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RESEARCH PAPER

ON THE GENERALIZED FRACTIONAL LAPLACIAN

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Abstract

The objective of this paper is, for the first time, to extend the fractional Laplacian $(-\triangle)^s u(x)$ over the space $C_k(R^n)$ (which contains $S(R^n)$ as a proper subspace) for all $s > 0$ and $s \neq 1, 2, \ldots$, based on the normalization in distribution theory, Pizzetti's formula and surface integrals in R^n. We further present two theorems showing that our extended fractional Laplacian is continuous at the end points $1, 2, \ldots$. Two illustrative examples are provided to demonstrate computational techniques for obtaining the fractional Laplacian using special functions, Cauchy's residue theorem and integral identities. An application to defining the Riesz derivative in the classical sense at odd numbers is also considered at the end.

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Key Words and Phrases: fractional Laplacian; normalization; distribution; Pizzetti's formula; Gamma function

1. Introduction

During the past few decades, fractional calculus (for details, see [1], [2], [3], [4]) has been exploring as a tool for developing more sophisticated mathematical models that can accurately describe complex systems. Fractional powers of the Laplacian operator arise naturally in the study of anomalous diffusion, where the fractional operator plays an analogous role to that of the integer-order Laplacian for ordinary diffusion ([5], [6]). By replacing Brownian motion of particles with Lévy flights [7], one obtains a fractional diffusion equation (or fractional kinetic equation) in terms of the fractional

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Laplacian of order $s \in (0,1)$ via the Cauchy principal value integral [8], given as

$$(-\triangle)^s u(x) = C_{n,s}\text{P.V.}\int_{R^n} \frac{u(x)-u(\zeta)}{|x-\zeta|^{n+2s}}d\zeta, \tag{1.1}$$

where u is a function from R^n to R, $\triangle = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2$, and the constant $C_{n,s}$ is given by

$$C_{n,s} = \left(\int_{R^n} \frac{1-\cos y_1}{|y|^{n+2s}}dy\right)^{-1} = \pi^{-n/2}2^{2s}\frac{\Gamma(\frac{n+2s}{2})}{\Gamma(1-s)}s.$$

Generally speaking, two main conditions are assumed on the function u to ensure the right-hand side integral in equation (1.1) exists:

(i) u needs to be sufficiently smooth near point x,

(ii) u must have a slow growth at infinity, for example

$$\int_{R^n} \frac{|u(x)|}{1+|x|^{n+2s}}dx < \infty.$$

In [9], Dipierro et al. defined the fractional Laplacian for functions which grow more than linearly at infinity. The basic idea for this is that, if the function grows too much at infinity, its fractional Laplacian diverges, but it can be written as a given function plus a diverging sequence of polynomials of a given degree.

In [10], Michelitsch et al. constructed the following formula for a certain function $u(x)$ and $s \geq 0$:

$$(-\triangle)^s u(x) = \frac{\Gamma\left(s+\dfrac{n}{2}\right)\Gamma(2s+1)}{\pi^{\frac{n+1}{2}}\Gamma\left(s+\dfrac{1}{2}\right)} \times \lim_{\epsilon\to 0^+} \text{Re}\left\{i^{2s+1}\int_{R^n} \frac{u(y)dy}{(|x-y|+i\epsilon)^{2s+1}|x-y|^{n-1}}\right\}, \tag{1.2}$$

where Re{..} denotes the real part of {..}.

We would like to point out the above holds for all positive exponents $s \geq 0$, including integers (zero as well) that represent the powers of the conventional Laplacian $\triangle$. The construction of equation (1.2) is based on the fact that the identity

$$-|\zeta|^{-2s-1}\sin s\pi = \lim_{\epsilon\to 0^+} \text{Re}(\epsilon - i\zeta)^{-2s-1}$$

is true in the distributional sense [11]. For instance, it can be regarded as the following integral for $\zeta_0 > 0$ and $s > 0$

$$\lim_{\epsilon\to 0^+} \text{Re}\int_0^{\zeta_0}(\epsilon - i\zeta)^{-2s-1}d\zeta = \frac{\sin \pi s}{2s}\zeta_0^{-2s},$$

which is certainly unsatisfied in the classical sense.

In particular for $n = 1$, we imply from equation (1.2)

$$\left(-\frac{d^2}{dx^2}\right)^s u(x) = -\frac{\Gamma(2s+1)\sin \pi s}{\pi}\int_{-\infty}^{\infty}\frac{u(x+y)}{|y|^{2s+1}}dy,$$

for $s \geq 0$.

Recently, Lischke et al. [12] provided both a theoretical and numerical overview of various contents of the fractional Laplacian of order in $(0, 1)$, as well as different kinds of partial differential equations involving this operator.

We begin to introduce some basic notations which are soon-to-be used. Let $x = (x_1, x_2, \cdots, x_n) \in R^n$. For a given n-tuple $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ of nonnegative integers (or called a multi-index), we define

$$\begin{aligned}
&|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n, \quad \alpha! = \alpha_1!\alpha_2!\cdots\alpha_n!,\\
&x^\alpha = x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n},\\
&\partial^\alpha u = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_n^{\alpha_n}u = \frac{\partial^{|\alpha|}u}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\cdots\partial x_n^{\alpha_n}}.
\end{aligned}$$

The Schwartz space $S(R^n)$ (space of rapidly decreasing functions on R^n) is the function space [13] defined as

$$S(R^n) = \left\{u(x) \in C^\infty(R^n) : \ \|u(x)\|_{\alpha,k} \leq C_{\alpha,k}(\text{const}) \ \ \forall \alpha, k \in N_0^n\right\},$$

where $N_0 = \{0\} \cup N$ is the set of nonnegative integers and

$$\|u(x)\|_{\alpha,k} = \sup_{x\in R^n}\left|x^\alpha \partial^k u(x)\right|.$$

Let $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$. Then, clearly $e^{-|x|^2} \in S(R^n)$.

On the other hand, the fractional Laplacian is widely considered as the Riesz fractional derivative, which is defined for a suitably smooth function $u(x)$ $(x \in R^n)$ by ([1], [14])

$${}_{RZ}D_x^\alpha u(x) = \frac{1}{d_{n,l}(\alpha)}\int_{R^n}\frac{(\triangle_y^l u)(x)}{|y|^{n+\alpha}}dy, \ \ 0 < \alpha < l,$$

where l can be arbitrary integer bigger than α, and $(\triangle_y^l u)(x)$ denotes the centred difference

$$(\triangle_y^l u)(x) = \sum_{k=0}^{l}(-1)^k\binom{l}{k}u(x + (l/2 - k)y),$$

or non-centred differences

$$(\triangle_y^l u)(x) = \sum_{k=0}^{l}(-1)^k\binom{l}{k}u(x - ky).$$

The $d_{n,l}(\alpha)$ are normalizing constants and analytic functions with respect to the parameter α. Cai and Li [15] showed that

$$(-\triangle)^s u(x) =_{RZ} D_x^{2s} u(x),$$

for $u(x) \in S(R^n)$ with $n > 1$ and $s \in (0, 1)$.

Kwaśnicki presented ten equivalent definitions for defining $(-\triangle)^s$ over certain function spaces [16]. For example, it can be defined:

(i) either as a Fourier multiplier given by the formula

$$\mathcal{F}((-\triangle)^s u)(\zeta) = |\zeta|^{2s} \mathcal{F}(u)(\zeta),$$

where the Fourier transform $\mathcal{F}(u)$ of a function u is given by

$$\mathcal{F}(u)(\zeta) = \int_{R^n} u(x) e^{-ix\zeta} dx,$$

(ii) or by singular integral definition

$$(-\triangle)^s u(\zeta) = -\lim_{r \to 0^+} \frac{2^{2s}\Gamma\left(\frac{n}{2}+s\right)}{\pi^{n/2}|\Gamma(-s)|} \int_{R^n \setminus B(x,r)} \frac{u(\zeta+z)-u(\zeta)}{|z|^{n+2s}} dz,$$

with the limit in Lebesgue spaces.

The Cauchy principal value integral for defining the fractional Laplacian in equation (1.1) has been normalized over the space $S(R^n)$ for $0 < s < 1$, and is given as [16, 17]:

$$(-\triangle)^s u(x) = -\frac{1}{2} C_{n,s} \int_{R^n} \frac{u(x+y) - 2u(x) + u(x-y)}{|y|^{n+2s}} dy. \tag{1.3}$$

As outlined in the abstract, the goal of this paper is to find a fresh approach to normalizing and defining the fractional Laplacian $(-\triangle)^s$ over a new function space which contains $S(R^n)$ as a proper subspace, for all $s > 0$ and $s \neq 1, 2, \ldots$, using distributional techniques. Furthermore, we obtain two theorems showing that the extended fractional Laplacian $(-\triangle)^s$ is continuous at the end points $1, 2, 3, \ldots$. To move forward, we start constructing an identity sequence $I_m(|x|)$ satisfying certain conditions, and introduce the Schwartz space of test functions for the concept of normalization of the distribution x_+^{-1-2s} in Section 2. Then, we further normalize the integral on the right-hand side of equation (1.3) in Section 3, by surface integrals on R^n as well as Pizzetti's formula. In Section 4, we present an interesting example computing the fractional Laplacian of a function that is not in the Schwartz space using Green's theorem. Moreover, we define the generalized fractional Laplacian $(-\triangle)^s$ on R for $s > 0$ and $s \neq 1, 2, \ldots$ by the normalization of the distribution $|x|^\lambda$, and present one example to demonstrate computational skills by Cauchy's theorem, as well as an application to extending the classical Riesz derivative to odd numbers in Section 5. Finally, we summarize the entire paper in Section 6.

2. **Preliminaries**

We begin by introducing an identity sequence $I_m(|x|)$ on R^n and the Schwartz space of test functions, which will be used in the following sections. Let $\tau(x)$ be an infinitely differentiable function on $[0,+\infty)$ satisfying the following conditions:

(i) $0 \leq \tau(x) \leq 1$,
(ii) $\tau(x) = 1 \quad \text{if} \quad 0 \leq x \leq 1/2$,
(iii) $\tau(x) = 0 \quad \text{if} \quad x \geq 1$.

We construct the sequence $I_m(|x|)$ for $m = 1, 2, \ldots$ as:

$$I_m(|x|) = \begin{cases} 1 & \text{if } |x| \leq m, \\ \tau\left(\dfrac{m^{2m}}{1+2m^{1+m}}|x|^2 - \dfrac{m^{2m+2}}{1+2m^{1+m}}\right) & \text{if } |x| > m. \end{cases}$$

Clearly, $I_m(|x|)$ is infinitely differentiable with respect to $x_1, x_2, \cdots, x_n$ and $|x|$, and $I_m(|x|) = 0$ if $|x| \geq m + m^{-m}$, as

$$\frac{m^{2m}}{1+2m^{1+m}}(m+m^{-m})^2 - \frac{m^{2m+2}}{1+2m^{1+m}} = 1.$$

Furthermore,

$$0 \leq I_m(|x|) \leq 1.$$

Let $\mathcal{D}(R^n)$ be the Schwartz space [11] of infinitely differentiable functions (or so-called the Schwartz space of test functions) with compact support in R^n, and $\mathcal{D}'(R^n)$ be the space of distributions (linearly continuous functionals) defined on $\mathcal{D}(R^n)$. In addition, we shall define a sequence $\phi_1(x)$, $\phi_2(x)$, $\cdots$, $\phi_m(x), \ldots$ which converges to zero in $\mathcal{D}(R^n)$ if all these functions vanish outside a certain fixed and bounded interval in R^n, and converge uniformly to zero (in the usual sense) together with their derivatives of any order. We further define $\mathcal{D}'(R^+)$ as the subspace of $\mathcal{D}'(R)$ $(n = 1)$ with support contained in R^+. Let $f \in \mathcal{D}'(R^n)$. It is conventional to write (f, ϕ) for the value of f acting on a test function $\phi \in \mathcal{D}(R^n)$. The functional δ defined as

$$(\delta, \phi) = \phi(0),$$

is a linear and continuous functional on $\mathcal{D}(R^n)$. Hence, $\delta \in \mathcal{D}'(R^n)$.

Let $f \in \mathcal{D}'(R^n)$ and $k = (k_1, k_2, \cdots, k_n)$ be an n-tuple of nonnegative integers. Then the distributional derivative $\dfrac{\partial^{|k|}}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}} f$ on $\mathcal{D}(R^n)$ is defined as:

$$\left(\frac{\partial^{|k|}}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}} f,\, \phi\right) = (-1)^{|k|}\left(f,\, \frac{\partial^{|k|}}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}} \phi\right), \quad k_i \geq 0$$

for $\phi \in \mathcal{D}(R^n)$. In particular for $n = 1$,

$$(\delta^{(m)}(x), \phi(x)) = (-1)^m \phi^{(m)}(0),$$

where m is a nonnegative integer.

The distribution x_+^λ on $\mathcal{D}(R)$ is normalized in [11] as:

$$(x_+^\lambda,\, \phi(x)) = \int_0^\infty x^\lambda \times \left[\phi(x) - \phi(0) - x\phi'(0) - \cdots - \frac{x^{m-1}}{(m-1)!}\phi^{(m-1)}(0)\right] dx, \quad (2.1)$$

where $-m-1 < \lambda < -m$ ($m \in Z^+ = N$) and $\phi \in \mathcal{D}(R)$. Note that the integral

$$\int_0^\infty x^\lambda \phi(x)$$

is undefined in the classical sense for $-m-1 < \lambda < -m$ with $m = 1, 2, \cdots$.

The distributions P.V.x^{-2m} (or x^{-2m} in short) for $m = 1, 2, \ldots$ and P.V.x^{-2m-1} (or x^{-2m-1}) for $m = 0, 1, \ldots$ are given in [11] as:

$$(x^{-2m},\, \phi) = \int_0^\infty x^{-2m}\{\phi(x) + \phi(-x) -2\left[\phi(0) + \frac{x^2}{2!}\phi''(0) + \cdots + \frac{x^{2m-2}}{(2m-2)!}\phi^{(2m-2)}(0)\right]\}dx, \quad (2.2)$$

$$(x^{-2m-1},\, \phi) = \int_0^\infty x^{-2m-1}\{\phi(x) - \phi(-x) -2\left[x\phi'(0) + \frac{x^3}{3!}\phi^{(3)}(0) + \cdots + \frac{x^{2m-1}}{(2m-1)!}\phi^{(2m-1)}(0)\right]\}dx.$$

In particular,

$$(x^{-2},\, \phi) = \int_0^\infty \frac{\phi(x) + \phi(-x) - 2\phi(0)}{x^2}dx, \quad \text{and}$$

$$(x^{-1},\, \phi) = \int_0^\infty \frac{\phi(x) - \phi(-x)}{x}dx.$$

Let $S_\phi(r)$ be the mean value of $\phi(x) \in \mathcal{D}(R^n)$ on the sphere of radius r given by

$$S_\phi(r) = \frac{1}{\Omega_n}\int_\Omega \phi(r\sigma)d\sigma, \quad (2.3)$$

where $\Omega_n = 2\pi^{\frac{n}{2}}/\Gamma(\frac{n}{2})$ is the area of the unit sphere Ω. We can write out the Taylor's series for $S_\phi(r)$ [18] in the space of analytic functions, namely

$$\begin{aligned} S_\phi(r) &= \phi(0) + \frac{1}{2!}S''_\phi(0)r^2 + \cdots + \frac{1}{(2k)!}S_\phi^{(2k)}(0)r^{2k} + \cdots \\ &= \sum_{k=0}^\infty \frac{\triangle^k \phi(0) r^{2k}}{2^k\, k!\, n(n+2)\cdots(n+2k-2)}, \end{aligned}$$

which is the well-known Pizzetti's formula [19]. It plays an important role in the work of Li, Aguirre and Fisher ([20], [21], [22], [23]) for defining distributional products on R^n.

REMARK 2.1. Pizzetti's formula is not a convergent series for all $\phi \in \mathcal{D}(R^n)$ from the counterexample below.

$$\phi(x) = \begin{cases} \exp\{-\frac{1}{r^2(1-r^2)}\} & \text{if } 0 < r < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\phi(x) \in \mathcal{D}(R^n)$ and $S_\phi(r) \neq 0$ for $0 < r < 1$, but the series in the formula is identically equal to zero. Obviously, $S_\phi(r) \to 0$ as $r \to 0$.

Clearly, we have from equation (2.3) and Pizzetti's formula that

$$\frac{d^{2k}}{dr^{2k}} \int_\Omega \phi(r\sigma) d\sigma \bigg|_{r=0} = \frac{\Omega_n (2k)! \triangle^k \phi(0)}{2^k\, k!\, n(n+2) \cdots (n+2k-2)}, \tag{2.4}$$

for $k = 1, 2, \ldots$. Evidently for $k = 0$,

$$\int_\Omega \phi(r\sigma) d\sigma \bigg|_{r=0} = \Omega_n \phi(0).$$

It follows from [27] that $\Phi_\lambda = \dfrac{x_+^{\lambda-1}}{\Gamma(\lambda)}$ is an entire function of λ on the complex plane, and

$$\frac{x_+^{\lambda-1}}{\Gamma(\lambda)} \bigg|_{\lambda=-m} = \delta^{(m)}(x), \quad \text{for } m = 0, 1, \ldots. \tag{2.5}$$

For the distribution $\Phi_\lambda = \dfrac{x_+^{\lambda-1}}{\Gamma(\lambda)}$, the (distributional) derivative formula is simpler than that for x_+^λ. In fact,

$$\frac{d}{dx} \Phi_\lambda = \frac{d}{dx} \frac{x_+^{\lambda-1}}{\Gamma(\lambda)} = \frac{(\lambda-1) x_+^{\lambda-2}}{\Gamma(\lambda)} = \frac{x_+^{\lambda-2}}{\Gamma(\lambda-1)} = \Phi_{\lambda-1}. \tag{2.6}$$

Let λ and μ be arbitrary complex numbers. Then we have from [27]

$$\Phi_\lambda * \Phi\mu = \Phi_{\lambda+\mu}. \tag{2.7}$$

Let λ be an arbitrary complex number and $g(x)$ be the distribution concentrated on $x \geq 0$. We define the primitive of order λ of g as a convolution in the distributional sense:

$$g_\lambda(x) = g(x) * \frac{x_+^{\lambda-1}}{\Gamma(\lambda)} = g(x) * \Phi_\lambda. \tag{2.8}$$

Clearly, the convolution on the right-hand side of equation (2.8) is well defined, as supports of g and Φ_λ are bounded on the same side.

Thus, equation (2.8) with various λ will not only give the fractional derivatives but also the fractional integrals of $g(x) \in \mathcal{D}'(R^+)$ when $\lambda \notin Z$. It also reduces to integer-order derivatives or integrals when $\lambda \in Z$. We define the convolution

$$g_{-\lambda} = g(x) * \Phi_{-\lambda},$$

as the fractional derivative of the distribution $g(x)$ with order λ, writing it as

$$g_{-\lambda} = \frac{d^\lambda}{dx^\lambda} g,$$

for Re$\lambda \geq 0$. Similarly, $\frac{d^\lambda}{dx^\lambda} g$ is interpreted as the fractional integral if Re$\lambda < 0$.

Replacing λ by $-\lambda$ in equation (2.7), we get

$$\frac{d^\lambda}{dx^\lambda}\left(\frac{x_+^{\mu-1}}{\Gamma(\mu)}\right) = \frac{x_+^{\mu-\lambda-1}}{\Gamma(\mu-\lambda)}.$$

In particular for $\mu = 0$, we have

$$\delta^{(\lambda)}(x) = \frac{x_+^{-\lambda-1}}{\Gamma(-\lambda)}, \tag{2.9}$$

which will be used in the following sections.

The following formula will be utilized in Section 4, which can be found on page 292 in [24].

Let $\phi(x)$ and $\psi(x)$ be infinitely differentiable functions. Then for $k = 0, 1, 2, \ldots,$

$$\triangle^k(\phi\psi) = \sum_{m+i+l=k} 2^i \binom{m+l}{m}\binom{k}{m+l} \nabla^i \triangle^m \phi \cdot \nabla^i \triangle^l \psi, \tag{2.10}$$

where

$$\nabla^i\phi \cdot \nabla^i\psi = \begin{cases} \phi\psi & \text{if } i = 0, \\ \displaystyle\sum_{j=1}^n \frac{\partial^i}{\partial x_j^i}\phi \frac{\partial^i}{\partial x_j^i}\psi & \text{if } i > 0. \end{cases}$$

3. The generalized fractional Laplacian on $C^\infty(R^n)$ $(n > 1)$

It is well known that (see page 53 in [11])

$$\Gamma(\lambda) = \int_0^\infty x^{\lambda-1}\left[e^{-x} - \sum_{j=0}^k (-1)^j \frac{x^j}{j!}\right] dx,$$

where $-k-1 < \lambda < -k$ $(k \in Z^+)$.

Let $s > 0$, $s \neq 1, 2, \ldots,$ and $C^\infty(R^n)$ $(n > 1)$ be the space of infinitely differentiable functions on R^n, which clearly contains $S(R^n)$ as a proper subspace. Based on equation (1.3), we define the generalized fractional Laplacian $(-\triangle)^s u(x)$ over the space $C^\infty(R^n)$ as:

$$\begin{aligned}(-\triangle)^s u(x) &= -\frac{1}{2}C_{n,s}\\ &\quad\times \lim_{m\to\infty}\int_{R^n}\frac{[u(x+y)-2u(x)+u(x-y)]I_m(|y|)}{|y|^{n+2s}}dy, \quad (3.1)\end{aligned}$$

if the limit exists, and where

$$C_{n,s}=\pi^{-n/2}2^{2s}\frac{\Gamma(\frac{n+2s}{2})}{\Gamma(1-s)}s$$

is well defined for $s>0, \quad s\neq 1,2,\cdots$. In particular, equation (3.1) becomes equation (1.3) for any function $u(x)\in S(R^n)$, as the integral

$$\int_{R^n}\frac{[u(x+y)-2u(x)+u(x-y)]I_m(|y|)}{|y|^{n+2s}}dy,$$

uniformly converges with respect to m and

$$\lim_{m\to\infty} I_m(|y|)=1.$$

Indeed, for any smooth function u, a second order of Taylor's expansion derives

$$\frac{|[u(x+y)-2u(x)+u(x-y)]I_m(|y|)|}{|y|^{n+2s}}\leq\frac{\left\|D^2u\right\|_{L^\infty}}{|y|^{n+2s-2}}, \quad 0<s<1,$$

which is integrable near zero by noting that $0\leq I_m(|y|)\leq 1$.

Using the spherical coordinates below

$$\begin{aligned}&y_1=r\cos\theta_1,\\ &y_2=r\sin\theta_1\cos\theta_2,\\ &y_3=r\sin\theta_1\sin\theta_2\cos\theta_3,\\ &\cdots\\ &y_{n-1}=r\sin\theta_1\cdots\sin\theta_{n-2}\cos\theta_{n-1},\\ &y_n=r\sin\theta_1\cdots\sin\theta_{n-2}\sin\theta_{n-1},\end{aligned}$$

where the angles $\theta_1,\theta_2,\cdots,\theta_{n-2}$ range over $[0,\pi]$ and θ_{n-1} ranges over $[0,2\pi]$, equation (3.1) turns out to be

$$\begin{aligned}(-\triangle)^s u(x) &= -\frac{1}{2}C_{n,s}\lim_{m\to\infty}\int_0^{m+m^{-m}}\frac{r^{n-1}S_m(r)}{r^{n+2s}}dr\\ &= -\frac{1}{2}C_{n,s}\lim_{m\to\infty}\int_0^{m+m^{-m}}\frac{S_m(r)}{r^{1+2s}}dr, \quad (3.2)\end{aligned}$$

where

$$\begin{aligned} S_m(r) &= \int_\Omega I_m(r)[u(x+r\sigma)-2u(x)+u(x-r\sigma)]d\sigma \\ &= I_m(r)\int_\Omega [u(x+r\sigma)-2u(x)+u(x-r\sigma)]d\sigma = I_m(r)S(r), \end{aligned}$$

and $d\sigma$ is the hypersurface area element on the unit sphere Ω given by

$$d\sigma = \sin^{n-2}\theta_1 \sin^{n-3}\theta_2 \cdots \sin\theta_{n-2} d\theta_1 d\theta_2 \cdots d\theta_{n-1}.$$

Clearly, $S_m(r)$ is an infinitely differentiable function of r with compact support $[0, m+m^{-m}]$ and

$$S_m(0) = 0.$$

Furthermore for $i = 1, 2, \ldots,$

$$\frac{d^{2i}}{dr^{2i}} S_m(r)\bigg|_{r=0} = I_m(0)\ \frac{d^{2i}}{dr^{2i}} S(r)\bigg|_{r=0} = \frac{d^{2i}}{dr^{2i}} S(r)\bigg|_{r=0},$$

since

$$\frac{d^j}{dr^j} I_m(r)\bigg|_{r=0} = 0,$$

for all $j = 1, 2, \ldots,$ by the construction of $I_m(r)$. Obviously, the integral

$$S(r) = \int_\Omega [u(x+r\sigma)-2u(x)+u(x-r\sigma)]d\sigma$$

is an even function with respect to r. Therefore,

$$\frac{d^{2i+1}}{dr^{2i+1}} S_m(r)\bigg|_{r=0} = 0,$$

for $i = 0, 1, \ldots.$

From equation (2.4), we get for $i = 1, 2, \ldots,$

$$\begin{aligned} \frac{d^{2i}}{dr^{2i}} S(r)\bigg|_{r=0} &= \frac{\Omega_n (2i)! \triangle^i [u(x+r\sigma)-2u(x)+u(x-r\sigma)]\big|_{r=0}}{2^i\, i!\, n(n+2)\cdots(n+2i-2)} \\ &= \frac{2\Omega_n (2i)! \triangle^i u(x)}{2^i\, i!\, n(n+2)\cdots(n+2i-2)}, \end{aligned}$$

since we can still consider the function

$$u(x+r\sigma) - 2u(x) + u(x-r\sigma),$$

has compact support $|r| \le m + m^{-m}$ due to the factor $I_m(r)$. In particular,

$$\frac{d^2}{dr^2} S_m(r)\bigg|_{r=0} = \frac{2\Omega_n \triangle u(x)}{n} \quad \text{and} \quad \frac{d^4}{dr^4} S_m(r)\bigg|_{r=0} = \frac{3!\Omega_n \triangle^2 u(x)}{n(n+2)}.$$

Applying equation (2.1) for $\frac{k-1}{2} < s < \frac{k}{2}$ ($k \in Z^+$), equation (3.2) becomes

$$\begin{aligned}(-\triangle)^s u(x) &= -\frac{1}{2}C_{n,s}\lim_{m\to\infty}\int_0^{m+m^{-m}} r^{-1-2s} \\ &\qquad \times \left[S_m(r) - rS'_m(0) - \cdots - \frac{r^{k-1}}{(k-1)!}S_m^{(k-1)}(0)\right] dr \\ &= -\frac{1}{2}C_{n,s}\lim_{m\to\infty}\int_0^{m+m^{-m}} r^{-1-2s} \\ &\qquad \times \left[S_m(r) - \frac{r^2}{2!}S''_m(0) - \cdots - \frac{r^{k-1}}{(k-1)!}S_m^{(k-1)}(0)\right] dr,\end{aligned}$$

using that $S'_m(0) = 0$.

Then for $k = 2i+1$ and $i = 0, 1, \ldots,$

$$\begin{aligned}(-\triangle)^s u(x) &= -\frac{1}{2}C_{n,s}\lim_{m\to\infty}\int_0^{m+m^{-m}} r^{-1-2s} \\ &\qquad \times \left[S_m(r) - \frac{r^2}{2!}S''_m(0) - \cdots - \frac{r^{2i}}{(2i)!}S_m^{(2i)}(0)\right] dr, \qquad (3.3)\end{aligned}$$

as well as

$$\begin{aligned}(-\triangle)^s u(x) &= -\frac{1}{2}C_{n,s}\lim_{m\to\infty}\int_0^{m+m^{-m}} r^{-1-2s} \\ &\qquad \times \left[S_m(r) - \frac{r^2}{2!}S''_m(0) - \cdots - \frac{r^{2i-2}}{(2i-2)!}S_m^{(2i-2)}(0)\right] dr, \qquad (3.4)\end{aligned}$$

for $k = 2i$ and $i = 1, 2, \ldots$.

We are going to introduce a new function space $C_k(R^n)$, on which we can define the fractional Laplacian $(-\triangle)^s$. Let $k = (k_1, k_2, \cdots, k_n)$ be an n-tuple of nonnegative integers, and

$$\begin{aligned}C_k(R^n) &= \Big\{u(x) \text{ is bounded and } \partial^{2k}u(x) \in C(R^n) : \\ &\qquad \exists\, M_k(\text{const}) > 0, \text{ such that } \left|\partial^{2k}u(x)\right| \le \frac{M_k}{|x|^2} \text{ as } |x| \to \infty\Big\}.\end{aligned}$$

Clearly,

$$\mathcal{D}(R^n) \subset S(R^n) \subset C_k(R^n) \subset C(R^n),$$

for any tuple $k \in N_0^n$, and $\partial^{2k}u(x)$ is a bounded function on R^n. We also must add that equation (3.1) turns out to be equation (1.3) for any function $u(x) \in C_k(R^n)$ and the proof is identical to the previous case for the space $S(R^n)$.

Assume that

$$\phi(x) = \frac{1}{|x|^2+1}, \quad x \in R^n.$$

Then $\phi(x) \in C_k(R^n)$ for all $k \in N_0^n$, but $\phi(x) \notin S(R^n)$. Note that the condition

$$\left|\partial^{2k} u(x)\right| \leq \frac{M_k}{|x|^2} \text{ as } |x| \to \infty,$$

is equivalent to

$$\sup_{x \in R^n} |x|^2 \left|\partial^{2k} u(x)\right| < \infty,$$

for every tuple $k \in N_0^n$.

We are ready to present the following main theorem.

THEOREM 3.1. *Let $i = 0, 1, \ldots$ and $i < s < i+1$. Then the generalized fractional Laplacian $(-\triangle)^s$ is defined over the space $C_k(R^n)$ as:*

$$(-\triangle)^s u(x) = -\frac{1}{2} C_{n,s} \int_0^\infty r^{-1-2s}$$
$$\times \left[S(r) - \frac{r^2 \Omega_n \triangle u(x)}{n} - \cdots - \frac{2r^{2i} \Omega_n \triangle^i u(x)}{2^i\, i!\, n(n+2) \cdots (n+2i-2)} \right] dr,$$

where $k = (k_1, k_2, \ldots, k_n)$ and $k_1 + \cdots + k_n = i+1$. Note that for $i = 0$, we define

$$\frac{r^2 \Omega_n \triangle u(x)}{n} + \cdots + \frac{2r^{2i} \Omega_n \triangle^i u(x)}{2^i\, i!\, n(n+2) \cdots (n+2i-2)} = 0.$$

P r o o f. Let $i < s < i+1$ for $i = 0, 1, \cdots$. Using Taylor's expansion, we derive for $u(x) \in C_k(R^n)$ that

$$\begin{aligned}
&u(x+r\sigma) - 2u(x) + u(x-r\sigma) \\
&= \sum_{|\alpha|=2} \frac{2\partial^\alpha u(x)}{\alpha!} (r\sigma)^\alpha + \cdots + \sum_{|\alpha|=2i} \frac{2\partial^\alpha u(x)}{\alpha!} (r\sigma)^\alpha \\
&+ \sum_{|\alpha|=2i+2} \frac{\partial^\alpha [u(x+\theta r\sigma) + u(x-\theta r\sigma)]}{\alpha!} (r\sigma)^\alpha \\
&= r^2 \sum_{|\alpha|=2} \frac{2\partial^\alpha u(x)}{\alpha!} \sigma^\alpha + \cdots + r^{2i} \sum_{|\alpha|=2i} \frac{2\partial^\alpha u(x)}{\alpha!} \sigma^\alpha \\
&+ r^{2i+2} \sum_{|\alpha|=2i+2} \frac{\partial^\alpha [u(x+\theta r\sigma) + u(x-\theta r\sigma)]}{\alpha!} \sigma^\alpha,
\end{aligned}$$

where $\theta \in (0,1)$ and $r\sigma \in R^n$. It follows from Pizzetti's formula that

$$
\begin{aligned}
S(r) &= \int_\Omega [u(x+r\sigma) - 2u(x) + u(x-r\sigma)]d\sigma \\
&= r^2 \sum_{|\alpha|=2} \frac{2\partial^\alpha u(x)}{\alpha!} \int_\Omega \sigma^\alpha d\sigma + \cdots + r^{2i} \sum_{|\alpha|=2i} \frac{2\partial^\alpha u(x)}{\alpha!} \int_\Omega \sigma^\alpha d\sigma \\
&\quad + r^{2i+2} \sum_{|\alpha|=2i+2} \frac{1}{\alpha!} \int_\Omega \partial^\alpha [u(x+\theta r\sigma) + u(x-\theta r\sigma)]\sigma^\alpha d\sigma \\
&= \frac{r^2 \Omega_n \triangle u(x)}{n} + \cdots + \frac{2r^{2i}\Omega_n \triangle^i u(x)}{2^i\, i!\, n(n+2)\cdots(n+2i-2)} \\
&\quad + r^{2i+2} \sum_{|\alpha|=2i+2} \frac{1}{\alpha!} \int_\Omega \partial^\alpha [u(x+\theta r\sigma) + u(x-\theta r\sigma)]\sigma^\alpha d\sigma.
\end{aligned}
$$

Since $k_1 + \cdots + k_n = i+1$ and $u(x) \in C_k(R^n)$, there exists a constant $M_k > 0$ such that for a fixed $x \in R^n$

$$
\begin{aligned}
|\partial^\alpha u(x+\theta r\sigma)| &\le \frac{M_k}{|x+\theta r\sigma|^2} \sim \frac{M_k}{r^2}, \quad \text{and} \\
|\partial^\alpha u(x-\theta r\sigma)| &\le \frac{M_k}{|x-\theta r\sigma|^2} \sim \frac{M_k}{r^2},
\end{aligned}
$$

as $r \to \infty$. Thus,

$$
\begin{aligned}
&\left| r^{-1-2s} \left[S(r) - \frac{r^2\Omega_n \triangle u(x)}{n} - \cdots - \frac{2r^{2i}\Omega_n \triangle^i u(x)}{2^i\, i!\, n(n+2)\cdots(n+2i-2)} \right] \right| \\
&\le 2M_k r^{-1-2s+2i} \sum_{|\alpha|=2i+2} \frac{1}{\alpha!} \int_\Omega |\sigma^\alpha| d\sigma, \quad \text{when } \ r \to \infty,
\end{aligned}
$$

which is integrable with respect to r at infinity, as $-1-2s+2i < -1$. On the other hand,

$$
\begin{aligned}
& r^{-1-2s} \left[S(r) - \frac{r^2\Omega_n \triangle u(x)}{n} - \cdots - \frac{2r^{2i}\Omega_n \triangle^i u(x)}{2^i\, i!\, n(n+2)\cdots(n+2i-2)} \right] \\
&= r^{-1-2s+2i+2} \sum_{|\alpha|=2i+2} \frac{1}{\alpha!} \int_\Omega \partial^\alpha [u(x+\theta r\sigma) + u(x-\theta r\sigma)]\sigma^\alpha d\sigma \\
&\sim r^{-1-2s+2i+2} \sum_{|\alpha|=2i+2} \frac{2\partial^\alpha u(x)}{\alpha!} \int_\Omega \sigma^\alpha d\sigma, \ \text{ as } \ r \to 0^+.
\end{aligned}
$$

Hence,

$$
r^{-1-2s} \left[S(r) - \frac{r^2\Omega_n \triangle u(x)}{n} - \cdots - \frac{2r^{2i}\Omega_n \triangle^i u(x)}{2^i\, i!\, n(n+2)\cdots(n+2i-2)} \right],
$$

is integrable near the origin as $-1-2s+2i+2>-1$. In summary, the integral

$$\int_0^\infty r^{-1-2s}\left[S(r)-\frac{r^2\Omega_n\triangle u(x)}{n}-\cdots-\frac{2r^{2i}\Omega_n\triangle^i u(x)}{2^i\, i!\, n(n+2)\cdots(n+2i-2)}\right]dr,$$

exists and converges for $i<s<i+1$. Clearly,

$$\begin{aligned}&\lim_{m\to\infty}\int_{m+m^{-m}}^\infty r^{-1-2s}\\ &\times\left[S_m(r)-\frac{r^2\Omega_n\triangle u(x)}{n}-\cdots-\frac{2r^{2i}\Omega_n\triangle^i u(x)}{2^i\, i!\, n(n+2)\cdots(n+2i-2)}\right]dr=0,\end{aligned}$$

since $-1-2s+2i<-1$ and $S_m(r)=0$ for $r\geq m+m^{-m}$.

Then it follows from equation (3.3) that the generalized fractional Laplacian can be normalized for $i<s<i+0.5$ and $i=0,1,\ldots$ as:

$$\begin{aligned}&(-\triangle)^s u(x)=-\frac{1}{2}C_{n,s}\lim_{m\to\infty}\int_0^{m+m^{-m}} r^{-1-2s}\\ &\quad\times\left[S_m(r)-\frac{r^2\Omega_n\triangle u(x)}{n}-\cdots-\frac{2r^{2i}\Omega_n\triangle^i u(x)}{2^i\, i!\, n(n+2)\cdots(n+2i-2)}\right]dr\\ &=-\frac{1}{2}C_{n,s}\lim_{m\to\infty}\int_0^\infty r^{-1-2s}\\ &\quad\times\left[S_m(r)-\frac{r^2\Omega_n\triangle u(x)}{n}-\cdots-\frac{2r^{2i}\Omega_n\triangle^i u(x)}{2^i\, i!\, n(n+2)\cdots(n+2i-2)}\right]dr\\ &=-\frac{1}{2}C_{n,s}\int_0^\infty r^{-1-2s}\\ &\quad\times\left[S(r)-\frac{r^2\Omega_n\triangle u(x)}{n}-\cdots-\frac{2r^{2i}\Omega_n\triangle^i u(x)}{2^i\, i!\, n(n+2)\cdots(n+2i-2)}\right]dr,\end{aligned}$$

since

$$\lim_{m\to\infty}\int_0^\infty r^{-1-2s}[S_m(y)-S(y)]dy=\lim_{m\to\infty}\int_m^\infty r^{-1-2s}[S_m(y)-S(y)]dy=0,$$

by noting that $S_m(y)$ and $S(y)$ are bounded functions.

Additionally, if $i+0.5<s<i+1$ for $i=0,1,\ldots$, then the generalized fractional Laplacian is normalized by equation (3.4) as

$$
\begin{aligned}
&(-\triangle)^s u(x) = -\frac{1}{2}C_{n,s}\lim_{m\to\infty}\int_0^{m+m^{-m}} r^{-1-2s} \\
&\times\left[S_m(r) - \frac{r^2\Omega_n\triangle u(x)}{n} - \cdots - \frac{2r^{2i}\Omega_n\triangle^i u(x)}{2^i\, i!\, n(n+2)\cdots(n+2i-2)}\right]dr \\
&= -\frac{1}{2}C_{n,s}\int_0^{\infty} r^{-1-2s} \\
&\times\left[S(r) - \frac{r^2\Omega_n\triangle u(x)}{n} - \cdots - \frac{2r^{2i}\Omega_n\triangle^i u(x)}{2^i\, i!\, n\cdots(n+2i-2)}\right]dr. \qquad (3.5)
\end{aligned}
$$

If $s = i + 0.5$ for $i = 0, 1, \ldots$, then $-1 - 2s = -2i - 2$. Equation (3.5) still holds by applying equation (2.2) to equation (3.2) and the fact that $S_m(r)$ is an even function with respect to r. □

Remark 3.1. As previously mentioned in Remark 2.1, Pizzetti's formula is not a convergent series for $\phi \in \mathcal{D}(R^n)$. However, we can derive a new Taylor's expansion for $S_\phi(r)$ with an explicit integral remainder. Clearly,

$$
\begin{aligned}
\phi(r\sigma) \;&=\; \phi(0) + \sum_{|\alpha|=1}\frac{\partial^\alpha\phi(0)}{\alpha!}(r\sigma)^\alpha + \cdots + \sum_{|\alpha|=2i}\frac{\partial^\alpha\phi(0)}{\alpha!}(r\sigma)^\alpha \\
&\quad + \sum_{|\alpha|=2i+1}\frac{\partial^\alpha\phi(0)}{\alpha!}(r\sigma)^\alpha + \sum_{|\alpha|=2i+2}\frac{\partial^\alpha\phi(\theta r\sigma)}{\alpha!}(r\sigma)^\alpha,
\end{aligned}
$$

where $\theta \in (0, 1)$, and

$$
r^{2i+1}\sum_{|\alpha|=2i+1}\frac{\partial^\alpha\phi(0)}{\alpha!}\int_\Omega \sigma^\alpha d\sigma = 0,
$$

for $i = 0, 1, \ldots$, due to the fact that each odd number of factors of the σ_j, where $\sigma = (\sigma_1, \ldots, \sigma_j, \ldots, \sigma_n)$, in the integrand fails to contribute to the integral. Hence,

$$
\begin{aligned}
S_\phi(r) \;&=\; \phi(0) + r^2\sum_{|\alpha|=2}\frac{\partial^\alpha\phi(0)}{\alpha!}\frac{1}{\Omega_n}\int_\Omega \sigma^\alpha d\sigma + \cdots \\
&\quad + r^{2i}\sum_{|\alpha|=2i}\frac{\partial^\alpha\phi(0)}{\alpha!}\frac{1}{\Omega_n}\int_\Omega \sigma^\alpha d\sigma \\
&\quad + r^{2i+2}\sum_{|\alpha|=2i+2}\frac{1}{\alpha!}\frac{1}{\Omega_n}\int_\Omega \partial^\alpha\phi(\theta r\sigma)\sigma^\alpha d\sigma.
\end{aligned}
$$

Let $\alpha = 2j$. Then,

$$\begin{aligned} S_\phi(r) &= \phi(0) + r^2 \sum_{|j|=1} \frac{\partial^{2j}\phi(0)}{(2j)!}\frac{1}{\Omega_n}\int_\Omega \sigma^{2j} d\sigma + \cdots \\ &+ r^{2i} \sum_{|j|=i} \frac{\partial^{2j}\phi(0)}{(2j)!}\frac{1}{\Omega_n}\int_\Omega \sigma^{2j} d\sigma \\ &+ r^{2i+2} \sum_{|j|=i+1} \frac{1}{(2j)!}\frac{1}{\Omega_n}\int_\Omega \partial^{2j}\phi(\theta r\sigma)\sigma^{2j} d\sigma. \end{aligned}$$

Using the formulas from [25],

$$\int_\Omega \sigma^{2j} d\sigma = \frac{2\Gamma\left(\frac{1}{2}+j_1\right)\cdots\Gamma\left(\frac{1}{2}+j_n\right)}{\Gamma\left(|j|+\frac{n}{2}\right)},$$

$$\Gamma\left(\frac{1}{2}+j_1\right) = \frac{(2j_1)!\sqrt{\pi}}{4^{j_1} j_1!},$$

$$\triangle^i = \left(\frac{\partial^2}{\partial x_1^2}+\cdots+\frac{\partial^2}{\partial x_n^2}\right)^i = \sum_{|j|=i} \frac{i!}{j_1!\cdots j_n!}\left(\frac{\partial^2}{\partial x_1^2}\right)^{j_1}\cdots\left(\frac{\partial^2}{\partial x_n^2}\right)^{j_n},$$

we derive that

$$\begin{aligned} & r^{2i} \sum_{|j|=i} \frac{\partial^{2j}\phi(0)}{(2j)!}\frac{1}{\Omega_n}\int_\Omega \sigma^{2j} d\sigma \\ &= r^{2i} \sum_{|j|=i} \frac{\partial^{2j}\phi(0)}{(2j_1)!\cdots(2j_n)!}\frac{1}{\Omega_n}\frac{2\Gamma\left(\frac{1}{2}+j_1\right)\cdots\Gamma\left(\frac{1}{2}+j_n\right)}{\Gamma\left(i+\frac{n}{2}\right)} \\ &= \frac{2\pi^{n/2}}{2^{2i}\ i!\ \Omega_n\Gamma\left(i+\frac{n}{2}\right)}\triangle^i\phi(0)r^{2i} = \frac{\Gamma(n/2)}{2^{2i}\ i!\ \Gamma\left(i+\frac{n}{2}\right)}\triangle^i\phi(0)r^{2i} \\ &= \frac{\triangle^i\phi(0)r^{2i}}{2^i\ i! n(n+2)\cdots(n+2i-2)}. \end{aligned}$$

Therefore,

$$\begin{aligned} S_\phi(r) &= \sum_{k=0}^{i} \frac{\triangle^k\phi(0)r^{2k}}{2^k\, k!\, n(n+2)\cdots(n+2k-2)} \\ &+ r^{2i+2} \sum_{|j|=i+1} \frac{1}{(2j)!}\frac{1}{\Omega_n}\int_\Omega \partial^{2j}\phi(\theta r\sigma)\sigma^{2j} d\sigma, \end{aligned}$$

for $i = 0, 1, \ldots$. Note that for $k = 0$, we define

$$\frac{\triangle^k \phi(0) r^{2k}}{2^k\, k!\, n(n+2)\cdots(n+2k-2)} = \phi(0).$$

In addition, we have the following theorem regarding the limits at the end points for the fractional Laplacian $(-\triangle)^s u(x)$ over the space $C_k(R^n)$.

THEOREM 3.2. *Let $u(x) \in C_k(R^n)$ with $n > 1$ and $i < s < i+1$ for $i = 0, 1, \ldots$. Then for any $x \in R^n$ (pointwise limit),*

$$\lim_{s \to (i+1)^-} (-\triangle)^s u(x) = (-1)^{i+1} \triangle^{i+1} u(x), \quad \textit{and}$$
$$\lim_{s \to i^+} (-\triangle)^s u(x) = (-1)^i \triangle^i u(x),$$

where $k = (k_1, k_2, \ldots, k_n) \in N_0^n$ and $k_1 + k_2 + \cdots + k_n = i + 1$.

P r o o f. It follows from Theorem 3.1 that

$$(-\triangle)^s u(x) = -\frac{1}{2} C_{n,s} \int_0^\infty r^{-1-2s}$$
$$\times \left[S(r) - \frac{r^2 \Omega_n \triangle u(x)}{n} - \cdots - \frac{2r^{2i} \Omega_n \triangle^i u(x)}{2^i\, i!\, n(n+2)\cdots(n+2i-2)} \right] dr.$$

Using equation (2.5), we have

$$\lim_{s \to (i+1)^-} \frac{r^{-1-2s}}{\Gamma(-2s)} = \frac{r^{-1-2(i+1)}}{\Gamma(-2i-2)} = \delta^{(2i+2)}(r).$$

This implies that

$$\lim_{s \to (i+1)^-} (-\triangle)^s u(x) = -\frac{1}{2} \pi^{-n/2} \lim_{s \to (i+1)^-} 2^{2s} \frac{\Gamma(-2s)}{\Gamma(1-s)} \Gamma\left(\frac{n+2s}{2}\right)$$
$$\times\, s \int_0^\infty \delta^{(2i+2)}(r)$$
$$\times \left[S(r) - \frac{r^2 \Omega_n \triangle u(x)}{n} - \cdots - \frac{2r^{2i} \Omega_n \triangle^i u(x)}{2^i\, i!\, n(n+2)\cdots(n+2i-2)} \right] dr.$$

Since

$$\frac{\partial^{2i+2}}{\partial r^{2i+2}} \left[\frac{r^2 \Omega_n \triangle u(x)}{n} + \cdots + \frac{2r^{2i} \Omega_n \triangle^i u(x)}{2^i\, i!\, n(n+2)\cdots(n+2i-2)} \right] = 0,$$

we arrive at

$$\lim_{s\to(i+1)^-}(-\triangle)^s u(x) = -\frac{1}{2}\pi^{-n/2}2^{2i+2}\,\Gamma\left(\frac{n+2+2i}{2}\right) \times (i+1)\,S^{(2i+2)}(0)\lim_{s\to(i+1)^-}\frac{\Gamma(-2s)}{\Gamma(1-s)}.$$

Applying the formula for $z \notin Z$,

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)},$$

we deduce

$$\begin{aligned}
&\lim_{s\to(i+1)^-}\frac{\Gamma(-2s)}{\Gamma(1-s)} = -\lim_{s\to(i+1)^-}\frac{\Gamma(s)\sin(\pi s)}{\Gamma(1+2s)\sin(2\pi s)}\\
&= -\frac{\Gamma(i+1)}{\Gamma(3+2i)}\lim_{s\to(i+1)^-}\frac{\pi\cos(\pi s)}{2\pi\cos(2\pi s)}\\
&= -\frac{\Gamma(i+1)}{\Gamma(3+2i)}\frac{(-1)^{i+1}}{2} = -\frac{i!}{(2i+2)!}\frac{(-1)^{i+1}}{2}.
\end{aligned}$$

Therefore,

$$\lim_{s\to(i+1)^-}(-\triangle)^s u(x) = \frac{1}{2}\pi^{-n/2}2^{2i+2}\,\Gamma\left(\frac{n+2+2i}{2}\right)(i+1) \times\frac{i!}{(2i+2)!}\frac{(-1)^{i+1}}{2}\frac{2\Omega_n(2i+2)!\triangle^{i+1}u(x)}{2^{i+1}\,(i+1)!\,n(n+2)\cdots(n+2i)}.$$

Noting that

$$\begin{aligned}
&\Omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)},\\
&\Gamma\left(\frac{n+2}{2}+i\right) = \left(\frac{n+2}{2}+i-1\right)\left(\frac{n}{2}+i-2\right)\cdots\frac{n}{2}\Gamma(n/2)\\
&= \frac{1}{2^{i+1}}(n+2i)\cdots n\,\Gamma(n/2).
\end{aligned}$$

Thus, we finally get

$$\lim_{s\to(i+1)^-}(-\triangle)^s u(x) = (-1)^{i+1}\triangle^{i+1}u(x).$$

It remains to show that

$$\lim_{s\to i^+}(-\triangle)^s u(x) = (-1)^i\triangle^i u(x).$$

Indeed, this formula follows immediately from equation (2.9) and the following identities for $s > i$:

$$\frac{r^{-1-2s}}{\Gamma(-2s)} = \delta^{(2s)}(r),$$
$$\frac{\partial^{2s}}{\partial r^{2s}}\left[\frac{r^2\Omega_n\triangle u(x)}{n} + \cdots + \frac{2r^{2i}\Omega_n\triangle^i u(x)}{2^i\, i!\, n(n+2)\cdots(n+2i-2)}\right] = 0,$$
$$\frac{1}{2}\pi^{-n/2}2^{2i}\,\Gamma\left(\frac{n+2i}{2}\right)\, i\frac{(i-1)!}{(2i)!}\frac{(-1)^i}{2}\frac{2\Omega_n(2i)!\triangle^i u(x)}{2^i\, i!\, n(n+2)\cdots(n+2i-2)}$$
$$= (-1)^i\triangle^i u(x),$$

where $\delta^{(2s)}(r)$ is the fractional $(2s)$-order derivative of $\delta(r)$. □

Remark 3.2. Di Nezza et al. [17] showed that the following statements for $u \in \mathcal{D}(R^n)$

$$\lim_{s\to 0^+}(-\triangle)^s u(x) = u(x),$$
$$\lim_{s\to 1^-}(-\triangle)^s u(x) = -\triangle u(x).$$

Obviously they are a special case of Theorem 3.2, since $\mathcal{D}(R^n) \subset C_k(R^n)$ for all $k \in N_0^n$.

Moreover, the following theorem can be derived from equations (3.3), (3.4), (3.2) and (2.2).

Theorem 3.3. *Let $i = 0, 1, \cdots$ and $i < s < i+1$. Then the generalized fractional Laplacian $(-\triangle)^s$ is normalized over the space $C^\infty(R^n)$ as*

$$(-\triangle)^s u(x) = -\frac{1}{2}C_{n,s}\lim_{m\to\infty}\int_0^{m+m^{-m}} r^{-1-2s}$$
$$\times\left[S_m(r) - \frac{r^2\Omega_n\triangle u(x)}{n} - \cdots - \frac{2r^{2i}\Omega_n\triangle^i u(x)}{2^i\, i!\, n(n+2)\cdots(n+2i-2)}\right] dr,$$

if the limit exists.

4. An illustrative example

In this section, we are going to present computation for the fractional Laplacian of a function that does not reside in the Schwartz space based on the results obtained in the previous section.

Theorem 4.1. *Let $s > 1$ and $n > 1$. Then $(-\triangle)^s(r^2 x_1) = 0$ on R^n.*

P r o o f. Clearly, we have by equation (2.10)

$$\begin{aligned}\triangle^2(r^2x_1) &= \sum_{m+i+l=2} 2^i \binom{m+l}{m}\binom{2}{m+l} \nabla^i \triangle^m r^2 \cdot \nabla^i \triangle^l x_1 \\ &= \sum_{m=0}^{1} 2^{2-m}\binom{2}{m} \nabla^{2-m}\triangle^m r^2 \cdot \nabla^{2-m} x_1 \\ &= 2\binom{2}{1}\nabla(2n)\cdot\nabla x_1 = 0,\end{aligned}$$

by noting that $\triangle r^2 = 2n$ and $\triangle^2 r^2 = 0$. Therefore, $(-\triangle)^s(r^2x_1) = 0$ for $s = 2, 3, \ldots$. Similarly,

$$\triangle(r^2x_1) = 2nx_1 + 4x_1.$$

Let $1 < s < 2$. By Theorem 3.3 (note that r^2x_1 is not a bounded function), we get

$$\begin{aligned}(-\triangle)^s(r^2x_1) &= -\frac{1}{2}C_{n,s}\lim_{m\to\infty}\int_0^{m+m^{-m}} r^{-1-2s}\left[S_m(r) - \frac{r^2\Omega_n\triangle(r^2x_1)}{n}\right]dr \\ &= -\frac{1}{2}C_{n,s}\lim_{m\to\infty}\int_0^{m+m^{-m}} r^{-1-2s}\left[S_m(r) - 2x_1r^2\Omega_n - \frac{4x_1r^2\Omega_n}{n}\right]dr \\ &= -\frac{1}{2}C_{n,s}\lim_{m\to\infty}\int_0^{m} r^{-1-2s}\left[S_m(r) - 2x_1r^2\Omega_n - \frac{4x_1r^2\Omega_n}{n}\right]dr \\ &\quad -\frac{1}{2}C_{n,s}\lim_{m\to\infty}\int_m^{m+m^{-m}} r^{-1-2s}\left[S_m(r) - 2x_1r^2\Omega_n - \frac{4x_1r^2\Omega_n}{n}\right]dr = T_1 + T_2.\end{aligned}$$

As for T_1, we note that $I_m(r) = 1$ and

$$S_m(r) = S(r) = \int_\Omega [u(x+r\sigma) - 2u(x) + u(x-r\sigma)]d\sigma.$$

Applying the spherical coordinates in the previous section, we get

$$\begin{aligned}&u(x+r\sigma) - 2u(x) + u(x-r\sigma) \\ &= \left[(x_1 + r\cos\theta_1)^2 + (x_2 + r\sin\theta_1\cos\theta_2)^2 + \cdots + (x_n + r\sin\theta_1\cdots\sin\theta_{n-1})^2\right] \\ &(x_1 + r\cos\theta_1) - 2(x_1^2 + \cdots + x_n^2)x_1 \\ &+ \left[(x_1 - r\cos\theta_1)^2 + (x_2 - r\sin\theta_1\cos\theta_2)^2 + \cdots + (x_n - r\sin\theta_1\cdots\sin\theta_{n-1})^2\right] \\ &(x_1 - r\cos\theta_1) \\ &= 2x_1r^2 + 4x_1r^2\cos^2\theta_1 + 4x_2r^2\cos\theta_1\sin\theta_1\cos\theta_2 + \cdots \\ &\quad + 4x_nr^2\cos\theta_1\sin\theta_1\cdots\sin\theta_{n-1} \\ &= 2x_1r^2 + 4x_1r^2y_1^2 + 4x_2r^2y_1y_2 + \cdots + 4x_nr^2y_1y_n.\end{aligned}$$

Thus,

$$S(r) \quad = \quad 2x_1r^2\Omega_n + 4x_1r^2\int_\Omega y_1^2 d\sigma + 4r^2\sum_{j=2}^n x_j\int_\Omega y_1y_jd\sigma.$$

Note that

$$\begin{aligned}
&\int_\Omega y_1y_jd\sigma = 0, \quad \text{and} \\
&\int_\Omega y_1^2 d\sigma = \text{Volume of the unit ball on } R^n = \frac{\pi^{n/2}}{\Gamma(1+n/2)} = \frac{\Omega_n}{n}.
\end{aligned}$$

The first comes from the integral cancellation over the unit sphere due to single factor of the y_1 or y_j. The second follows from the special case of Green's theorem [26]

$$2\int_\Omega y_1^2 d\sigma = \int_\Omega \frac{\partial y_1^2}{\partial n}d\sigma = \int_{\text{unit ball}} \triangle^2 y_1^2 dy = 2\frac{\pi^{n/2}}{\Gamma(1+n/2)}.$$

So,

$$S(r) = 2x_1r^2\Omega_n + \frac{4x_1r^2\Omega_n}{n}.$$

This infers that

$$T_1 = 0.$$

Regarding T_2, we deduce that

$$\begin{aligned}
T_2 \quad &= \quad -\frac{1}{2}C_{n,s}\lim_{m\to\infty}\int_m^{m+m^{-m}} r^{-1-2s} \\
&\quad \times t\left[2x_1r^2\Omega_nI_m(r) + \frac{4x_1r^2\Omega_n}{n}I_m(r) - 2x_1r^2\Omega_n - \frac{4x_1r^2\Omega_n}{n}\right]dr \\
&= \quad C_{n,s}x_1\Omega_n\left(1+\frac{2}{n}\right)\lim_{m\to\infty}\int_m^{m+m^{-m}} r^{1-2s}\left[1 - I_m(r)\right]dr = 0.
\end{aligned}$$

□

Remark 4.1. Clearly,

$$\lim_{s\to 1^+}(-\triangle)^s(r^2x_1) = 0 \neq (-\triangle)(r^2x_1) = -2nx_1 - 4x_1,$$

and $r^2x_1 \notin C_k(R^n)$, where $k = (k_1, k_2, \cdots, k_n) \in N_0^n$ and $k_1+k_2+\cdots+k_n = 2$. Hence, $(-\triangle)^s(r^2x_1) = 0$ can not be continuously extended to the values of $s \leq 1$.

5. The generalized fractional Laplacian on $C^\infty(R)$

In this section, we define the generalized fractional Laplacian $(-\triangle)^s$ over the space $C_k(R)$ for $s > 0$ and $s \neq 1, 2, \cdots$ by the normalization of the distribution $|x|^\lambda$, and present one example showing that

$$(-\triangle)^s \arctan x = \operatorname{sgn}(x) \frac{\Gamma(2s) \sin\left(\pi s - 2s \arctan \frac{1}{|x|}\right)}{(x^2+1)^s},$$

for $0 < s < 1$. Note that the function $\arctan x$ is not in the Schwartz space either.

Clearly, we have for $n = 1$ that

$$I_m(|x|) = \begin{cases} 1 & \text{if } |x| \leq m, \\ \tau\left(\frac{m^{2m}}{1+2m^{1+m}} x^2 - \frac{m^{2m+2}}{1+2m^{1+m}}\right) & \text{if } |x| > m, \end{cases}$$

and $I_m(|x|)$ is even and infinitely differentiable with respect to $x \in R$ with compact support $[-m - m^{-m},\ m + m^m]$.

The distribution $|x|^\lambda$ is given as [11]

$$\begin{aligned}(|x|^\lambda,\, \phi) &= \int_0^\infty x^\lambda \\ &\times \left\{\phi(x) + \phi(-x) - 2\left[\phi(0) + \cdots + \frac{x^{2k-2}}{(2k-2)!}\phi^{(2k-2)}(0)\right]\right\} dx, \quad (5.1)\end{aligned}$$

where $\phi \in \mathcal{D}(R)$ and $-2k - 1 < \lambda < -2k + 1$ for $k = 1, 2, \cdots$. Obviously, equation (2.2) is a special case of equation (5.1) for $\lambda = -2k$.

Let $s > 0$, $s \neq 1, 2, \cdots$, and $C^\infty(R)$ be the space of infinitely differentiable functions on R, which contains $S(R)$ as a proper subspace. Based on equation (1.3), we define the generalized fractional Laplacian $(-\triangle)^s u(x)$ over the space $C^\infty(R)$ as:

$$\begin{aligned}&(-\triangle)^s u(x) \\ &= -\frac{1}{2} C_{1,s} \lim_{m\to\infty} \int_{-m-m^{-m}}^{m+m^{-m}} \frac{[u(x+y) - 2u(x) + u(x-y)] I_m(|y|)}{|y|^{1+2s}} dy \\ &= -C_{1,s} \lim_{m\to\infty} \int_0^{m+m^{-m}} \frac{[u(x+y) - 2u(x) + u(x-y)] I_m(y)}{y^{1+2s}} dy, (5.2)\end{aligned}$$

if the limit exists.

For any fixed $x \in R$, the function

$$S_m(y) = [u(x+y) - 2u(x) + u(x-y)] I_m(y) = S(y) I_m(|y|),$$

is even and infinitely differentiable with respect to y since $u \in C^\infty(R)$, and has compact support.

In particular, equation (5.2) becomes equation (1.3) for any function $u(x) \in S(R)$ as the integral

$$\int_0^{m+m^{-m}} \frac{[u(x+y)-2u(x)+u(x-y)]I_m(y)}{y^{1+2s}}dy,$$

uniformly converges with respect to m and

$$\lim_{m\to\infty} I_m(y) = 1.$$

In fact, for any smooth function u, a second order of Taylor's expansion derives

$$\frac{|[u(x+y)-2u(x)+u(x-y)]I_m(y)|}{y^{1+2s}} \le \frac{\left\|u^{(2)}\right\|_{L^\infty}}{y^{2s-1}}, \quad 0<s<1,$$

which is integrable near zero, by noting that $0 \le I_m(y) \le 1$. From $u \in S(R)$, we deduce that

$$|y^2 u^{(2)}(y)| \le \text{C (const)}, \quad \text{as } |y| \to \infty.$$

This ensures

$$\frac{|[u(x+y)-2u(x)+u(x-y)]I_m(y)|}{y^{1+2s}},$$

is integrable at infinity.

Clearly,

$$\left.\frac{d^i}{dy^i}S_m(y)\right|_{y=0} = \begin{cases} 0 & \text{if } i = 0,1,3,\ldots, \\ 2u^{(2j)}(x), & \text{if } i = 2j \text{ and } j = 1,2,\ldots, \end{cases}$$

by noting that $I_m(0) = 1$ and $I_m^{(j)}(0) = 0$.

Evidently for $n = 1$, $C_k(R^n)$ turns out to be

$$\begin{aligned} C_k(R) \; = \; \Big\{ & u(x) \text{ is bounded and } u^{(2k)}(x) \in C(R) : \\ & \exists\, M_k(\text{const}) > 0, \text{ such that } \left|u^{(2k)}(x)\right| \le \frac{M_k}{x^2} \text{ as } |x| \to \infty \Big\}, \end{aligned}$$

and

$$\mathcal{D}(R) \subset S(R) \subset C_k(R) \subset C(R),$$

for all $k = 0, 1 \cdots$. We claim that $u^{(2k)}(x)$ is bounded for every k as

$$\lim_{|x|\to\infty} |u^{(2k)}(x)| = 0.$$

Further, equation (5.2) becomes equation (1.3) for any function $u(x) \in C_k(R)$ and the proof is identical to the above.

Applying equation (5.1), we have the following theorem.

THEOREM 5.1. *Let $k-1 < s < k$ and $k = 1, 2, \cdots$. Then the fractional Laplacian $(-\triangle)^s$ is defined over $C_k(R)$ as:*

$$(-\triangle)^s u(x) = -C_{1,s} \int_0^\infty y^{-1-2s} \times \left[S(y) - u^{(2)}(x) y^2 - \cdots - \frac{2y^{2k-2}}{(2k-2)!} u^{(2k-2)}(x) \right] dy. \quad (5.3)$$

Note that if $k = 1$, we define

$$u^{(2)}(x) y^2 + \cdots + \frac{2y^{2k-2}}{(2k-2)!} u^{(2k-2)}(x) = 0.$$

P r o o f. Let $u(x) \in C_k(R)$. It follows from Taylor's expansion that

$$\begin{aligned} S(y) &= u(x+y) - 2u(x) + u(x-y) = u^{(2)}(x) y^2 + \cdots \\ &\quad + \frac{2y^{2k-2}}{(2k-2)!} u^{(2k-2)}(x) + \frac{y^{2k}}{(2k)!} (u^{(2k)}(x+\theta y) + u^{(2k)}(x - \theta y)), \end{aligned}$$

where $\theta \in (0, 1)$. Clearly, there exists a constant $M_k > 0$ such that for a fixed $x \in R$

$$\left| u^{(2k)}(x+\theta y) \right| \le \frac{M_k}{(x+\theta y)^2} \sim M_k y^{-2}, \quad \text{and}$$
$$\left| u^{(2k)}(x-\theta y) \right| \le \frac{M_k}{(x-\theta y)^2} \sim M_k y^{-2},$$

when $|y| \to \infty$. Therefore, the integral

$$\int_0^\infty y^{-1-2s} \left[S(y) - u^{(2)}(x) y^2 - \cdots - \frac{2y^{2k-2}}{(2k-2)!} u^{(2k-2)}(x) \right] dy,$$

is well defined and converges by noting that

$$y^{-1-2s} \left[S(y) - u^{(2)}(x) y^2 - \cdots - \frac{2y^{2k-2}}{(2k-2)!} u^{(2k-2)}(x) \right] \sim \frac{2y^{-1-2s+2k}}{(2k)!} u^{(2k)}(x) \ \text{ as } \ y \to 0^+,$$

and $-1 - 2s + 2k > -1$, as well as

$$y^{-1-2s} \left[S(y) - u^{(2)}(x) y^2 - \cdots - \frac{2y^{2k-2}}{(2k-2)!} u^{(2k-2)}(x) \right] \sim \frac{2M_k}{(2k)!} y^{-1-2s+2k-2} \ \text{ as } y \to \infty,$$

and $-1 - 2s + 2k - 2 < -1$.

Furthermore,

$$\lim_{m\to\infty}\int_{m+m^{-m}}^{\infty} y^{-1-2s}\left[S_m(y)-u^{(2)}(x)y^2-\cdots-\frac{2y^{2k-2}}{(2k-2)!}u^{(2k-2)}(x)\right]dy$$
$$=\lim_{m\to\infty}\int_{m+m^{-m}}^{\infty} y^{-1-2s}\left[-u^{(2)}(x)y^2-\cdots-\frac{2y^{2k-2}}{(2k-2)!}u^{(2k-2)}(x)\right]dy=0.$$

From equations (5.1) and (5.2), we further derive that

$$(-\triangle)^s u(x)=-C_{1,s}\lim_{m\to\infty}\int_0^{m+m^{-m}} y^{-1-2s}$$
$$\times\left[S_m(y)-u^{(2)}(x)y^2-\cdots-\frac{2y^{2k-2}}{(2k-2)!}u^{(2k-2)}(x)\right]dy$$
$$=-C_{1,s}\lim_{m\to\infty}\int_0^{\infty} y^{-1-2s}\left[S_m(y)-u^{(2)}(x)y^2-\cdots-\frac{2y^{2k-2}}{(2k-2)!}u^{(2k-2)}(x)\right]dy$$
$$=-C_{1,s}\lim_{m\to\infty}\int_0^{\infty} y^{-1-2s}\left[S(y)-u^{(2)}(x)y^2-\cdots-\frac{2y^{2k-2}}{(2k-2)!}u^{(2k-2)}(x)\right]dy,$$

as

$$\lim_{m\to\infty}\int_0^{\infty} y^{-1-2s}\left[S_m(y)-S(y)\right]dy$$
$$=\lim_{m\to\infty}\int_m^{\infty} y^{-1-2s}\left[S_m(y)-S(y)\right]dy=0,$$

by noting that $S_m(y)$ and $S(y)$ are bounded functions due to the fact $u(x)\in C_k(R)$. □

Similarly, we have the following theorem regarding the limits at the end points for the fractional Laplacian $(-\triangle)^s u(x)$ over the space $C_k(R)$.

THEOREM 5.2. *Let $k-1<s<k$ and $k=1,2,\cdots$. Then in pointwise convergence,*

$$\lim_{s\to k^-}(-\triangle)^s u(x)=(-1)^k u^{(2k)}(x), \quad \text{and}$$
$$\lim_{s\to(k-1)^+}(-\triangle)^s u(x)=(-1)^{k-1}u^{(2k-2)}(x),$$

for $u(x)\in C_k(R)$.

P r o o f. It follows from Theorem 5.1 that

$$(-\triangle)^s u(x)=-C_{1,s}\int_0^{\infty} y^{-1-2s}$$
$$\times\left[S(y)-u^{(2)}(x)y^2-\cdots-\frac{2y^{2k-2}}{(2k-2)!}u^{(2k-2)}(x)\right]dy,$$

for $k-1<s<k$ and $k=1,2,\ldots$. Thus, we deduce that

$$\begin{aligned}
\lim_{s\to k^-}(-\triangle)^s u(x) &= -\lim_{s\to k^-} C_{1,s}\Gamma(-2s)\int_0^\infty \delta^{(2k)}(y) \\
&\quad\times\left[S(y)-u^{(2)}(x)y^2-\cdots-\frac{2y^{2k-2}}{(2k-2)!}u^{(2k-2)}(x)\right]dy \\
&= -\lim_{s\to k^-} C_{1,s}\Gamma(-2s)S^{(2k)}(0) = -2\lim_{s\to k^-} C_{1,s}\Gamma(-2s)u^{(2k)}(x) \\
&= -\pi^{-1/2}2^{2k+1}\,k\,u^{(2k)}(x)\Gamma\left(k+\frac{1}{2}\right)\lim_{s\to k^-}\frac{\Gamma(-2s)}{\Gamma(1-s)} \\
&= \pi^{-1/2}2^{2k+1}\,k\,u^{(2k)}(x)\Gamma\left(k+\frac{1}{2}\right)\frac{(k-1)!}{(2k)!}\frac{(-1)^k}{2},
\end{aligned}$$

by applying

$$\int_0^\infty \delta^{(2k)}(y)\left[S(y)-u^{(2)}(x)y^2-\cdots-\frac{2y^{2k-2}}{(2k-2)!}u^{(2k-2)}(x)\right]dy = S^{(2k)}(0).$$

Using the formula

$$\Gamma\left(k+\frac{1}{2}\right)=\frac{(2k)!}{2^{2k}k!}\sqrt{\pi},$$

we finally reach

$$\lim_{s\to k^-}(-\triangle)^s u(x) = (-1)^k u^{(2k)}(x).$$

It remains to prove that

$$\lim_{s\to (k-1)^+}(-\triangle)^s u(x) = (-1)^{k-1} u^{(2k-2)}(x).$$

This immediately follows from the following identities for $s>k-1$:

$$\begin{aligned}
&\frac{y_+^{-1-2s}}{\Gamma(-2s)} = \delta^{(2s)}(y), \\
&\frac{\partial^{2s}}{\partial y^{2s}}\left[u^{(2)}(x)y^2+\cdots+\frac{2y^{2k-2}}{(2k-2)!}u^{(2k-2)}(x)\right]dy = 0, \\
&\Gamma\left(k-\frac{1}{2}\right)=\frac{(2k-2)!}{2^{2k-2}(k-1)!}\sqrt{\pi}, \\
&\pi^{-1/2}2^{2k-1}\,(k-1)\,u^{(2k-2)}(x)\Gamma\left(k-\frac{1}{2}\right)\frac{(k-2)!}{(2k-2)!}\frac{(-1)^{k-1}}{2} \\
&= (-1)^{k-1}u^{(2k-2)}(x).
\end{aligned}$$

□

Similarly, the following theorem can be obtained from equations (5.1) and (5.2).

THEOREM 5.3. *Let $k-1 < s < k$ and $k = 1, 2, \cdots$. Then the fractional Laplacian $(-\triangle)^s$ is defined over the space $C^\infty(R)$ as:*

$$\begin{aligned}(-\triangle)^s u(x) &= -C_{1,s} \lim_{m\to\infty} \int_0^{m+m^{-m}} y^{-1-2s} \\ &\quad \times \left[S_m(y) - u^{(2)}(x) y^2 - \cdots - \frac{2y^{2k-2}}{(2k-2)!} u^{(2k-2)}(x) \right] dy,\end{aligned}$$

if the limit exists.

Evidently, Theorem 5.3 turns out to be Theorem 5.1 if $u(x) \in C_k(R)$.

To end off this section, we present the following example using Theorem 5.1 and Cauchy's residue theorem.

EXAMPLE 5.1. Let $0 < s < 1$. Then,

$$(-\triangle)^s \arctan x = \operatorname{sgn}(x) \frac{\Gamma(2s) \sin\left(\pi s - 2s \arctan \frac{1}{|x|}\right)}{(x^2+1)^s},$$

where sgn (x) is the sign function. In particular,

$$(-\triangle)^{\frac{1}{2}} \arctan x = \frac{x}{x^2+1}.$$

P r o o f. Clearly, $u(x) = \arctan x \in C_1(R)$ (but not in $S(R)$) since it is bounded and

$$\left|(\arctan x)^{(2)}\right| = \left|\frac{-2x}{(1+x^2)^2}\right| = \frac{2|x|}{(1+x^2)(1+x^2)} \le \frac{1}{1+x^2} \le \frac{1}{x^2},$$

for $x \neq 0$. By Theorem 5.1 for $0 < s < 1$ and integration by parts,

$$\begin{aligned}&(-\triangle)^s \arctan x \\ &= -C_{1,s} \int_0^\infty \frac{\arctan(x+y) - 2\arctan x + \arctan(x-y)}{y^{1+2s}} dy \\ &= \frac{C_{1,s}}{2s} \frac{\arctan(x+y) - 2\arctan x + \arctan(x-y)}{y^{2s}} \Bigg|_{y=0}^{\infty} \\ &\quad - \frac{C_{1,s}}{2s} \int_0^\infty y^{-2s} \left[\frac{1}{1+(x+y)^2} - \frac{1}{1+(x-y)^2} \right] dy \\ &= \frac{2xC_{1,s}}{s} \int_0^\infty \frac{y^{1-2s}}{(1+(x+y)^2)(1+(x-y)^2)} dy,\end{aligned}$$

by applying

$$\lim_{y\to\infty}\frac{\arctan(x+y)-2\arctan x+\arctan(x-y)}{y^{2s}}=0,\quad \text{and}$$

$$\lim_{y\to 0^+}\frac{\arctan(x+y)-2\arctan x+\arctan(x-y)}{y^{2s}}=0.$$

In particular,

$$(-\triangle)^s \arctan x|_{x=0}=0,$$

since the integral

$$\int_0^\infty \frac{y^{1-2s}}{(1+y^2)^2}dy,$$

is well defined and converges for $0<s<1$.

Assuming $x\neq 0$, we are going to use the following contour

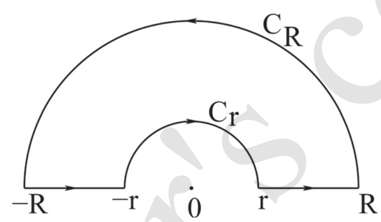

FIGURE 1.

and Cauchy's residue theorem to evaluate the integral

$$\int_0^\infty \frac{y^{1-2s}}{(1+(x+y)^2)(1+(x-y)^2)}dy.$$

Clearly,

$$\begin{aligned}
&\int_r^R \frac{y^{1-2s}}{(1+(x+y)^2)(1+(x-y)^2)}dy+\int_{C_R}\frac{z^{1-2s}}{(1+(x+z)^2)(1+(x-z)^2)}dz\\
&+\int_{-R}^{-r}\frac{y^{1-2s}}{(1+(x+y)^2)(1+(x-y)^2)}dy\\
&+\int_{C_r}\frac{z^{1-2s}}{(1+(x+z)^2)(1+(x-z)^2)}dz\\
&=2\pi i\mathrm{Res}\left\{\frac{z^{1-2s}}{(1+(x+z)^2)(1+(x-z)^2)},\ x+i\right\}\\
&\quad+2\pi i\mathrm{Res}\left\{\frac{z^{1-2s}}{(1+(x+z)^2)(1+(x-z)^2)},\ -x+i\right\},
\end{aligned}$$

where $0<r<1/2$ and R is positively large in Figure 1.

Direct computations imply that

$$\operatorname{Res}\left\{\frac{z^{1-2s}}{(1+(x+z)^2)(1+(x-z)^2)},\, x+i\right\} = \frac{\pi(x+i)^{1-2s}}{1+(2x+i)^2},$$
$$\operatorname{Res}\left\{\frac{z^{1-2s}}{(1+(x+z)^2)(1+(x-z)^2)},\, -x+i\right\} = \frac{\pi(-x+i)^{1-2s}}{1+(2x-i)^2},$$
$$\lim_{R\to\infty}\int_{C_R}\frac{z^{1-2s}}{(1+(x+z)^2)(1+(x-z)^2)}dz = 0, \quad \text{and}$$
$$\lim_{r\to 0^+}\int_{C_r}\frac{z^{1-2s}}{(1+(x+z)^2)(1+(x-z)^2)}dz = 0,$$

as $-1 < 1-2s < 1$. Making the variable change,

$$\begin{aligned}
&\int_{-R}^{-r}\frac{y^{1-2s}}{(1+(x+y)^2)(1+(x-y)^2)}dy \\
&= \int_r^R \frac{(-1)^{1-2s}y^{1-2s}}{(1+(x+y)^2)(1+(x-y)^2)}dy \\
&= e^{i\pi(1-2s)}\int_r^R \frac{y^{1-2s}}{(1+(x+y)^2)(1+(x-y)^2)}dy,
\end{aligned}$$

as

$$\frac{1}{(1+(x+y)^2)(1+(x-y)^2)}$$

is even with respect to y. Hence,

$$\begin{aligned}
&\int_0^\infty \frac{y^{1-2s}}{(1+(x+y)^2)(1+(x-y)^2)}dy \\
&= \frac{\pi}{1+e^{i\pi(1-2s)}}\left[\frac{(x+i)^{1-2s}}{1+(2x+i)^2} + \frac{(-x+i)^{1-2s}}{1+(2x-i)^2}\right] \\
&= \frac{4x\pi}{1+e^{i\pi(1-2s)}}\frac{(x-i)(x+i)^{1-2s}+(x+i)(-x+i)^{1-2s}}{16x^4+16x^2}.
\end{aligned}$$

Assuming $x > 0$, we have

$$\begin{aligned}
x+i &= \sqrt{x^2+1}e^{i\arctan\frac{1}{x}}, \\
x-i &= \sqrt{x^2+1}e^{-i\arctan\frac{1}{x}}, \\
-x+i &= \sqrt{x^2+1}e^{-i\arctan\frac{1}{x}+i\pi}.
\end{aligned}$$

Thus we come to

$$\begin{aligned}
(x-i)(x+i)^{1-2s} &= (x^2+1)^{1-s}e^{-i2s\arctan\frac{1}{x}}, \\
(x+i)(-x+i)^{1-2s} &= (x^2+1)^{1-s}e^{i2s\arctan\frac{1}{x}+i\pi(1-2s)}.
\end{aligned}$$

Therefore,

$$(-\triangle)^s \arctan x = \frac{C_{1,s}\pi}{2s(1+e^{i\pi(1-2s)})}\frac{e^{-i2s\arctan\frac{1}{x}} + e^{i2s\arctan\frac{1}{x}+i\pi(1-2s)}}{(x^2+1)^s}$$
$$= \frac{C_{1,s}\pi e^{-\frac{i\pi(1-2s)}{2}}}{2s(1+e^{i\pi(1-2s)})e^{-\frac{i\pi(1-2s)}{2}}}\frac{e^{-i2s\arctan\frac{1}{x}} + e^{i2s\arctan\frac{1}{x}+i\pi(1-2s)}}{(x^2+1)^s}.$$

Obviously,

$$(1+e^{i\pi(1-2s)})e^{-\frac{i\pi(1-2s)}{2}} = 2\cos\left(\frac{\pi}{2} - s\pi\right) = 2\sin\pi s,$$
$$e^{-\frac{i\pi(1-2s)}{2}}(e^{-i2s\arctan\frac{1}{x}} + e^{i2s\arctan\frac{1}{x}+i\pi(1-2s)})$$
$$= 2\cos\left(2s\arctan\frac{1}{x} + \frac{\pi(1-2s)}{2}\right).$$

In summary,

$$(-\triangle)^s \arctan x = \frac{C_{1,s}\pi\cos\left(2s\arctan\frac{1}{x} + \frac{\pi(1-2s)}{2}\right)}{2s\sin\pi s(x^2+1)^s}$$
$$= \frac{\pi^{1/2}2^{2s-1}\Gamma\left(\frac{1}{2}+s\right)\cos\left(2s\arctan\frac{1}{x} + \frac{\pi(1-2s)}{2}\right)}{\Gamma(1-s)\ \sin\pi s\ (x^2+1)^s}$$
$$= \frac{\Gamma(2s)\cos\left(2s\arctan\frac{1}{x} + \frac{\pi(1-2s)}{2}\right)}{(x^2+1)^s}$$
$$= -\frac{\Gamma(2s)\sin\left(2s\arctan\frac{1}{x} - \pi s\right)}{(x^2+1)^s} = \frac{\Gamma(2s)\sin\left(\pi s - 2s\arctan\frac{1}{x}\right)}{(x^2+1)^s},$$

by the formulas

$$\Gamma(1-s)\Gamma(s) = \frac{\pi}{\sin\pi s}, \quad \text{and}$$
$$\Gamma(s)\Gamma(s+\frac{1}{2}) = 2^{1-2s}\sqrt{\pi}\Gamma(2s).$$

Similarly for $x < 0$,

$$(-\triangle)^s \arctan x = -\frac{\Gamma(2s)\sin\left(\pi s + 2s\arctan\frac{1}{x}\right)}{(x^2+1)^s}.$$

In particular,

$$(-\triangle)^{\frac{1}{2}} \arctan x = \frac{x}{x^2+1}, \quad \text{and} \tag{5.4}$$
$$\lim_{s\to 1^-}(-\triangle)^s \arctan x = (-\triangle)^1 \arctan x = \frac{2x}{(x^2+1)^2},$$

for all $x \in R$. □

REMARK 5.1. Li et al. [28] recently studied the half-order Laplacian operator $(-\triangle)^{\frac{1}{2}}$ on the dual space of the Schwartz test functions based on the generalized convolution and Temple's delta sequence. They showed that

$$(-\triangle)^{\frac{1}{2}} \arctan x = \frac{x}{1+x^2}. \tag{5.5}$$

Clearly, equation (5.4) coincides with equation (5.5). Furthermore, the function with respect to s

$$\operatorname{sgn}(x) \frac{\Gamma(2s) \sin\left(\pi s - 2s \arctan \frac{1}{|x|}\right)}{(x^2+1)^s},$$

is well defined beyond the open interval $(0,1)$. This inspires us to investigate the fractional Laplacian $(-\triangle)^s \arctan x$ for other values such as $-1/2 < s < 0$ by possible analytic continuation.

Let $k = 1, 2, \ldots$. We define the normed space $W_k(R)$ as

$$W_k(R) = \left\{u(x) : u^{(2k)}(x) \text{ is continuous on } R \text{ and } \|u\|_k < \infty\right\},$$

where

$$\|u\|_k = \max\left\{\sup_{x\in R} |xu(x)|,\ \sup_{x\in R} |xu'(x)|,\ \sup_{x\in R} \left|(x^2+1)u^{(2k)}(x)\right|\right\}.$$

To conclude this paper, we would like to mention there is an application of Theorem 5.1 in the reference [29], where Li and Beaudin constructed an integral representation for the generalized Riesz derivative ${}_{RZ}D_x^{2s}u(x)$ for $k < s < k+1$ with $k = 0, 1, \ldots$, and obtained for $u(x) \in W_{k+1}(R)$

$$\begin{aligned} {}_{RZ}D_x^{2k+1}u(x) &= \frac{2^{2k+1}(k+1/2)(2k)!}{(-4)^k \pi} \int_0^\infty y^{-2-2k} \\ &\times \left[u(x+y) - 2u(x) + u(x-y) - u^{(2)}(x)y^2 - \cdots - \frac{2y^{2k}}{(2k)!}u^{(2k)}(x)\right] dy, \end{aligned}$$

for $k = 0, 1, \ldots$. The above integral clearly extends the α-order Riesz derivative in the classical sense, given as

$${}_{RZ}D_x^{\alpha}u(x) = -\Psi_\alpha \left({}_{RL}D_{-\infty,x}^{\alpha} + {}_{RL}D_{x,\infty}^{\alpha}\right) u(x),$$

where

$$\Psi_\alpha = \frac{1}{2\cos\dfrac{\alpha\pi}{2}}, \quad \alpha \neq 1, 3, \ldots,$$

to all odd numbers $1, 3, \ldots$, for $u(x) \in W_{k+1}(R)$.

6. **Conclusion**

For the first time, we have extended the fractional Laplacian $(-\triangle)^s$ over the space $C^\infty(R^n)$ for all $s > 0$ and $s \neq 1, 2, \cdots$, based on the normalization in distribution theory, Pizzetti's formula as well as surface integrals. Additionally, we have showed that such an extension is continuous at the end points $s = 1, 2, \cdots$ on the space $C_k(R^n) \supsetneq S(R^n)$, and further presented a couple of examples to demonstrate computations using the main results obtained, with the help of special functions and Cauchy's residue theorem. Finally, an application to extending the classical Riesz derivative to odd numbers was also mentioned.

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