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The Hausdorff–Pompeiu Distance in *Gn*-Menger Fractal Spaces

Donal O'Regan [1,†], Reza Saadati [2,*,†], Chenkuan Li [3,†] and Fahd Jarad [4,5,†]

1 School of Mathematical and Statistical Science, National University of Ireland, University Road, H91 TK33 Galway, Ireland
2 School of Mathematics, Iran University of Science and Technology, Narmak, Tehran 13114-16846, Iran
3 Department of Mathematics and Computer Science, Brandon University, Brandon, MB R7A 6A9, Canada
4 Department of Mathematics, Cankaya University, Etimesgut, Ankara 06790, Turkey
5 Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan
* Correspondence: rsaadati@eml.cc
† These authors contributed equally to this work.

Abstract: This paper introduces a complete *Gn*-Menger space and defines the Hausdorff–Pompeiu distance in the space. Furthermore, we show a novel fixed-point theorem for *Gn*-Menger-θ-contractions in fractal spaces.

Keywords: fixed point; generalized contraction; Hausdorff–Pompeiu distance; iterated function system; *Gn*-Menger fractal space

MSC: 54C40; 14E20; 46E25

1. Introduction and Preliminaries

We begin with the concept of a *Gn*-Menger space using distributional maps (DMs) and triangular norms. Throughout the entire paper, we let $\mathbb{I} = [0,1]$, $\mathbb{I}^\circ = (0,1)$, $\mathbb{R}^\bullet = [-\infty, +\infty]$, $\mathbb{J} = [0, +\infty)$ and $\mathbb{J}^\circ = (0, +\infty)$. Define the set of distributional maps $\mho^+$ as the set of all functions $\jmath : \mathbb{R}^\bullet \to \mathbb{I}$, denoting $\jmath_\imath = \jmath(\imath)$, which are left continuous and nondecreasing on $\mathbb{R}$ with $\jmath_0 = 0$ and $\jmath_{+\infty} = 1$. In addition, let $\partial^+ \subseteq \mho^+$ consist of all (proper) mappings $\jmath \in \mho^+$ for which $\ell^- \jmath_{+\infty} = 1$, where $\ell^- \jmath_\imath$ means the left limit at the point $\imath$. Please refer to [1–3] for more details. Note all proper DMs are the DMs of real random variables (namely, we have $P(|g| = \infty) = 0$ for any random variable g).

In $\mho^+$, we define "$\leq$" as follows:

$$\jmath \leq \hbar \iff \jmath_\tau \leq \hbar_\tau$$

for each τ in $\mathbb{R}$ (partially ordered). For example,

$$\hbar_\tau = \begin{cases} 0, & \text{if } \tau \in \mathbb{R} - \mathbb{J}^\circ, \\ 1 - e^{-\tau}, & \text{if } \tau \in \mathbb{J}^\circ, \end{cases}$$

for $\hbar \in \partial^+$. Note that the function $\wp^u_\tau$ defined by

$$\wp^u_\tau = \begin{cases} 0, & \text{if } \tau \leq u, \\ 1, & \text{if } \tau > u, \end{cases}$$

is an element of $\mho^+$, and $\wp^0_\tau$ is the maximal element in this space (for more information, see [1–3]).

Definition 1 ([1,4])**.** *A continuous triangular norm (CTN) is a continuous binary operation $*$ from $\mathbb{I}^2$ to $\mathbb{I}$, such that*

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(a) $\vartheta * \text{ł} = \text{ł} * \vartheta$ *and* $\vartheta * (\text{ł} * \text{ß}) = (\vartheta * \text{ł}) * \text{ß}$ *for all* $\vartheta, \text{ł}, \text{ß} \in \mathbb{I}$*;*
(b) $\vartheta * 1 = \vartheta$ *for all* $\vartheta \in \mathbb{I}$*;*
(c) $\vartheta * \text{ł} \leq \vartheta' * \text{ł}'$ *whenever* $\vartheta \leq \vartheta'$ *and* $\text{ł} \leq \text{ł}'$ *for all* $\vartheta, \text{ł}, \vartheta', \text{ł}' \in \mathbb{I}$*.*

Some examples of t-norms are:
(1) $\vartheta *_P \text{ł} = \vartheta\text{ł}$ (the product CTN);
(2) $\vartheta *_M \text{ł} = \min\{\vartheta, \text{ł}\}$ (the minimum CTN);
(3) $\vartheta *_L \text{ł} = \max\{\vartheta + \text{ł} - 1, 0\}$ (the Lukasiewicz CTN).

Assume that, for every $\vartheta \in \mathbb{I}^\circ$, there exists a $\text{ł} \in \mathbb{I}^\circ$ (which is independent of ℓ, but depends on ϑ) such that the following inequality holds

$$\overbrace{(1-\text{ł}) * \cdots * (1-\text{ł})}^{\ell} > 1 - \vartheta, \quad \text{for each } \ell \in \{2, 3, \ldots\}. \tag{1}$$

In this case, we say the CTN $*$ has the (D) property (CTND for short).

Definition 2. *Let $*$ be a CTN, $U \neq \emptyset$ and ζ be a mapping from U^n to ∂^+. The ordered tuple $(U, \zeta, *)$ is called a Gn-Menger space if the following conditions are satisfied:*

(ζ1) $\zeta_\tau^{u_1,\ldots,u_n} = \wp_\tau^0$ for $\tau \in \mathbb{J}^\circ$, if and only if $u_1 = u_2 = \cdots = u_n$ and $\tau \in \mathbb{J}^\circ$;

(ζ2) $\zeta_\tau^{u_1,\ldots,u_n}$ is invariant under any permutation of $u_1, \ldots, u_n \in U$ and $\tau \in \mathbb{J}^\circ$;

(ζ3) $\zeta_\tau^{u_1,u_1\ldots,u_1,u_2} \geq \zeta_\tau^{u_1,u_2,\ldots,u_n}$ for every $u_1, \ldots, u_n \in U$ and $\tau \in \mathbb{J}^\circ$;

(ζ4) $\zeta_{\tau+\varsigma}^{u_1,u_2,\ldots,u_n} \geq \zeta_\varsigma^{u_1,u_{n+1},\ldots,u_{n+1}} * \zeta_\tau^{u_{n+1},u_2,\ldots,u_n}$ for every $u_1, \ldots, u_n, u_{n+1} \in U$ and $\tau, \varsigma \in \mathbb{J}^\circ$.

Moreover, ζ is called a Gn-Menger distance.

For more details about *Gn*-Menger space and distance, see [5–15]. Our results improve and generalize recent results in [16–18].

Example 1. Define $\zeta : \mathbb{R}^n \to \partial^+$ by

$$\zeta_\tau^{u_1,\ldots,u_n} = \begin{cases} 0, & \text{if } \tau \in \mathbb{R} - \mathbb{J}^\circ, \\ \exp(-\max\limits_{i \neq j, i,j \in \{1,2,\ldots,n\}} \{|u_i - u_j|\}/\tau), & \text{if } \tau \in \mathbb{J}^\circ. \end{cases}$$

Then, the ordered tuple $(\mathbb{R}, \zeta, *_P)$ is a *Gn*-Menger space.

Clearly, (ζ1) and (ζ2) are straightforward. For (ζ3), let $\tau \in \mathbb{J}^\circ$, and since

$$\frac{|u_1 - u_2|}{\tau} \leq \frac{\max\limits_{i \neq j, i,j \in \{1,2,\ldots,n\}} \{|u_i - u_j|\}}{\tau},$$

we get

$$\begin{aligned} \zeta_\tau^{u_1,u_1\ldots,u_1,u_2} &= \exp\left(-\frac{|u_1 - u_2|}{\tau}\right) \\ &\geq \exp\left(-\frac{\max\limits_{i \neq j, i,j \in \{1,2,\ldots,n\}} \{|u_i - u_j|\}}{\tau}\right) \\ &= \zeta_\tau^{u_1,\ldots,u_n}. \end{aligned}$$

Regarding (ζ4), let $\tau, \varsigma \in \mathbb{J}^\circ$, and note

$$\begin{aligned}
& \zeta_\varsigma^{u_1,u_{n+1},\ldots,u_{n+1}} *_P \zeta_\tau^{u_{n+1},u_2,\ldots,u_n} \\
= \; & \exp\left(-\frac{|u_1 - u_{n+1}|}{\varsigma}\right). \exp\left(-\frac{\max\limits_{i\neq j, i,j\in\{2,\ldots,n,n+1\}}\{|u_i - u_j|\}}{\tau}\right) \\
\leq \; & \exp\left(-\frac{|u_1 - u_{n+1}|}{\varsigma+\tau}\right). \exp\left(-\frac{\max\limits_{i\neq j, i,j\in\{2,\ldots,n,n+1\}}\{|u_i - u_j|\}}{\varsigma+\tau}\right) \\
= \; & \exp\left(-\frac{|u_1 - u_{n+1}| + \max\limits_{i\neq j, i,j\in\{2,\ldots,n,n+1\}}\{|u_i - u_j|\}}{\varsigma+\tau}\right) \\
\leq \; & \exp\left(-\frac{\max\limits_{i\neq j, i,j\in\{1,2,\ldots,n,n+1\}}\{|u_i - u_j|\}}{\varsigma+\tau}\right) \\
\leq \; & \exp\left(-\frac{\max\limits_{i\neq j, i,j\in\{1,2,\ldots,n\}}\{|u_i - u_j|\}}{\varsigma+\tau}\right) \\
= \; & \zeta_{\tau+\varsigma}^{u_1,u_2,\ldots,u_n}.
\end{aligned}$$

We would like to point out that the above example also holds for CTN $*_M$. In the following, we show every Gn-Menger space induces a Menger metric space in the sense of Schweizer and Sklar.

Example 2. Let $(U, \zeta, *)$ be a Gn-Menger space. Define the distributional function η on U^2 as

$$\eta_\tau^{u,v} = \zeta_\tau^{u,v,\ldots,v} * \zeta_\tau^{v,u,\ldots,u},$$

for every $u, v \in U$ and $\tau \in \mathbb{J}^\circ$. Then, $(U, \eta, *)$ is a Menger metric space. In fact, it is easy to check that η is a Menger metric (for more references, see [1,9,19]).

(I) Let $\tau \in \mathbb{J}^\circ$ and

$$\begin{aligned}
\wp_\tau^0 &= \eta_\tau^{u,v} \\
&= \zeta_\tau^{u,v,\ldots,v} * \zeta_\tau^{v,u,\ldots,u},
\end{aligned}$$

so we have

$$\wp_\tau^0 = \zeta_\tau^{u,v,\ldots,v}$$

and

$$\wp_\tau^0 = \zeta_\tau^{v,u,\ldots,u}.$$

Using (ζ1), we get $u = v$. Obviously, the converse is also true.

(II) From (ζ2), we have $\eta_\tau^{u,v} = \eta_\tau^{v,u}$ for every $u, v \in U$ and $\tau \in \mathbb{J}^\circ$.

(III) Let $u, v, w \in U$ and $\tau, \varsigma \in \mathbb{J}^\circ$. From ($\zeta$4), we have

$$\begin{aligned}
\eta_{\tau+\varsigma}^{u,v} &= \zeta_{\tau+\varsigma}^{u,v,\ldots,v} * \zeta_{\tau+\varsigma}^{v,u,\ldots,u} \\
&\geq \left[\zeta_\tau^{u,w,\ldots,w} * \zeta_\varsigma^{w,v,\ldots,v}\right] * \left[\zeta_\varsigma^{v,w,\ldots,w} * \zeta_\tau^{w,u,\ldots,u}\right] \\
&= \left[\zeta_\tau^{u,w,\ldots,w} * \zeta_\tau^{w,u,\ldots,u}\right] * \left[\zeta_\varsigma^{w,v,\ldots,v} * \zeta_\varsigma^{v,w,\ldots,w}\right] \\
&= \eta_\tau^{u,w} * \eta_\varsigma^{w,v}.
\end{aligned}$$

It now follows that $(U, \eta, *)$ is a Menger metric space from (I), (II) and (III).

Definition 3. *Let $(U, \zeta, *)$ be a Gn-Menger space. Assume $\rho \in \mathbb{I}^\circ$, $\tau \in \mathbb{J}^\circ$ and $u_0 \in U$. We define the open ball with center u_0 and radius ρ as*

$$O^{u_0}_{\rho,\tau} = \{u \in U : \zeta^{u_0,u,\ldots,u}_{\tau} > 1 - \rho \text{ and } \zeta^{u,u_0,\ldots,u_0}_{\tau} > 1 - \rho\}.$$

Definition 4. *Let $(U, \zeta, *)$ be a Gn-Menger space.*

(1) A sequence $\{u_k\}$ in U is said to be convergent *to u in U if, for every $\lambda \in \mathbb{I}^\circ$, there exists a positive integer N such that $\zeta^{u,u_k,\ldots,u_k}_{\tau} > 1 - \lambda$ for every $\tau \in \mathbb{J}^\circ$ whenever $k \geq N$.*

(2) A sequence $\{u_k\}$ in U is called a Cauchy sequence *if, for every $\lambda \in \mathbb{I}^\circ$, there exists a positive integer N such that $\zeta^{u_{k_1},u_{k_2},\ldots,u_{k_n}}_{\tau} > 1 - \lambda$ for every $\tau \in \mathbb{J}^\circ$ whenever $k_1, \ldots, k_n \geq N$.*

(3) A Gn-Menger space $(U, \zeta, *)$ is said to be complete*, if and only if every Cauchy sequence in U is convergent to a point in U.*

Lemma 1. *Let $(U, \zeta, *)$ be a Gn-Menger space. Then, ζ is continuous on U^n.*

Proof. For a fixed n, we let $(u_1, \ldots, u_n) \in U^n$ and $\tau \in \mathbb{J}^\circ$. Let $\{(u_{1,k}, \ldots, u_{n,k})\}$ be a sequence in U^n converging to $(u_1, \ldots, u_n)$. Consider a fixed number $\alpha \in \mathbb{J}^\circ$ such that $\alpha < \frac{\tau}{n+1}$. Using ($\zeta$4) we derive

$$\begin{aligned}
\zeta^{u_{1,k},\ldots,u_{n,k}}_{\tau} &\geq \zeta^{u_{1,k},u_1,\ldots,u_1}_{\alpha} * \zeta^{u_1,u_{2,k},\ldots,u_{n,k}}_{\tau-\alpha} \\
&= \zeta^{u_{1,k},u_1,\ldots,u_1}_{\alpha} * \zeta^{u_1,u_{2,k},\ldots,u_{n,k}}_{\frac{\alpha}{2}+\tau-\frac{3}{2}\alpha} \\
&\geq \zeta^{u_{1,k},u_1,\ldots,u_1}_{\alpha} * \zeta^{u_{2,k},u_2,\ldots,u_2}_{\frac{\alpha}{2}} * \zeta^{u_1,u_2,u_{3,k},\ldots,u_{n,k}}_{\tau-\frac{3}{2}\alpha} \\
&= \zeta^{u_{1,k},u_1,\ldots,u_1}_{\alpha} * \zeta^{u_{2,k},u_2,\ldots,u_2}_{\frac{\alpha}{2}} * \zeta^{u_1,u_2,u_{3,k},\ldots,u_{n,k}}_{\frac{\alpha}{2}+\tau-\frac{4}{2}\alpha} \\
&\geq \zeta^{u_{1,k},u_1,\ldots,u_1}_{\alpha} * \zeta^{u_{2,k},u_2,\ldots,u_2}_{\frac{\alpha}{2}} * \zeta^{u_{3,k},u_3,\ldots,u_3}_{\frac{\alpha}{2}} * \zeta^{u_1,u_2,u_3,u_{4,k},\ldots,u_{n,k}}_{\tau-\frac{4}{2}\alpha} \\
&\cdot \\
&\cdot \\
&\cdot \\
&\geq \zeta^{u_{1,k},u_1,\ldots,u_1}_{\alpha} * \zeta^{u_{2,k},u_2,\ldots,u_2}_{\frac{\alpha}{2}} * \zeta^{u_{3,k},u_3,\ldots,u_3}_{\frac{\alpha}{2}} \\
&\quad * \cdots * \zeta^{u_{n,k},u_n,\ldots,u_n}_{\frac{\alpha}{2}} * \zeta^{u_1,u_2,u_3,u_4,\ldots,u_n}_{\tau-\frac{n+1}{2}\alpha},
\end{aligned}$$

and

$$\begin{aligned}
\zeta^{u_1,\ldots,u_n}_{\tau} &\geq \zeta^{u_1,u_{1,k},\ldots,u_{1,k}}_{\alpha} * \zeta^{u_{1,k},u_2,\ldots,u_n}_{\tau-\alpha} \\
&= \zeta^{u_1,u_{1,k},\ldots,u_{1,k}}_{\alpha} * \zeta^{u_{1,k},u_2,\ldots,u_n}_{\frac{\alpha}{2}+\tau-\frac{3}{2}\alpha} \\
&\geq \zeta^{u_1,u_{1,k},\ldots,u_{1,k}}_{\alpha} * \zeta^{u_2,u_{2,k},\ldots,u_{2,k}}_{\frac{\alpha}{2}} * \zeta^{u_{1,k},u_{2,k},u_3,\ldots,u_n}_{\tau-\frac{3}{2}\alpha} \\
&= \zeta^{u_1,u_{1,k},\ldots,u_{1,k}}_{\alpha} * \zeta^{u_2,u_{2,k},\ldots,u_{2,k}}_{\frac{\alpha}{2}} * \zeta^{u_{1,k},u_{2,k},u_3,\ldots,u_n}_{\frac{\alpha}{2}+\tau-\frac{4}{2}\alpha} \\
&\geq \zeta^{u_1,u_{1,k},\ldots,u_{1,k}}_{\alpha} * \zeta^{u_2,u_{2,k},\ldots,u_{2,k}}_{\frac{\alpha}{2}} * \zeta^{u_3,u_{3,k},\ldots,u_{3,k}}_{\frac{\alpha}{2}} * \zeta^{u_{1,k},u_{2,k},u_{3,k},u_4,\ldots,u_n}_{\tau-\frac{4}{2}\alpha} \\
&\cdot \\
&\cdot \\
&\cdot \\
&\geq \zeta^{u_1,u_{1,k},\ldots,u_{1,k}}_{\alpha} * \zeta^{u_2,u_{2,k},\ldots,u_{2,k}}_{\frac{\alpha}{2}} * \zeta^{u_3,u_{3,k},\ldots,u_{3,k}}_{\frac{\alpha}{2}} \\
&\quad * \cdots * \zeta^{u_n,u_{n,k},\ldots,u_{n,k}}_{\frac{\alpha}{2}} * \zeta^{u_{1,k},u_{2,k},u_{3,k},u_{4,k},\ldots,u_{n,k}}_{\tau-\frac{n+1}{2}\alpha}.
\end{aligned}$$

We can do this for any n. Letting $k \to \infty$ in the above, we imply by the continuity property of a CTN that

$$\lim_{k\to\infty} \zeta^{u_{1,k},\ldots,u_{n,k}}_{\tau} \geq \zeta^{u_1,u_2,u_3,u_4,\ldots,u_n}_{\tau-\frac{n+1}{2}\alpha}, \tag{2}$$

and

$$\zeta_{\tau}^{u_1,\ldots,u_n} \geq \lim_{k\to\infty} \zeta_{\tau-\frac{n+1}{2}\alpha}^{u_{1,k},u_{2,k},u_{3,k},u_{4,k},\ldots,u_{n,k}}. \tag{3}$$

From (2) and (3), we get by letting α tend to zero that

$$\lim_{k\to\infty} \zeta_{\tau}^{u_{1,k},\ldots,u_{n,k}} = \zeta_{\tau}^{u_1,\ldots,u_n}, \tag{4}$$

for every $\tau > 0$, which shows the continuity of ζ. □

2. Fixed-Point Theorem

Lemma 2. *Consider the Gn-Menger space $(U, \zeta, *)$ in which $*$ is a CTND. Define $\Xi_{\vartheta,\zeta} : U^n \longrightarrow \mathbb{J}$ by*

$$\Xi_{\vartheta,\zeta}(u_1, \ldots, u_n) = \inf\{\tau \in \mathbb{J}^{\circ} : \zeta_{\tau}^{u_1,\ldots,u_n} > 1-\vartheta\},$$

for each $\vartheta \in \mathbb{I}^{\circ}$ and $u_1, \ldots, u_n \in U$. Then, we have the following:

(I) Let $u_1, \ldots, u_n, w_1, \ldots, w_n \in U$. For every $ł \in \mathbb{J}^{\circ}$, there exists $\vartheta \in \mathbb{J}^{\circ}$ such that

$$\Xi_{ł,\zeta}(u_1, \ldots, u_n) \leq \sum_{j=1}^{n} \Xi_{\vartheta,\zeta}(u_j, w_j, w_j, \ldots, w_j) + \Xi_{\vartheta,\zeta}(w_1, \ldots, w_n);$$

(II) The sequence $\{u_k\}$ is convergent with respect to the Gn-Menger metric ζ, if and only if $\Xi_{\vartheta,\zeta}(u, u_k, \ldots, u_k) \to 0$. Moreover, the sequence $\{u_k\}$ is a Cauchy sequence with respect to the Gn-Menger metric ζ, if and only if it is a Cauchy sequence in $\Xi_{\vartheta,\zeta}$;

(III) Let $u_{k_1}, u_{k_2}, \ldots, u_{k_n} \in U$, where $k_1, \ldots, k_n \in \mathbb{N}$. For every $ł \in \mathbb{J}^{\circ}$ there exists $\vartheta \in \mathbb{J}^{\circ}$ such that for $n \geq 3$,

$$\Xi_{ł,\zeta}(u_{k_1}, u_{k_2}, \ldots, u_{k_n}) \leq \sum_{j=1}^{n-2} j\Xi_{\vartheta,\zeta}(u_{k_j}, u_{k_{j+1}}, \ldots, u_{k_{j+1}}) + \Xi_{\vartheta,\zeta}(u_{k_{n-1}}, u_{k_n}, \ldots, u_{k_n});$$

(IV) A sequence $\{u_k\}$ in the Gn-Menger space U is Cauchy, if and only if, for every $\epsilon \in \mathbb{J}^{\circ}$, there exists a positive integer N such that for every $\epsilon > 0$,

$$\Xi_{ł,\zeta}(u_{k_1}, u_{k_2}, \ldots, u_{k_2}) \leq \epsilon, \tag{5}$$

for all $k_1, k_2 \geq N$.

Proof. (I). For every $ł \in \mathbb{I}^{\circ}$, we can find a $\vartheta \in \mathbb{I}^{\circ}$ such that

$$\overbrace{(1-\vartheta) * \cdots * (1-\vartheta)}^{n+1} > 1 - ł,$$

due to the (D) property. Using (ζ4), we infer

$$\begin{aligned}
& \zeta^{u_1,\ldots,u_n}_{\sum_{j=1}^{n} \Xi_{\vartheta,\zeta}(u_j,w_j,w_j,\ldots,w_j)+\Xi_{\vartheta,\zeta}(w_1,\ldots,w_n)+(n+1)\omega} \\
\geq\ & \zeta^{u_1,w_1,\ldots,w_1}_{\Xi_{\vartheta,\zeta}(u_1,w_1,\ldots,w_1)+\omega} * \zeta^{u_2,w_2,\ldots,w_2}_{\Xi_{\vartheta,\zeta}(u_2,w_2,\ldots,w_2)+\omega} \cdots * \zeta^{u_n,w_n,\ldots,w_n}_{\Xi_{\vartheta,\zeta}(u_n,w_n,\ldots,w_n)+\omega} * \zeta^{w_1,w_2,\ldots,w_n}_{\Xi_{\vartheta,\zeta}(w_1,w_2,\ldots,w_n)+\omega} \\
\geq\ & \overbrace{(1-\vartheta) * \cdots * (1-\vartheta)}^{n+1} \\
>\ & 1 - ł.
\end{aligned}$$

for each $\omega \in \mathbb{J}^\circ$. Hence,

$$\Xi_{\text{ł},\zeta}(u_1, ..., u_n) \leq \sum_{j=1}^{n} \Xi_{\vartheta,\zeta}(u_j, w_j, w_j, ..., w_j) + \Xi_{\vartheta,\zeta}(w_1, ..., w_n) + (n+1)\omega.$$

Letting ω tend to 0, we get

$$\Xi_{\text{ł},\zeta}(u_1, ..., u_n) \leq \sum_{j=1}^{n} \Xi_{\vartheta,\zeta}(u_j, w_j, w_j, ..., w_j) + \Xi_{\vartheta,\zeta}(w_1, ..., w_n).$$

(II). We have $\zeta_\tau^{u_1,...,u_n} > 1 - \text{ł} \Longleftrightarrow \Xi_{\vartheta,\zeta}(u_1, ..., u_n) < \text{ł}$ for every $\text{ł} \in \mathbb{J}^\circ$.

(III). For every $\text{ł} \in \mathbb{I}^\circ$, we can find a $\vartheta \in \mathbb{I}^\circ$ such that for $n \geq 3$,

$$\overbrace{(1-\vartheta) * \cdots * (1-\vartheta)}^{\frac{n(n-1)}{2}} > 1 - \text{ł}.$$

Then, we use a similar method in (I) to complete the proof.

(IV). It follows immediately from (II) and (III). □

We let Θ be the family of all onto and strictly increasing mappings $\theta : \mathbb{J}^\circ \to \mathbb{J}^\circ$ such that $\theta(\rho) < \rho$ for all $\rho \in \mathbb{J}^\circ$, and let all distributional maps be in ∂_+^+. Since $\zeta \in \partial^+$ and (ζ1), we get in a *Gn*-Menger space $(U, \zeta, *)$ that

$$\zeta_\tau^{u_1,...,u_n} = C, \quad \text{for all } \tau \in \mathbb{J}^\circ \text{ implies } C = \wp_\tau^0.$$

Lemma 3. *Consider the Gn-Menger space $(U, \zeta, *)$ in which $*$ is a CTND. Assume that $\theta \in \Theta$. Then, for $\tau \in \mathbb{J}^\circ$*

$$\inf\{\theta^k(\tau) \in \mathbb{J}^\circ : \zeta_\tau^{u_1,...,u_n} > 1 - \vartheta\} \leq \theta^k(\inf\{\tau \in \mathbb{J}^\circ : \zeta_\tau^{u_1,...,u_n} > 1 - \vartheta\}),$$

for each $u_1, ..., u_n \in U$, $\vartheta \in \mathbb{I}^\circ$ and $k \in \mathbb{N}$.

Proof. Let $\tau \in \mathbb{J}^\circ$ be arbitrary and fixed with $\zeta_\tau^{u_1,...,u_n} > 1 - \vartheta$. Then, $\theta^k(\tau) \in \mathbb{J}^\circ$, and

$$\theta^k(\tau) \geq \inf\{\theta^k(\text{ł}) \in \mathbb{J}^\circ : \ \zeta_{\text{ł}}^{u_1,...,u_n} > 1 - \vartheta\}.$$

This implies that

$$\tau \geq (\theta^k)^{-1}(\inf\{\theta^k(\text{ł}) \in \mathbb{J}^\circ : \ \zeta_{\text{ł}}^{u_1,...,u_n} > 1 - \vartheta\}),$$

as θ^k is onto and strictly increasing. Thus,

$$\inf\{\tau \in \mathbb{J}^\circ : \zeta_{\text{ł}}^{u_1,...,u_n} > 1 - \vartheta\} \geq (\theta^k)^{-1}(\inf\{\theta^k(\text{ł}) \in \mathbb{J}^\circ : \ \zeta_{\text{ł}}^{u_1,...,u_n} > 1 - \vartheta\}),$$

which shows that

$$\inf\{\theta^k(\tau) \in \mathbb{J}^\circ : \ \zeta_\tau^{u_1,...,u_n} > 1 - \vartheta\} \leq \theta^k(\inf\{\tau \in \mathbb{J}^\circ : \ \zeta_\tau^{u_1,...,u_n} > 1 - \vartheta\}).$$

□

Lemma 4. *Consider the Gn-Menger space $(U, \zeta, *)$ in which $*$ is a CTND. Assume that $\theta \in \Theta$ and $\{u_k\} \subseteq U$ such that*

$$\zeta_{\theta^k(\tau)}^{u_k,u_{k+1},...,u_{k+1}} \geq \zeta_\tau^{u_1,u_2,...,u_2},$$

for all $\tau \in \mathbb{J}^\circ$. Then, $\{u_k\}$ is a Cauchy sequence.

Proof. From Lemma 3 and our assumption, we arrive at

$$\begin{aligned}\Xi_{\l,\zeta}(u_k, u_{k+1}, \ldots, u_{k+1}) &= \inf\{\theta^k(\tau) \in \mathbb{J}^\circ : \zeta_{\theta^k(\tau)}^{u_k,u_{k+1},\ldots,u_{k+1}} > 1 - \l\} \\ &\leq \inf\{\theta^k(\tau) \in \mathbb{J}^\circ : \zeta_\tau^{u_1,u_2,\ldots,u_2} > 1 - \l\} \\ &\leq \theta^k(\inf\{\tau \in \mathbb{J}^\circ : \zeta_\tau^{u_1,u_2,\ldots,u_2} > 1 - \l\}) \\ &= \theta^k(\Xi_{\l,\zeta}(u_1, u_2, \ldots, u_2)) \to 0,\end{aligned}$$

for every $\l \in \mathbb{I}^\circ$. Applying Lemma 2 (II), (III) and (IV), we conclude that $\{u_k\}$ is a Cauchy sequence. □

We are now ready to present a fixed-point (FP) theorem, with a controller $\theta \in \Theta$, in a complete Gn-Menger space $(U, \zeta, *)$ in which $*$ is a CTND. We say a mapping $\Omega : U \to U$ is a Gn-Menger-θ-contraction if

$$\zeta_\rho^{\Omega(\alpha_1),\ldots,\Omega(\alpha_n)} \geq \zeta_{\theta(\rho)}^{\alpha_1,\ldots,\alpha_n}, \tag{6}$$

for every $\rho \in \mathbb{J}^\circ$.

Theorem 1. *Consider the complete Gn-Menger space $(U, \zeta, *)$ in which $*$ is a CTND. Let the Gn-Menger-θ-contraction Ω satisfy (6) in which $\theta \in \Theta$. Then, Ω has a unique fixed point in U.*

Proof. From Lemma 4 and inequality (6), we have that, for each $\alpha \in U$, the sequence $\{\Omega^n(\alpha)\}_{n=1}^{+\infty}$ is Cauchy and $\lim_{k\to+\infty} \Omega^k(\alpha) = \delta \in U$ since U is complete. Applying the following inequality

$$\begin{aligned}\zeta_\rho^{\Omega(\alpha_1),\ldots,\Omega(\alpha_n)} &\geq \zeta_{\theta(\rho)}^{\alpha_1,\ldots,\alpha_n} \\ &\geq \zeta_\rho^{\alpha_1,\ldots,\alpha_n}\end{aligned}$$

for all $\alpha_1, \ldots, \alpha_n \in U$ and $\rho \in \mathbb{J}^\circ$, we conclude the continuity of Ω and so we get

$$\delta = \lim_{n\to+\infty} \Omega^{n+1}(\alpha) = \lim_{n\to+\infty} \Omega(\Omega^n(\alpha)) = \Omega(\lim_{n\to+\infty} \Omega^n(\alpha)) = \Omega(\delta).$$

In addition, inequality (6) also infers the uniqueness. □

3. Application to the *Gn*-Menger-fractal space

In [20], Hutchinson considered fractal theory, which was further investigated and generalized by Barnsley [21], Bisht [22], Imdad [23], and Ri [24]. The basic concept of fractal theory is that the iterated function system (IFS) serves as the main generator of fractals. This consists of a finite set of Gn-Menger-θ-contractions $\{\Omega_1, \Omega_2, \ldots, \Omega_m\}$ with $(m \geq 2)$, defined in a complete Gn-Menger space $(U, \zeta, *)$, satisfying inequality (6). For such an IFS, there is always a unique nonempty compact subset Γ of the complete Gn-Menger space $(U, \zeta, *)$, such that $\Gamma = \bigcup_{i=1}^{m} \Omega_i(\Gamma)$, wherein Γ is a fractal set called the attractor of the respective IFS.

Now, we denote $\mathcal{H}(U)$ as the set of all nonempty compact subsets of the Gn-Menger space $(U, \zeta, *)$.

Let $V_j \neq \emptyset$ $(j = 1, \ldots, n-1)$ be subsets of the Gn-Menger space $(U, \zeta, *)$, $u \in U$ and $\tau \in \mathbb{J}^\circ$. We define the Gn-Menger distance between u and $\{V_1, \ldots, V_{n-1}\}$ as

$$\zeta_\tau^{u,V_1,\ldots,V_{n-1}} = \sup_{v_j \in V_j, j=1,2,\ldots,n-1} \zeta_\tau^{u,v_1,\ldots,v_{n-1}}. \tag{7}$$

Lemma 5. *Consider the Gn-Menger space $(U, \zeta, *)$. Then, for every $u \in U$, $V_j \subset \mathcal{H}(U)$ $(j = 1, ..., n-1)$ and $\tau \in \mathbb{J}^\circ$, we can find $v_{j,0} \in V_j$ such that*

$$\zeta_\tau^{u,V_1,...,V_{n-1}} = \zeta_\tau^{u,v_{1,0},...,v_{n-1,0}}. \tag{8}$$

Proof. Suppose that $u \in U$, $V_j \subset \mathcal{H}(U)$ $(j = 1, ..., n-1)$ and $\tau \in \mathbb{J}^\circ$. Since ζ is continuous from Lemma 1, the compactness of V_j $(j = 1, ..., n-1)$ implies that we can find $v_{j,0} \in V_j$ such that

$$\sup_{v_j \in V_j, j=1,2,...,n-1} \zeta_\tau^{u,v_1,...,v_{n-1}} = \zeta_\tau^{u,v_{1,0},...,v_{n-1,0}}, \tag{9}$$

so

$$\zeta_\tau^{u,V_1,...,V_{n-1}} = \zeta_\tau^{u,v_{1,0},...,v_{n-1,0}}.$$

□

Lemma 6. *Consider the Gn-Menger space $(U, \zeta, *)$. Let $u \in U$, $V_j \subset \mathcal{H}(U)$ $(j = 1, ..., n-1)$, $\emptyset \neq W \subseteq U$ and $\tau, \varsigma \in \mathbb{J}^\circ$. Then,*

$$\zeta_{\tau+\varsigma}^{u,V_1,...,V_{n-1}} \geq \zeta_\tau^{u,W,W,...,W} * \zeta_\varsigma^{w_u,V_1,...,V_{n-1}}, \tag{10}$$

where $w_u \in W$ satisfies $\zeta_\tau^{u,W,V_2,...,V_{n-1}} = \zeta_\tau^{u,w_u,V_2,...,V_{n-1}}$.

Proof. From Lemma 5, we can find a $w_u \in W$ such that

$$\zeta_\tau^{u,W,...,W} = \zeta_\tau^{u,w_u,...,w_u},$$

for every $\tau \in \mathbb{J}^\circ$. From Lemma 5 again and (ζ4), we have

$$\begin{aligned} \zeta_{\tau+\varsigma}^{u,V_1,...,V_{n-1}} &= \zeta_{\tau+\varsigma}^{u,v_1,v_2,...,v_{n-1}} \\ &\geq \zeta_\tau^{u,w_u,...,w_u} * \zeta_\varsigma^{w_u,v_1,...,v_{n-1}} \\ &= \zeta_\tau^{u,W,...,W} * \zeta_\varsigma^{w_u,v_1,...,v_{n-1}}. \end{aligned} \tag{11}$$

Then, the result follows immediately from taking the supremum over $v_j \in V_j$, $j = 1, 2, ..., n-1$ and inequality (11). □

We now define the *Gn*-Menger Hausdorff–Pompeiu distance among E_j, $j = 1, ..., n$, in $\mathcal{H}(U)$ as:

$$\begin{aligned} & \Upsilon_{\zeta\,\rho}^{E_1,...,E_n} \\ = & \inf_{\alpha_1 \in E_1} \sup_{\alpha_j \in E_j, j=2,3,...,n} \zeta_\rho^{\alpha_1,...,\alpha_n} \\ *_M & \inf_{\alpha_2 \in E_2} \sup_{\alpha_j \in E_j, j=1,3,4,...,n} \zeta_\rho^{\alpha_1,...,\alpha_n} \\ *_M & \cdots \\ *_M & \inf_{\alpha_n \in E_n} \sup_{\alpha_j \in E_j, j=1,2,...,n-1} \zeta_\rho^{\alpha_1,...,\alpha_n}, \end{aligned} \tag{12}$$

for every $\rho \in \mathbb{J}^{\circ}$, which is equivalent to

$$\begin{aligned} & \Upsilon^{E_1,\dots,E_n}_{\substack{\zeta\\ \rho}} \\ = \quad & \inf_{\alpha_1 \in E_1} \zeta_{\rho}^{\alpha_1,E_2,E_3,\dots,E_n} \\ *_M \quad & \inf_{\alpha_2 \in E_2} \zeta_{\rho}^{\alpha_2,E_1,E_3,\dots,E_n} \\ *_M \quad & \cdots \\ *_M \quad & \inf_{\alpha_n \in E_n} \zeta_{\rho}^{E_1,E_2,\dots,E_{n-1},\alpha_n}, \end{aligned} \tag{13}$$

for every $\rho \in \mathbb{J}^{\circ}$.

Example 3. Consider Example 1 in which $U = \mathbb{R}$. Let $* = *_M$, $E_1 = [e_1, f_1]$, $E_2 = [e_2, f_2]$ and $E_3 = [e_3, f_3]$. Define the *Gn*-Menger Hausdorff distance as

$$\Upsilon^{E_1,E_2,E_3}_{\substack{\zeta\\ \rho}} = \exp\left(-\frac{\max_{i,j\in\{1,2,3\}}\{|e_i - e_j|, |f_i - f_j|\}}{\rho}\right),$$

for all $\rho \in \mathbb{J}^{\circ}$. Then, $(\mathcal{H}(U), \Upsilon^{\cdot}_{\substack{\zeta\\ \cdot}}, *)$ is a *Gn*-Menger space.

Clearly, the classical Hausdorff–Pompeiu distance for compact sets $E_1 = [e_1, f_1]$, $E_2 = [e_2, f_2]$ and $E_3 = [e_3, f_3]$ is

$$\max_{i,j\in\{1,2,3\}}\{|e_i - e_j|, |f_i - f_j|\}.$$

Now, using (12), (13), Example 1 and a similar method in ([25] Proposition 3), we have that the *Gn*-Menger Hausdorff distance $\Upsilon^{E_1,E_2,E_3}_{\substack{\zeta\\ \rho}}$ is a *Gn*-Menger distance.

Lemma 7. *Consider the Gn-Menger space* $(U, \zeta, *)$. *Then,* $(\mathcal{H}(U), \Upsilon^{\cdot}_{\substack{\zeta\\ \cdot}}, *)$ *is a Gn-Menger space.*

Proof. Clearly, $(\zeta 1)$, $(\zeta 2)$ and $(\zeta 3)$ are straightforward. It only remains to prove $(\zeta 4)$.

Suppose that $E_j \in \mathcal{H}(U)$, $j = 1, \dots, n$, $u \in E_1$, and $\varsigma, \tau \in \mathbb{J}^{\circ}$. Let $\emptyset \neq W \subseteq U$. From Lemma 6, we have

$$\zeta_{\tau+\varsigma}^{u,E_2,\dots,E_n} \geq \zeta_{\varsigma}^{u,W,W,\dots,W} * \zeta_{\tau}^{w_u,E_2,\dots,E_n}, \tag{14}$$

where $w_u \in W$ satisfies $\zeta_{\tau}^{u,W,E_2,\dots,E_n} = \zeta_{\tau}^{u,w_u,E_2,\dots,E_n}$. Let $\alpha_j \in E_j$, $j = 1, 2, \dots, n$, and from $(\zeta 4)$ we have

$$
\begin{aligned}
& \Upsilon^{E_1,\ldots,E_n}_{\substack{\zeta\\ \varsigma+\tau}} && (15)\\
= \quad & \inf_{\alpha_1\in E_1} \zeta^{\alpha_1,E_2,E_3,\ldots,E_n}_{\varsigma+\tau} \\
*_M \quad & \inf_{\alpha_2\in E_2} \zeta^{\alpha_2,E_1,E_3,\ldots,E_n}_{\varsigma+\tau} \\
*_M \quad & \cdots \\
*_M \quad & \inf_{\alpha_n\in E_n} \zeta^{E_1,E_2,\ldots,E_{n-1},\alpha_n}_{\varsigma+\tau} \\
\geq \quad & \inf_{\alpha_1\in E_1} \big[\zeta^{\alpha_1,W,W,\ldots,W}_{\varsigma} * \zeta^{w_{\alpha_1},E_2,E_3,\ldots,E_n}_{\tau}\big] \\
*_M \quad & \inf_{\alpha_2\in E_2} \big[\zeta^{\alpha_2,W,W,\ldots,W}_{\varsigma} * \zeta^{w_{\alpha_2},E_1,E_3,\ldots,E_n}_{\tau}\big] \\
*_M \quad & \cdots \\
*_M \quad & \inf_{\alpha_n\in E_n} \big[\zeta^{W,W,\ldots,W,\alpha_n}_{\varsigma} * \zeta^{E_1,E_2,\ldots,E_{n-1},w_{\alpha_n}}_{\tau}\big] \\
\geq \quad & \big[\inf_{\alpha_1\in E_1} \zeta^{\alpha_1,W,W,\ldots,W}_{\varsigma} * \inf_{\alpha_2\in E_2} \zeta^{\alpha_2,W,W,\ldots,W}_{\varsigma} * \cdots * \inf_{\alpha_n\in E_n} \zeta^{W,W,\ldots,W,\alpha_n}_{\varsigma}\big] \\
*_M \quad & \big[\zeta^{w_{\alpha_1},E_2,E_3,\ldots,E_n}_{\tau} * \zeta^{w_{\alpha_2},E_1,E_3,\ldots,E_n}_{\tau} * \cdots * \zeta^{w_{\alpha_2},E_1,E_3,\ldots,E_n}_{\tau}\big],
\end{aligned}
$$

which gives

$$
\begin{aligned}
& \Upsilon^{E_1,\ldots,E_n}_{\substack{\zeta\\ \varsigma+\tau}} && (16)\\
\geq \quad & \big[\Upsilon^{E_1,W,\ldots,W}_{\substack{\zeta\\ \varsigma}}\big] \\
*_M \quad & \big[\zeta^{w_{\alpha_1},E_2,E_3,\ldots,E_n}_{\tau} * \zeta^{w_{\alpha_2},E_1,E_3,\ldots,E_n}_{\tau} * \cdots * \zeta^{w_{\alpha_2},E_1,E_3,\ldots,E_n}_{\tau}\big].
\end{aligned}
$$

Taking the supremum over (16) for all $w \in W$, we arrive at

$$
\begin{aligned}
& \Upsilon^{E_1,\ldots,E_n}_{\substack{\zeta\\ \varsigma+\tau}} && (17)\\
\geq \quad & \Upsilon^{E_1,W,\ldots,W}_{\substack{\zeta\\ \varsigma}} *_M \Upsilon^{W,E_2,\ldots,E_n}_{\substack{\zeta\\ \tau}} \\
\geq \quad & \Upsilon^{E_1,W,\ldots,W}_{\substack{\zeta\\ \varsigma}} * \Upsilon^{W,E_2,\ldots,E_n}_{\substack{\zeta\\ \tau}}.
\end{aligned}
$$

□

Lemma 8. *Assume that $(U, \zeta, *)$ is a complete Gn-Menger space. Suppose that $\theta \in \Theta$ and Ω is a Gn-Menger-θ-contraction. Then,*

$$
\Upsilon^{\Gamma_\Omega(E_1),\ldots,\Gamma_\Omega(E_n)}_{\substack{\zeta\\ \rho}} \geq \Upsilon^{E_1,\ldots,E_n}_{\substack{\zeta\\ \theta(\rho)}},
$$

for every $E_1, \ldots, E_n \in \mathcal{H}(U)$ and $\rho \in \mathbb{J}^\circ$, and $\Gamma_\Omega : \mathcal{H}(U) \to \mathcal{H}(U)$ is also a Gn-Menger-θ-contraction, where $\Gamma_\Omega(G) := \Omega(G)$ for every $G \in \mathcal{H}(U)$.

Proof. Consider $E_1, \dots, E_n$ in $\mathcal{H}(U)$. Using inequality (6) and definition (12), we get

$$\begin{aligned}
\Upsilon^{\Gamma_\Omega(E_1),\dots,\Gamma_\Omega(E_n)}_{\substack{\zeta\\ \rho}} &= \Upsilon^{\Omega(E_1),\dots,\Omega(E_n)}_{\substack{\zeta\\ \rho}} \\
&= \inf_{\Omega(\alpha_1)\in\Omega(E_1)} \sup_{\Omega(\alpha_j)\in\Omega(E_j), j=2,3,\dots,n} \zeta_\rho^{\Omega(E_1),\dots,\Omega(E_n)} \\
&*_M \inf_{\Omega(\alpha_2)\in\Omega(E_2)} \sup_{\Omega(\alpha_j)\in\Omega(E_j), j=1,3,4,\dots,n} \zeta_\rho^{\Omega(E_1),\dots,\Omega(E_n)} \\
&*_M \cdots \\
&*_M \inf_{\Omega(\alpha_n)\in\Omega(E_n)} \sup_{\Omega(\alpha_j)\in\Omega(E_j), j=1,2,\dots,n-1} \zeta_\rho^{\Omega(E_1),\dots,\Omega(E_n)} \\
&= \inf_{\alpha_1\in E_1} \sup_{\alpha_j\in E_j, j=2,3,\dots,n} \zeta_\rho^{\Omega(E_1),\dots,\Omega(E_n)} \\
&*_M \inf_{\alpha_2\in E_2} \sup_{\Omega(\alpha_j)\in\Omega(E_j), j=1,3,4,\dots,n} \zeta_\rho^{\Omega(E_1),\dots,\Omega(E_n)} \\
&*_M \cdots \\
&*_M \inf_{\alpha_n\in E_n} \sup_{\Omega(\alpha_j)\in\Omega(E_j), j=1,2,\dots,n-1} \zeta_\rho^{\Omega(E_1),\dots,\Omega(E_n)} \\
&\geq \inf_{\alpha_1\in E_1} \sup_{\alpha_j\in E_j, j=2,3,\dots,n} \zeta_{\theta(\rho)}^{\alpha_1,\dots,\alpha_n} \\
&*_M \inf_{\alpha_2\in E_2} \sup_{\alpha_j\in E_j, j=1,3,4,\dots,n} \zeta_{\theta(\rho)}^{\alpha_1,\dots,\alpha_n} \\
&*_M \cdots \\
&*_M \inf_{\alpha_n\in E_n} \sup_{\alpha_j\in E_j, j=1,2,\dots,n-1} \zeta_{\theta(\rho)}^{\alpha_1,\dots,\alpha_n} \\
&= \Upsilon^{E_1,\dots,E_n}_{\substack{\zeta\\ \theta(\rho)}},
\end{aligned}$$

for every $\rho \in \mathbb{J}^\circ$. □

Theorem 2. *Assume that $(U, \zeta, *)$ is a complete Gn-Menger space in which $*$ is a CTND. Suppose that $\theta \in \Theta$ and Ω is Gn-Menger-θ-contractive. Then, $\Gamma_\Omega : \mathcal{H}(U) \to \mathcal{H}(U)$ has a unique fixed point.*

Proof. From Lemma 8, Γ_Ω is *Gn*-Menger-θ-contractive on $\mathcal{H}(U)$ and so by Theorem 1, Γ_Ω has a unique fixed point. □

Example 4. Consider the complete *Gn*-Menger space defined in Example 1. Suppose that $\theta(\tau) = \frac{\tau}{1+\tau}$, $\Omega(u) = \frac{u}{3}$ and $\Gamma_\Omega[-u, u] = [-\frac{u}{3}, \frac{u}{3}]$. It is easy to show that Ω is *Gn*-Menger-θ-contractive. Furthermore, Γ_Ω has a unique fixed point $\{0\}$.

4. Conclusions

We defined a new version of the probabilistic Hausdorff–Pompeiu distance using the concept of *Gn*-Menger space and we presented a new fixed-point theorem for *Gn*-Menger-θ-contractions in *Gn*-Menger fractal spaces. In the future, we hope to consider our results to get more common fixed-point theorems to investigate the existence and uniqueness of solutions for differential and integral equations.

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