions vanish outside a certain fixed and bounded set, and converge to zero uniformly together with their derivatives of any order. Clearly, $\mathcal{D}(R)$ is not empty since it contains the following function

$$\varphi(x, a) = \begin{cases} e^{-\frac{a^2}{a^2 - x^2}} & \text{if } |x| < a, \\ 0 & \text{otherwise} \end{cases}$$

where $a > 0$. Evidently any locally integrable function $f(x)$ on R is a (regular) distribution in $\mathcal{D}'(R)$ as

$$(f(x), \varphi(x)) = \int_{-\infty}^{\infty} f(x)\varphi(x)dx$$

is well defined. Hence, f is linear and continuous on $\mathcal{D}(R)$. In particular, the unit step function $\theta(x)$ defined as

$$\theta(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x < 0. \end{cases}$$

is a member of $\mathcal{D}'(R)$. Furthermore, the functional $\delta(x - x_0)$ on $\mathcal{D}(R)$ given by

$$(\delta(x - x_0), \varphi(x)) = \varphi(x_0)$$

is linear and continuous on $\mathcal{D}(R)$, according to the topological structure of the Schwartz testing function space.

Let $f \in \mathcal{D}'(R)$. The distributional derivative f' (or df/dx), is defined as

$$(f', \varphi) = -(f, \varphi')$$

for $\varphi \in \mathcal{D}(R)$. Therefore,

$$(\delta^{(n)}(x - x_0), \varphi(x)) = (-1)^n(\delta(x - x_0), \varphi^{(n)}(x)) = (-1)^n\varphi^{(n)}(x_0).$$

Let

$$g(x) = x_+^{1/2} = \begin{cases} \sqrt{x} & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

As an example, we will find the distributional derivative of g (note that this function is not differentiable at $x = 0$ in the classical sense). Indeed, using integration by parts we derive

$$(g'(x),\, \varphi(x)) = -(g(x),\, \varphi'(x)) = -\int_0^\infty \sqrt{x}\varphi'(x)dx = \frac{1}{2}\int_{-\infty}^\infty x_+^{-1/2}\varphi(x)dx,$$

which infers that

$$g'(x) = \frac{x_+^{-1/2}}{2}.$$

Following Gel'fand and Shilov [23], we define

$$(x_+^\lambda,\, \varphi(x)) = \int_0^\infty x^\lambda \left[\varphi(x) - \varphi(0) - \cdots - \frac{x^{n-1}}{(n-1)!}\varphi^{(n-1)}(0)\right] dx$$

where $-n-1 < \lambda < -n$. This implies for $n = 1$ that

$$(x_+^{-1.5}, \varphi(x)) = \int_0^\infty x^{-1.5}[\varphi(x) - \varphi(0)]dx.$$

Clearly, $x_+^{1.5} \in \mathcal{D}'(R^+)$ (the set of all distributions concentrated on R^+, which is a subspace of $\mathcal{D}'(R)$).

Assume that f and g are distributions in $\mathcal{D}'(R^+)$. Then the convolution $f * g$ is well defined by the equation [23]

$$((f * g)(x),\, \varphi(x)) = (g(x),\, (f(y),\, \varphi(x+y)))$$

for $\varphi \in \mathcal{D}(R)$. This also implies that

$$f * g = g * f \quad \text{and} \quad (f * g)' = g' * f = g * f'.$$

It follows from [23, 24, 25] that $\Phi_\lambda = \dfrac{x_+^{\lambda-1}}{\Gamma(\lambda)} \in \mathcal{D}'(R^+)$ is an entire analytic function of λ on the complex plane, and

$$\left.\frac{x_+^{\lambda-1}}{\Gamma(\lambda)}\right|_{\lambda=-n} = \delta^{(n)}(x), \quad \text{for} \quad n = 0, 1, 2, \ldots \tag{5}$$

which plays an important role in solving fractional differential equations by using the distributional convolutions. Let λ and μ be arbitrary numbers, then the following identity

$$\Phi_\lambda * \Phi_\mu = \Phi_{\lambda+\mu} \tag{6}$$

is satisfied [26].

Let λ be an arbitrary complex number and $g(x)$ be a distribution in $\mathcal{D}'(R^+)$. We define the primitive of order λ of g as the distributional convolution

$$g_\lambda(x) = g(x) * \frac{x_+^{\lambda-1}}{\Gamma(\lambda)} = g(x) * \Phi_\lambda. \tag{7}$$

Note that this is well defined since the distributions g and Φ_λ are in $\mathcal{D}'(R^+)$. We shall write the convolution

$$g_{-\lambda} = \frac{d^\lambda}{dx^\lambda} g = g(x) * \Phi_{-\lambda}$$

as the fractional derivative of the distribution g of order λ if Re$\lambda \geq 0$, and $\frac{d^\lambda}{dx^\lambda} g$ is interpreted as the fractional integral if Re$\lambda < 0$.

As an application, we are going to solve the following integral equation, which cannot be done in the classical sense, by the fractional derivatives of distributions.

Example 2.1 Let

$$g(x) = \begin{cases} 1 & \text{if } 0 \leq a < x < b, \\ 0 & \text{otherwise.} \end{cases}$$

Then the integral equation for $m = 0, 1, 2, \cdots$

$$g(x) = \int_0^x u(\tau)(x-\tau)^{m-0.5} d\tau$$

has the solution in the space $\mathcal{D}'(R^+)$

$$u(x) = (-1)^m \frac{(x-a)_+^{-m-0.5} - (x-b)_+^{-m-0.5}}{\pi}.$$

In fact, we have

$$g(x) = \frac{\Gamma(m+0.5)}{\Gamma(m+0.5)} \int_0^x u(\tau)(x-\tau)^{m-0.5} d\tau = \Gamma(m+0.5)(\Phi_{m+0.5} * u)(x).$$

This implies that

$$u(x) = \frac{1}{\Gamma(m+0.5)} (\Phi_{-m-0.5} * g)(x) = \frac{1}{\Gamma(m+0.5)} (\theta^{(m+0.5)}(x-a) - \theta^{(m+0.5)}(x-b))$$

by noting that $\Phi_0(x) = \delta(x)$ and $g(x) = \theta(x-a) - \theta(x-b)$.

Clearly,

$$\theta^{(m+0.5)}(x-a) = \frac{d^{m+0.5}}{dx^{m+0.5}} \theta(x-a) = \frac{d^{m+0.5}}{dx^{m+0.5}} \Phi_1(x-a) = \frac{(x-a)_+^{-m-0.5}}{\Gamma(-m+0.5)},$$
$$\theta^{(m+0.5)}(x-b) = \frac{(x-b)_+^{-m-0.5}}{\Gamma(-m+0.5)}$$

which derive that

$$u(x) = (-1)^m \frac{(x-a)_+^{-m-0.5} - (x-b)_+^{-m-0.5}}{\pi}$$

by

$$\Gamma(m+0.5)\Gamma(-m+0.5) = (-1)^m \pi$$

based on the well known formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin z\pi}.$$

In this paper, we shall extend the techniques used by Babenko in his book [27], for solving various types of fractional differential and integral equations in the classical sense, to distributions, and obtain the fractional Green's function for the first time in the distributional space $\mathcal{D}'(R^+)$ for the following fractional-order integro-differential equation with constant coefficients

$$a_n u^{(\beta_n)}(x) + a_{n-1} u^{(\beta_{n-1})}(x) + \cdots + a_1 u^{(\beta_1)}(x) + a_0 u^{(\beta_0)}(x) = g(x).$$

Our approach is much simpler than the classical one as it does not require any integral transforms along with complicated properties of the Mittag-Leffler function. Furthermore, the Green's function derived is much more general since it deals with generalized functions such as non-locally integrable function $g(x) = x_+^{-1.5}$, and goes back to the classical Green's formula when $g(x)$ has the well-defined Laplace transform. We would also like to add that Babenko's method itself is close to the Laplace transform method in the ordinary sense, but it can be used in more cases [4] such as solving integral or fractional differential equations with distributions whose Laplace transforms do not exist, as indicated below.

To illustrate Babenko's approach in detail, we consider the following Abel's integral equation of the second kind in the space $\mathcal{D}'(R^+)$ for $a \neq 0$ and $\lambda > -0.5$

$$x_+^{-1.5} = au(x) + \int_0^x (x-\tau)^\lambda u(\tau)d\tau.$$

We should point out at the beginning that this equation cannot be solved by the Laplace transform since the distribution $x_+^{-1.5}$ is not locally integrable. Clearly, it can be converted into

$$\begin{aligned} \frac{1}{a}x_+^{-1.5} &= u(x) + \frac{1}{a}\int_0^x (x-\tau)^\lambda u(\tau)d\tau = u(x) + \frac{\Gamma(\lambda+1)}{a\Gamma(\lambda+1)}\int_0^x (x-\tau)^\lambda u(\tau)d\tau \\ &= u(x) + \frac{\Gamma(\lambda+1)}{a}(\Phi_{\lambda+1} * u)(x) = \left(\delta + \frac{\Gamma(\lambda+1)}{a}\Phi_{\lambda+1}\right) * u. \end{aligned}$$

This implies that by Babenko's approach (differential or integral operators act like ordinary variables)

$$\begin{aligned} u(x) &= \frac{1}{a}\left(\delta+\frac{\Gamma(\lambda+1)}{a}\Phi_{\lambda+1}\right)^{-1} * x_+^{-1.5} \\ &= \frac{1}{a}\left(\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma^n(\lambda+1)}{a^n}\Phi_{\lambda+1}^n\right) * \frac{\Gamma(-0.5)x_+^{-1.5}}{\Gamma(-0.5)} \\ &= \frac{-2\sqrt{\pi}}{a}\left(\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma^n(\lambda+1)}{a^n}\Phi_{n(\lambda+1)}\right) * \Phi_{-0.5} \\ &= \frac{-2\sqrt{\pi}}{a}\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma^n(\lambda+1)}{a^n}\Phi_{n(\lambda+1)-0.5} \\ &= \frac{1}{a}x_+^{-1.5}+\frac{2\sqrt{\pi}\,\Gamma(\lambda+1)}{a^2}x_+^{\lambda-0.5}\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma^n(\lambda+1)}{a^n}\frac{x_+^{n(\lambda+1)}}{\Gamma(n(\lambda+1)+\lambda+0.5)} \\ &= \frac{1}{a}x_+^{-1.5}+\frac{2\sqrt{\pi}\,\Gamma(\lambda+1)}{a^2}x_+^{\lambda-0.5}E_{\lambda+1,\,\lambda+0.5}\left(\frac{-\Gamma(\lambda+1)x_+^{\lambda+1}}{a}\right). \end{aligned}$$

We note that $\frac{1}{a}x_+^{-1.5}$ is a singular distribution, while the term

$$\frac{2\sqrt{\pi}\,\Gamma(\lambda+1)}{a^2}x_+^{\lambda-0.5}E_{\lambda+1,\,\lambda+0.5}\left(\frac{-\Gamma(\lambda+1)x_+^{\lambda+1}}{a}\right)$$

is regular (locally integrable).

3 Three-term Green's function in distribution

We start with the three-term fractional order differential equation in the distributional space $\mathcal{D}'(R^+)$

$$au^{(\beta)}(x)+bu^{(\alpha)}(x)+cu(x)=\delta(x), \tag{8}$$

where $a\neq 0,\ \ \beta>\alpha$, and b, c are not zero simultaneously. Using equation (6), equation (8) can be converted into

$$\left(\delta+\frac{b}{a}\Phi_{\beta-\alpha}+\frac{c}{a}\Phi_{\beta}\right) * u(x)=\frac{1}{a}\Phi_{\beta}.$$

Using Babenko's approach, we obtain

$$
\begin{aligned}
u(x) &= \frac{1}{a}\frac{1}{\delta+\frac{b}{a}\Phi_{\beta-\alpha}+\frac{c}{a}\Phi_\beta} * \Phi_\beta \\
&= \frac{1}{a}\sum_{j=0}^{\infty}(-1)^j\left(\frac{b}{a}\Phi_{\beta-\alpha}+\frac{c}{a}\Phi_\beta\right)^j * \Phi_\beta \\
&= \frac{1}{a}\sum_{j=0}^{\infty}(-1)^j\sum_{i=0}^{j}\binom{j}{i}\left(\frac{b}{a}\right)^i\Phi_{i(\beta-\alpha)}\left(\frac{c}{a}\right)^{j-i} * \Phi_{(j-i)\beta} * \Phi_\beta \\
&= \frac{1}{a}\sum_{j=0}^{\infty}\sum_{i=0}^{j}(-1)^j\binom{j}{i}\left(\frac{b}{a}\right)^i\left(\frac{c}{a}\right)^{j-i}\Phi_{i(\beta-\alpha)+(j-i)\beta+\beta}.
\end{aligned}
$$

Using the following formula

$$\sum_{p=0}^{\infty}\sum_{q=0}^{p}a_{q,\,p-q}=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}a_{k,\,j}$$

we drive that

$$
\begin{aligned}
u(x) &= \frac{1}{a}\sum_{j=0}^{\infty}\sum_{i=0}^{j}(-1)^{j-i+i}\frac{(j-i+i)!}{i!(j-i)!}\left(\frac{b}{a}\right)^i\left(\frac{c}{a}\right)^{j-i}\Phi_{i(\beta-\alpha)+(j-i)\beta+\beta} \\
&= \frac{1}{a}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{j+k}\frac{(j+k)!}{k!\,j!}\left(\frac{b}{a}\right)^k\left(\frac{c}{a}\right)^j\Phi_{k(\beta-\alpha)+j\beta+\beta} \\
&= \frac{1}{a}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\left(\frac{-b}{a}\right)^k(-1)^j\left(\frac{c}{a}\right)^j\frac{(j+k)!}{k!\,j!}\Phi_{(k+j)\beta+\beta-k\alpha} \\
&= \frac{1}{a}\sum_{j=0}^{\infty}\left(\frac{-b}{a}\right)^j\sum_{k=0}^{\infty}(-1)^k\left(\frac{c}{a}\right)^k\frac{(j+k)!}{k!\,j!}\Phi_{(k+j)\beta+\beta-j\alpha} \\
&= \frac{1}{a}\sum_{j=0}^{\infty}\left(\frac{-b}{a}\right)^j\sum_{k=0}^{\infty}\left(\frac{-c}{a}\right)^k\frac{(j+k)!}{k!\,j!}\Phi_{(k+j)\beta+\beta-j\alpha} \\
&= \frac{1}{a}\,x_+^{\beta-1}\sum_{j=0}^{\infty}\left(\frac{-b}{a}\right)^j\frac{x_+^{(\beta-\alpha)j}}{j!}\sum_{k=0}^{\infty}\left(\frac{-c}{a}\right)^k\frac{(j+k)!}{k!\,\Gamma(k\beta+\beta+(\beta-\alpha)j)}x_+^{k\beta}. \qquad (9)
\end{aligned}
$$

Clearly, if $b=0$ then the fractional differential equation for nonzero a and c

$$au^{(\beta)}(x)+cu(x)=\delta(x) \qquad (10)$$

has the solution

$$
\begin{aligned}
u(x) &= \frac{1}{a}\,x_+^{\beta-1}\sum_{k=0}^{\infty}\left(\frac{-c}{a}\right)^k\frac{x_+^{k\beta}}{\Gamma(k\beta+\beta)} \\
&= \frac{1}{a}\,x_+^{\beta-1}E_{\beta,\,\beta}(-cx_+^{\beta}/a)
\end{aligned}
$$

where $\beta > 0$.

Moreover, if $\beta < 0$ then we have

$$\left(\frac{a}{c}\Phi_{-\beta} + \delta\right) * u = \frac{1}{c}\delta(x)$$

from equation (10). By Babenko's approach, we come to

$$\begin{aligned} u(x) &= \frac{1}{c}\frac{1}{\frac{a}{c}\Phi_{-\beta} + \delta} = \frac{1}{c}\sum_{k=0}^{\infty}(-1)^k\left(\frac{a}{c}\right)^k\Phi_{-k\beta} \\ &= \frac{1}{c}\sum_{k=0}^{\infty}(-1)^k\left(\frac{a}{c}\right)^k\frac{x_+^{-k\beta-1}}{\Gamma(-k\beta)} \\ &= \frac{1}{c}\delta(x) + \frac{1}{c}\sum_{k=0}^{\infty}(-1)^{k+1}\left(\frac{a}{c}\right)^{k+1}\frac{x_+^{-(k+1)\beta-1}}{\Gamma(-(k+1)\beta)} \\ &= \frac{1}{c}\delta(x) - \frac{a}{c^2}x_+^{-\beta-1}E_{-\beta,\,-\beta}(-ax_+^{-\beta}/c) \end{aligned}$$

which is the solution for equation (10).

On the other hand, if $c = 0$ then the fractional differential equation for nonzero a and b

$$au^{(\beta)}(x) + bu^{(\alpha)}(x) = \delta(x), \quad \beta > \alpha \tag{11}$$

has the solution

$$\begin{aligned} u(x) &= \frac{1}{a}\,x_+^{\beta-1}\sum_{j=0}^{\infty}\left(\frac{-b}{a}\right)^j\frac{x_+^{j(\beta-\alpha)}}{\Gamma((\beta-\alpha)j+\beta)} \\ &= \frac{1}{a}\,x_+^{\beta-1}\,E_{\beta-\alpha,\,\beta}(-bx_+^{\beta-\alpha}/a). \end{aligned}$$

This implies that the fractional and integral equation (mixed type) for $\beta, \alpha > 0$

$$au^{(\beta)}(x) + \frac{b}{\Gamma(\alpha)}\int_0^x (x-\tau)^{\alpha-1}u(\tau)d\tau = \delta(x) \tag{12}$$

where a and b are nonzero, has the solution in the space $\mathcal{D}'(R^+)$

$$u(x) = \frac{1}{a}\,x_+^{\beta-1}\,E_{\beta+\alpha,\,\beta}(-bx_+^{\beta+\alpha}/a).$$

We would like to mention that the above approach to finding the fractional Green's function is much easier than the existing classical method, which uses the inverse Laplace transform, as well as complex properties of the Mittag-Leffler function with its derivatives [4]. Furthermore, we derive the Green's function in distribution first in the space $\mathcal{D}'(R^+)$, which is an extension of the classical Green's function as it deals with generalized functions rather than only ordinary functions in the normal sense. We also note that the Laplace transform works for equations (10), (11) and (12), but it fails if we replace the right-hand side function $\delta(x)$ by a more singular distribution, such as $\Phi_{-0.75}(x)$. Clearly, our method is workable in this case.

For example, the fractional differential equation

$$au^{(2)}(x) + bu^{(\alpha)}(x) = \Phi_{-0.75}(x), \quad \alpha < 2$$

has the solution

$$\begin{aligned} u(x) &= \frac{1}{a}\sum_{j=0}^{\infty}\left(\frac{-b}{a}\right)^j \frac{x_+^{j(2-\alpha)+1}}{\Gamma(j(2-\alpha)+2)} * \Phi_{-0.75}(x) \\ &= \frac{1}{a}x_+^{0.25}E_{2-\alpha,\,1.25}(-bx_+^{2-\alpha}/a). \end{aligned}$$

We shall show that the double series in equation (9) is convergent in the next section, which deals with the general Green's function.

Theorem 3.1 Assume that b and c are not zero simultaneously. Then the fractional differential and integral equation

$$au^{(2)}(x) + bu^{(\alpha)}(x) + cu(x) = \delta(x), \quad (a \neq 0),$$

where $\alpha \leq 1$ and $\alpha \neq 0$, has the convergent solution

$$u(x) = \frac{x_+}{a}\sum_{j=0}^{\infty}\left(\frac{-b}{a}\right)^j \frac{x_+^{(2-\alpha)j}}{j!}\sum_{k=0}^{\infty}\left(\frac{-c}{a}\right)^k \frac{(j+k)!}{k!\,\Gamma(2k+(2-\alpha)j+2)}x_+^{2k} \tag{13}$$

in the distributional space $\mathcal{D}'(R^+)$.

Proof. By equation (9), we have equation (13) for $\beta = 2$ and $0 \neq \alpha \leq 1$. We can directly show that this double series is convergent. Note that

$$\left|\frac{(j+k)!}{\Gamma(2k+2+(2-\alpha)j)}\right| = \left|\frac{\Gamma(j+k+1)}{\Gamma(2k+2+(2-\alpha)j)}\right| \leq 1$$

since $2 - \alpha \geq 1$.

Furthermore, both

$$\sum_{j=0}^{\infty}\frac{\left(\frac{-b}{a}\right)^j}{j!}x_+^{(2-\alpha)j}, \quad \text{and} \quad \sum_{k=0}^{\infty}\frac{\left(\frac{-c}{a}\right)^k}{k!}x_+^{2k}$$

absolutely converge by the ratio test. Therefore,

$$u(x) = \frac{x_+}{a}\sum_{j=0}^{\infty}\left(\frac{-b}{a}\right)^j \frac{x_+^{(2-\alpha)j}}{j!}\sum_{k=0}^{\infty}\left(\frac{-c}{a}\right)^k \frac{(j+k)!}{k!\,\Gamma(2k+(2-\alpha)j+2)}x_+^{2k}$$

is convergent. This completes the proof of Theorem 3.1. □

From the proof of Theorem 3.1, we can see that if $\beta - \alpha \geq 1$ and $\alpha > 0$ then the fractional differential equation

$$au^{(\beta)}(x) + bu^{(\alpha)}(x) + cu(x) = \delta(x)$$

has the convergent solution

$$u(x) = \frac{1}{a}\, x_+^{\beta-1} \sum_{j=0}^{\infty} \left(\frac{-b}{a}\right)^j \frac{x_+^{(\beta-\alpha)j}}{j!} \sum_{k=0}^{\infty} \left(\frac{-c}{a}\right)^k \frac{(j+k)!}{k!\,\Gamma(k\beta+\beta+(\beta-\alpha)j)} x_+^{k\beta}.$$

This result contains Caputo's work of $\beta = 2$ and $0 < \alpha < 1$ as a special case [28].

In particular, the fractional differential equation for $0 < \alpha \leq 1$

$$u^{(2)}(x) - u^{(\alpha)}(x) - u(x) = x_+^{-1.5}$$

can only be solved in the distributional sense. Indeed,

$$x_+^{-1.5} = -2\sqrt{\pi}\frac{x_+^{-1.5}}{\Gamma(-0.5)} = -2\sqrt{\pi}\Phi_{-0.5}.$$

Hence the solution

$$\begin{aligned} u(x) &= -2\sqrt{\pi}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(j+k)!}{k!\,j!}\Phi_{2(k+j)+1.5-j\alpha}(x) \\ &= -2\sqrt{\pi}\, x_+^{0.5} \sum_{j=0}^{\infty} \frac{1}{j!}\, x_+^{(2-\alpha)j} \sum_{k=0}^{\infty} \frac{(j+k)!}{k!\,\Gamma(2k+(2-\alpha)j+1.5)}\, x_+^{2k} \end{aligned}$$

is absolutely convergent by Theorem 3.1.

Using the formula

$$\Gamma(n+0.5) = \frac{(2n)!}{4^n n!}\sqrt{\pi},$$

we can infer the solution for the differential equation

$$aD^2u(x) + bDu(x) + cu(x) = x_+^{-1.5},$$

where a, b and c are all nonzero constants, as

$$u(x) = -\frac{2}{a}x_+^{0.5} \sum_{j=0}^{\infty} \left(\frac{-b}{a}\right)^j \frac{x_+^j}{j!} \sum_{k=0}^{\infty} \left(\frac{-c}{a}\right)^k \frac{(j+k)!\,4^{2k+j+1}(2k+j+1)!}{k!\,(4k+2j+2)!} x_+^{2k}.$$

In particular, we have the solution

$$u(x) = -\frac{2}{a}x_+^{0.5} \sum_{j=0}^{\infty} \left(\frac{-b}{a}\right)^j \frac{4^{j+1}(j+1)!}{(2j+2)!} x_+^j = -\frac{2\sqrt{\pi}\,x_+^{0.5}}{a}\, E_{1,\,1.5}(-bx_+/a)$$

for the differential equation ($a \neq 0$)

$$aD^2u(x) + bDu(x) = x_+^{-1.5},$$

which is not doable in the classical sense.

By equation (9), we have the following theorem for $\beta = 2$ and $\alpha = 1.5$.

Theorem 3.2 The fractional differential equation

$$au^{(2)}(x) + bu^{(1.5)}(x) + cu(x) = \delta(x)$$

has the convergent solution in the space $\mathcal{D}'(R^+)$

$$u(x) = \frac{1}{a}\, x_+ \sum_{j=0}^{\infty} \left(\frac{-b}{a}\right)^j \frac{x_+^{0.5j}}{j!} \sum_{k=0}^{\infty} \left(\frac{-c}{a}\right)^k \frac{(j+k)!}{k!\,\Gamma(2k+2+0.5j)} x_+^{2k}. \tag{14}$$

Moreover, if $\beta \geq 2$ then the fractional differential equation

$$au^{(\beta)}(x) + bu^{(1.5)}(x) + cu(x) = \delta(x)$$

has the convergent solution

$$u(x) = \frac{1}{a}\, x_+^{\beta-1} \sum_{j=0}^{\infty} \left(\frac{-b}{a}\right)^j \frac{x_+^{(\beta-1.5)j}}{j!} \sum_{k=0}^{\infty} \left(\frac{-c}{a}\right)^k \frac{(j+k)!}{k!\,\Gamma(k\beta+\beta+(\beta-1.5)j)} x_+^{k\beta}.$$

This result contains Bagley and Torvik's work of $\beta = 2$ and $\alpha = 1.5$ as a special case [29], which is in the classical sense.

4 General Green's function

In order to obtain Green's function for the n-term of fractional integro-differential equation with constant coefficients ($a_n \neq 0$ and $\beta_n > \cdots > \beta_0$ with $\beta_n > 0$)

$$a_n u^{(\beta_n)}(x) + a_{n-1}u^{(\beta_{n-1})}(x) + \cdots + a_1 u^{(\beta_1)}(x) + a_0 u^{(\beta_0)}(x) = \delta(x), \tag{15}$$

we need the multinomial theorem which states as

$$(x_1 + x_2 + \cdots + x_m)^k = \sum_{k_1+k_2+\cdots+k_m=k} \binom{k}{k_1,\, k_2,\, \cdots,\, k_m} x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m}$$

where the summation is taken over all sequences of nonnegative integer indices k_1 through k_m such as the sum of all k_i is k. The coefficients $\binom{k}{k_1,\, k_2,\, \cdots,\, k_m}$ are known as multinomial coefficients, and are defined by the formula

$$\binom{k}{k_1,\, k_2,\, \cdots,\, k_m} = \frac{k!}{k_1!\, k_2! \cdots k_m!}.$$

Combinatorially, the multinomial coefficient $\binom{k}{k_1,\, k_2,\, \cdots,\, k_m}$ counts the number of different ways to partition an n-element set into disjoint subsets of sizes $k_1,\, k_2,\, \cdots, k_m$.

The substitution of $x_i = 1$ for all i into the multinomial theorem implies that

$$\sum_{k_1+k_2+\cdots+k_m=k} \binom{k}{k_1,\, k_2,\, \cdots,\, k_m} = m^k.$$

The number of terms in the multinomial theorem is

$$\binom{k+m-1}{m-1}$$

which is equal to the number of monomials of degree k on the variables $x_1, \cdots, x_m$.

Using equation (6), equation (15) can be converted into

$$\left(\delta + \frac{a_{n-1}}{a_n}\Phi_{\beta_n-\beta_{n-1}} + \cdots + \frac{a_1}{a_n}\Phi_{\beta_n-\beta_1} + \frac{a_0}{a_n}\Phi_{\beta_n-\beta_0}\right) * u(x) = \frac{1}{a_n}\Phi_{\beta_n}.$$

Using Babenko's approach, we derive by the multinomial theorem that

$$\begin{aligned}
u(x) &= \frac{1}{a_n}\,\frac{1}{\delta + \frac{a_{n-1}}{a_n}\Phi_{\beta_n-\beta_{n-1}} + \cdots + \frac{a_1}{a_n}\Phi_{\beta_n-\beta_1} + \frac{a_0}{a_n}\Phi_{\beta_n-\beta_0}} * \Phi_{\beta_n} \\
&= \frac{1}{a_n}\sum_{k=0}^{\infty}(-1)^k\left(\frac{a_{n-1}}{a_n}\Phi_{\beta_n-\beta_{n-1}} + \cdots + \frac{a_1}{a_n}\Phi_{\beta_n-\beta_1} + \frac{a_0}{a_n}\Phi_{\beta_n-\beta_0}\right)^k * \Phi_{\beta_n} \\
&= \frac{1}{a_n}\sum_{k=0}^{\infty}(-1)^k \sum_{k_1+k_2+\cdots+k_n=k}\binom{k}{k_1,\, k_2,\, \cdots,\, k_n}\left(\frac{a_{n-1}}{a_n}\right)^{k_1}\Phi^{k_1}_{\beta_n-\beta_{n-1}} \\
&\quad * \left(\frac{a_{n-2}}{a_n}\right)^{k_2}\Phi^{k_2}_{\beta_n-\beta_{n-2}} * \cdots * \left(\frac{a_1}{a_n}\right)^{k_{n-1}}\Phi^{k_{n-1}}_{\beta_n-\beta_1} * \left(\frac{a_0}{a_n}\right)^{k_n}\Phi^{k_n}_{\beta_n-\beta_0} * \Phi_{\beta_n} \\
&= \frac{1}{a_n}\sum_{k=0}^{\infty}\frac{(-1)^k}{a_n^k}\sum_{k_1+k_2+\cdots+k_n=k}\binom{k}{k_1,\, k_2,\, \cdots,\, k_n} a_{n-1}^{k_1}a_{n-2}^{k_2}\cdots a_0^{k_n} \\
&\quad \times \Phi_{k_1(\beta_n-\beta_{n-1})+k_2(\beta_n-\beta_{n-2})+\cdots+k_n(\beta_n-\beta_0)+\beta_n} \\
&= \frac{x_+^{\beta_n-1}}{a_n}\sum_{k=0}^{\infty}\frac{(-1)^k}{a_n^k}\sum_{k_1+k_2+\cdots+k_n=k}\binom{k}{k_1,\, k_2,\, \cdots,\, k_n} a_{n-1}^{k_1}a_{n-2}^{k_2}\cdots a_0^{k_n} \\
&\quad \times \frac{x_+^{k_1(\beta_n-\beta_{n-1})+k_2(\beta_n-\beta_{n-2})+\cdots+k_n(\beta_n-\beta_0)}}{\Gamma(k_1(\beta_n-\beta_{n-1})+k_2(\beta_n-\beta_{n-2})+\cdots+k_n(\beta_n-\beta_0)+\beta_n)} \\
&= \frac{x_+^{\beta_n-1}}{a_n}\sum_{k=0}^{\infty}\frac{(-1)^k}{a_n^k}\sum_{k_1+k_2+\cdots+k_n=k}\frac{a_{n-1}^{k_1}a_{n-2}^{k_2}\cdots a_0^{k_n}}{k_1!\,k_2!\,\cdots\,k_n!} \\
&\quad \times \frac{k!\;x_+^{k_1(\beta_n-\beta_{n-1})+k_2(\beta_n-\beta_{n-2})+\cdots+k_n(\beta_n-\beta_0)}}{\Gamma(k_1(\beta_n-\beta_{n-1})+k_2(\beta_n-\beta_{n-2})+\cdots+k_n(\beta_n-\beta_0)+\beta_n)}.
\end{aligned} \tag{16}$$

We would like to mention that one obtains Green's function in the classical sense [4] for equation (15) by complicated inverse operations of the Laplace transform as well as the Mittage-Leffler functions, as

$$\begin{aligned} u(x) &= \frac{1}{a_n}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\sum_{k_2+k_3+\cdots+k_n=k}\binom{k}{k_2,\, k_3,\, \cdots ,\, k_n} \\ &\quad \times \Pi_{i=2}^{n}\left(\frac{a_{n-i}}{a_n}\right)^{ki} x_+^{(\beta_n-\beta_{n-1})k+\beta_n+\sum_{j=2}^{n}(\beta_{n-1}-\beta_{n-j})\,k_j-1} \\ &\quad \times E^{(k)}_{\beta_n-\beta_{n-1},\,\beta_n+\sum_{j=2}^{n}(\beta_{n-1}-\beta_{n-j})\,k_j}\left(-\frac{a_{n-1}}{a_n}x_+^{\beta_n-\beta_{n-1}}\right). \end{aligned} \tag{17}$$

Comparing the above classical Green's function, our result is not only much easier to deduce but also simpler in structure, without involving any Mittag-Leffler functions and their derivatives at all. Furthermore, it can deal with distributions instead of only ordinary functions. Using equation (4) we can see

$$\begin{aligned} & E^{(k)}_{\beta_n-\beta_{n-1},\,\beta_n+\sum_{j=2}^{n}(\beta_{n-1}-\beta_{n-j})k_j}\left(-\frac{a_{n-1}}{a_n}x^{\beta_n-\beta_{n-1}}\right) \\ &= \sum_{k_1=0}^{\infty}\frac{(k_1+k)!\left(-\frac{a_{n-1}}{a_n}x^{\beta_n-\beta_{n-1}}\right)^{k_1}}{\Gamma(k_1(\beta_n-\beta_{n-1})+(\beta_n-\beta_{n-1})k+\beta_n+\sum_{j=2}^{n}(\beta_{n-1}-\beta_{n-j})k_j)}. \end{aligned}$$

This implies that equations (16) and (17) are equivalent. Indeed, let $k_2+\cdots k_n=l$ for multi-index $k=(k_1,k_2,\cdots,k_n)$. Then $k=|k|=k_1+l$. We can change equation (16) into

$$\begin{aligned} u(x) &= \frac{1}{a_n}\sum_{k=0}^{\infty}\frac{(-1)^k}{a_n^k}\sum_{k_1+k_2+\cdots+k_n=k}\frac{a_{n-1}^{k_1}a_{n-2}^{k_2}\cdots a_0^{k_n}}{k_1!\,k_2!\,\cdots\,k_n!} \\ &\quad \times\frac{k!\,x_+^{k_1(\beta_n-\beta_{n-1})+k_2(\beta_n-\beta_{n-2})+\cdots+k_n(\beta_n-\beta_0)+\beta_n-1}}{\Gamma(k_1(\beta_n-\beta_{n-1})+k_2(\beta_n-\beta_{n-2})+\cdots+k_n(\beta_n-\beta_0)+\beta_n)} \\ &= \frac{1}{a_n}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\sum_{k_1=0}^{\infty}(-1)^{k_1}\sum_{k_2+k_3+\cdots+k_n=l}\frac{l!\,(k_1+l)!\,a_{n-1}^{k_1}a_{n-2}^{k_2}\cdots a_0^{k_n}}{a_n^{k_1+l}\,k_1!\,k_2!\,\cdots\,k_n!} \\ &\quad \times\frac{x_+^{k_1(\beta_n-\beta_{n-1})+k_2(\beta_n-\beta_{n-2})+\cdots+k_n(\beta_n-\beta_0)+\beta_n-1}}{\Gamma(k_1(\beta_n-\beta_{n-1})+k_2(\beta_n-\beta_{n-2})+\cdots+k_n(\beta_n-\beta_0)+\beta_n)}. \end{aligned}$$

Using the identity

$$\begin{aligned} & k_1(\beta_n-\beta_{n-1})+k_2(\beta_n-\beta_{n-2})+\cdots+k_n(\beta_n-\beta_0)+\beta_n \\ =\; & k_1(\beta_n-\beta_{n-1})+(\beta_n-\beta_{n-1})l+\beta_n+\sum_{j=2}^{n}(\beta_{n-1}-\beta_{n-j})k_j \end{aligned}$$

and changing l into k, we complete the proof.

Our main theorem is stated as follows.

Theorem 4.1 The solution $u(x)$ given in equation (16) is continuous on the interval $[0, \infty)$ if $\beta_n \geq 1$, and belongs to $L(0, T)$ for all $T > 0$ if $0 < \beta_n < 1$.

Proof. Clearly,

$$\begin{aligned} |u(x)| &\leq \frac{x_+^{\beta_n - 1}}{|a_n|} \sum_{k=0}^{\infty} \frac{1}{|a_n|^k} \sum_{k_1 + k_2 + \cdots + k_n = k} \frac{|a_{n-1}|^{k_1} |a_{n-2}|^{k_2} \cdots |a_0|^{k_n}}{k_1! \, k_2! \, \cdots \, k_n!} \\ &\quad \times \frac{k! \, x_+^{k_1(\beta_n - \beta_{n-1}) + k_2(\beta_n - \beta_{n-2}) + \cdots + k_n(\beta_n - \beta_0)}}{\Gamma(k_1(\beta_n - \beta_{n-1}) + k_2(\beta_n - \beta_{n-2}) + \cdots + k_n(\beta_n - \beta_0) + \beta_n)} \\ &= \frac{x_+^{\beta_n - 1}}{|a_n|} \sum_{k=0}^{\infty} \sum_{k_1 + k_2 + \cdots + k_n = k} \frac{k! \left| \frac{a_{n-1}}{a_n} \right|^{k_1} \left| \frac{a_{n-2}}{a_n} \right|^{k_2} \cdots \left| \frac{a_0}{a_n} \right|^{k_n}}{k_1! \, k_2! \, \cdots \, k_n!} \\ &\quad \times \frac{x_+^{k_1(\beta_n - \beta_{n-1}) + k_2(\beta_n - \beta_{n-2}) + \cdots + k_n(\beta_n - \beta_0)}}{\Gamma(k_1(\beta_n - \beta_{n-1}) + k_2(\beta_n - \beta_{n-2}) + \cdots + k_n(\beta_n - \beta_0) + \beta_n)}. \end{aligned}$$

For all $x \in [0, T]$ (where T is positive), we come to

$$\begin{aligned} u(x) &\leq \frac{T^{\beta_n - 1}}{|a_n|} \sum_{k=0}^{\infty} \sum_{k_1 + k_2 + \cdots + k_n = k} \frac{k! \left| \frac{a_{n-1}}{a_n} \right|^{k_1} \left| \frac{a_{n-2}}{a_n} \right|^{k_2} \cdots \left| \frac{a_0}{a_n} \right|^{k_n}}{k_1! \, k_2! \, \cdots \, k_n!} \\ &\quad \times \frac{T^{k_1(\beta_n - \beta_{n-1}) + k_2(\beta_n - \beta_{n-2}) + \cdots + k_n(\beta_n - \beta_0)}}{\Gamma(k_1(\beta_n - \beta_{n-1}) + k_2(\beta_n - \beta_{n-2}) + \cdots + k_n(\beta_n - \beta_0) + \beta_n)} \\ &= \frac{T_+^{\beta_n - 1}}{|a_n|} E_{(\beta_n - \beta_{n-1}, \cdots, \beta_n - \beta_0), \, \beta_n} \left(\left| \frac{a_{n-1}}{a_n} \right| T^{\beta_n - \beta_{n-1}}, \cdots, \left| \frac{a_0}{a_n} \right| T^{\beta_n - \beta_0} \right), \end{aligned}$$

where

$$E_{(\beta_n - \beta_{n-1}, \cdots, \beta_n - \beta_0), \, \beta_n} \left(\left| \frac{a_{n-1}}{a_n} \right| T^{\beta_n - \beta_{n-1}}, \cdots, \left| \frac{a_0}{a_n} \right| T^{\beta_n - \beta_0} \right)$$

is the value at $z_1 = \left| \frac{a_{n-1}}{a_n} \right| T^{\beta_n - \beta_{n-1}}, \cdots, z_n = \left| \frac{a_0}{a_n} \right| T^{\beta_n - \beta_0}$ of the multivariate Mittag-Leffler function $E_{(\beta_n - \beta_{n-1}, \cdots, \beta_n - \beta_0), \, \beta_n}(z_1, \cdots, z_n)$ given in [2]. This implies that the solution $u(x)$ given in equation (16) is absolutely and uniformly convergent on the interval $[0, T]$. Since T is arbitrary,

$u(x)$ is continuous on $[0,\infty)$. Assume $0<\beta_n<1$. Then for all $T>0$,

$$\begin{aligned}\int_0^T |u(x)|dx &\leq \frac{1}{|a_n|}\sum_{k=0}^{\infty}\sum_{k_1+k_2+\cdots+k_n=k}\frac{k!\,\left|\frac{a_{n-1}}{a_n}\right|^{k_1}\left|\frac{a_{n-2}}{a_n}\right|^{k_2}\cdots\left|\frac{a_0}{a_n}\right|^{k_n}}{k_1!\,k_2!\,\cdots\,k_n!}\\ &\quad\times\int_0^T\frac{x^{\beta_n-1+k_1(\beta_n-\beta_{n-1})+k_2(\beta_n-\beta_{n-2})+\cdots+k_n(\beta_n-\beta_0)}dx}{\Gamma(k_1(\beta_n-\beta_{n-1})+k_2(\beta_n-\beta_{n-2})+\cdots+k_n(\beta_n-\beta_0)+\beta_n)}\\ &= \frac{1}{|a_n|}\sum_{k=0}^{\infty}\sum_{k_1+k_2+\cdots+k_n=k}\frac{k!\,\left|\frac{a_{n-1}}{a_n}\right|^{k_1}\left|\frac{a_{n-2}}{a_n}\right|^{k_2}\cdots\left|\frac{a_0}{a_n}\right|^{k_n}}{k_1!\,k_2!\,\cdots\,k_n!}\\ &\quad\times\frac{T^{\beta_n+k_1(\beta_n-\beta_{n-1})+k_2(\beta_n-\beta_{n-2})+\cdots+k_n(\beta_n-\beta_0)}dx}{\Gamma(k_1(\beta_n-\beta_{n-1})+k_2(\beta_n-\beta_{n-2})+\cdots+k_n(\beta_n-\beta_0)+\beta_n+1)}\\ &= \frac{T^{\beta_n}}{|a_n|}E_{(\beta_n-\beta_{n-1},\cdots,\beta_n-\beta_0),\,\beta_n+1}\left(\left|\frac{a_{n-1}}{a_n}\right|T^{\beta_n-\beta_{n-1}},\cdots,\left|\frac{a_0}{a_n}\right|T^{\beta_n-\beta_0}\right),\end{aligned}$$

which is a finite value. This completes the proof of Theorem 4.1. □

Using representation of the integral equation and successive approximations, Kim and O [18] also obtained equation (16) under certain initial conditions in the classical sense, which does not deal with the equation like

$$a_n u^{(\beta_n)}(x)+a_{n-1}u^{(\beta_{n-1})}(x)+\cdots+a_1u^{(\beta_1)}(x)+a_0u^{(\beta_0)}(x)=x_+^{-1.2}.$$

As an application, we can solve the following fractional differential equation

$$u^{(4.5)}(x)+u^{(3.5)}(x)+u^{(2.5)}(x)+u^{(1.5)}(x)+u^{(0.5)}(x)=\Phi_{-3.5}(x)$$

by equation (16), but the classical result fails to do so since the Laplace transform of $\Phi_{-3.5}(x)$ does not exist.

Indeed,

$$\begin{aligned}u(x) &= \sum_{k=0}^{\infty}(-1)^k\sum_{k_1+k_2+k_3+k_4=k}\binom{k}{k_1,\,k_2,\,k_3,\,k_4}\Phi_{k_1+2k_2+3k_3+4k_4+4.5}*\Phi_{-3.5}\\ &= \sum_{k=0}^{\infty}(-1)^k k!\sum_{k_1+k_2+k_3+k_4=k}\frac{x_+^{k_1+2k_2+3k_3+4k_4}}{k_1!\,k_2!k_3!\,k_4!\,(k_1+2k_2+3k_3+4k_4)!}\end{aligned}$$

which is clearly convergent as

$$\left|\frac{k!}{(k_1+2k_2+3k_3+4k_4)!}\right|\leq 1$$

and

$$\sum_{k_1+k_2+k_3+k_4=k}\frac{x_+^{k_1+2k_2+3k_3+4k_4}}{k_1!\,k_2!k_3!\,k_4!}$$

has a finite number of terms with factorials as denominators.

We let $\delta_{i,j}$ be the Kronecker delta function defined as

$$\delta_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Then the multinomial theorem can be restated as

$$(x_1 + x_2 + \cdots + x_m)^k = \sum_{k_1=0}^{k} \sum_{k_2=0}^{k} \cdots \sum_{k_m=0}^{k} \delta_{k_1+k_2+\cdots+k_m,\, k} \binom{k}{k_1,\, k_2,\, \cdots,\, k_m} x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m}.$$

Hence, the solution of equation (15) can be written as

$$\begin{aligned} u(x) &= \frac{x_+^{\beta_n-1}}{a_n} \sum_{k=0}^{\infty} \frac{(-1)^k}{a_n^k} \sum_{k_1=0}^{k} \sum_{k_2=0}^{k} \cdots \sum_{k_n=0}^{k} \delta_{k_1+k_2+\cdots+k_n,\, k} \frac{a_{n-1}^{k_1} a_{n-2}^{k_2} \cdots a_0^{k_n}}{k_1!\, k_2! \cdots k_n!} \\ &\quad \times \frac{k!\, x_+^{k_1(\beta_n-\beta_{n-1})+k_2(\beta_n-\beta_{n-2})+\cdots+k_n(\beta_n-\beta_0)}}{\Gamma(k_1(\beta_n-\beta_{n-1})+k_2(\beta_n-\beta_{n-2})+\cdots+k_n(\beta_n-\beta_0)+\beta_n)}. \end{aligned}$$

We would like to mention that if $0 \ge \beta_n > \cdots > \beta_0$, then equation (15) is an integral equation and has the solution

$$\begin{aligned} u(x) &= \frac{1}{a_n} \sum_{k=0}^{\infty} \frac{(-1)^k}{a_n^k} \sum_{k_1=0}^{k} \sum_{k_2=0}^{k} \cdots \sum_{k_n=0}^{k} \delta_{k_1+k_2+\cdots+k_n,\, k} \frac{a_{n-1}^{k_1} a_{n-2}^{k_2} \cdots a_0^{k_n}}{k_1!\, k_2! \cdots k_n!} \\ &\quad \times \frac{k!\, x_+^{k_1(\beta_n-\beta_{n-1})+k_2(\beta_n-\beta_{n-2})+\cdots+k_n(\beta_n-\beta_0)+\beta_n-1}}{\Gamma(k_1(\beta_n-\beta_{n-1})+k_2(\beta_n-\beta_{n-2})+\cdots+k_n(\beta_n-\beta_0)+\beta_n)} \end{aligned}$$

by noting that $x_+^{\beta_n-1}$ cannot be put in the front as it is not defined when $\beta_n = -1, -2, \cdots$ in distribution.

The above solution can further be written into two parts: $u(x) =$ a singular distribution + a locally integrable function. We use the following example to illustrate this in detail.

Consider the integral equation

$$u^{(-0.5)}(x) + u^{(-1.5)}(x) + u^{(-2)}(x) = \delta(x).$$

Then the solution is

$$\begin{aligned} u(x) &= \sum_{k=0}^{\infty} (-1)^k \sum_{k_1+k_2=k} \frac{k!}{k_1!\, k_2!} \frac{x_+^{-1.5+k_1+1.5k_2}}{\Gamma(k_1+1.5k_2-0.5)} \\ &= \frac{x_+^{-1.5}}{\Gamma(-0.5)} + \sum_{k=1}^{\infty} (-1)^k \sum_{k_1+k_2=k} \frac{k!}{k_1!\, k_2!} \frac{x_+^{-1.5+k_1+1.5k_2}}{\Gamma(k_1+1.5k_2-0.5)}. \end{aligned}$$

Clearly,

$$\frac{x_+^{-1.5}}{\Gamma(-0.5)} = \delta^{(0.5)}(x)$$

is a singular distribution, while

$$\sum_{k=1}^{\infty}(-1)^k \sum_{k_1+k_2=k} \frac{k!}{k_1!\,k_2!} \frac{x_+^{-1.5+k_1+1.5k_2}}{\Gamma(k_1+1.5k_2-0.5)}$$

is locally integrable as the function $x_+^{-0.5}$ is locally integrable.

To prove

$$u(x) = \text{a singular distribution} + \text{a locally integrable function}$$

in general. Choose the minimum k such that

$$\min_{k\geq 1}\{k_1(\beta_n-\beta_{n-1}) + k_2(\beta_n-\beta_{n-2}) + \cdots + k_n(\beta_n-\beta_0) + \beta_n - 1\} > -1,$$

and denote it as k_0. Then,

$$\begin{aligned} u(x) \;=\;& \frac{1}{a_n}\sum_{k=0}^{k_0-1}\frac{(-1)^k}{a_n^k}\sum_{k_1=0}^{k}\sum_{k_2=0}^{k}\cdots\sum_{k_n=0}^{k}\delta_{k_1+k_2+\cdots+k_n,\,k}\frac{a_{n-1}^{k_1}a_{n-2}^{k_2}\cdots a_0^{k_n}}{k_1!\,k_2!\,\cdots\,k_n!} \\ &\times\frac{k!\;x_+^{k_1(\beta_n-\beta_{n-1})+k_2(\beta_n-\beta_{n-2})+\cdots+k_n(\beta_n-\beta_0)+\beta_n-1}}{\Gamma(k_1(\beta_n-\beta_{n-1})+k_2(\beta_n-\beta_{n-2})+\cdots+k_n(\beta_n-\beta_0)+\beta_n)} \\ +\;& \frac{1}{a_n}\sum_{k=k_0}^{\infty}\frac{(-1)^k}{a_n^k}\sum_{k_1=0}^{k}\sum_{k_2=0}^{k}\cdots\sum_{k_n=0}^{k}\delta_{k_1+k_2+\cdots+k_n,\,k}\frac{a_{n-1}^{k_1}a_{n-2}^{k_2}\cdots a_0^{k_n}}{k_1!\,k_2!\,\cdots\,k_n!} \\ &\times\frac{k!\;x_+^{k_1(\beta_n-\beta_{n-1})+k_2(\beta_n-\beta_{n-2})+\cdots+k_n(\beta_n-\beta_0)+\beta_n-1}}{\Gamma(k_1(\beta_n-\beta_{n-1})+k_2(\beta_n-\beta_{n-2})+\cdots+k_n(\beta_n-\beta_0)+\beta_n)}. \end{aligned}$$

The first part is a distribution as the minimum power is less than or equal to -1, and the second is locally integrable by following the proof of Theorem 4.1.

5 The applications in the wave reaction-diffusion equation

Let $\Phi(x,t)$ be the concentration of a substance distributed in space (one dimensional space) and $\varphi(x,t)$ be a nonlinear function, where t is the time variable and x is the space variable. Figueiredo Camargo et al. [13] introduced the so-called generalized wave reaction-diffusion equation given as

$$aD_t^{2\alpha}\Phi(x,t) + bD_t^{\alpha}\Phi(x,t) = c\ {}_{-\infty}D_x^{2\beta}\Phi(x,t) - \nu^2\Phi(x,t) + \varphi(x,t),$$

where $t > 0$ and $x \in R$, with $0 < \alpha \leq 1$, $0 < \beta \leq 1$, and a, b, c, and ν^2 are real constants. We are interested in a special case where $c = 0$, $\Phi(x,t) = \Phi(t)$ and $\varphi(x,t) = \varphi(t)$. Then the above equation is converted into

$$aD_t^{2\alpha}\Phi(t) + bD_t^{\alpha}\Phi(t) = -\nu^2\Phi(t) + \varphi(t), \tag{18}$$

with $\nu^2 = w^2$, where w is the frequency of the harmonic oscillator. From our equation (9), the corresponding Green's function, which is the solution of

$$aD_t^{2\alpha}\Phi(t) + bD_t^{\alpha}\Phi(t) = -w^2\Phi(t) + \delta(t),$$

is

$$\Phi(t) = \frac{1}{a}\, t_+^{2\alpha-1} \sum_{j=0}^{\infty} \left(\frac{-b}{a}\right)^j \frac{t_+^{\alpha j}}{j!} \sum_{k=0}^{\infty} \left(\frac{-w^2}{a}\right)^k \frac{(j+k)!}{k!\,\Gamma(2k\alpha + 2\alpha + \alpha j)} t_+^{2k\alpha}.$$

This expression is clearer than one involving the Mittag-Leffler functions, given in [13] as

$$\Phi(t) = \frac{1}{a} \sum_{j=0}^{\infty} \left(\frac{-b}{a}\right)^j t_+^{\alpha j + 2\alpha - 1}\, E_{2\alpha, \alpha j + 2\alpha}^{(j+1)} \left(-\frac{w^2}{a} t_+^{2\alpha}\right).$$

Furthermore, if $\varphi(t) = t_+^{-1.5}$, then the fractional differential equation

$$aD_t^{2\alpha}\Phi(t) + bD_t^{\alpha}\Phi(t) = -w^2\Phi(t) + t_+^{-1.5}$$

has the solution

$$\begin{aligned}
\Phi(t) &= \frac{1}{a}\, t_+^{2\alpha-1} \sum_{j=0}^{\infty} \left(\frac{-b}{a}\right)^j \frac{t_+^{\alpha j}}{j!} \sum_{k=0}^{\infty} \left(\frac{-w^2}{a}\right)^k \frac{(j+k)!}{k!\,\Gamma(2k\alpha + 2\alpha + \alpha j)} t_+^{2k\alpha} * (-2\sqrt{\pi}\Phi_{-0.5}) \\
&= \frac{-2\sqrt{\pi}}{a} \sum_{j=0}^{\infty} \left(\frac{-b}{a}\right)^j \frac{1}{j!} \sum_{k=0}^{\infty} \left(\frac{-w^2}{a}\right)^k \frac{(j+k)!}{k!\,\Gamma(2k\alpha + 2\alpha + \alpha j) - 0.5} t_+^{2k\alpha + 2\alpha + \alpha j - 1.5}.
\end{aligned}$$

We should note that this solution is well defined distributionally, but not classically as $\varphi(t) = t_+^{-1.5}$ is not locally integrable.

In addition, the following fractional differential equation associated with the driven harmonic oscillator [30] is a particular case of equation (18) with $a = 1$, $b = 0$, $1/2 < \alpha \le 1$ and $\nu^2 = \omega^{2\alpha}$,

$$D_t^{2\alpha}\Phi(t) = -\omega^{2\alpha}\Phi(t) + \varphi(t).$$

Then the corresponding Green's function is

$$\Phi(t) = t_+^{2\alpha-1} \sum_{k=0}^{\infty} (-\omega^{2\alpha})^k \frac{t_+^{2k\alpha}}{\Gamma(2k\alpha + 2\alpha)} = t_+^{2\alpha-1}\, E_{2\alpha,\,2\alpha}(-\omega^{2\alpha} t_+^{2\alpha}).$$

Clearly, the fractional differential equation

$$D_t^{2\alpha}\Phi(t) = -\omega^{2\alpha}\Phi(t) + t_+^{-1.5}$$

has the solution

$$\Phi(t) = -2\sqrt{\pi} \sum_{k=0}^{\infty} (-\omega^{2\alpha})^k \frac{t_+^{2k\alpha + 2\alpha - 1.5}}{\Gamma(2k\alpha + 2\alpha - 0.5)} = -2\sqrt{\pi} t_+^{2\alpha - 1.5} E_{2\alpha, 2\alpha - 0.5}(-\omega^{2\alpha} t_+^{2\alpha})$$

where $t_+^{2\alpha-1.5}$ is locally integrable and $2\alpha - 0.5 > 0$.

Finally, the fractional relaxation equation is a particular case of equation (18) with $a = 0$

$$bD_t^\alpha \Phi(t) = -w^2\Phi(t) + \delta(t).$$

The Corresponding Green's function is

$$\Phi(t) = \frac{1}{b} t_+^{\alpha-1} E_{\alpha,\,\alpha}(-w^2 t_+^\alpha / b)$$

by equation (10). The solution for the fractional differential equation

$$bD_t^\alpha \Phi(t) = -w^2\Phi(t) + t_+^{-1.5}$$

can be obtained easily, which is not doable in the classical sense.

6 Conclusion

Applying Babenko's approach, we have deduced a simpler Green's function for the n-term fractional differential (or integral) equation

$$a_n u^{(\beta_n)}(x) + a_{n-1} u^{(\beta_{n-1})}(x) + \cdots + a_1 u^{(\beta_1)}(x) + a_0 u^{(\beta_0)}(x) = g(x)$$

in the distributional space $\mathcal{D}'(R^+)$ for the first time, which is an extension of the classical result as it deals with generalized functions. Several interesting examples of solving fractional differential and integral equations, some which are not doable in the classical sense, were presented. Finally, we demonstrated nice applications of our results in the wave reaction-diffusion equation.

Acknowledgments

This work is partially supported by NSERC (Canada 2019-03907) and NSFC (China 11671251). The authors are grateful to Dr. H.M. Srivastava for his careful reading of the paper with several productive suggestions and corrections.

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