Rezaei Aderyani et al. *Journal of Inequalities and Applications* (2023) 2023:35
https://doi.org/10.1186/s13660-023-02942-0

Journal of Inequalities and Applications
a SpringerOpen Journal

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Special functions and multi-stability of the Jensen type random operator equation in C^*-algebras via fixed point

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Abstract

In this paper, we apply some special functions to introduce a new class of control functions that help us define the concept of multi-stability. Further, we investigate the multi-stability of homomorphisms in C^*-algebras and Lie C^*-algebras, multi-stability of derivations in C^*-algebras, and Lie C^*-algebras for the following random operator equation via fixed point methods:

$$\mu f\left(\eth, \frac{x+y}{2}\right) + \mu f\left(\eth, \frac{x-y}{2}\right) = f(\eth, \mu x).$$

In particular, for $\mu = 1$, the above equation turns out to be Jensen's random operator equation.

MSC: Primary 54H20; 46L05; secondary 39B62

Keywords: Multi control functions; Mittag–Leffler function; $\mathbb{H}$-Fox function; Hypergeometric function; Wright function; C^*-algebras

1 Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam, posed in 1940, concerning the stability of group homomorphisms. In the next year, Hyers gave a partial affirmative answer to the question of Ulam in the context of Banach spaces in the case of additive mappings; that was the first significant breakthrough and a step toward more solutions in this area. Since then, many papers have been published in connection with various generalizations of Ulam's problem and Hyers's theorem. In 1978, Rassias succeeded in extending Hyers's theorem for mappings between Banach spaces by considering an unbounded Cauchy difference subject to a continuity condition upon the mapping. He was the first to prove the stability of linear mapping. This result of Rassias attracted several mathematicians worldwide who began to be stimulated to investigate the stability problems of functional equations. In the present paper, we apply some special functions to introduce a new class of control functions which help us define the concept of multi-stability.

Let $\mho_1$ and $\mho_2$ be Banach algebras and (∂, Π) be a probability measure space. Assume $(\mho_1, \mathfrak{B}_{\mho_1})$ and $(\mho_2, \mathfrak{B}_{\mho_2})$ are Borel measurable spaces. Clearly, a map $f : \partial \times \mho_1 \to \mho_2$ is well defined if $\{\eth : f(\eth, \mathsf{S}) \in \mathcal{C}\} \in \Pi$ for all S in $\mho_1$ and $\mathcal{C} \in \mathfrak{B}_{\mho_2}$. We are going to investigate a vector valued generalized metric spaces. Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ and $\gamma = (\gamma_1, \ldots, \gamma_m)$, $m \in \mathbb{N}$. Then we define

$$\lambda \leq \gamma \quad \Longleftrightarrow \quad \lambda_i \leq \gamma_i, \quad i = 1, \ldots, m;$$

and

$$\lambda \to 0 \quad \Longleftrightarrow \quad \lambda_i \to 0, \quad i = 1, \ldots, m.$$

Definition 1.1 ([1]) Suppose $\mathcal{G}$ is a nonempty set, and $d : \mathcal{G}^2 \to [0, +\infty]^m$ (with $m \in \mathbb{N}$) is a given mapping. We say that d is a generalized metric on $\mathcal{G}$ if the following conditions satisfy:

(1) for every $(\Psi, \Phi) \in \mathcal{G} \times \mathcal{G}$, we have

$$d(\Psi, \Phi) = \underbrace{(0, \ldots, 0)}_{m} \quad \Longleftrightarrow \quad \Psi = \Phi;$$

(2) for every $(\Psi, \Phi) \in \mathcal{G} \times \mathcal{G}$,

$$d(\Phi, \Psi) = d(\Psi, \Phi) \quad \Longleftrightarrow \quad \Psi = \Phi;$$

(3) for every $\Psi, \Phi, \Upsilon \in \mathcal{G}$,

$$d(\Psi, \Upsilon) + d(\Upsilon, \Phi) \succeq d(\Phi, \Psi).$$

Theorem 1.2 ([1]) *Let $(\mathcal{G}, d)$ be a complete generalized metric space, and let $\Gamma : \mathcal{G} \to \mathcal{G}$ be strictly contractive, i.e.,*

$$d(\Gamma\Psi, \Gamma\Phi) \leq (L_1, \ldots, L_m) d(\Psi, \Phi), \quad \forall \Psi, \Phi \in \mathcal{G}$$

for some Lipschitz constants $L_i < 1$, for $i = 1, \ldots, m \in \mathbb{N}$. Then

(1) *the mapping Γ has a unique fixed point $\Psi^* = \Gamma\Psi^*$;*

(2) *the fixed point Ψ^* is globally attractive, i.e.,*

$$\lim_{n\to\infty} \Gamma^n \Psi = \Psi^*,$$

for any starting point $\Psi \in \mathcal{G}$;

(3) *the following three inequalities hold*:

$$d\big(\Gamma^n\Psi, \Psi^*\big) \leq \big(L_1^n, \ldots, L_m^n\big)\, d\big(\Psi, \Psi^*\big),$$
$$d\big(\Gamma^n\Psi, \Psi^*\big) \leq \left(\frac{1}{1-L_1}, \ldots, \frac{1}{1-L_m}\right) d\big(\Gamma^n\Psi, \Gamma^{n+1}\Psi\big),$$
$$d\big(\Psi, \Psi^*\big) \leq \left(\frac{1}{1-L_1}, \ldots, \frac{1}{1-L_m}\right) d(\Psi, \Gamma\Psi),$$

for all nonnegative integers n and all $\Psi \in \mathcal{G}$ and $m \in \mathbb{N}$.

Now, we generalize Theorem 1.2.

Theorem 1.3 ([1]) *Suppose $d:\mathcal{G}^2 \to [0,+\infty]^m$, $m \in \mathbb{N}$, and $(\mathcal{G},d)$ is a complete generalized metric space. Suppose $\Gamma:\mathcal{G}\to\mathcal{G}$ is a strictly contractive mapping with Lipschitz constant $\mathcal{Z}<1$. Then for any given element $\Psi\in\mathcal{G}$, either*

$$d\big(\Gamma^n\Psi,\Gamma^{n+1}\Psi\big) = \underbrace{(+\infty,\dots,+\infty)}_{m},$$

for any $n\in\mathbb{N}\cup\{0\}$ or there exists an $n_0\in\mathbb{N}$ such that

(1) $d(\Gamma^n\Psi,\Gamma^{n+1}\Psi) \preceq \underbrace{(+\infty,\dots,+\infty)}_{m}$, $\forall n\ge n_0$;

(2) *The fixed point Φ^* of Γ is a convergent point of sequence $\{\Gamma^n\Psi\}$;*

(3) *Φ^* is the unique fixed point of Γ in the set $\mathcal{Q}=\{\Phi\in\mathcal{G}|d(\Gamma^{n_0}\Psi,\Phi)\preceq \underbrace{(+\infty,\dots,+\infty)}_{m}\}$;*

(4) *$d(\Phi,\Phi^*)\preceq \frac{1}{1-\mathcal{Z}}d(\Phi,\Gamma\Phi)$ for all $\Phi\in\mathcal{Q}$.*

For more applications of Theorem 1.3 in stability analysis see references [2–4]. We now consider the infinite contour $\mathcal{Z}$ having one of the following forms:

- ▶ $\mathcal{Z}=\mathcal{Z}_{-\infty}$ is a left loop starting at $-\infty$ and ending at $-\infty$, enclosing all the poles of $\Gamma(Y)$.
- ▶ $\mathcal{Z}=\mathcal{Z}_{+\infty}$ is a left loop starting at $+\infty$ and ending at $+\infty$, enclosing all the poles of $\Gamma(d_j-Y)$, for $j=1,\dots,s$, situated in a horizontal strip starting at the point $+\infty+i\mathcal{P}_1$ and terminating at the point $+\infty+i\mathcal{P}_2$ with $-\infty<\mathcal{P}_1<\mathcal{P}_2<+\infty$, and $d_j\in\mathbb{C}$.
- ▶ $\mathcal{Z}=\mathcal{Z}_{i\hbar\infty}$ is a contour starting at the point $\hbar-i\infty$ and terminating at the point $\hbar+i\infty$, where $\hbar\in\mathbb{R}$.

We now introduce some special functions as follows. For more details please see [5–9]. The standard Lie algebraic techniques are important methods for studying special functions. There are some operators defined on Lie algebras for the purpose of deriving properties of some special functions [10–12].

▶ *Exponential function:*

We first define the complex exponential function as

$${}_0\mathbb{H}_0[X] := \exp(X) = \sum_{k=0}^{\infty}\frac{X^k}{\Gamma(k+1)},\quad X\in\mathbb{C}.$$

▶ *Mittag–Leffler function (generalized exponential function):*

The function

$${}_0\mathbb{H}_1[e_1;X] := \mathbb{E}_{e_1}(X) = \sum_{k=0}^{\infty}\frac{X^k}{\Gamma(1+e_1k)},\quad e_1\in\mathbb{C},\Re(e_1)>0, X\in\mathbb{C}$$

is said to be the Mittag–Leffler function of one-parameter.

▶ *Hypergeometric function (the Gauss Hypergeometric series):*

The series given as

$${}_2\mathbb{H}_1[d_1,d_2;e_1;X] = \sum_{w=0}^{\infty}\frac{(d_1)_w(d_2)_w}{(e_1)_w}\frac{X^w}{w!} = \frac{\Gamma(e_1)}{\Gamma(d_1)\Gamma(d_2)}\sum_{w=0}^{\infty}\frac{\Gamma(d_1+w)\Gamma(d_2+w)}{\Gamma(e_1+w)}\frac{X^w}{w!}$$

is called the Hypergeometric function, where $d_1, d_2, e_1 \in \mathbb{C}$, $\Re(d_1), \Re(d_2), \Re(e_1) > 0$. Furthermore, the Hypergeometric function can be represented in terms of the Mellin–Barnes integral of the form

$$ {}_2\mathbb{H}_1[d_1, d_2; e_1; X] = \frac{\Gamma(e_1)}{\Gamma(d_1)\Gamma(d_2)} \frac{1}{2\pi i} \int_{\mathcal{Z}} \frac{\Gamma(Y)\Gamma(d_1 - Y)\Gamma(d_2 - Y)}{\Gamma(e_1 - Y)} (-X)^{-Y} \, dY, $$

where $e_1 \neq 0, -1, -2, -3, \dots$.

► *Wright function (Bessel–Maitland function):*

The series representation

$$ {}_1\mathbb{H}_1[d_1; e_1; X] := \mathbb{W}_{d_1, e_1}(X) = \sum_{n=0}^{\infty} \frac{X^n}{n!\Gamma(d_1 n + e_1)} $$

is called Wright function, where $d_1 > -1$, $e_1, X \in \mathbb{C}$.

► *Fox–Wright function (the generalized Wright function):*

Consider positive vectors $\mathbf{D} = (D_1, \dots, D_s)$, $\mathbf{E} = (E_1, \dots, E_r)$, complex vectors $\mathbf{d} = (d_1, \dots, d_s)$, and $\mathbf{e} = (e_1, \dots, e_r)$. The Fox–Wright function or the generalized Wright function is defined by the series

$$ {}_s\mathbb{H}_r\left[X\Big|^{(d_1,D_1),\dots,(d_s,D_s)}_{(e_1,E_1),\dots,(e_r,E_r)}\right] = {}_s\mathbb{H}_r\left[X\Big|^{(\mathbf{d},\mathbf{D})}_{(\mathbf{e},\mathbf{E})}\right] = \sum_{n=0}^{\infty} \frac{\Gamma(\mathbf{D}n + \mathbf{d})}{\Gamma(\mathbf{E}n + \mathbf{e})} \frac{X^n}{n!}, \tag{1.1} $$

where

$$ \Gamma(\mathbf{D}n + \mathbf{d}) = \prod_{j=1}^{s} \Gamma(Dn + d), \tag{1.2} $$

and $\Gamma(\mathbf{E}n + \mathbf{e})$ follows similarly.

The series (1.1) has a nonzero radius of convergence if

$$ \mathcal{N} := \sum_{j=1}^{r} E_j - \sum_{j=1}^{s} D_j \geq -1. \tag{1.3} $$

Moreover, if $\mathcal{N} > -1$ then the series converges for all finite values of X (hence it is an entire function), and if $\mathcal{N} = -1$, its radius of convergence equals

$$ \mathcal{M} := \prod_{k=1}^{s} D_k^{-D_k} \prod_{j=1}^{r} E_j^{E_j}. \tag{1.4} $$

The Convergence on the boundary $|X| = \mathcal{M}$, however, depends on the value of

$$ \mathcal{W} := \sum_{j=1}^{r} e_j - \sum_{k=1}^{s} d_k + \frac{s - r - 1}{2}, \tag{1.5} $$

by noting that series (1.1) converges absolutely for $|X| = \mathcal{M}$ if $\Re(\mathcal{W}) > 0$.

The function ${}_s\mathbb{H}_r$ is an extension of the generalized hypergeometric function (which we will present later). In addition, ${}_1\mathbb{H}_1$ and ${}_0\mathbb{H}_1$ are the Wright (the Bessel–Maitland) function and Mittag–Leffler function with $D_1 = d_1 = 1$, respectively.

► *Fox's $\mathbb{H}$–function (generalized Fox–Wright function):*
We now present Fox's $\mathbb{H}$–function as

$$ {}_{s}^{v}\mathbb{H}_{r}^{w}\left[X\Big|_{(e_j,E_j)_{1,r}}^{(d_j,D_j)_{1,s}}\right] := \frac{1}{2\pi i}\int_{\mathcal{Z}} \mathcal{O}(Y)X^{Y}\,dY, \tag{1.6}$$

where $i^2 = -1$, $X \in \mathbb{C}\backslash\{0\}$, $X^Y = \exp(Y[\log|X| + i\arg(X)])$, $\log|X|$ denotes the natural logarithm of $|X|$, and $\arg(X)$ is not necessarily the principal value. For convenience,

$$\mathcal{O}(Y) := \frac{\prod_{j=1}^{v}\Gamma(e_j - E_jY)\prod_{j=1}^{w}\Gamma(1-d_j+D_jY)}{\prod_{j=v+1}^{r}\Gamma(1-e_j+E_jY)\prod_{j=w+1}^{s}\Gamma(d_j-D_jY)},$$

where an empty product is interpreted as 1, and the integers v, w, s, r satisfy the inequalities $0 \le w \le s$ and $1 \le v \le r$. Assume the coefficients

$$D_j > 0 \quad (j = 1,\dots,s) \quad \text{and} \quad E_j > 0 \quad (j = 1,\dots,r),$$

and the complex parameters

$$d_j \quad (j = 1,\dots,s) \quad \text{and} \quad e_j \quad (j = 1,\dots,r)$$

are constrained such that no poles of integrand in (1.6) coincide, and $\mathcal{Z}$ is a suitable contour of the Mellin–Barnes type (in the complex Y-plane), which separates the poles of one product from the others. Further, if we assume

$$\ell := \sum_{j=1}^{w} D_j - \sum_{j=w+1}^{s} D_j + \sum_{j=1}^{v} E_j - \sum_{j=\mathrm{Q}+1}^{r} E_j > 0,$$

then the integral in (1.6) converges absolutely and defines the $\mathbb{H}$-function, which is analytic in the sector:

$$\left|\arg(X)\right| < \frac{1}{2}\ell\pi$$

and with the point $X = 0$ being tacitly excluded. Actually, the $\mathbb{H}$-function makes sense and also defines an analytic function of X when either

$$\mathcal{E} := \sum_{j=1}^{s} D_j - \sum_{j=1}^{r} E_j < 0 \quad \text{and} \quad 0 < |X| < \infty,$$

or

$$\mathcal{E} = 0 \quad \text{and} \quad 0 < |X| < R := \prod_{j=1}^{s} D_j^{-D_j} \prod_{j=1}^{r} E_j^{E_j}.$$

► *Meijer $\mathbb{G}$-function:*
The Meijer $\mathbb{G}$-function is a special case of the $\mathbb{H}$-function, that is,

$$ {}_{s}^{v}\mathbb{G}_{r}^{w}\left[X\Big|_{e_1,\dots,e_r}^{d_1,\dots,d_s}\right] \equiv {}_{s}^{v}\mathbb{H}_{r}^{w}\left[X\Big|_{(e_1,1),\dots,(e_r,1)}^{(d_1,1),\dots,(d_s,1)}\right] \tag{1.7}$$

$$= \frac{1}{2\pi i}\int_{\mathcal{Z}} \mathcal{O}'(Y)X^{-Y}\,dY,$$

where

$$\mathcal{O}'(Y) := \frac{\prod_{j=1}^{v}\Gamma(e_j+Y)\prod_{i=1}^{w}\Gamma(1-d_i-Y)}{\prod_{i=w+1}^{s}\Gamma(d_j+Y)\prod_{j=v+1}^{r}\Gamma(1-d_j-Y)}, \tag{1.8}$$

$X^{-Y} = \exp(-Y[\log|X| + i\arg(X)])$, $X \neq 0$ and $i^2 = -1$, and also $\log|X|$ represents the natural logarithm of $|X|$, and $\arg(X)$ is not necessarily the principle value as mentioned before.

Notice that an empty product in (1.8) is defined to be one, and the poles

$$e_{j\wp} = -(e_j + \wp), \quad j = 1,\dots,v, \wp \in \mathbb{N}_0, \tag{1.9}$$

of the gamma functions $\Gamma(e_j + Y)$ and the poles

$$d_{i\wp'} = 1 - d_i + \wp', \quad j = 1,\dots,w, \wp' \in \mathbb{N}_0, \tag{1.10}$$

of the gamma functions $\Gamma(1 - d_i - Y)$ do not coincide, that is,

$$e_j + \wp \neq d_i - \wp' - 1, \quad i = 1,\dots,w, j = 1,\dots,v, \wp, \wp' \in \mathbb{N}_0. \tag{1.11}$$

Further, $\mathcal{Z}$ is one of the contours defined above, which separates all poles $e_{j\wp}$ in (1.9) on the left from all poles $d_{i\wp}$ in (1.10) on the right of $\mathcal{Z}$.

► $\mathbb{G}$*-function (generalized Hypergeometric function):*

The generalized hypergeometric function is defined by the following generalized hypergeometric series

$${}_s\mathbb{H}_r[d_1,\dots,d_s;e_1,\dots,e_s;X] = \sum_{k=0}^{\infty}\frac{\prod_{i=1}^{s}(d_i)_k}{\prod_{j=1}^{r}(e_j)_k}\frac{X^k}{k!}, \tag{1.12}$$

where $X \in \mathbb{C}$, $s, r \in \mathbb{N}_0$, and $d_i, e_j \in \mathbb{C}$, for $i = 1,\dots,s$ and $j = 1,\dots,r$. For $z \in \mathbb{C}$, we denote

$$(z)_0 = 1, z \neq 0,$$
$$(z)_k = z(z+1)\dots(z+k-1), \quad k \in \mathbb{N}.$$

If $d_j \neq -\wp$, $j = 1,\dots,r$ and $\wp \in \mathbb{N}_0$, then the generalized hypergeometric series (1.12) can be represented in terms of the Mellin–Barnes integral of the form

$$\begin{aligned}&{}_s\mathbb{H}_r[d_1,\dots,d_s;e_1,\dots,e_r;X]\\ &\quad = \frac{\prod_{j=1}^{r}\Gamma(e_j)}{\prod_{i=1}^{s}\Gamma(d_i)}\frac{1}{2\pi i}\int_{\mathcal{Z}}\frac{\Gamma(Y)\prod_{i=1}^{s}\Gamma(d_i-Y)}{\prod_{j=1}^{r}\Gamma(e_j-Y)}(-X)^{-Y}\,dY, \quad X \neq 0,\end{aligned}$$

where $e_j \neq 0,-1,-2,\dots j = 1,\dots r$, $e_j \neq 0,-1,-2,\dots j = 1,\dots s$, and with the special contour $\mathcal{Z}$.

Such a formula converts representation (1.12) as the Meijer $\mathbb{G}$-function given by:

$${}_s\mathbb{H}_r[d_1,\dots,d_s;e_1,\dots,e_r;X] = \frac{\prod_{j=1}^{r}\Gamma(e_j)}{\prod_{i=1}^{s}\Gamma(d_i)}{}^{1}_{s}\mathbb{G}^{\mathsf{E}}_{r+1}\left[-X\Big|^{1-d_1,\dots,1-d_s}_{0,1-e_1,\dots,1-e_r}\right].$$

► *Generalized Hyperbolic function:*
The generalizations of the Hyperbolic Functions are defined by

$$\mathbb{H}[X] := \mathbb{J}^{\Bbbk}_{A,R}(X) = \sum_{k=0}^{\infty} \frac{\Bbbk^k}{(Ak+R)!} X^{Ak+R}, \tag{1.13}$$

for $R = 0, \dots, A-1$, where $\Bbbk \in \mathbb{C}$. Also, we have

$$\mathbb{J}^{\Bbbk}_{A,0}(0) = 1, \tag{1.14}$$

$$\mathbb{J}^{k}(X) = \Bbbk\mathbb{J}(X), \tag{1.15}$$

where

$$\mathbb{J}^{(k)}(0) = 0, \quad \text{if } k \neq R, 0 \leq k \leq n-1, \tag{1.16}$$

$$\mathbb{J}^{(k)}(0) = 1, \quad \text{if } k = R. \tag{1.17}$$

We would like to point out that the special case $\mathbb{J}^{1}_{A,0}(X)$ is the Mittag–Leffler function.
Now let

$$\operatorname{diag}[\zeta_1, \dots, \zeta_n]_{n\times n} = \begin{bmatrix} \zeta_1 & 0 & \dots & 0 \\ 0 & \zeta_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \zeta_n \end{bmatrix}_{n\times n}.$$

Note that $\boldsymbol{\zeta} := \operatorname{diag}[\zeta_1, \dots, \zeta_n] \preceq \boldsymbol{\xi} := \operatorname{diag}[\xi_1, \dots, \xi_n]$ if $\zeta_i \leq \xi_i$ for all $1 \leq i \leq n$.
Consider the following matrix valued control function given by

$$\begin{aligned} \mathfrak{W}[X] = \operatorname{diag}\bigl[&{}_0\mathbb{H}_0[X], {}_0\mathbb{H}_1[e_1;X], {}_1\mathbb{H}_1[d_1;e_1;X], {}_2\mathbb{H}_1[d_1,d_2;e_1;X], \\ &{}_s\mathbb{H}_r[d_1,\dots,d_s;e_1,\dots,e_r;X], {}_s\mathbb{H}_r\bigl[X\big|^{(d_1,D_1),\dots,(d_s,D_s)}_{(e_1,E_1),\dots,(e_r,E_r)}\bigr], {}^{v}_{s}\mathbb{H}^{w}_{r}\bigl[X\big|^{(d_1,1),\dots,(d_s,1)}_{(e_1,1),\dots,(e_r,1)}\bigr], \\ &{}^{v}_{s}\mathbb{H}^{w}_{r}\bigl[X\big|^{(d_j,D_j)_{1,s}}_{(e_j,E_j)_{1,r}}\bigr], \mathbb{H}[X]\bigr]_{9\times 9}. \end{aligned}$$

Let a mapping Θ from vector space U to normed linear space V have Hyers–Ulam–Rassias stability. If we replace the control function of Hyers–Ulam–Rassias stability with $\mathfrak{W}[X]$, we say Θ has multi-stability property.
Clearly, if we have

$$\operatorname{diag}\bigl[\varphi_1(x,y), \dots, \varphi_9(x,y)\bigr]_{9\times 9} := \theta\mathfrak{W}\bigl(\|x\|^r_A + \|y\|^r_A\bigr)$$

then the following are satisfied:
► for any $r < 1$, and $\theta > 0$, there exist constants $L_i < 1$ s.t.

$$\varphi_i(x,0) \leq 2L_i\varphi_i\left(\frac{x}{2}, 0\right), \quad i = 1, \dots, 9;$$

► for any $r > 1$, and $\theta > 0$, there exist constants $L_i < 1$ s.t.

$$\varphi_i(x,0) \leq \frac{1}{2}L_i\varphi_i(2x,0), \quad i = 1,\dots,9.$$

2 Multi-stability of homomorphisms in C^*-algebras

Throughout this entire section, let A be a C^*-algebra with norm $\|\cdot\|_A$ and that B be a C^*-algebra with norm $\|\cdot\|_B$.

For a given mapping $f : \partial \times A \to B$, we define

$$D_\mu f(\eth,x,y) := \mu f\left(\eth, \frac{x+y}{2}\right) + \mu f\left(\eth, \frac{x-y}{2}\right) - f(\eth, \mu x),$$

for $\mu \in \mathbb{T}^1 := \{\nu \in \mathbb{C} : |\nu| = 1\}$ and $x, y \in A$ and $\eth \in \partial$.

Notice that a $\mathbb{C}$-linear mapping $H : \partial \times A \to B$ is a *homomorphism* in C^*-algebras if H satisfies $H(\eth,xy) = H(\eth,x)H(\eth,y)$ and $H(\eth,x^*) = H(\eth,x)^*$ for $x, y \in A$ and $\eth \in \partial$.

We investigate the generalized Hyers–Ulam stability of homomorphisms in C^*-algebras for the functional equation $D_\mu f(\eth,x,y) = 0$.

Theorem 2.1 *Assume $f : \partial \times A \to B$ is a mapping for which there exist functions $\varphi_i : A^2 \to [0,\infty)$, for $i = 1,\dots,n \in \mathbb{N}$, s.t.*

$$\begin{aligned} &\operatorname{diag}\bigl[\bigl\|D_\mu f(\eth,x,y)\bigr\|_B,\dots,\bigl\|D_\mu f(\eth,x,y)\bigr\|_B\bigr]_{n\times n} \\ &\quad\preceq \operatorname{diag}\bigl[\varphi_1(x,y),\dots,\varphi_n(x,y)\bigr]_{n\times n}, \end{aligned} \tag{2.1}$$

$$\begin{aligned} &\operatorname{diag}\bigl[\bigl\|f(\eth,xy) - f(\eth,x)f(\eth,y)\bigr\|_B,\dots,\bigl\|f(\eth,xy) - f(\eth,x)f(\eth,y)\bigr\|_B\bigr]_{n\times n} \\ &\quad\preceq \operatorname{diag}\bigl[\varphi_1(x,y),\dots,\varphi_n(x,y)\bigr]_{n\times n}, \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} &\operatorname{diag}\bigl[\bigl\|f\bigl(\eth,x^*\bigr) - f(\eth,x)^*\bigr\|_B,\dots,\bigl\|f\bigl(\eth,x^*\bigr) - f(\eth,x)^*\bigr\|_B\bigr]_{n\times n} \\ &\quad\preceq \operatorname{diag}\bigl[\varphi_1(x,x),\dots,\varphi_n(x,x)\bigr]_{n\times n}, \end{aligned} \tag{2.3}$$

for $\mu \in \mathbb{T}^1$ and $x, y \in A$ and $\eth \in \partial$. If there exist constants $L_i < 1$ s.t. $\varphi_i(x,0) \leq 2L_i\varphi_i(\frac{x}{2},0)$ for $x \in A$ and $i = 1,\dots,n$, then there exists a unique C^*-algebra homomorphism $H : \partial \times A \to B$ s.t.

$$\begin{aligned} &\operatorname{diag}\bigl[\bigl\|f(\eth,x) - H(\eth,x)\bigr\|_B,\dots,\bigl\|f(\eth,x) - H(\eth,x)\bigr\|_B\bigr]_{n\times n} \\ &\quad\preceq \operatorname{diag}\left[\frac{L_1}{1-L_1}\varphi_1(x,0),\dots,\frac{L_n}{1-L_n}\varphi_n(x,0)\right]_{n\times n}, \end{aligned} \tag{2.4}$$

for $x \in A$ and $\eth \in \partial$.

Proof Assume the set

$$X := \{g : \partial \times A \to B\},$$

and define the *generalized metric* d on X:

$$\begin{aligned} d(g,h) := \inf\bigl\{&(C_1,\dots,C_n)\in\mathbb{R}_+ : \\ &\operatorname{diag}\bigl[\|g(\eth,x)-h(\eth,x)\|_B,\dots,\|g(\eth,x)-h(\eth,x)\|_B\bigr]_{n\times n} \\ &\preceq [C_1,\dots,C_n]^T_{n\times 1}\operatorname{diag}\bigl[\varphi_1(x,0),\dots,\varphi_n(x,0)\bigr]_{n\times n}, \forall x\in A, \eth\in\partial\bigr\}. \end{aligned}$$

Then (X,d) is complete.

Let the linear mapping $J : X\to X$ s.t.

$$Jg(\eth,x) := \frac{1}{2}g(\eth,2x),$$

for $x\in A$ and $\eth\in\partial$.

According to Theorem 3.1 of [13] and [1],

$$d(Jg,Jh)\preceq (L_1,\dots,L_n)d(g,h),$$

for $g,h\in X$.

Setting $\mu = 1$ and $y = 0$ in (2.1), we have

$$\begin{aligned} &\operatorname{diag}\left[\left\|2f\left(\eth,\frac{x}{2}\right)-f(\eth,x)\right\|_B,\dots,\left\|2f\left(\eth,\frac{x}{2}\right)-f(\eth,x)\right\|_B\right]_{n\times n} \\ &\quad\preceq \operatorname{diag}\bigl[\varphi_1(x,0),\dots,\varphi_n(x,0)\bigr]_{n\times n}, \end{aligned} \tag{2.5}$$

for $x\in A$ and $\eth\in\partial$. So,

$$\begin{aligned} &\operatorname{diag}\left[\left\|f(\eth,x)-\frac{1}{2}f(\eth,2x)\right\|_B,\dots,\left\|f(\eth,x)-\frac{1}{2}f(\eth,2x)\right\|_B\right]_{n\times n} \\ &\quad\preceq \operatorname{diag}\left[\frac{1}{2}\varphi_1(2x,0),\dots,\frac{1}{2}\varphi_n(2x,0)\right]_{n\times n} \\ &\quad\preceq \operatorname{diag}\bigl[L_1\varphi_1(x,0),\dots,L_n\varphi_n(x,0)\bigr]_{n\times n}, \end{aligned}$$

for $x\in A$ and $\eth\in\partial$. Hence $d(f,Jf)\preceq (L_1,\dots,L_n)$.

According to Theorem 1.3, there exists a mapping $H : \partial\times A\to B$ s.t.

(1) H is a fixed point of J, i.e.,

$$H(\eth,2x) = 2H(\eth,x), \tag{2.6}$$

for $x\in A$ and $\eth\in\partial$. The mapping H is a unique fixed point of J in the set

$$Y = \bigl\{g\in X : d(f,g) < \overbrace{(\infty,\dots,\infty)}^{n}\bigr\}.$$

This concludes that H is a unique mapping satisfying (2.6) s.t. there exists $(C_1,\dots,C_n)\in(0,\infty)^n$ satisfying

$$\begin{aligned}&\operatorname{diag}\big[\|H(\eth,x)-f(\eth,x)\|_B,\dots,\|H(\eth,x)-f(\eth,x)\|_B\big]_{n\times n}\\&\quad\preceq\operatorname{diag}\big[C_1\varphi_1(x,0),\dots,C_n\varphi_n(x,0)\big]_{n\times n}\end{aligned}$$

for $x\in A$ and $\eth\in\partial$.

(2) $d(J^kf,H)\to\overbrace{(0,\dots,0)}^{n}$ as $k\to\infty$. This concludes the equality

$$\lim_{k\to\infty}\frac{f(\eth,2^kx)}{2^k}=H(\eth,x), \tag{2.7}$$

for $x\in A$ and $\eth\in\partial$.

(3) $d(f,H)\le(\frac{1}{1-L_1},\dots,\frac{1}{1-L_n})d(f,Jf)$, which implies the inequality

$$d(f,H)\le\left(\frac{L_1}{1-L_1},\dots,\frac{L_n}{1-L_n}\right).$$

This claims that the inequality (2.4) holds.

According to (2.1) and (2.7),

$$\begin{aligned}&\operatorname{diag}\left[\left\|H\left(\eth,\frac{x+y}{2}\right)+H\left(\eth,\frac{x-y}{2}\right)-H(\eth,x)\right\|_B,\dots,\right.\\&\qquad\left.\left\|H\left(\eth,\frac{x+y}{2}\right)+H\left(\eth,\frac{x-y}{2}\right)-H(\eth,x)\right\|_B\right]_{n\times n}\\&\quad=\operatorname{diag}\left[\lim_{k\to\infty}\frac{1}{2^k}\big\|f\big(\eth,2^{k-1}(x+y)\big)+f\big(\eth,2^{k-1}(x-y)\big)-f\big(\eth,2^kx\big)\big\|_B,\dots,\right.\\&\qquad\left.\lim_{k\to\infty}\frac{1}{2^k}\big\|f\big(\eth,2^{k-1}(x+y)\big)+f\big(\eth,2^{k-1}(x-y)\big)-f\big(\eth,2^kx\big)\big\|_B\right]_{n\times n}\\&\quad\preceq\operatorname{diag}\left[\lim_{k\to\infty}\frac{1}{2^k}\varphi_1\big(2^kx,2^ky\big),\dots,\lim_{k\to\infty}\frac{1}{2^k}\varphi_n\big(2^kx,2^ky\big)\right]_{n\times n}\\&\quad=\operatorname{diag}[0,\dots,0]_{n\times n},\end{aligned}$$

for $x,y\in A$ and $\eth\in\partial$. So,

$$H\left(\eth,\frac{x+y}{2}\right)+H\left(\eth,\frac{x-y}{2}\right)=H(\eth,x), \tag{2.8}$$

for $x,y\in A$ and $\eth\in\partial$. Letting $z=\frac{x+y}{2}$ and $w=\frac{x-y}{2}$ in (2.8), we have

$$H(\eth,z)+H(\eth,w)=H(\eth,z+w),$$

for $z,w\in A$ and $\eth\in\partial$. So the mapping $H:\partial\times A\to B$ is Cauchy additive, i.e., $H(\eth,z+w)=H(\eth,z)+H(\eth,w)$ for $z,w\in A$ and $\eth\in\partial$.

Letting $y=x$ in (2.1), we have

$$\mu f(\eth,x)=f(\eth,\mu x),$$

for $\mu \in \mathbb{T}^1$ and $x \in A$ and $\eth \in \partial$. Similarly, we have

$$\mu H(\eth, x) = H(\eth, \mu x),$$

for $\mu \in \mathbb{T}^1$ and $x \in A$ and $\eth \in \partial$. Thus, one can prove that the mapping $H : \partial \times A \to B$ is $\mathbb{C}$-linear.

According to (2.2),

$$\begin{aligned}
&\operatorname{diag}\bigl[\bigl\|H(\eth,xy) - H(\eth,x)H(\eth,y)\bigr\|_B, \dots, \bigl\|H(\eth,xy) - H(\eth,x)H(\eth,y)\bigr\|_B\bigr]_{n\times n} \\
&\quad = \operatorname{diag}\biggl[\lim_{k\to\infty} \frac{1}{4^k} \bigl\|f\bigl(\eth, 4^k xy\bigr) - f\bigl(\eth, 2^k x\bigr) f\bigl(\eth, 2^k y\bigr)\bigr\|_B, \dots, \\
&\qquad \lim_{k\to\infty} \frac{1}{4^k} \bigl\|f\bigl(\eth, 4^k xy\bigr) - f\bigl(\eth, 2^k x\bigr) f\bigl(\eth, 2^k y\bigr)\bigr\|_B\biggr]_{n\times n} \\
&\quad \preceq \operatorname{diag}\biggl[\lim_{k\to\infty} \frac{1}{4^k} \varphi_1\bigl(2^k x, 2^k y\bigr), \dots, \lim_{k\to\infty} \frac{1}{4^k} \varphi_n\bigl(2^k x, 2^k y\bigr)\biggr]_{n\times n} \\
&\quad \preceq \operatorname{diag}\biggl[\lim_{k\to\infty} \frac{1}{2^k} \varphi_1\bigl(2^k x, 2^k y\bigr), \dots, \lim_{k\to\infty} \frac{1}{2^k} \varphi_n\bigl(2^k x, 2^k y\bigr)\biggr]_{n\times n} \\
&\quad = \operatorname{diag}[0, \dots, 0]_{n\times n},
\end{aligned}$$

for $x, y \in A$ and $\eth \in \partial$. So,

$$H(\eth, xy) = H(\eth, x)H(\eth, y),$$

for $x, y \in A$ and $\eth \in \partial$.

According to (2.3),

$$\begin{aligned}
&\operatorname{diag}\bigl[\bigl\|H\bigl(\eth,x^*\bigr) - H(\eth,x)^*\bigr\|_B, \dots, \bigl\|H\bigl(\eth,x^*\bigr) - H(\eth,x)^*\bigr\|_B\bigr]_{n\times n} \\
&\quad = \operatorname{diag}\biggl[\lim_{k\to\infty} \frac{1}{2^k} \bigl\|f\bigl(\eth, 2^k x^*\bigr) - f\bigl(\eth, 2^k x\bigr)^*\bigr\|_B, \dots, \\
&\qquad \lim_{k\to\infty} \frac{1}{2^k} \bigl\|f\bigl(\eth, 2^k x^*\bigr) - f\bigl(\eth, 2^k x\bigr)^*\bigr\|_B\biggr]_{n\times n} \\
&\quad \preceq \operatorname{diag}\biggl[\lim_{k\to\infty} \frac{1}{2^k} \varphi_1\bigl(2^k x, 2^k x\bigr), \dots, \lim_{k\to\infty} \frac{1}{2^k} \varphi_n\bigl(2^k x, 2^k x\bigr)\biggr]_{n\times n} \\
&\quad = \operatorname{diag}[0, \dots, 0]_{n\times n},
\end{aligned}$$

for $x \in A$ and $\eth \in \partial$. So,

$$H\bigl(\eth, x^*\bigr) = H(\eth, x)^*,$$

for $x \in A$ and $\eth \in \partial$.

Thus, $H : \partial \times A \to B$ is a C^*-algebra homomorphism satisfying (2.4), as desired. □

Theorem 2.2 *Assume $f : \partial \times A \to B$ is a mapping for which there exist functions $\varphi_i : A^2 \to [0, \infty)$ satisfying (2.1), (2.2) and (2.3) for $i = 1, \dots, n$. Furthermore, if there exist constants*

$L_i < 1$ s.t. $\varphi_i(x,0) \preceq \frac{1}{2}L_i\varphi_i(2x,0)$ for each $x \in A$ and $i = 1,\dots,n$, then there exists a unique C^*-algebra homomorphism $H : \partial \times A \to B$ s.t.

$$\begin{aligned} &\operatorname{diag}\big[\|f(\eth,x) - H(\eth,x)\|_B,\dots,\|f(\eth,x) - H(\eth,x)\|_B\big]_{n\times n} \\ &\quad\preceq \operatorname{diag}\left[\frac{L_1}{2-2L_1}\varphi_1(x,0),\dots,\frac{L_n}{2-2L_n}\varphi_n(x,0)\right]_{n\times n}, \end{aligned} \tag{2.9}$$

for $x \in A$ and $\eth \in \partial$.

Proof We consider the linear mapping $J : X \to X$ s.t.

$$Jg(\eth,x) := 2g\left(\eth,\frac{x}{2}\right),$$

for $x \in A$ and $\eth \in \partial$.

It follows from (2.5) that

$$\begin{aligned} &\left[\left\|f(\eth,x) - 2f\left(\eth,\frac{x}{2}\right)\right\|_B,\dots,\left\|f(\eth,x) - 2f\left(\eth,\frac{x}{2}\right)\right\|_B\right]_{n\times n} \\ &\quad\preceq \left[\varphi_1\left(\frac{x}{2},0\right),\dots,\varphi_n\left(\frac{x}{2},0\right)\right]_{n\times n} \\ &\quad\preceq \left[\frac{L_1}{2}\varphi_1(x,0),\dots,\frac{L_n}{2}\varphi_n(x,0)\right]_{n\times n}, \end{aligned}$$

for $x \in A$ and $\eth \in \partial$. Hence $d(f,Jf) \preceq (\frac{L_1}{2},\dots,\frac{L_n}{2})$.

According to Theorem 1.3, there exists a mapping $H : \partial \times A \to B$ s.t.

(1) H is a fixed point of J, i.e.,

$$H(\eth,2x) = 2H(\eth,x), \tag{2.10}$$

for $x \in A$. Moreover, the mapping H is a unique fixed point of J in the set

$$Y = \big\{g \in X : d(f,g) < \underbrace{(\infty,\dots,\infty)}_{n}\big\}.$$

This implies that H is a unique mapping satisfying (2.10) s.t. there exists $(C_1,\dots,C_n) \in (0,\infty)^n$ satisfying

$$\begin{aligned} &\operatorname{diag}\big[\|H(\eth,x) - f(\eth,x)\|_B,\dots,\|H(\eth,x) - f(\eth,x)\|_B\big]_{n\times n} \\ &\quad\preceq \operatorname{diag}\big[C_1\varphi_1(x,0),\dots,C_n\varphi_n(x,0)\big]_{n\times n} \end{aligned}$$

for $x \in A$ and $\eth \in \partial$.

(2) $d(J^kf,H) \to \underbrace{(0,\dots,0)}_{n}$ as $k \to \infty$. This deduces the equality

$$\lim_{k\to\infty} 2^kf\left(\eth,\frac{x}{2^k}\right) = H(\eth,x)$$

for $x \in A$ and $\eth \in \partial$.

(3) $d(f,H) \leq (\frac{1}{1-L_1},\dots,\frac{1}{1-L_n})d(f,Jf)$ claims the inequality

$$d(f,H) \leq \left(\frac{L_1}{2-2L_1},\dots,\frac{L_n}{2-2L_n}\right),$$

which infers that the inequality (2.9) holds.

The rest of the proof is similar to the proof of Theorem 2.1. □

Theorem 2.3 *Assume $f:\partial\times A\to B$ is an odd mapping for which there exist functions $\varphi_i : A^2 \to [0,\infty)$ satisfying (2.1), (2.2) and (2.3) for $i=1,\dots,n$. Moreover, if there exist constants $L_i<1$ s.t. $\varphi_i(x,3x)\leq 2L_i\varphi_i(\frac{x}{2},\frac{3x}{2})$ for $x\in A$ and $i=1,\dots,n$, then there exists a unique C^*-algebra homomorphism $H:\partial\times A\to B$ s.t.*

$$\begin{aligned}&\operatorname{diag}\big[\|f(\eth,x)-H(\eth,x)\|_B,\dots,\|f(\eth,x)-H(\eth,x)\|_B\big]_{n\times n}\\&\quad\preceq \operatorname{diag}\left[\frac{1}{2-2L_1}\varphi_1(x,3x),\dots,\frac{1}{2-2L_n}\varphi_n(x,3x)\right]_{n\times n},\end{aligned}\tag{2.11}$$

for $x\in A$ and $\eth\in\partial$.

Proof Assume the set

$$X:=\{g:\partial\times A\to B\},$$

and introduce the *generalized metric d* on X as

$$\begin{aligned}d(g,h):=\inf\big\{&(C_1,\dots,C_n)\in\mathbb{R}_+:\operatorname{diag}\big[\|g(\eth,x)-h(\eth,x)\|_B,\dots,\|g(\eth,x)-h(\eth,x)\|_B\big]_{n\times n}\\&\preceq [C_1,\dots,C_n]^T_{n\times1}\operatorname{diag}\big[\varphi_1(x,3x),\dots,\varphi_n(x,3x)\big]_{n\times n},\forall x\in A,\eth\in\partial\big\}.\end{aligned}$$

Then the space (X,d) is complete.

Now we assume the linear mapping $J:X\to X$ s.t.

$$Jg(\eth,x):=\frac{1}{2}g(\eth,2x),$$

for $x\in A$ and $\eth\in\partial$.

According to Theorem 3.1 of [1],

$$d(Jg,Jh)\leq(L_1,\dots,L_n)d(g,h),$$

for $g,h\in X$.

Letting $\mu=1$ and replacing y by $3x$ in (2.1), we have

$$\begin{aligned}&\operatorname{diag}\big[\|f(\eth,2x)-2f(\eth,x)\|_B,\dots,\|f(\eth,2x)-2f(\eth,x)\|_B\big]_{n\times n}\\&\quad\preceq\operatorname{diag}\big[\varphi_1(x,3x),\dots,\varphi_n(x,3x)\big]_{n\times n},\end{aligned}\tag{2.12}$$

for $x\in A$ and $\eth\in\partial$. So,

$$\operatorname{diag}\left[\left\|f(\eth,x)-\frac{1}{2}f(\eth,2x)\right\|_B,\dots,\left\|f(\eth,x)-\frac{1}{2}f(\eth,2x)\right\|_B\right]_{n\times n}$$

$$\preceq \operatorname{diag}\left[\frac{1}{2}\varphi_1(x,3x),\ldots,\frac{1}{2}\varphi_n(x,3x)\right]_{n\times n},$$

for $x \in A$ and $\eth \in \partial$. Hence $d(f,Jf) \preceq (\frac{1}{2},\ldots,\frac{1}{2})$.

According to Theorem 1.3, there exists a mapping $H : \partial \times A \to B$ s.t.

(1) H is a fixed point of J, i.e.,

$$H(\eth,2x) = 2H(\eth,x), \tag{2.13}$$

for $x \in A$ and $\eth \in \partial$. In addition, the mapping H is a unique fixed point of J in the set

$$Y = \{g \in X : d(f,g) < \underbrace{(\infty,\ldots,\infty)}_{n}\}.$$

This concludes that H is a unique mapping satisfying (2.13) s.t. there exists $(C_1,\ldots,C_n) \in (0,\infty)^n$ satisfying

$$\begin{aligned}&\operatorname{diag}\left[\left\|H(\eth,x)-f(\eth,x)\right\|_B,\ldots,\left\|H(\eth,x)-f(\eth,x)\right\|_B\right]_{n\times n}\\&\quad\preceq \operatorname{diag}\left[C_1\varphi_1(x,3x),\ldots,C_n\varphi_n(x,3x)\right]_{n\times n},\end{aligned}$$

for $x \in A$ and $\eth \in \partial$.

(2) $d(J^nf,H) \to \underbrace{(0,\ldots,0)}_{n}$ as $n \to \infty$. This infers the equality

$$\lim_{n\to\infty}\frac{f(\eth,2^nx)}{2^n} = H(\eth,x),$$

for $x \in A$ and $\eth \in \partial$.

(3) $d(f,H) \preceq (\frac{1}{1-L_1},\ldots,\frac{1}{1-L_n})d(f,Jf)$ confirms the inequality

$$d(f,H) \preceq \left(\frac{1}{2-2L_1},\ldots,\frac{1}{2-2L_n}\right).$$

Therefore, the inequality (2.11) holds.

The rest follows immediately from the proof of Theorem 2.1. □

Theorem 2.4 *Assume $f : \partial \times A \to B$ is an odd mapping for which there exist functions $\varphi_i : A^2 \to [0,\infty)$ satisfying (2.1), (2.2) and (2.3), for $i = 1,\ldots,n$. In addition, if there exist constants $L_i < 1$ s.t. $\varphi_i(x,3x) \leq \frac{1}{2}L_i\varphi_i(2x,6x)$ for $x \in A$ and $i = 1,\ldots,n$, then there exists a unique C^*-algebra homomorphism $H : \partial \times A \to B$ s.t.*

$$\begin{aligned}&\operatorname{diag}\left[\left\|f(\eth,x)-H(\eth,x)\right\|_B,\ldots,\left\|f(\eth,x)-H(\eth,x)\right\|_B\right]_{n\times n}\\&\quad\preceq \operatorname{diag}\left[\frac{L_1}{2-2L_1}\varphi_1(x,3x),\ldots,\frac{L_n}{2-2L_n}\varphi_n(x,3x)\right]_{n\times n},\end{aligned} \tag{2.14}$$

for $x \in A$ and $\eth \in \partial$.

Proof We assume the linear mapping $J : X \to X$ s.t.

$$Jg(\eth, x) := 2g\left(\eth, \frac{x}{2}\right),$$

for $x \in A$ and $\eth \in \partial$.

According to (2.12),

$$\begin{aligned}
&\operatorname{diag}\left[\left\|f(\eth,x) - 2f\left(\eth,\frac{x}{2}\right)\right\|_B, \dots, \left\|f(\eth,x) - 2f\left(\eth,\frac{x}{2}\right)\right\|_B\right]_{n\times n} \\
&\quad\preceq \operatorname{diag}\left[\varphi_1\left(\frac{x}{2},\frac{3x}{2}\right), \dots, \varphi_n\left(\frac{x}{2},\frac{3x}{2}\right)\right]_{n\times n} \\
&\quad\preceq \operatorname{diag}\left[\frac{L_1}{2}\varphi_1(x,3x), \dots, \frac{L_n}{2}\varphi_n(x,3x)\right]_{n\times n},
\end{aligned}$$

for $x \in A$ and $\eth \in \partial$. Hence $d(f, Jf) \preceq (\frac{L_1}{2}, \dots, \frac{L_n}{2})$.

Using Theorem 1.3, there exists a mapping $H : \partial \times A \to B$ s.t.

(1) H is a fixed point of J, i.e.,

$$H(\eth, 2x) = 2H(\eth, x), \tag{2.15}$$

for $x \in A$ and $\eth \in \partial$. Further, the mapping H is a unique fixed point of J in the set

$$Y = \{g \in X : d(f, g) \prec \underbrace{(\infty, \dots, \infty)}_{n}\}.$$

This indicates that H is a unique mapping satisfying (2.15) s.t. there exists $(C_1, \dots, C_n) \in (0, \infty)^n$ satisfying

$$\begin{aligned}
&\operatorname{diag}\left[\left\|H(\eth,x) - f(\eth,x)\right\|_B, \dots, \left\|H(\eth,x) - f(\eth,x)\right\|_B\right]_{n\times n} \\
&\quad\preceq \operatorname{diag}\left[C_1\varphi_1(x,3x), \dots, C_n\varphi_n(x,3x)\right]_{n\times n}
\end{aligned}$$

for $x \in A$ and $\eth \in \partial$.

(2) The condition $d(J^k f, H) \to \underbrace{(0, \dots, 0)}_{n}$ as $k \to \infty$ derives the equality

$$\lim_{k\to\infty} 2^k f\left(\eth, \frac{x}{2^k}\right) = H(\eth, x),$$

for $x \in A$ and $\eth \in \partial$.

(3) The inequality $d(f, H) \preceq (\frac{1}{1-L_1}, \dots, \frac{1}{1-L_n}) d(f, Jf)$ claims

$$d(f, H) \preceq \left(\frac{L_1}{2 - 2L_1}, \dots, \frac{L_n}{2 - 2L_n}\right),$$

which concludes that the inequality (2.14) holds.

The rest of the proof follows from the proof of Theorem 2.1. □

3 Multi-stability of derivations on C^*-algebras

In this section, we let A be a C^*-algebra with norm $\|\cdot\|_A$.

Recall that a $\mathbb{C}$-linear mapping $\delta : \partial \times A \to A$ is a *derivation* on A if δ satisfies $\delta(\eth, xy) = \delta(\eth, x)y + x\delta(\eth, y)$ for $x, y \in A$ and $\eth \in \partial$.

We are going to present the generalized Hyers-Ulam stability of derivations on C^*-algebras for the functional equation $D_\mu f(\eth, x, y) = 0$.

Theorem 3.1 *Assume $f : \partial \times A \to A$ is a mapping for which there exist functions $\varphi_i : A^2 \to [0,\infty)$ for $i = 1, \dots, n$, such that*

$$\operatorname{diag}\big[\big\|D_\mu f(\eth, x, y)\big\|_A, \dots, \big\|D_\mu f(\eth, x, y)\big\|_A\big]_{n\times n} \preceq \operatorname{diag}\big[\varphi_1(x,y), \dots, \varphi_n(x,y)\big], \tag{3.1}$$

$$\begin{aligned}&\operatorname{diag}\big[\big\|f(\eth, xy) - f(\eth, x)y - xf(\eth, y)\big\|_A, \dots, \big\|f(\eth, xy) - f(\eth, x)y - xf(\eth, y)\big\|_A\big]_{n\times n} \\ &\quad\preceq \operatorname{diag}\big[\varphi_1(x,y), \dots, \varphi_n(x,y)\big]_{n\times n},\end{aligned} \tag{3.2}$$

for $\mu \in \mathbb{T}^1$ and $x, y \in A$, $\eth \in \partial$. Additionally, suppose there exist constants $L_i < 1$ such that $\varphi_i(x, 0) \le 2L_i\varphi_i(\frac{x}{2}, 0)$ for $x \in A$ and $i = 1, \dots, n$. Then there exists a unique derivation $\delta : \partial \times A \to A$ satisfying

$$\begin{aligned}&\operatorname{diag}\big[\big\|f(\eth, x) - \delta(\eth, x)\big\|_A, \dots, \big\|f(\eth, x) - \delta(\eth, x)\big\|_A\big]_{n\times n} \\ &\quad\preceq \operatorname{diag}\left[\frac{L_1}{1-L_1}\varphi_1(x,0), \dots, \frac{L_n}{1-L_n}\varphi_n(x,0)\right]_{n\times n},\end{aligned} \tag{3.3}$$

for $x \in A$ and $\eth \in \partial$.

Proof By the same reasoning as the proof of Theorem 2.1, there exists a unique evolutive $\mathbb{C}$-linear mapping $\delta : \partial \times A \to A$ satisfying (3.3). The mapping $\delta : \partial \times A \to A$ is given by

$$\delta(\eth, x) = \lim_{k\to\infty} \frac{f(\eth, 2^k x)}{2^k},$$

for $x \in A$ and $\eth \in \partial$.

Applying (3.2),

$$\begin{aligned}&\operatorname{diag}\big[\big\|\delta(\eth, xy) - \delta(\eth, x)y - x\delta(\eth, y)\big\|_A, \dots, \big\|\delta(\eth, xy) - \delta(\eth, x)y - x\delta(\eth, y)\big\|_A\big]_{n\times n} \\ &\quad= \operatorname{diag}\bigg[\lim_{k\to\infty}\frac{1}{4^k}\big\|f\big(\eth, 4^k xy\big) - f\big(\eth, 2^k x\big)\cdot 2^k y - 2^k x f\big(\eth, 2^k y\big)\big\|_A, \dots, \\ &\qquad\lim_{k\to\infty}\frac{1}{4^k}\big\|f\big(\eth, 4^k xy\big) - f\big(\eth, 2^k x\big)\cdot 2^k y - 2^k x f\big(2^k y\big)\big\|_A\bigg]_{n\times n} \\ &\quad\preceq \operatorname{diag}\bigg[\lim_{k\to\infty}\frac{1}{4^k}\varphi_1\big(2^k x, 2^k y\big), \dots, \lim_{k\to\infty}\frac{1}{4^k}\varphi_n\big(2^k x, 2^k y\big)\bigg]_{n\times n} \\ &\quad\preceq \operatorname{diag}\bigg[\lim_{k\to\infty}\frac{1}{2^k}\varphi_1\big(2^k x, 2^k y\big), \dots, \lim_{k\to\infty}\frac{1}{2^k}\varphi_n\big(2^k x, 2^k y\big)\bigg]_{n\times n} \\ &\quad= \operatorname{diag}[0, \dots, 0]_{n\times n}\end{aligned}$$

for $x, y \in A$ and $\eth \in \partial$. So,

$$\delta(\eth, xy) = \delta(\eth, x)y + x\delta(\eth, y)$$

for $x, y \in A$ and $\eth \in \partial$. Thus $\delta : \partial \times A \to A$ is a derivation satisfying (3.3). □

Theorem 3.2 *Assume $f : \partial \times A \to A$ is a mapping for which there exist functions $\varphi_i : A^2 \to [0, \infty)$ satisfying (3.1) and (3.2) for $i = 1, \ldots, n$. Also assume there exist constants $L_i < 1$ s.t. $\varphi_i(x, 0) \leq \frac{1}{2} L_i \varphi_i(2x, 0)$ for $x \in A$ and $i = 1, \ldots, n$. Then there exists a unique derivation $\delta : \partial \times A \to A$ s.t.*

$$\begin{aligned} &\operatorname{diag}\big[\big\|f(\eth, x) - \delta(\eth, x)\big\|_A, \ldots, \big\|f(\eth, x) - \delta(\eth, x)\big\|_A\big]_{n\times n} \\ &\quad\preceq \operatorname{diag}\left[\frac{L_1}{2 - 2L_1}\varphi_1(x, 0), \ldots, \frac{L_n}{2 - 2L_n}\varphi_n(x, 0)\right]_{n\times n}, \end{aligned} \tag{3.4}$$

for $x \in A$ and $\eth \in \partial$.

Proof The proof is similar to the proofs of Theorems 2.2 and 3.1. □

Remark 3.3 For the inequalities controlled by the product of powers of norms, one can obtain results similar to Theorems 2.3 and 2.4.

4 Multi-stability of homomorphisms in Lie C^*-algebras

A C^*-algebra $\mathcal{C}$, endowed with the Lie product $[x, y] := \frac{xy - yx}{2}$ on $\mathcal{C}$, is called a *Lie C^*-algebra*.

Definition 4.1 ([14]) Assume A and B are Lie C^*-algebras. A $\mathbb{C}$-linear mapping $H : \partial \times A \to B$ is called a *Lie C^*-algebra homomorphism* if $H([x, y]) = [H(x), H(y)]$ for $x, y \in A$.

Throughout this section, we assume A is a Lie C^*-algebra with norm $\|\cdot\|_A$, and B is a Lie C^*-algebra with norm $\|\cdot\|_B$.

We show the generalized Hyers–Ulam stability of homomorphisms in Lie C^*-algebras for the functional equation $D_\mu f(x, y) = 0$.

Theorem 4.2 *Assume $f : \partial \times A \to B$ is a mapping for which there exist functions $\varphi_i : A^2 \to [0, \infty)$ for $i = 1, \ldots, n$ satisfying (2.1) and*

$$\begin{aligned} &\operatorname{diag}\big[\big\|f\big(\eth, [x, y]\big) - \big[f(\eth, x), f(\eth, y)\big]\big\|_B, \ldots, \big\|f\big(\eth, [x, y]\big) - \big[f(\eth, x), f(\eth, y)\big]\big\|_B\big]_{n\times n} \\ &\quad\preceq \operatorname{diag}\big[\varphi_1(x, y), \ldots, \varphi_n(x, y)\big]_{n\times n} \end{aligned} \tag{4.1}$$

for $x, y \in A$, $\eth \in \partial$. Furthermore, if there exist constants $L_i < 1$ s.t. $\varphi_i(x, 0) \leq 2L_i \varphi_i(\frac{x}{2}, 0)$ for $x \in A$, and $i = 1, \ldots, n$, then there exists a unique Lie C^*-algebra homomorphism $H : \partial \times A \to B$ satisfying (2.4).

Proof By the same arguments as the proof of Theorem 2.1, there exists a unique $\mathbb{C}$-linear mapping $\delta : \partial \times A \to A$ satisfying (2.4). The mapping $H : \partial \times A \to B$ is defined by

$$H(\eth, x) = \lim_{k\to\infty} \frac{f(\eth, 2^k x)}{2^k},$$

for $x \in A$ and $\eth \in \partial$.

Utilizing (4.1),

$$\begin{aligned}
&\operatorname{diag}\big[\big\|H\big(\eth,[x,y]\big)-\big[H(\eth,x),H(\eth,y)\big]\big\|_B,\dots,\big\|H\big(\eth,[x,y]\big)-\big[H(\eth,x),H(\eth,y)\big]\big\|_B\big]_{n\times n}\\
&\quad=\operatorname{diag}\bigg[\lim_{k\to\infty}\frac{1}{4^k}\big\|f\big(\eth,4^k[x,y]\big)-\big[f\big(\eth,2^kx\big),f\big(\eth,2^ky\big)\big]\big\|_B,\dots,\\
&\qquad\lim_{k\to\infty}\frac{1}{4^k}\big\|f\big(\eth,4^k[x,y]\big)-\big[f\big(\eth,2^kx\big),f\big(\eth,2^ky\big)\big]\big\|_B\bigg]_{n\times n}\\
&\preceq\operatorname{diag}\bigg[\lim_{k\to\infty}\frac{1}{4^k}\varphi_1\big(2^kx,2^ky\big),\dots,\lim_{k\to\infty}\frac{1}{4^k}\varphi_n\big(2^kx,2^ky\big)\bigg]_{n\times n}\\
&\preceq\operatorname{diag}\bigg[\lim_{k\to\infty}\frac{1}{2^k}\varphi_1\big(2^kx,2^ky\big),\dots,\lim_{k\to\infty}\frac{1}{2^k}\varphi_n\big(2^kx,2^ky\big)\bigg]_{n\times n}\\
&=\operatorname{diag}[0,\dots,0]_{n\times n},
\end{aligned}$$

for $x,y\in A$ and $\eth\in\partial$. Hence,

$$H\big(\eth,[x,y]\big)=\big[H(\eth,x),H(\eth,y)\big],$$

for $x,y\in A$ and $\eth\in\partial$.

In summary, $H:\partial\times A\to B$ is a Lie C^*-algebra homomorphism satisfying (2.4), as desired. □

Theorem 4.3 *Assume $f:\partial\times A\to B$ is a mapping for which there exist functions $\varphi_i:A^2\to[0,\infty)$ satisfying (2.1) and (4.1) for $i=1,\dots,n$. Further, if there exist constants $L_i<1$ s.t. $\varphi_i(x,0)\le\frac{1}{2}L_i\varphi_i(2x,0)$ for $x\in A$, and $i=1,\dots,n$ then there exists a unique Lie C^*-algebra homomorphism $H:\partial\times A\to B$ satisfying (2.9).*

Proof The proof follows similarly from the proofs of Theorems 2.2 and 2.3. □

Remark 4.4 For the inequalities controlled by the product of powers of norms, one can derive results similar to Theorems 2.3 and 2.4.

5 Multi-stability of Lie derivations on C^*-algebras

Definition 5.1 ([15]) Let A be a Lie C^*-algebra. A $\mathbb{C}$-linear mapping $\delta:\partial\times A\to A$ is called a *Lie derivation* if $\delta(\eth,[x,y])=[\delta(\eth,x),y]+[x,\delta(\eth,y)]$ for $x,y\in A$ and $\eth\in\partial$.

Throughout this section, we assume A is a Lie C^*-algebra with norm $\|\cdot\|_A$.

We prove the generalized Hyers–Ulam stability of derivations on Lie C^*-algebras for the functional equation $D_\mu f(\eth,x,y)=0$.

Theorem 5.2 *Let $f:\partial\times A\to A$ be a mapping for which there exist functions $\varphi_i:A^2\to[0,\infty)$ satisfying (3.1) and*

$$\begin{aligned}
&\operatorname{diag}\big[\big\|f\big(\eth,[x,y]\big)-\big[f(\eth,x),y\big]-\big[x,f(\eth,y)\big]\big\|_A,\dots,\\
&\qquad\big\|f\big(\eth,[x,y]\big)-\big[f(\eth,x),y\big]-\big[x,f(\eth,y)\big]\big\|_A\big]_{n\times n}\\
&\quad\preceq\operatorname{diag}\big[\varphi_1(x,y),\dots,\varphi_n(x,y)\big]_{n\times n}
\end{aligned}\tag{5.1}$$

for $x,y \in A$, $\eth \in \partial$ and $i = 1,\dots,n$. Besides, there exist constants $L_i < 1$ s.t. $\varphi_i(x,0) \leq 2L_i\varphi_i(\frac{x}{2},0)$ for $x \in A$ and $i = 1,\dots,n$. Then there exists a unique Lie derivation $\delta : \partial \times A \to A$ satisfying (3.3).

Proof By the same reasoning as the proof of Theorem 2.1, there exists a unique evolutive $\mathbb{C}$-linear mapping $\delta : \partial \times A \to A$ satisfying (3.3). The mapping $\delta : \partial \times A \to A$ is further defined by

$$\delta(\eth,x) = \lim_{n\to\infty} \frac{f(\eth,2^n x)}{2^n},$$

for $x \in A$ and $\eth \in \partial$.

According to (5.1),

$$\begin{aligned}
&\operatorname{diag}\bigl[\bigl\|\delta\bigl(\eth,[x,y]\bigr) - \bigl[\delta(\eth,x),y\bigr] - \bigl[x,\delta(\eth,y)\bigr]\bigr\|_A,\dots,\\
&\qquad\bigl\|\delta\bigl(\eth,[x,y]\bigr) - \bigl[\delta(\eth,x),y\bigr] - \bigl[x,\delta(\eth,y)\bigr]\bigr\|_A\bigr]_{n\times n}\\
&\quad= \operatorname{diag}\biggl[\lim_{k\to\infty}\frac{1}{4^k}\bigl\|f\bigl(\eth,4^k[x,y]\bigr) - \bigl[f\bigl(\eth,2^k x\bigr),2^k y\bigr] - \bigl[2^k x,f\bigl(\eth,2^k y\bigr)\bigr]\bigr\|_A,\dots,\\
&\qquad\lim_{k\to\infty}\frac{1}{4^k}\bigl\|f\bigl(\eth,4^k[x,y]\bigr) - \bigl[f\bigl(\eth,2^k x\bigr),2^k y\bigr] - \bigl[2^k x,f\bigl(\eth,2^k y\bigr)\bigr]\bigr\|_A\biggr]_{n\times n}\\
&\quad\leq \operatorname{diag}\biggl[\lim_{k\to\infty}\frac{1}{4^k}\varphi\bigl(2^k x,2^k y\bigr),\dots,\lim_{k\to\infty}\frac{1}{4^k}\varphi\bigl(2^k x,2^k y\bigr)\biggr]_{n\times n}\\
&\quad\leq \operatorname{diag}\biggl[\lim_{k\to\infty}\frac{1}{2^k}\varphi\bigl(2^k x,2^k y\bigr),\dots,\lim_{k\to\infty}\frac{1}{2^k}\varphi\bigl(2^k x,2^k y\bigr)\biggr]_{n\times n}\\
&\quad= \operatorname{diag}[0,\dots,0]_{n\times n}
\end{aligned}$$

for $x,y \in A$ and $\eth \in \partial$. So,

$$\delta\bigl(\eth,[x,y]\bigr) = \bigl[\delta(\eth,x),y\bigr] + \bigl[x,\delta(\eth,y)\bigr],$$

for $x,y \in A$ and $\eth \in \partial$. Thus, $\delta : \partial \times A \to A$ is a derivation satisfying (3.3). □

Theorem 5.3 *Assume $f : \partial \times A \to A$ is a mapping for which there exist functions $\varphi_i : A^2 \to [0,\infty)$ satisfying* (3.1) *and* (5.1) *for $i = 1,\dots,n$. In addition, if there exist constants $L_i < 1$ s.t. $\varphi_i(x,0) \leq \frac{1}{2}L_i\varphi_i(2x,0)$ for $x \in A$ and $i = 1,\dots,n$, then there exists a unique Lie derivation $\delta : \partial \times A \to A$ satisfying* (3.4).

Proof The proof is similar to the proofs of Theorems 2.2 and 2.3. □

Remark 5.4 For the inequalities controlled by the product of powers of norms, one can obtain results similar to Theorems 2.3 and 2.4.

6 Multi-stability by matrix valued multicontrol functions

Corollary 6.1 *Assume $r < 1$, $\theta > 0$ and $f : \partial \times A \to B$ is a mapping such that*

$$\operatorname{diag}\bigl[\bigl\|D_\mu f(\eth,x,y)\bigr\|_B,\dots,\bigl\|D_\mu f(\eth,x,y)\bigr\|_B\bigr]_{9\times 9} \preceq \theta\mathfrak{W}\bigl(\|x\|_A^r + \|y\|_A^r\bigr), \tag{6.1}$$

$$\begin{aligned}&\operatorname{diag}\big[\big\|f(\eth,xy)-f(\eth,x)f(\eth,y)\big\|_B,\dots,\big\|f(\eth,xy)-f(\eth,x)f(\eth,y)\big\|_B\big]_{9\times 9} && (6.2)\\&\quad\preceq \theta\mathfrak{W}\big(\|x\|_A^r+\|y\|_A^r\big),\\&\operatorname{diag}\big[\big\|f\big(\eth,x^*\big)-f(\eth,x)^*\big\|_B,\dots,\big\|f\big(\eth,x^*\big)-f(\eth,x)^*\big\|_B\big]_{9\times 9}\preceq\theta\mathfrak{W}\big(2\|x\|_A^r\big), && (6.3)\end{aligned}$$

for $\mu\in\mathbb{T}^1$ and $x,y\in A$ and $\eth\in\partial$. Then there exists a unique C^*-algebra homomorphism $H:\partial\times A\to B$ s.t.

$$\operatorname{diag}\big[\big\|f(\eth,x)-H(\eth,x)\big\|_B,\dots,\big\|f(\eth,x)-H(\eth,x)\big\|_B\big]_{9\times 9}\preceq\frac{2^r\theta}{2-2^r}\mathfrak{W}\big(\|x\|_A^r\big), \tag{6.4}$$

for $x\in A$ and $\eth\in\partial$.

Proof The proof follows from Theorem 2.1 by taking

$$\operatorname{diag}\big[\varphi_1(x,y),\dots,\varphi_9(x,y)\big]_{9\times 9}:=\theta\mathfrak{W}\big(\|x\|_A^r+\|y\|_A^r\big)$$

for $x,y\in A$. Setting $L_i=2^{r-1}$, $i=1,\dots,9$, we have the desired result. □

Corollary 6.2 *Assume $r>2$, $\theta>0$ and $f:\partial\times A\to B$ is a mapping satisfying* (6.1), (6.2) *and* (6.3). *Then there exists a unique C^*-algebra homomorphism $H:\partial\times A\to B$ s.t.*

$$\operatorname{diag}\big[\big\|f(\eth,x)-H(\eth,x)\big\|_B,\dots,\big\|f(\eth,x)-H(\eth,x)\big\|_B\big]_{9\times 9}\preceq\frac{\theta}{2^r-2}\mathfrak{W}\big(\|x\|_A^r\big), \tag{6.5}$$

for $x\in A$ and $\eth\in\partial$.

Proof The proof follows from Theorem 2.2 by taking

$$\operatorname{diag}\big[\varphi_1(x,y),\dots,\varphi_9(x,y)\big]_{9\times 9}:=\theta\mathfrak{W}\big(\|x\|_A^r+\|y\|_A^r\big),$$

for $x,y\in A$. Choosing $L_i=2^{1-r}$, $i=1,\dots,9$, we get the desired result. □

Corollary 6.3 *Assume $r<\frac{1}{2}$, $\theta>0$ and $f:\partial\times A\to B$ is an odd mapping satisfying*

$$\begin{aligned}&\operatorname{diag}\big[\big\|D_\mu f(\eth,x,y)\big\|_B,\dots,\big\|D_\mu f(\eth,x,y)\big\|_B\big]_{9\times 9}\preceq\theta\cdot\mathfrak{W}\big(\|x\|_A^r\cdot\|y\|_A^r\big), && (6.6)\\&\operatorname{diag}\big[\big\|f(\eth,xy)-f(\eth,x)f(\eth,y)\big\|_B,\dots,\big\|f(\eth,xy)-f(\eth,x)f(\eth,y)\big\|_B\big]_{9\times 9} && (6.7)\\&\quad\preceq\theta\cdot\mathfrak{W}\big(\|x\|_A^r\cdot\|y\|_A^r\big),\\&\operatorname{diag}\big[\big\|f\big(\eth,x^*\big)-f(\eth,x)^*\big\|_B,\dots,\big\|f\big(\eth,x^*\big)-f(\eth,x)^*\big\|_B\big]_{9\times 9}\preceq\theta\cdot\mathfrak{W}\big(\|x\|_A^{2r}\big), && (6.8)\end{aligned}$$

for $\mu\in\mathbb{T}^1$ and $x,y\in A$ and $\eth\in\partial$. Then there exists a unique C^*-algebra homomorphism $H:\partial\times A\to B$ s.t.

$$\operatorname{diag}\big[\big\|f(\eth,x)-H(\eth,x)\big\|_B,\dots,\big\|f(\eth,x)-H(\eth,x)\big\|_B\big]_{9\times 9}\preceq\frac{3^r\theta}{2-2^{2r}}\mathfrak{W}\big(\|x\|_A^{2r}\big), \tag{6.9}$$

for $x\in A$ and $\eth\in\partial$.

Proof The proof follows from Theorem 2.3 by taking

$$\operatorname{diag}\big[\varphi_1(x,y),\dots,\varphi_9(x,y)\big]_{9\times 9} := \theta\cdot\mathfrak{W}\big(\|x\|_A^r\cdot\|y\|_A^r\big),$$

for $x,y\in A$. Picking $L_i = 2^{2r-1}$, $i=1,\dots,9$, we come to the desired result. □

Corollary 6.4 *Assume $r>1$, $\theta>0$ and $f:\partial\times A\to B$ is an odd mapping satisfying (6.6), (6.7) and (6.8). Then there exists a unique C^*-algebra homomorphism $H:\partial\times A\to B$ s.t.*

$$\operatorname{diag}\big[\big\|f(\eth,x)-H(\eth,x)\big\|_B,\dots,\big\|f(\eth,x)-H(\eth,x)\big\|_B\big]_{9\times 9} \preceq \frac{\theta}{2^{2r}-2}\mathfrak{W}\big(\|x\|_A^{2r}\big), \tag{6.10}$$

for $x\in A$ and $\eth\in\partial$.

Proof The proof can be derived from Theorem 2.4 by taking

$$\operatorname{diag}\big[\varphi_1(x,y),\dots,\varphi_9(x,y)\big]_{9\times 9} := \theta\cdot\mathfrak{W}\big(\|x\|_A^r\cdot\|y\|_A^r\big),$$

for $x,y\in A$. Letting $L_i = 2^{1-2r}$, $i=1,\dots,9$, we get the desired result. □

Corollary 6.5 *Assume $r<1$, $\theta>0$ and $f:\partial\times A\to A$ is a mapping s.t.*

$$\operatorname{diag}\big[\big\|D_\mu f(\eth,x,y)\big\|_A,\dots,\big\|D_\mu f(\eth,x,y)\big\|_A\big]_{9\times 9} \preceq \theta\mathfrak{W}\big(\|x\|_A^r+\|y\|_A^r\big), \tag{6.11}$$

$$\operatorname{diag}\big[\big\|f(\eth,xy)-f(\eth,x)y-xf(\eth,y)\big\|_A,\dots,\big\|f(\eth,xy)-f(\eth,x)y-xf(\eth,y)\big\|_A\big]_{9\times 9} \preceq \theta\mathfrak{W}\big(\|x\|_A^r+\|y\|_A^r\big), \tag{6.12}$$

for $\mu\in\mathbb{T}^1$ and $x,y\in A$ and $\eth\in\partial$. Then there exists a unique derivation $\delta:\partial\times A\to A$ s.t.

$$\operatorname{diag}\big[\big\|f(\eth,x)-\delta(\eth,x)\big\|_A,\dots,\big\|f(\eth,x)-\delta(\eth,x)\big\|_A\big]_{9\times 9} \preceq \frac{2^r\theta}{2-2^r}\mathfrak{W}\big(\|x\|_A^r\big), \tag{6.13}$$

for $x\in A$ and $\eth\in\partial$.

Proof The proof follows from Theorem 3.1 by taking

$$\operatorname{diag}\big[\varphi_1(x,y),\dots,\varphi_9(x,y)\big]_{9\times 9} := \theta\mathfrak{W}\big(\|x\|_A^r+\|y\|_A^r\big),$$

for $x,y\in A$. Setting $L_i = 2^{r-1}$, $i=1,\dots 9$, we come to the conclusion. □

Corollary 6.6 *Assume $r>2$, $\theta>0$ and $f:\partial\times A\to A$ is a mapping satisfying (6.11) and (6.12). Then there exists a unique derivation $\delta:\partial\times A\to A$ s.t.*

$$\operatorname{diag}\big[\big\|f(\eth,x)-\delta(\eth,x)\big\|_A,\dots,\big\|f(\eth,x)-\delta(\eth,x)\big\|_A\big]_{9\times 9} \preceq \frac{\theta}{2^r-2}\mathfrak{W}\big(\|x\|_A^r\big), \tag{6.14}$$

for $x\in A$ and $\eth\in\partial$.

Proof The proof comes directly from Theorem 3.2 by taking

$$\operatorname{diag}\big[\varphi_1(x,y),\ldots,\varphi_9(x,y)\big]_{9\times 9} := \theta\mathfrak{W}\big(\|x\|_A^r + \|y\|_A^r\big)$$

for $x, y \in A$. Choosing $L_i = 2^{1-r}$, $i = 1,\ldots,9$, we get the desired result. □

Remark 6.7 For the inequalities controlled by the product of powers of norms, one can obtain results similar to Corollaries 6.3 and 6.4.

Corollary 6.8 *Assume $r < 1$, $\theta > 0$ and $f : \partial \times A \to B$ is a mapping satisfying* (6.1) *s.t.*

$$\begin{aligned} &\operatorname{diag}\big[\big\|f\big(\eth,[x,y]\big) - \big[f(\eth,x),f(\eth,y)\big]\big\|_B,\ldots,\big\|f\big(\eth,[x,y]\big) - \big[f(\eth,x),f(\eth,y)\big]\big\|_B\big]_{9\times 9} \\ &\quad\preceq \theta\mathfrak{W}\big(\|x\|_A^r + \|y\|_A^r\big), \end{aligned} \tag{6.15}$$

for $x, y \in A$ and $\eth \in \partial$. Then there exists a unique Lie C^*-algebra homomorphism $H : \partial \times A \to B$ satisfying (6.4).

Proof The proof follows from Theorem 5.2 by taking

$$\operatorname{diag}\big[\varphi_1(x,y),\ldots,\varphi_n(x,y)\big]_{9\times 9} := \theta\mathfrak{W}\big(\|x\|_A^r + \|y\|_A^r\big),$$

for $x, y \in A$. Then setting $L_i = 2^{r-1}$, $i = 1,\ldots,9$, we deduce the desired result. □

Corollary 6.9 *Assume $r > 2$, $\theta > 0$ and $f : \partial \times A \to B$ is a mapping satisfying* (6.1) *and* (6.15). *Then there exists a unique Lie C^*-algebra homomorphism $H : \partial \times A \to B$ satisfying* (6.5).

Proof The proof follows from Theorem 5.3 by taking

$$\operatorname{diag}\big[\varphi_1(x,y),\ldots,\varphi_n(x,y)\big]_{9\times 9} := \theta\mathfrak{W}\big(\|x\|_A^r + \|y\|_A^r\big),$$

for $x, y \in A$. Letting $L_i = 2^{1-r}$, $i = 1,\ldots,9$, we get the desired result. □

Remark 6.10 For the inequalities controlled by the product of powers of norms, one can obtain results similar to Corollaries 6.3 and 6.4.

Corollary 6.11 *Assume $r < 1$, $\theta > 0$ and $f : \partial \times A \to A$ is a mapping satisfying* (6.11) *s.t.*

$$\begin{aligned} &\operatorname{diag}\big[\big\|f\big(\eth,[x,y]\big) - \big[f(\eth,x),y\big] - \big[x,f(\eth,y)\big]\big\|_A,\ldots, \\ &\qquad \big\|f\big(\eth,[x,y]\big) - \big[f(\eth,x),y\big] - \big[x,f(\eth,y)\big]\big\|_A\big]_{9\times 9} \\ &\quad\preceq \theta\mathfrak{W}\big(\|x\|_A^r + \|y\|_A^r\big), \end{aligned} \tag{6.16}$$

for $x, y \in A$ and $\eth \in \partial$. Then there exists a unique Lie derivation $\delta : \partial \times A \to A$ satisfying (6.13).

Proof The proof comes directly from Theorem 5.2 by taking

$$\operatorname{diag}\big[\varphi_1(x,y),\ldots,\varphi_9(x,y)\big]_{9\times 9} := \theta\mathfrak{W}\big(\|x\|_A^r + \|y\|_A^r\big),$$

for $x, y \in A$. Then setting $L_i = 2^{r-1}$, $i = 1,\ldots,9$, we have the desired result. □

Corollary 6.12 *Assume $r > 2$, $\theta > 0$ and $f : \eth \times A \to A$ is a mapping satisfying* (6.11) *and* (6.16). *Then there exists a unique Lie derivation $\delta : \eth \times A \to A$ satisfying* (6.14).

Proof It follows from Theorem 5.3 by taking

$$\operatorname{diag}\big[\varphi_1(x,y),\ldots,\varphi_9(x,y)\big]_{9\times 9} := \theta\mathfrak{W}\big(\|x\|_A^r + \|y\|_A^r\big),$$

for $x, y \in A$. Then choosing $L_i = 2^{1-r}$, $i = 1,\ldots,9$, we imply the desired result. □

Remark 6.13 For the inequalities controlled by the product of powers of norms, we can derive results similar to Corollaries 6.3 and 6.4.

7 Conclusion

Using a new class of control functions defined by some special function, we study the generalized Hyers–Ulam stability of homomorphisms and multi-stability of derivations in C^*-algebras and Lie C^*-algebras for the following random operator equation based on fixed point methods:

$$\mu f\left(\eth, \frac{x+y}{2}\right) + \mu f\left(\eth, \frac{x-y}{2}\right) = f(\eth, \mu x),$$

where μ a complex number with $|\mu| = 1$.

Acknowledgements
The authors are thankful to the area editor and referees for giving valuable comments and suggestions. Chenkuan Li is supported by NSERC Discovery Grant number 2019-03907.

Funding
This research does not receive specific funding, the corresponding author is a full-time member of the School of Mathematics, Iran University of Science and Technology, Narmak, Tehran, Iran.

Availability of data and materials
There are no data that we needed for this manuscript.

Declarations

Competing interests
The authors declare no competing interests.

Author contributions
S.R.A., methodology, writing–original draft preparation. R.S., supervision and project administration. CL, project administration. T.M.R., supervision and project administration. C.P., methodology. All authors read and approved the final manuscript.

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Received: 3 May 2022 Accepted: 20 February 2023 Published online: 28 February 2023

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