JOURNAL OF INTEGRAL
EQUATIONS AND APPLICATIONS
Volume 33 (2021), No. 4, 463–475

UNIQUENESS OF THE PARTIAL INTEGRO-DIFFERENTIAL EQUATIONS

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We study the uniqueness of solutions for certain partial integro-differential equations with the initial conditions in a Banach space. The results derived are new and based on Babenko's approach, convolution and Banach's contraction principle. We also include several examples for the illustration of main theorems.

1. Introduction

Let $0 < \omega_i < \infty$ for $i = 1, 2, \dots, n$ and $x \in \Omega = [0, \omega_1] \times [0, \omega_2] \times \cdots \times [0, \omega_n] \subset R^n$. Let I_k^β be the partial Riemann–Liouville fractional integral of order $\beta \in R^+$ with respect to $x_k \in [0, \omega_k]$, with initial point zero [6],

$$(I_k^\beta u)(x) = \frac{1}{\Gamma(\beta)} \int_0^{x_k} (x_k - s)^{\beta-1} u(x_1, \dots, x_{k-1}, s, x_{k+1}, \dots, x_n)\, ds$$

for $k = 1, 2, \dots, n$.

In particular for $\beta = 0$,

$$(I_k^0 u)(x) = u(x).$$

Clearly for $\beta_{ik} \geq 0$,

$$I_1^{\beta_{1k}} I_2^{\beta_{2k}} \cdots I_n^{\beta_{nk}} u(x) = \frac{1}{\Gamma(\beta_{1k}) \cdots \Gamma(\beta_{nk})} \int_0^{x_1} \cdots \int_0^{x_n} (x_1 - s_1)^{\beta_{1k}-1} \cdots (x_n - s_n)^{\beta_{nk}-1} u(s_1, \dots, s_n)\, ds_n \cdots ds_1,$$

which is regarded as the partial Riemann–Liouville fractional integral with order $\beta_{1k} + \cdots + \beta_{nk}$ (β_{ik}-th order in x_i-direction for $i = 1, 2, \dots, n$); see [12].

Let $t \in [0, t_0]$ with $0 < t_0 < \infty$. The space $L([0, t_0] \times \Omega)$ of Lebesgue integrable functions on $[0, t_0] \times \Omega$ is defined as

$$L([0, t_0] \times \Omega) = \left\{ u(t, x) : \|u\|_L = \int_{[0,t_0]\times\Omega} |u(t, x)|\, dt dx < \infty \right\}.$$

The space $C([0, t_0] \times \Omega)$ of continuous functions on $[0, t_0] \times \Omega$ is given by

$$C([0, t_0] \times \Omega) = \{u(t, x) : \|u\|_C = \max_{t\in[0,t_0], x\in\Omega} |u(t, x)| < \infty\}.$$

This work is supported by the Natural Sciences and Engineering Research Council of Canada (Grant No. 2019-03907).
2020 AMS *Mathematics subject classification:* 26A33, 34A12, 45E10.
Keywords and phrases: partial Riemann–Liouville fractional integral, Banach's fixed point theorem, Babenko's approach, Mittag–Leffler function, gamma function.
Received by the editors on October 18, 2020.

DOI: 10.1216/jie.2021.33.463

Clearly,

$$C([0, t_0] \times \Omega) \subset L([0, t_0] \times \Omega).$$

We further define the space $S_m([0, t_0] \times \Omega)$, for $m \in N = \{1, 2, \ldots\}$, of those functions on $[0, t_0] \times \Omega$ with up to m-th order continuous partial derivatives with respect to t by

$$S_m([0, t_0] \times \Omega) = \left\{ u(t, x) : \frac{\partial^m}{\partial t^m} u(t, x) \in C([0, t_0] \times \Omega) \text{ and } \|u\|_m < \infty \right\},$$

where

$$\|u\|_m = \max\left\{ \|u(t, x)\|_C, \left\| \frac{\partial}{\partial t} u(t, x) \right\|_C, \ldots, \left\| \frac{\partial^m}{\partial t^m} u(t, x) \right\|_C \right\}.$$

Obviously,

$$S_m([0, t_0] \times \Omega) \subset C([0, t_0] \times \Omega) \subset L([0, t_0] \times \Omega).$$

Furthermore, $S_m([0, t_0] \times \Omega)$ is a Banach space using Theorem 7.17 in [15] stated as:

Theorem. *If $\{u_n\}$ is a sequence of differentiable functions on $[a, b]$ such that $\lim_{n\to\infty} u_n(x_0)$ exists (and is finite) for some $x_0 \in [a, b]$ and the sequence $\{u_n'\}$ converges uniformly on $[a, b]$, then u_n converges uniformly to a function u on $[a, b]$, and $u'(x) = \lim_{n\to\infty} u_n'(x)$ for $x \in [a, b]$.*

Let $\lambda_k(x)$ be continuous on Ω for $k = 1, 2, \ldots, l$. In this paper, we shall consider the partial integro-differential equation in the space $S_m([0, t_0] \times \Omega)$ with a given function $g(t, x)$

$$\frac{\partial^m}{\partial t^m} u(t, x) - \sum_{k=1}^{l} \lambda_k(x) I_1^{\beta_{1k}} \cdots I_n^{\beta_{nk}} u(t, x) = g(t, x) \in C([0, t_0] \times \Omega), \tag{1}$$

with the initial conditions

$$u(0, x) = 0, \ldots, \frac{\partial^{m-1}}{\partial t^{m-1}} u(0, x) = 0. \tag{2}$$

by Babenko's approach [2], which treats integral operators like variables in solving differential and integral equations. The method itself is close to the Laplace transform method in the ordinary sense, but it can be used in more cases [13; 14], such as dealing with integral or fractional differential equations with distributions whose Laplace transforms do not exist in the classical sense. Clearly, it is always necessary to show convergence of the series obtained as solutions. Recently, Li studied the generalized Abel's integral equations of the second kind with variable coefficients by Babenko's technique [8; 11].

In addition, we study uniqueness of solutions for the partial integro-differential equation by Banach's contraction principle with condition (2)

$$\frac{\partial^m}{\partial t^m} u(t, x) - \sum_{k=1}^{l} \lambda_k(x) I_1^{\beta_{1k}} \cdots I_n^{\beta_{nk}} u(t, x) = f\left(t, x, u(t, x), \ldots, \frac{\partial^{m-1}}{\partial t^{m-1}} u(t, x)\right), \tag{3}$$

where $f(t, x, y_0, \ldots, y_{m-1})$ is continuous on $[0, t_0] \times \Omega \times R^m$ with a Lipchitz condition.

To the best of the author's knowledge, (1) and (3) are new in current studies. There are many research works on fractional differential and integral equations involving Riemann–Liouville or Caputo operators with boundary value problems or initial conditions. For example, Bai et al. [3] considered the existence and uniqueness for the following fractional differential equation with the initial conditions

$$\begin{aligned} D^q_{0+}u(t) &= f(t, u(t), D^{q-1}_{0+}u(t)), \quad t \in (0, T) \\ u(0) = 0, &\quad D^{q-1}_{0+}u(0) = u_0, \end{aligned}$$

where $f \in C([0, T] \times R^2)$ and $D^q_{0+}u(t)$ is the standard Riemann–Liouville fractional derivative with $1 < q < 2$, based on fixed point theorems, and lower and upper solution method.

In [4], Eshaghi Gordji et al. proved the existence and uniqueness of the solutions of the following nonhomogeneous nonlinear Volterra integral equation:

$$u(x) = f(x) + \phi\left(\int_a^x F(x, t, u(t))\,dt\right), \quad u \in X = C([a, b], R^n)$$

where $x, t \in [a, b]$ with $-\infty < a < \infty$, $f : [a, b] \to R^n$ is a mapping, F is a continuous function on $[a, b] \times [a, x) \times X$, and $\phi : X \to X$ is a bounded linear transformation.

Let $X_\mu[a, b]$ be the space of those Lebesgue measurable functions u on $[a, b]$ for which $x^{\mu-1}u(x)$ is absolutely integrable:

$$X_\mu[a, b] = \left\{u : [a, b] \to C \text{ and } \|u\|_{X_\mu} = \int_a^b x^{\mu-1}|u(x)|\,dx < \infty\right\}.$$

Then it is a Banach space. In addition, the fractional version of the Hadamard-type integral and derivative are defined by

$$(\mathscr{J}^\alpha_{a+,\mu}u)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \left(\frac{t}{x}\right)^\mu \left(\log\frac{x}{t}\right)^{\alpha-1} u(t)\frac{dt}{t} \quad \alpha > 0, x \in [a, b],$$

and

$$(\mathscr{D}^\alpha_{a+,\mu}u)(x) = x^{-\mu}\delta^n x^\mu (\mathscr{J}^{n-\alpha}_{a+,\mu}u)(x), \quad \delta = x\frac{d}{dx},$$

where $n = [\alpha] + 1$ and $[\alpha]$ being integral part of α. Very recently, Li [10] obtained uniqueness of solutions for the nonlinear coupled system of integral equations given below:

$$\begin{aligned} a_n(\mathscr{J}^{\alpha_n}_{a+,\mu}u)(x) + \cdots + a_1(\mathscr{J}^{\alpha_1}_{a+,\mu}u)(x) + u(x) &= g_1(x, u(x), v(x)), \\ b_n(\mathscr{J}^{\beta_n}_{a+,\mu}v)(x) + \cdots + b_1(\mathscr{J}^{\beta_1}_{a+,\mu}v)(x) + v(x) &= g_2(x, u(x), v(x)), \end{aligned}$$

on the product space $X_\mu[a, b] \times X_\mu[a, b]$, based on Babenko's approach and Banach's contraction principle. Furthermore, Li [9] derived uniqueness of solutions for the following nonlinear Hadamard-type integro-differential equation for all $\mu \in R$, in the space $AC_0[a, b]$

$$\mathscr{D}^{\alpha_n}_{a+,\mu}u + a_{n-1}\mathscr{D}^{\alpha_{n-1}}_{a+,\mu}u + \cdots + a_0\mathscr{D}^{\alpha_0}_{a+,\mu}u + b_{n+1}\mathscr{J}^{\beta_{n+1}}_{a+,\mu}u + \cdots + b_m\mathscr{J}^{\beta_m}_{a+,\mu}u = \int_a^x f(\tau, u'(\tau))\,d\tau,$$

where

$$AC_0[a, b] = \left\{u : u(x) \in AC[a, b] \text{ with } u(a) = 0 \text{ and } \|u\|_0 = \int_a^b |u'(x)|\,dx < \infty\right\}$$

is a Banach space.

In a wide range of mathematical and engineering problems, the existence of a solution to a differential or integral equation is equivalent to the existence of a fixed point for a suitable operator. Fixed points are therefore of paramount importance in studying differential or integral equations arising from the real world. There are new and interesting studies on fixed point theorems for different operators on metric spaces as well as applications dealing with the existence of a solution for systems either of functional equations or of nonlinear matrix equations [1; 5].

2. Main results

We begin to present our first theorem which shows the solution for (1) with condition (2) as a convergent series in the space $S_m([0, t_0] \times \Omega)$ by Babenko's approach.

Theorem 1. *Let $\beta_{ij} \geq 0$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, l$, and $\lambda_i(x)$ be continuous on Ω for $i = 1, 2, \ldots, l$. Then (1) with condition (2) has a unique solution*

$$(4)\quad u(t,x) = I_t^m \sum_{j=0}^{\infty} I_t^{jm} \sum_{j_1+j_2+\cdots+j_l=j} \binom{j}{j_1, j_2, \ldots, j_l} (\lambda_1(x) I_1^{\beta_{11}} \cdots I_n^{\beta_{n1}})^{j_1} \cdots (\lambda_l(x) I_1^{\beta_{1l}} \cdots I_n^{\beta_{nl}})^{j_l} g(t,x)$$

in $S_m([0, t_0] \times \Omega)$.

Proof. Clearly,

$$I_t^m \lambda_k(x) I_1^{\beta_{1k}} \cdots I_n^{\beta_{nk}} = \lambda_k(x) I_1^{\beta_{1k}} \cdots I_n^{\beta_{nk}} I_t^m,$$

and

$$u(t,x) - \sum_{k=1}^{l} \lambda_k(x) I_t^m I_1^{\beta_{1k}} \cdots I_n^{\beta_{nk}} u(t,x) = I_t^m g(t,x)$$

by applying the integral operator I_t^m to both sides of (1) and the initial conditions. By Babenko's approach,

$$\begin{aligned}
u(t,x) &= \left(1 - \sum_{k=1}^{l} \lambda_k(x) I_t^m I_1^{\beta_{1k}} \cdots I_n^{\beta_{nk}} u(t,x)\right)^{-1} I_t^m g(t,x) \\
&= \sum_{j=0}^{\infty} \left(\sum_{k=1}^{l} \lambda_k(x) I_t^m I_1^{\beta_{1k}} \cdots I_n^{\beta_{nk}} \right)^j I_t^m g(t,x) \\
&= \sum_{j=0}^{\infty} \sum_{j_1+j_2+\cdots+j_l=j} \binom{j}{j_1, j_2, \ldots, j_l} (\lambda_1(x) I_t^m I_1^{\beta_{11}} \cdots I_n^{\beta_{n1}})^{j_1} \cdots (\lambda_l(x) I_t^m I_1^{\beta_{1l}} \cdots I_n^{\beta_{nl}})^{j_l} I_t^m g(t,x) \\
&= I_t^m \sum_{j=0}^{\infty} I_t^{jm} \sum_{j_1+j_2+\cdots+j_l=j} \binom{j}{j_1, j_2, \ldots, j_l} (\lambda_1(x) I_1^{\beta_{11}} \cdots I_n^{\beta_{n1}})^{j_1} \cdots (\lambda_l(x) I_1^{\beta_{1l}} \cdots I_n^{\beta_{nl}})^{j_l} g(t,x).
\end{aligned}$$

Obviously, $u(t,x)$ satisfies the initial conditions due to the integral operator I_t^m and the uniqueness immediately follows from the fact that the linear homogeneous integral equation

$$\frac{\partial^m}{\partial t^m} u(t,x) - \sum_{k=1}^{l} \lambda_k(x) I_1^{\beta_{1k}} \cdots I_n^{\beta_{nk}} u(t,x) = 0$$

with condition (2) has only zero solution. It remains to show that $u(t, x)$ is in the space $S_m([0, t_0] \times \Omega)$ and is a solution of (1).

Since $\lambda_k(x)$, for $k = 1, 2, \dots, l$, is continuous on Ω, therefore it is bounded and there exists $M > 0$ such that

$$\max_{x \in \Omega} |\lambda_k(x)| \leq M.$$

Let $\lambda = \max\{t_0, \omega_1, \dots, \omega_n\} > 0$. Then the norm of the integral operator $I_i^{\beta_{ir}}$ for $i = 1, 2, \dots, n$ and $r = 1, 2, \dots, l$ on the space $C([0, t_0] \times \Omega)$ is

$$\begin{aligned}
\|I_i^{\beta_{ir}}\|_C &= \max_{\|u\|_C \leq 1} |I_i^{\beta_{ir}} u| \\
&= \max_{\|u\|_C \leq 1} \left| \frac{1}{\Gamma(\beta_{ir})} \int_0^{x_i} (x_i - s)^{\beta_{ir}-1} u(t, x_1, \dots, x_{i-1}, s, x_{s+1}, \dots, x_n)\, ds \right| \\
&\leq \frac{1}{\Gamma(\beta_{ir})} \int_0^{x_i} (x_i - s)^{\beta_{ir}-1}\, ds \\
&= \frac{x_i^{\beta_{ir}}}{\Gamma(\beta_{ir} + 1)} \\
&\leq \frac{\lambda^{\beta_{ir}}}{\Gamma(\beta_{ir} + 1)}.
\end{aligned}$$

Similarly for $j = 0, 1, \dots,$

$$\|I_t^{jm+m}\|_C \leq \frac{\lambda^{jm+m}}{(jm+m)!}.$$

Therefore,

$$\begin{aligned}
&\left\| I_t^{mj+m} \sum_{j_1+j_2+\cdots+j_l=j} \binom{j}{j_1, j_2, \dots, j_l} (\lambda_1(x) I_1^{\beta_{11}} \cdots I_n^{\beta_{n1}})^{j_1} \cdots (\lambda_l(x) I_1^{\beta_{1l}} \cdots I_n^{\beta_{nl}})^{j_l} \right\|_C \\
&\qquad \leq \frac{\lambda^{jm+m}}{(jm+m)!} M^j \sum_{j_1+\cdots+j_l=j} \binom{j}{j_1, j_2, \dots, j_l} \|I_1^{\beta_{11} j_1+\cdots+\beta_{1l} j_l}\|_C \cdots \|I_n^{\beta_{n1} j_1+\cdots+\beta_{nl} j_l}\|_C.
\end{aligned}$$

Using

$$\sum_{j_1+\cdots+j_l=j} \binom{j}{j_1, j_2, \dots, j_l} = l^j,$$

$$\|I_1^{\beta_{11} j_1+\cdots+\beta_{1l} j_l}\|_C \leq \frac{\lambda^{\beta_{11} j_1+\cdots+\beta_{1l} j_l}}{\Gamma(\beta_{11} j_1 + \cdots + \beta_{1l} j_l + 1)} \leq \tfrac{5}{4} \lambda^{\beta_{11} j_1+\cdots+\beta_{1l} j_l},$$

$$\vdots$$

$$\|I_n^{\beta_{n1} j_1+\cdots+\beta_{nl} j_l}\|_C \leq \frac{\lambda^{\beta_{n1} j_1+\cdots+\beta_{nl} j_l}}{\Gamma(\beta_{n1} j_1 + \cdots + \beta_{nl} j_l + 1)} \leq \tfrac{5}{4} \lambda^{\beta_{n1} j_1+\cdots+\beta_{nl} j_l}$$

by noting that for $y \geq 0$,

$$\Gamma(y+1) \geq \tfrac{4}{5}$$

in [16], we arrive at

$$\left\| I_t^{mj+m} \sum_{j_1+j_2+\cdots+j_l=j} \binom{j}{j_1,\, j_2,\, \ldots,\, j_l} (\lambda_1(x) I_1^{\beta_{11}} \cdots I_n^{\beta_{n1}})^{j_1} \cdots (\lambda_l(x) I_1^{\beta_{1l}} \cdots I_n^{\beta_{nl}})^{j_l} \right\|_C \leq \left(\tfrac{5}{4}\right)^n (S^n)^j l^j M^j \frac{\lambda^{jm+m}}{(jm+m)!},$$

where

$$S = \max\{\lambda^{\beta_{ir}}\} > 0.$$

Hence,

$$\begin{aligned} \|u(t,x)\|_C &\leq \left(\tfrac{5}{4}\right)^n \lambda^m \|g(t,x)\|_C \sum_{j=0}^{\infty} \frac{(\lambda^m S^n l M)^j}{(jm+m)!} \\ &= \left(\tfrac{5}{4}\right)^n \lambda^m \|g(t,x)\|_C \sum_{j=0}^{\infty} \frac{(\lambda^m S^n l M)^j}{\Gamma(jm+m+1)} \\ &= \left(\tfrac{5}{4}\right)^n \lambda^m \|g(t,x)\|_C E_{m,m+1}(\lambda^m S^n l M) < \infty, \end{aligned}$$

where

$$E_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^n}{\Gamma(\alpha j + \beta)}, \quad \alpha, \beta > 0, z \in \mathbb{C},$$

is the Mittag–Leffler function.

Evidently,

$$\begin{aligned} \frac{\partial}{\partial t} u(t,x) &= I_t^{m-1} \sum_{j=0}^{\infty} I_t^{jm} \sum_{j_1+j_2+\cdots+j_l=j} \binom{j}{j_1,\, j_2,\, \ldots,\, j_l} (\lambda_1(x) I_1^{\beta_{11}} \cdots I_n^{\beta_{n1}})^{j_1} \cdots (\lambda_l(x) I_1^{\beta_{1l}} \cdots I_n^{\beta_{nl}})^{j_l} g(t,x), \\ &\vdots \\ \frac{\partial^m}{\partial t^m} u(t,x) &= \sum_{j=0}^{\infty} I_t^{jm} \sum_{j_1+j_2+\cdots+j_l=j} \binom{j}{j_1,\, j_2,\, \ldots,\, j_l} (\lambda_1(x) I_1^{\beta_{11}} \cdots I_n^{\beta_{n1}})^{j_1} \cdots (\lambda_l(x) I_1^{\beta_{1l}} \cdots I_n^{\beta_{nl}})^{j_l} g(t,x), \end{aligned}$$

and

$$\begin{aligned} \left\| \frac{\partial}{\partial t} u(t,x) \right\|_C &\leq \left(\tfrac{5}{4}\right)^n \lambda^{m-1} \|g(t,x)\|_C E_{m,m}(\lambda^m S^n l M) < \infty, \\ &\vdots \\ \left\| \frac{\partial^m}{\partial t^m} u(t,x) \right\|_C &\leq \left(\tfrac{5}{4}\right)^n \|g(t,x)\|_C E_{m,1}(\lambda^m S^n l M) < \infty. \end{aligned}$$

Thus, $u(t,x) \in S_m([0,t_0]\times\Omega)$ and the series in (4) is absolutely and uniformly convergent. Finally, we substitute $u(t,x)$ in (4) into the left-hand side of (1),

$$\sum_{j=0}^{\infty} I_t^{jm} \sum_{j_1+j_2+\cdots+j_l=j} \binom{j}{j_1,j_2,\ldots,j_l} (\lambda_1(x) I_1^{\beta_{11}}\cdots I_n^{\beta_{n1}})^{j_1}\cdots(\lambda_l(x) I_1^{\beta_{1l}}\cdots I_n^{\beta_{nl}})^{j_l} g(t,x) - \left(\sum_{k=1}^{l} \lambda_k(x) I_1^{\beta_{1k}}\cdots I_n^{\beta_{nk}}\right)$$
$$\times \sum_{j=0}^{\infty} I_t^{jm+m} \sum_{j_1+j_2+\cdots+j_l=j} \binom{j}{j_1,j_2,\ldots,j_l} (\lambda_1(x) I_1^{\beta_{11}}\cdots I_n^{\beta_{n1}})^{j_1}\cdots(\lambda_l(x) I_1^{\beta_{1l}}\cdots I_n^{\beta_{nl}})^{j_l} g(t,x)$$
$$= g(t,x) + \sum_{j=1}^{\infty} I_t^{jm} \sum_{j_1+j_2+\cdots+j_l=j} \binom{j}{j_1,j_2,\ldots,j_l} (\lambda_1(x) I_1^{\beta_{11}}\cdots I_n^{\beta_{n1}})^{j_1}\cdots(\lambda_l(x) I_1^{\beta_{1l}}\cdots I_n^{\beta_{nl}})^{j_l} g(t,x)$$
$$- \left(\sum_{k=1}^{l} \lambda_k(x) I_1^{\beta_{1k}}\cdots I_n^{\beta_{nk}}\right)$$
$$\times \left(I_t^m g(t,x) + \sum_{j=1}^{\infty} I_t^{mj+m} \sum_{j_1+\cdots+j_l=j} \binom{j}{j_1,j_2,\ldots,j_l} (\lambda_1(x) I_1^{\beta_{11}}\cdots I_n^{\beta_{n1}})^{j_1}\cdots(\lambda_l(x) I_1^{\beta_{1l}}\cdots I_n^{\beta_{nl}})^{j_l} g(t,x) \right).$$

Clearly,

$$\sum_{j=1}^{\infty} I_t^{jm} \sum_{j_1+j_2+\cdots+j_l=j} \binom{j}{j_1,\, j_2,\, \ldots,\, j_l} (\lambda_1(x) I_1^{\beta_{11}}\cdots I_n^{\beta_{n1}})^{j_1}\cdots(\lambda_l(x) I_1^{\beta_{1l}}\cdots I_n^{\beta_{nl}})^{j_l} g(t,x)$$
$$= \left(\sum_{k=1}^{l} \lambda_k(x) I_1^{\beta_{1k}}\cdots I_n^{\beta_{nk}}\right) I_t^m g(t,x)$$
$$+ \sum_{j=2}^{\infty} I_t^{mj} \sum_{j_1+j_2+\cdots+j_l=j} \binom{j}{j_1,\, j_2,\, \ldots,\, j_l} (\lambda_1(x) I_1^{\beta_{11}}\cdots I_n^{\beta_{n1}})^{j_1}\cdots(\lambda_l(x) I_1^{\beta_{1l}}\cdots I_n^{\beta_{nl}})^{j_l} g(t,x).$$

Therefore,

$$\sum_{j=1}^{\infty} I_t^{jm} \sum_{j_1+j_2+\cdots+j_l=j} \binom{j}{j_1,j_2,\ldots,j_l} (\lambda_1(x) I_1^{\beta_{11}}\cdots I_n^{\beta_{n1}})^{j_1}\cdots(\lambda_l(x) I_1^{\beta_{1l}}\cdots I_n^{\beta_{nl}})^{j_l} g(t,x) - \left(\sum_{k=1}^{l} \lambda_k(x) I_1^{\beta_{1k}}\cdots I_n^{\beta_{nk}}\right)$$
$$\times \left(I_t^m g(t,x) + \sum_{j=1}^{\infty} I_t^{mj+m} \sum_{j_1+\cdots+j_l=j} \binom{j}{j_1,j_2,\ldots,j_l} (\lambda_1(x) I_1^{\beta_{11}}\cdots I_n^{\beta_{n1}})^{j_1}\cdots(\lambda_l(x) I_1^{\beta_{1l}}\cdots I_n^{\beta_{nl}})^{j_l} g(t,x) \right)$$
$$= \sum_{j=2}^{\infty} I_t^{mj} \sum_{j_1+j_2+\cdots+j_l=j} \binom{j}{j_1,j_2,\ldots,j_l} (\lambda_1(x) I_1^{\beta_{11}}\cdots I_n^{\beta_{n1}})^{j_1}\cdots(\lambda_l(x) I_1^{\beta_{1l}}\cdots I_n^{\beta_{nl}})^{j_l} g(t,x) - \left(\sum_{k=1}^{l} \lambda_k(x) I_1^{\beta_{1k}}\cdots I_n^{\beta_{nk}}\right)$$
$$\times \left(\sum_{j=1}^{\infty} I_t^{mj+m} \sum_{j_1+\cdots+j_l=j} \binom{j}{j_1,j_2,\ldots,j_l} (\lambda_1(x) I_1^{\beta_{11}}\cdots I_n^{\beta_{n1}})^{j_1}\cdots(\lambda_l(x) I_1^{\beta_{1l}}\cdots I_n^{\beta_{nl}})^{j_l} g(t,x) \right)$$

$$=\sum_{j=3}^{\infty} I_t^{mj} \sum_{j_1+\cdots+j_l=j} \binom{j}{j_1, j_2, \ldots, j_l} (\lambda_1(x) I_1^{\beta_{11}} \cdots I_n^{\beta_{n1}})^{j_1} \cdots (\lambda_l(x) I_1^{\beta_{1l}} \cdots I_n^{\beta_{nl}})^{j_l} g(t,x) - \left(2\sum_{k=1}^{l} \lambda_k(x) I_1^{\beta_{1k}} \cdots I_n^{\beta_{nk}}\right)$$

$$\times \left(\sum_{j=2}^{\infty} I_t^{mj+m} \sum_{j_1+\cdots+j_l=j} \binom{j}{j_1, j_2, \ldots, j_l} (\lambda_1(x) I_1^{\beta_{11}} \cdots I_n^{\beta_{n1}})^{j_1} \cdots (\lambda_l(x) I_1^{\beta_{1l}} \cdots I_n^{\beta_{nl}})^{j_l} g(t,x)\right)$$

by noting that

$$I_t^{2m} \sum_{j_1+\cdots+j_l=2} \binom{2}{j_1, j_2, \ldots, j_l} (\lambda_1(x) I_1^{\beta_{11}} \cdots I_n^{\beta_{n1}})^{j_1} \cdots (\lambda_l(x) I_1^{\beta_{1l}} \cdots I_n^{\beta_{nl}})^{j_l} g(t,x)$$

$$=\left(\sum_{k=1}^{l} \lambda_k(x) I_1^{\beta_{1k}} \cdots I_n^{\beta_{nk}}\right) \left(I_t^{2m} \sum_{j_1+\cdots+j_l=1} \binom{1}{j_1, j_2, \ldots, j_l} (\lambda_1(x) I_1^{\beta_{11}} \cdots I_n^{\beta_{n1}})^{j_1} \cdots (\lambda_l(x) I_1^{\beta_{1l}} \cdots I_n^{\beta_{nl}})^{j_l}\right) g(t,x).$$

Repeating this pattern, we show that $u(t, x)$ is a solution of (1). This completes the proof of Theorem 1. □

The following is an example for demonstrating the use of Theorem 1.

Example 2. The partial integro-differential equation

$$\frac{\partial^2}{\partial t^2} t(t, x_1, x_2) - x_1 I_1^{0.5} t(t, x_1, x_2) - x_2^{1.5} I_2^{1.5} t(t, x_1, x_2) = \frac{t x_1^2 x_2^2}{4}$$

with the initial conditions

$$u(0, x_1, x_2) = \frac{\partial}{\partial t} u(0, x_1, x_2) = 0,$$

has a unique solution in the space $S_2([0, t_0] \times \Omega)$

$$u(t, x_1, x_2) = t^3 \sum_{j=0}^{\infty} \frac{t^{2j}}{(2j+3)!} \sum_{j_1+j_2=j} \binom{j}{j_1,\, j_2} A_{j_1} B_{j_2} \frac{x_1^{1.5 j_1+2}}{\Gamma(1.5 j_1+3)} \frac{x_2^{3 j_2+2}}{(3 j_2+2)!},$$

where

$$A_{j_1} = \begin{cases} 3.5 \cdot 5 \cdots (1.5 j_1 + 2) & \text{if } j_1 \geq 1, \\ 1 & \text{if } j_1 = 0, \end{cases}$$

and

$$B_{j_2} = \begin{cases} 5! \cdots (3 j_2 + 2)!/(\Gamma(4.5) \cdots \Gamma(3 j_2 + 1.5)) & \text{if } j_2 \geq 1, \\ 1 & \text{if } j_2 = 0. \end{cases}$$

Proof. Indeed by Theorem 1,

$$u(t, x_1, x_2) = \sum_{j=0}^{\infty} I_t^{2j+2} \sum_{j_1+j_2=j} \binom{j}{j_1,\, j_2} (x_1 I_1^{0.5})^{j_1} (x_2^{1.5} I_2^{1.5})^{j_2} \tfrac{1}{4} t x_1^2 x_2^2$$

$$= \sum_{j=0}^{\infty} I_t^{2j+2} t \sum_{j_1+j_2=j} \binom{j}{j_1,\, j_2} (x_1 I_1^{0.5})^{j_1} \frac{x_1^2}{2} (x_2^{1.5} I_2^{1.5})^{j_2} \frac{x_2^2}{2}.$$

Let α and β be arbitrary complex numbers. Then it follows from [7] that

$$\Phi_\alpha * \Phi_\beta = \Phi_{\alpha+\beta}$$

where

$$\Phi_\alpha = \frac{x_+^{\alpha-1}}{\Gamma(\alpha)}.$$

So,

$$I_t^{2j+2} t = \Phi_{2j+2}(t) * \frac{t_+^{2-1}}{\Gamma(2)} = \Phi_{2j+2}(t) * \Phi_2(t) = \Phi_{2j+4}(t) = \frac{t_+^{2j+3}}{(2j+3)!} = \frac{t^{2j+3}}{(2j+3)!}$$

since $t \geq 0$.

Let us work on the term $(x_1 I_1^{0.5})^{j_1} x_1^2/2$. Clearly,

$$x_1 I_1^{0.5} \frac{x_1^2}{2} = x_1(\Phi_{0.5}(x_1) * \Phi_3(x_1)) = x_1 \Phi_{3.5}(x_1) = \frac{x_1^{3.5}}{\Gamma(3.5)},$$

$$x_1 I_1^{0.5} \frac{x_1^{3.5}}{\Gamma(3.5)} = \frac{\Gamma(4.5)}{\Gamma(3.5)} x_1(\Phi_{0.5} * \Phi_{4.5}) = \frac{\Gamma(4.5)}{\Gamma(3.5)} \frac{x_1^5}{\Gamma(5)},$$

$$\vdots$$

$$\begin{aligned}
(x_1 I_1^{0.5})^{j_1} \frac{x_1^2}{2} &= \frac{\Gamma(4.5)\Gamma(6)\cdots\Gamma(1.5j_1+3)}{\Gamma(3.5)\Gamma(5)\cdots\Gamma(1.5j_1+2)} \frac{x_1^{1.5j_1+2}}{\Gamma(1.5j_1+3)} \\
&= 3.5 \cdot 5 \cdots (1.5j_1+2) \frac{x_1^{1.5j_1+2}}{\Gamma(1.5j_1+3)} = A_{j_1} \frac{x_1^{1.5j_1+2}}{\Gamma(1.5j_1+3)}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
(x_2^{1.5} I_2^{1.5})^{j_2} \frac{x_2^2}{2} &= \frac{\Gamma(6)\cdots\Gamma(3j_2+3)}{\Gamma(4.5)\cdots\Gamma(3j_2+1.5)} \frac{x_2^{3j_2+2}}{\Gamma(3j_2+3)} \\
&= \frac{5!\cdots(3j_2+2)!}{\Gamma(4.5)\cdots\Gamma(3j_2+1.5)} \frac{x_2^{3j_2+2}}{(3j_2+2)!} \\
&= B_{j_2} \frac{x_2^{3j_2+2}}{(3j_2+2)!}.
\end{aligned}$$

This completes the proof of Example 2. □

Remark 3. (i) Following the proof of Theorem 1, we can easily solve the following partial integro-differential equation with condition (2)

$$\frac{\partial^m}{\partial t^m} u(t,x) - \sum_{k=1}^{l} I_t^{\alpha_k} \lambda_k(x) I_1^{\beta_{1k}} \cdots I_n^{\beta_{nk}} u(t,x) = g(t,x) \in C([0,t_0] \times \Omega),$$

where $\alpha_k \geq 0$.

(ii) Similarly, we can work on the partial integral equation without any initial conditions

$$u(t,x) - \sum_{k=1}^{l} \lambda_k(x) I_1^{\beta_{1k}} \cdots I_n^{\beta_{nk}} u(t,x) = g(t,x) \in L([0,t_0] \times \Omega)$$

in the space $L([0,t_0] \times \Omega)$ using the same method.

We are ready to present the following theorem on the uniqueness of solutions for (3) by Banach's contraction principle.

Theorem 4. *Assume $f(t,x,y_0,\dots,y_{m-1})$ is continuous on $[0,t_0] \times \Omega \times R^m$ and there exist nonnegative constants $C_0, C_1, \dots, C_{m-1}$ such that*

$$|f(t,x,y_0,\dots,y_{m-1}) - f(t,x,z_0,\dots,z_{m-1})| \le C_0|y_0 - z_0| + \cdots + C_{m-1}|y_{m-1} - z_{m-1}|.$$

Furthermore, suppose

$$q = m \max\{C_0,\dots,C_{m-1}\}\left(\tfrac{5}{4}\right)^n \max\{\lambda^m E_{m,m+1}(\lambda^m S^n l M), \dots, E_{m,1}(\lambda^m S^n l M)\} < 1.$$

Then (3) with condition (2) has a unique solution in $S_m([0,t_0] \times \Omega)$.

Proof. If $u(t,x) \in S_m([0,t_0] \times \Omega)$, then $f\big(t,x,u(t,x),\dots,\frac{\partial^{m-1}}{\partial t^{m-1}}u(t,x)\big)$ is continuous on $[0,t_0] \times \Omega$. Construct a mapping on the space $S_m([0,t_0] \times \Omega)$ by

$$T(u) = I_t^m \sum_{j=0}^{\infty} I_t^{jm} \sum_{j_1+j_2+\cdots+j_l=j} \binom{j}{j_1, j_2, \dots, j_l} (\lambda_1(x) I_1^{\beta_{11}} \cdots I_n^{\beta_{n1}})^{j_1} \cdots (\lambda_l(x) I_1^{\beta_{1l}} \cdots I_n^{\beta_{nl}})^{j_l} \times f\left(t,x,u(t,x),\dots,\frac{\partial^{m-1}}{\partial t^{m-1}}u(t,x)\right).$$

We first show that T is a mapping from $S_m([0,t_0] \times \Omega)$ to itself. Indeed from the proof of Theorem 1,

$$\|T(u)\|_C \le \left(\tfrac{5}{4}\right)^n \lambda^m E_{m,m+1}(\lambda^m S^n l M) \left\| f(t,x,u(t,x),\dots,\frac{\partial^{m-1}}{\partial t^{m-1}}u(t,x)) \right\|_C < \infty,$$

$$\left\| \frac{\partial}{\partial t} T(u) \right\|_C \le \left(\tfrac{5}{4}\right)^n \lambda^{m-1} E_{m,m}(\lambda^m S^n l M) \|f\|_C < \infty,$$

$$\vdots$$

$$\left\| \frac{\partial^m}{\partial t^m} T(u) \right\|_C \le \left(\tfrac{5}{4}\right)^n E_{m,1}(\lambda^m S^n l M) \|f\|_C < \infty.$$

Thus,

$$\|T(u)\|_m = \max\left\{ \|T(u)\|_C, \left\| \frac{\partial}{\partial t} T(u) \right\|_C, \dots, \left\| \frac{\partial^m}{\partial t^m} T(u) \right\|_C \right\} < \infty.$$

It remains to prove T is contractive by Banach's contraction principle. Then for $u, v \in S_m([0,t_0] \times \Omega)$,

$$\|T(u) - T(v)\|_m = \max\left\{ \|T(u) - T(v)\|_C, \left\| \frac{\partial}{\partial t}(T(u) - T(v)) \right\|_C, \dots, \left\| \frac{\partial^m}{\partial t^m}(T(u) - T(v)) \right\|_C \right\}.$$

Clearly,

$$
\begin{aligned}
&\|T(u)-T(v)\|_C \\
&\quad\le \left(\tfrac{5}{4}\right)^n \lambda^m E_{m,m+1}(\lambda^m S^n l M) \\
&\qquad\qquad \times \left\| f\left(t,x,u(t,x),\ldots,\frac{\partial^{m-1}}{\partial t^{m-1}}u(t,x)\right) - f\left(t,x,v(t,x),\ldots,\frac{\partial^{m-1}}{\partial t^{m-1}}v(t,x)\right)\right\|_C \\
&\quad= \left(\tfrac{5}{4}\right)^n \lambda^m E_{m,m+1}(\lambda^m S^n l M) \\
&\qquad\qquad \times \max_{t\in[0,t_0],x\in\Omega} \left| f\left(t,x,u(t,x),\ldots,\frac{\partial^{m-1}}{\partial t^{m-1}}u(t,x)\right) - f\left(t,x,v(t,x),\ldots,\frac{\partial^{m-1}}{\partial t^{m-1}}v(t,x)\right)\right| \\
&\quad\le \left(\tfrac{5}{4}\right)^n \lambda^m E_{m,m+1}(\lambda^m S^n l M) \\
&\qquad\qquad \times \max_{t\in[0,t_0],x\in\Omega} \left\{ C_0|u(t,x)-v(t,x)| + \cdots + C_{m-1}\left|\frac{\partial^{m-1}}{\partial t^{m-1}}(u(t,x)-v(t,x))\right|\right\} \\
&\quad\le \left(\tfrac{5}{4}\right)^n \lambda^m \le m \max\{C_0,\ldots,C_{m-1}\}\left(\tfrac{5}{4}\right)^n \lambda^m E_{m,m+1}(\lambda^m S^n l M)\|u-v\|_m.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\left\|\frac{\partial}{\partial t}(T(u)-T(v))\right\|_C &\le m\max\{C_0,\ldots,C_{m-1}\}\left(\tfrac{5}{4}\right)^n \lambda^{m-1} E_{m,m}(\lambda^m S^n l M)\|u-v\|_m, \\
&\vdots \\
\left\|\frac{\partial^m}{\partial t^m}(T(u)-T(v))\right\|_C &\le m\max\{C_0,\ldots,C_{m-1}\}\left(\tfrac{5}{4}\right)^n E_{m,1}(\lambda^m S^n l M)\|u-v\|_m.
\end{aligned}
$$

Hence,

$$
\|T(u)-T(v)\|_m \le q\|u-v\|_m,
$$

which implies that (3) has a unique solution in the space $S_m([0,t_0]\times\Omega)$. This completes the proof of Theorem 4. □

Example 5. Let $(t,x_1,x_2)\in\left[0,\frac{1}{2}\right]\times\left[0,\frac{1}{3}\right]\times[0,1]$. Then the partial integro-differential equation

$$
\begin{aligned}
&\frac{\partial^2}{\partial t^2}u(t,x_1,x_2) - \tfrac{1}{2}\sin(x_1x_2) I_1^{0.75} I_2^{1.33} u(t,x_1,x_2) - \frac{1}{x_1^2+x_2^2+3} I_1^{0.5} I_2^{2.133} u(t,x_1,x_2) \\
&\qquad = \cos(t(x_1+x_2)) + \tfrac{1}{7}\sin u(t,x_1,x_2) + \tfrac{1}{5}\frac{\frac{\partial}{\partial t}u(t,x_1,x_2)}{1+\left(\frac{\partial}{\partial t}u(t,x_1,x_2)\right)^2}
\end{aligned}
$$

with the initial conditions

$$
u(0,x_1,x_2) = \frac{\partial}{\partial t}u(0,x_1,x_2) = 0,
$$

has a unique solution in the space $S_m([0,t_0]\times\Omega)$.

Proof. Let

$$
\lambda = \max\left\{\tfrac{1}{2},\tfrac{1}{3},1\right\} = 1.
$$

Hence,

$$S = \max\{\lambda^{\beta_{ir}}\} = 1$$

for $i = 1, 2$ and $r = 1, 2$. Obviously,

$$\left|\tfrac{1}{2}\sin(x_1 x_2)\right| \leq \tfrac{1}{2}, \quad \left|\frac{1}{x_1^2 + x_2^2 + 3}\right| \leq \tfrac{1}{3}$$

and $M = \frac{1}{2}$. The function

$$f(t, x, y_0, y_1) = \cos(t(x_1 + x_2)) + \tfrac{1}{7}\sin y_0 + \tfrac{1}{5}\frac{y_1}{1 + y_1^2}$$

is continuous and satisfies the condition

$$|f(t, x, y_0, y_1) - f(t, x, z_0, z_1)| \leq \tfrac{1}{7}|y_0 - z_0| + \tfrac{1}{5}|y_1 - z_1|.$$

Thus,

$$\max\{C_0, C_1\} = \frac{1}{5}$$

and

$$\begin{aligned}
&\max\{\lambda^2 E_{2,3}(\lambda^2 S^2 l M), \lambda E_{2,2}(\lambda^2 S^2 l M), E_{2,1}(\lambda^2 S^2 l M)\} \\
&\qquad = \sum_{j=0}^{\infty} \frac{1}{\Gamma(2j+1)} \\
&\qquad = \sum_{j=0}^{\infty} \frac{1}{(2j)!} \\
&\qquad = 1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \frac{1}{8!} + \cdots \\
&\qquad \leq \left(\tfrac{1}{4}\right)^0 + \left[\left(\tfrac{1}{4}\right)^1 + \left(\tfrac{1}{4}\right)^1\right] + \left(\tfrac{1}{4}\right)^2 + \left(\tfrac{1}{4}\right)^3 + \left(\tfrac{1}{4}\right)^4 + \cdots \\
&\qquad = \left(\tfrac{1}{4}\right)^1 + \tfrac{1}{1-1/4} \\
&\qquad = \tfrac{19}{12}.
\end{aligned}$$

Therefore,

$$q = m\max\{C_0, \ldots, C_{m-1}\}\left(\tfrac{5}{4}\right)^n \max\{\lambda^m E_{m,m+1}(\lambda^m S^n l M), \ldots, E_{m,1}(\lambda^m S^n l M)\} \leq 2\cdot\tfrac{1}{5}\left(\tfrac{5}{4}\right)^2 \tfrac{19}{12} = \tfrac{950}{960} < 1.$$

By Theorem 4, it has a unique nonzero solution as zero is clearly not a solution. This completes the proof of Example 5. □

3. Conclusion

Applying Babenko's approach and Banach's contraction principle, we have given sufficient conditions for uniqueness of solutions for several partial integro-differential equations with the initial conditions and variable coefficients in Banach space $S_m([0, t_0] \times \Omega)$. Both methods used and results derived are new. Furthermore, these approaches can be widely applied to solving many kinds of fractional differential and integral equations. We also presented several examples for the illustration of our main conclusions by gamma function, convolution as well as Mittag–Leffler functions.

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