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Generalized Solutions of Ordinary Differential Equations Related to the Chebyshev Polynomial of the Second Kind

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Abstract: In this work, we employed the Laplace transform of right-sided distributions in conjunction with the power series method to obtain distributional solutions to the modified Bessel equation and its related equation, whose coefficients contain the parameters ν and γ. We demonstrated that the solutions can be expressed as finite linear combinations of the Dirac delta function and its derivatives, with the specific form depending on the values of ν and γ.

Keywords: Dirac delta function; distributional solution; generalized solutions; Laplace transform; power series solution

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1. Introduction

In the framework of generalized functions, Kanwal [1] classified solution types of homogeneous linear ordinary differential equations (ODEs) of the form

$$\sum_{n=0}^{m} a_n(t)y^{(n)}(t) = 0, \tag{1}$$

where the coefficient $a_n(t)$ is an infinitely smooth function for each n and $t \in \mathbb{R}$. A solution $y(t)$ to Equation (1) can be classified by type as follows:

(i) The solution $y(t)$ is a classical solution if it is sufficiently smooth for differentiation to be performed in the usual sense in Equation (1) and the resulting equation is an identity.

(ii) The solution $y(t)$ is a weak solution if it is not sufficiently smooth, meaning that differentiation in Equation (1) cannot be performed in the usual sense; however, it is a regular distribution that satisfies Equation (1) in the sense of distribution.

(iii) The solution $y(t)$ is a distributional solution if it is a singular distribution and satisfies Equation (1) in the sense of distribution.

These are referred to as generalized solutions.

The only generalized solution in the sense of distribution for normal homogeneous linear ODEs with infinitely smooth coefficients is the classical solution. Equation (1) with singular coefficients might have a distributional solution. For instance, the distributional solution of the following differential equations is the Dirac delta function, $\delta(t)$:

$$ty''(t) + 2y'(t) + ty(t) = 0;$$

the Bessel equation

$$t^2y''(t) + ty'(t) + (t^2 - 1)y(t) = 0;$$

the confluent hypergeometric equation

$$ty''(t) + (2 - t)y'(t) - y(t) = 0;$$

and the second-order Cauchy–Euler equation

$$t^2y''(t) + 3ty'(t) + y(t) = 0.$$

This can be easily checked by using Formula (17). Furthermore, the distributional solutions of some classes of Cauchy–Euler equations have been studied by many researchers; see [2–9] for more details.

A distributional solution of an ODE is important because it provides a rigorous interpretation of a fundamental solution of a nonhomogeneous linear ODE when the nonhomogeneous term is the Dirac delta function. As the class of generalized functions includes the set of regular distributions, there are many singular distributions, one of which is the Dirac delta function. This calls into question the existence of weak solutions and singular distributions of ODEs with singular coefficients. In particular those singular distributions, which are linear combinations of Dirac delta functions and their derivatives.

$$y(t) = \sum_{n=0}^{m} x_n \delta^{(n)}(t). \tag{2}$$

In 1980, Wiener [10] considered solutions to linear systems of functional differential equations of the form in (2). He established two theorems regarding solutions in the space of finite-order distributions and applied the theorems to some important second-order ODEs. In 1982, Wiener [11] studied the criteria for the existence of m^{th} order distributional solutions, of the form in (2), to differential equations of the following forms:

$$\sum_{i=0}^{n} a_i(t)y^{(n-i)}(t) = 0,$$
$$\sum_{i=0}^{n} t^i a_i(t)y^{(i)}(t) = 0,$$

and

$$ty^{(n)}(t) + \sum_{i=1}^{n} a_i(t)y^{(n-i)}(t) = 0.$$

Cooke and Wiener [12] published the existence theorems for distributional and analytic solutions of functional differential equations in 1984. Littlejohn and Kanwal [13] investigated the distributional solutions of the hypergeometric differential equation, which has solutions of the form in (8). Wiener and Cooke [14] showed the necessary and sufficient conditions for the simultaneous existence of rational functions and solutions (2) to linear ODEs in 1990.

In 1999, Kananthai [2] considered generalized solutions of the third-order Cauchy–Euler equation of the form

$$t^3y'''(t) + t^2y''(t) + ty'(t) + my(t) = 0, \tag{3}$$

where m represents some integers and $t \in \mathbb{R}$. Generalized solutions of (3) are either distributional solutions or weak solutions, depending on the values of m.

In 2015, Nonlaopon et al. [3] studied generalized solutions of certain n^{th}-order differential equations with polynomial coefficients of the form

$$ty^{(n)}(t) + my^{(n-1)}(t) + ty'(t) + ty(t) = 0, \tag{4}$$

where m and n are any integers with $n \geq 2$ and $t \in \mathbb{R}$.

In 2019, Jhanthanam et al. [8] examined the generalized solution of the third-order Cauchy–Euler equation in the space of right-sided distributions via the Laplace transform of the form

$$t^3y'''(t) + at^2y''(t) + bty'(t) + cy(t) = 0, \tag{5}$$

where a, b, and $c \in \mathbb{Z}$ and $t \in \mathbb{R}$. The authors studied the type of solution in the space of right-sided distributions, and they found that it depended on the values of a, b, and c.

In 2020, Waiyaworn et al. [15] studied the distributional solutions of linear ODEs of the forms

$$t^2y''(t) + 2ty'(t) - [t^2 + \nu(\nu+1)]y(t) = 0 \tag{6}$$

and

$$t^2y''(t) + 3ty'(t) - (t^2 + \nu^2 - 1)y(t) = 0, \tag{7}$$

where $\nu \in \mathbb{N} \cup \{0\}$ and $t \in \mathbb{R}$, by using the Laplace transform and power series solution techniques, depending on the values of ν.

The infinite order distributional solution to various differential equations with singular coefficients of the form

$$y(t) = \sum_{n=0}^{\infty} x_n \delta^{(n)}(t) \tag{8}$$

has also been explored by many investigators; for additional information, see [12,16–20]. Kanwal [1] provided a short introduction to these ideas as well.

Motivated by the preceding work, we proposed distributional solutions of the following ODEs:

$$t^2y''(t) + ty'(t) - [t^2 + (\nu+1)^2]y(t) = 0 \tag{9}$$

and

$$t^2y''(t) + 2(1-\gamma)ty'(t) - [t^2 + 2\gamma + \nu(\nu + 2\gamma + 1)]y(t) = 0, \tag{10}$$

where $\nu \in \mathbb{N} \cup \{0\}$ and $\gamma, t \in \mathbb{R}$. We used the Laplace transform of right-sided distributions together with the power series method to search for the distributional solutions. We found that our new solutions were finite linear combinations of the Dirac delta function and its derivatives, depending on the values of ν and γ.

The remainder of the paper is divided into three parts. We provide related definitions and the lemmas required to derive our major results in Section 2. We prove our main results with supporting examples in Section 3. Finally, we summarize the entire work in Section 4.

2. Preliminaries

In this section, we introduce the fundamental definitions, lemmas, and useful examples that were required for this work.

Definition 1. *The space $\mathcal{D}$ of* test functions *consists of all real-valued functions $\varphi(t)$, defined on $\mathbb{R}$, having the following properties:*

(i) *$\varphi(t)$ is infinitely smooth;*

(ii) *$\varphi(t)$ has a compact support where the support of $\varphi(t)$ is the closure of the set of all numbers t such that $\varphi(t) \neq 0$.*

Definition 2. *A* distribution *T is a continuous linear functional on the space $\mathcal{D}$. The space of all such distributions is denoted by $\mathcal{D}'$.*

For every $T \in \mathcal{D}'$ and $\varphi(t) \in \mathcal{D}$, the value where T acts on $\varphi(t)$ is denoted by $\langle T, \varphi(t) \rangle$. Note that $\langle T, \varphi(t) \rangle \in \mathbb{R}$. Distributions are classified into two groups: *regular distributions* and *singular distributions*. A regular distribution is a distribution generated by a locally integrable function. That is, if $f(t)$ is a locally integrable function, then a regular distribution T_f generated by $f(t)$ is given by

$$\langle T_f, \varphi(t)\rangle = \int_{-\infty}^{\infty} f(t)\varphi(t)\,dt \quad \text{for any } \varphi(t) \in \mathcal{D}.$$

It is customary to use the same symbol, $f(t)$, for the corresponding distribution, T_f. A singular distribution is a distribution that is not a regular distribution.

Example 1.

(i) *The Heaviside function*

$$H(t) = \begin{cases} 1, & \text{if } t > 0; \\ 0, & \text{if } t \le 0. \end{cases}$$

is a regular distribution because it is locally integrable and

$$\langle H(t), \varphi(t)\rangle = \int_{0}^{\infty} \varphi(t)\,dt \quad \text{for any } \varphi(t) \in \mathcal{D}.$$

(ii) *The Dirac delta function $\delta(t)$ is a distribution defined by*

$$\langle \delta(t), \varphi(t)\rangle = \varphi(0) \quad \text{for any } \varphi(t) \in \mathcal{D}.$$

It is well-known that this can not be generated by any locally integrable function. Thus, it is a singular distribution. Note that its support is $\{0\}$.

Definition 3. *The k^{th}-*order derivative *of a distribution T, denoted by $T^{(k)}$, is defined by $\left\langle T^{(k)}, \varphi(t)\right\rangle = (-1)^k \left\langle T, \varphi^{(k)}(t)\right\rangle$ for all $\varphi(t) \in \mathcal{D}$.*

A simple illustration is the first order derivative of the Dirac delta function, $\delta'(t)$, which is defined by $\langle \delta'(t), \varphi(t)\rangle = -\langle \delta(t), \varphi'(t)\rangle = -\varphi'(0)$, whereas the k^{th}-order of the Dirac delta function, $\delta^{(k)}(t)$, is $\left\langle \delta^{(k)}(t), \varphi(t)\right\rangle = (-1)^k \left\langle \delta(t), \varphi^{(k)}(t)\right\rangle = (-1)^k \varphi^{(k)}(0)$.

Example 2. *In the sense of distribution, $H'(t) = \delta(t)$ because for any $\varphi(t) \in \mathcal{D}$, we have*

$$\langle H'(t), \varphi(t)\rangle = -\langle H(t), \varphi'(t)\rangle = \varphi(0) = \langle \delta, \varphi(t)\rangle.$$

Definition 4. *Let $\alpha(t)$ be an infinitely differentiable function. We define the* product *of $\alpha(t)$ with any distribution T in $\mathcal{D}'$ by $\langle \alpha(t)T, \varphi(t)\rangle = \langle T, \alpha(t)\varphi(t)\rangle$, for all $\varphi(t) \in \mathcal{D}$. We should note that $\alpha(t)\varphi(t) \in \mathcal{D}$ if $\varphi(t) \in \mathcal{D}$.*

Definition 5. *Let $y(t)$ be a singular distribution that satisfies the equation*

$$\sum_{m=0}^{n} a_m(t) y^{(n)}(t) = f(t) \tag{11}$$

in the sense of distribution, where $a_m(t)$ is an infinitely differentiable function for each $m = 0, 1, \ldots, n$ and $f(t)$ is an arbitrary known distribution. Function $y(t)$ is called a distributional solution of Equation (11).

Definition 6. *Let $M \in \mathbb{R}$ and $f(t)$ be a locally integrable function satisfying the following conditions:*

(i) *$f(t) = 0$ for all $t < M$;*

(ii) *There exists a real number, c, such that $e^{-ct} f(t)$ is absolutely integrable over $\mathbb{R}$.*

The Laplace transform *of $f(t)$ is defined by*

$$F(s) = \mathcal{L}\{f(t)\} = \int_M^{\infty} f(t)e^{-st}dt, \tag{12}$$

where s is a complex variable.

It is well known that if $f(t)$ is continuous then $F(s)$ is an analytic function on the half-plane $\Re(s) > \sigma_a$, where σ_a is an abscissa of absolute convergence for $\mathcal{L}\{f(t)\}$.

Recall that the Laplace transform, $G(s)$, of a locally integrable function, $g(t)$, satisfies the conditions of Definition 6, that is,

$$G(s) = \mathcal{L}\{g(t)\} = \int_M^{\infty} g(t)e^{-st}dt, \tag{13}$$

where $\Re(s) > \sigma_a$. Then, $G(s)$ can be written in the form $G(s) = \langle g(t), e^{-st}\rangle$.

Definition 7. *The space S of test functions of rapid decay consists of all complex-valued functions, $\varphi(t)$, defined on $\mathbb{R}$, that have the following properties:*

(i) *$\varphi(t)$ is infinitely smooth;*

(ii) *$\varphi(t)$, together with their derivatives of all orders, decrease to zero faster than every power of $|t|$, i.e., they satisfy the inequality*

$$|t^p \phi^{(k)}(t)| < C_{pk},$$

where C_{pk} is a constant that depends on non-negative integers p, k, and $\varphi(t)$;

(iii) *$\varphi(t)$ satisfies*

$$\|\varphi\|_{p,k} = \sup_{t\in\mathbb{R}} |t^p \varphi^{(k)}(t)| < +\infty,$$

where $\|\cdot\|_{p,k}$ is a collection of seminorms.

Definition 8. *A* distribution of slow growth*, or a* tempered distribution *T, is a continuous and linear functional over the space S. That is, the complex number, denoted by $\langle T, \varphi(t)\rangle$, that T assigns to each test function of rapid decay, $\varphi(t)$, has the following properties:*

(i) *For every $\varphi_1, \varphi_2 \in S$ and constants c_1, c_2,*

$$\langle T, c_1\varphi_1(t) + c_2\varphi_2(t)\rangle = c_1\langle T, \varphi_1(t)\rangle + c_2\langle T, \varphi_2(t)\rangle;$$

(ii) *For every null sequence $\{\varphi_m(t)\} \subset S$,*

$$\lim_{m\to\infty} \langle T, \varphi_m(t)\rangle = 0.$$

The set of all tempered distributions is denoted by S'.

Definition 9. *Let $f(t)$ be a distribution satisfying the following properties:*

(i) *$f(t)$ is a right-sided distribution, that is, $f(t) \in \mathcal{D}'_R$;*

(ii) *There exists a real number, c, such that $e^{-ct}f(t)$ is a tempered distribution.*

The Laplace transform of a right-sided distribution $f(t)$ *satisfying (ii) is defined by*

$$F(s) = \mathcal{L}\{f(t)\} = \left\langle e^{-ct}f(t), X(t)e^{-(s-c)t}\right\rangle, \tag{14}$$

where $X(t)$ is an infinitely differentiable function with a support bounded on the left, which equals 1 *over a neighbourhood of the support of $f(t)$.*

For $\Re(s) > c$, the function $X(t)e^{-(s-c)t}$ is a testing function in the space S and $e^{-ct}f(t)$ is in the space S'. Then, the Laplace transform (14) can be reduced to

$$F(s) = \mathcal{L}\{f(t)\} = \langle f(t), e^{-st} \rangle. \tag{15}$$

Now, $F(s)$ is a function of s defined over the right half-plane $\Re(s) > c$. Zemanian [21] proved that $F(s)$ is an analytic function in the region of convergence $\Re(s) > \sigma_1$, where σ_1 is the abscissa of convergence and $e^{-ct}f(t) \in S'$ for some real number $c > \sigma_1$.

Lemma 1. *Let $f(t)$ be a Laplace-transformable distribution in $\mathcal{D}'_R$. If k is a positive integer then the following hold:*

(i) $\mathcal{L}\{(t^{k-1}H(t))/(k-1)!\} = 1/s^k, \quad \Re(s) > 0;$
(ii) $\mathcal{L}\{\delta(t)\} = 1, \quad -\infty < \Re(s) < \infty;$
(iii) $\mathcal{L}\left\{\delta^{(k)}(t)\right\} = s^k, \quad -\infty < \Re(s) < \infty;$
(iv) $\mathcal{L}\left\{t^k f(t)\right\} = (-1)^k F^{(k)}(s), \quad \Re(s) > \sigma_1;$
(v) $\mathcal{L}\left\{f^{(k)}(t)\right\} = s^k F(s), \quad \Re(s) > \sigma_1.$

Lemma 2. *If $\alpha(t)$ is infinitely differentiable then*

$$\alpha(t)\delta^{(m)}(t) = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \alpha^{(m-k)}(0)\delta^{(k)}(t). \tag{16}$$

Applying Lemma 2 (see the proof in [1]) to any monomial $\alpha(t) = t^n$ with the observation that

$$(t^n)^{(m-k)}\Big|_{t=0} = \begin{cases} 0, & \text{if } m-k \neq n; \\ n!, & \text{if } m-k = n. \end{cases}$$

yields a useful formula:

$$t^n \delta^{(m)}(t) = \begin{cases} 0, & \text{if } m < n; \\ (-1)^n \dfrac{m!}{(m-n)!} \delta^{(m-n)}(t), & \text{if } m \geq n. \end{cases} \tag{17}$$

3. Main Results

In this section, we make use of the Laplace transform and the power series method to find distributional solutions of ODEs (9) and (10), as shown in the following Theorems 1 and 2.

Theorem 1. *Consider the modified Bessel equation*

$$t^2 y''(t) + t y'(t) - [t^2 + (\nu+1)^2] y(t) = 0, \tag{18}$$

where $\nu \in \mathbb{N} \cup \{0\}$ and $t \in \mathbb{R}$. The distributional solution of Equation (18) is given by

$$y(t) = U_\nu(D)\delta(t), \tag{19}$$

while

$$U_\nu(D) = \sum_{k=0}^{\lfloor \nu/2 \rfloor} (-1)^k \frac{2^{\nu-2k}(\nu-k)!}{k!(\nu-2k)!} D^{\nu-2k} \tag{20}$$

is the Chebyshev polynomial, in D, of the second kind, and $D = d/dt$ is the distributional derivative operator.

Proof. Let us denote $Y(s) = \mathcal{L}\{y(t)\}$. Appealing to (iv) and (v) of Lemma 1, the Laplace transformation of Equation (18) yields

$$(1-s^2)Y''(s) - 3sY'(s) + \nu(\nu+2)Y(s) = 0. \tag{21}$$

We assume a solution of Equation (21) takes the form of $Y(s) = \sum_{n=0}^{\infty} a_n s^n$ and calculate the derivatives

$$Y'(s) = \sum_{n=1}^{\infty} n a_n s^{n-1} \quad \text{and} \quad Y''(s) = \sum_{n=2}^{\infty} n(n-1)a_n s^{n-2}.$$

Substituting these forms into Equation (21) gives us

$$\begin{aligned}&[2(1)a_2 + \nu(\nu+2)a_0] + [3(2)a_3 - (3(1) - \nu(\nu+2))a_1]s \\ &\quad + \sum_{n=2}^{\infty}[(n+2)(n+1)a_{n+2} - n(n-1)a_n - 3na_n + \nu(\nu+2)a_n]s^n = 0.\end{aligned}$$

Since $s^n \neq 0$ for all $n \geq 0$, it follows that

$$\begin{gathered}2a_2 + \nu(\nu+2)a_0 = 0, \quad (3\cdot 2)a_3 - (3 - \nu(\nu+2))a_1 = 0, \\ (n+2)(n+1)a_{n+2} - [n(n-1) + 3n - \nu(\nu+2)]a_n = 0, \quad n \geq 2,\end{gathered} \tag{22}$$

which can be grouped into a recursion formula

$$a_{n+2} = -\frac{(\nu-n)(\nu+n+2)}{(n+2)(n+1)}a_n, \quad n \geq 0. \tag{23}$$

Calculation of the coefficients from the recurrence relation in (23) leads to the following:

$$\begin{aligned}a_2 &= -\frac{(\nu)(\nu+2)}{2!}a_0 \\ a_4 &= (-1)^2\frac{(\nu-2)(\nu)(\nu+2)(\nu+4)}{4!}a_0 \\ &\vdots \\ a_{2n} &= (-1)^n\frac{(\nu-2n+2)\cdots(\nu-2)(\nu)(\nu+2)(\nu+4)\cdots(\nu+2n)}{(2n)!}a_0.\end{aligned}$$

Similarly,

$$\begin{aligned}a_3 &= -\frac{(\nu-1)(\nu+3)}{3!}a_1 \\ a_5 &= (-1)^2\frac{(\nu-3)(\nu-1)(\nu+3)(\nu+5)}{5!}a_1 \\ &\vdots \\ a_{2n+1} &= (-1)^n\frac{(\nu-2n+1)\cdots(\nu-3)(\nu-1)(\nu+3)(\nu+5)\cdots(\nu+2n+1)}{(2n+1)!}a_1.\end{aligned}$$

With $a_0 = 1, a_1 = 0$ in one case and $a_0 = 0, a_1 = 1$ in another, we obtain two linearly independent solutions of Equation (21) as

$$Y_e(s) = 1 + \sum_{n=1}^{\infty}(-1)^n\frac{(\nu-2n+2)\cdots(\nu-2)(\nu)(\nu+2)(\nu+4)\cdots(\nu+2n)}{(2n)!}s^{2n}$$

and

$$Y_o(s) = s + \sum_{n=1}^{\infty} (-1)^n \frac{(\nu - 2n + 1) \cdots (\nu - 3)(\nu - 1)(\nu + 3)(\nu + 5) \cdots (\nu + 2n + 1)}{(2n + 1)!} s^{2n+1},$$

respectively.

Now, consider the case when ν is an even number. Writing $\nu = 2m$ for a non-negative integer, m, we have

$$(\nu - 2n + 2) \cdots (\nu - 2)\nu = \begin{cases} 0, & \text{if } m = 0; \\ \dfrac{2^n m!}{(m-n)!}, & \text{if } m \geq n \geq 1; \\ 0, & \text{if } n > m \geq 1. \end{cases}$$

and

$$(\nu + 2)(\nu + 4) \cdots (\nu + 2n) = \frac{2^n (m+n)!}{m!}.$$

Hence $Y_e(s)$ reduces to a finite series

$$Y_e(s) = \sum_{k=0}^{m} \frac{(-1)^k 2^{2k} (m+k)!}{(m-k)!(2k)!} s^{2k}. \tag{24}$$

Next consider the case when ν is an odd number. Writing $\nu = 2m + 1$ for a non-negative integer, m, we have

$$(\nu - 2n + 1) \cdots (\nu - 3)(\nu - 1) = \begin{cases} 0, & \text{if } m = 0; \\ \dfrac{2^n m!}{(m-n)!}, & \text{if } m \geq n \geq 1; \\ 0, & \text{if } n > m \geq 1, \end{cases}$$

and

$$(\nu + 3)(\nu + 5) \cdots (\nu + 2n + 1) = \frac{2^n (m+n+1)!}{(m+1)!}.$$

Such as in the case of an even integer, $Y_o(s)$ reduces to a finite series

$$Y_o(s) = \frac{1}{m+1} \sum_{k=0}^{m} \frac{(-1)^k 2^{2k} (m+k+1)!}{(m-k)!(2k+1)!} s^{2k+1}. \tag{25}$$

For $\nu = 0, 1, 2, \ldots$, we have $Y_\nu(s)$ as follows:

$$\begin{aligned}
Y_0(s) &= 1 = U_0(s), \\
Y_1(s) &= s = \frac{1}{2} U_1(s), \\
Y_2(s) &= 1 - 4s^2 = -U_2(s), \\
Y_3(s) &= s - 2s^3 = -\frac{1}{4} U_3(s), \\
Y_4(s) &= 1 - 12s^2 + 16s^4 = U_4(s), \\
Y_5(s) &= s - \frac{16}{3} s^3 + \frac{16}{3} s^5 = \frac{1}{6} U_5(s), \\
Y_6(s) &= 1 - 24s^2 + 80s^4 - 64s^6 = -U_6(s), \\
Y_7(s) &= s - 10s^3 + 24s^5 - 16s^7 = -\frac{1}{8} U_7(s),
\end{aligned}$$

etc. Here, $U_\nu(s)$ is the Chebyshev polynomial [22], in s, of the second kind. As Equation (21) is homogeneous and linear, the Chebyshev polynomial is also a solution. Appealing to (ii) and (iii) of Lemma 1, the inverse Laplace transform of $U_\nu(s)$ gives us the distributional solutions of Equation (18) in the form

$$\begin{aligned} y(t) &= \sum_{k=0}^{\lfloor \nu/2 \rfloor} (-1)^k \frac{2^{\nu-2k}(\nu-k)!}{k!(\nu-2k)!} D^{\nu-2k}\delta(t) \\ &= U_\nu(D)\delta(t), \end{aligned} \tag{26}$$

where $U_\nu(D)$ is the Chebyshev polynomial, in D, of the second kind and $D = d/dt$ is the distributional derivative operator. □

Example 3. *For $\nu = 1$, Equation (18) appears as*

$$t^2 y''(t) + t y'(t) - (t^2 + 4) y(t) = 0, \tag{27}$$

which has a solution, according to Theorem 1. According to Theorem 1, Equation (27) also has a solution,

$$y(t) = 2\delta'(t). \tag{28}$$

For $\nu = 4$, Equation (18) appears as

$$t^2 y''(t) + t y'(t) - (t^2 + 25) y(t) = 0, \tag{29}$$

which has a solution, according to Theorem 1,

$$y(t) = 16\delta^{(4)}(t) - 12\delta''(t) + \delta(t). \tag{30}$$

With the help of formula (17), it is straight forward to check that distributions in (28) and (30) satisfy Equations (27) and (29), respectively.

Theorem 2. *Consider the equation of the form*

$$t^2 y''(t) + 2(1-\gamma) t y'(t) - [t^2 + 2\gamma + \nu(\nu + 2\gamma + 1)] y(t) = 0, \tag{31}$$

where $\nu \in \mathbb{N} \cup \{0\}$ and $\gamma, t \in \mathbb{R}$. The distributional solution of Equation (31) is given by

$$y(t) = C_\nu^\gamma(D)\delta(t), \tag{32}$$

while

$$C_\nu^\gamma(D) = \sum_{k=0}^{\lfloor \nu/2 \rfloor} (-1)^k \frac{2^{\nu-2k}\Gamma(\nu+\lambda-k)}{k!(\nu-2k)!\Gamma(\lambda)} D^{\nu-2k} \tag{33}$$

is the Gegenbauer polynomial in terms of the distributional derivative operator, $D = d/dt$, Γ is the gamma function, and

$$\lambda = \gamma + \frac{1}{2} > 0. \tag{34}$$

Proof. Let us denote $Y(s) = \mathcal{L}\{y(t)\}$. Appealing to (iv) and (v) of Lemma 1 the Laplace transformation of both sides of Equation (31) yields

$$(1-s^2) Y''(s) - 2(1+\gamma) s Y'(s) + (\nu^2 + 2\gamma\nu + \nu) Y(s) = 0. \tag{35}$$

We assume a solution of Equation (35) takes the form of $Y(s) = \sum_{n=0}^{\infty} a_n s^n$ and calculate the derivatives:

$$Y'(s) = \sum_{n=1}^{\infty} n a_n s^{n-1} \quad \text{and} \quad Y''(s) = \sum_{n=2}^{\infty} n(n-1) a_n s^{n-2}.$$

Substituting these forms into Equation (35) gives us

$$\left[2(1)a_2 + (\nu^2 + 2\gamma\nu + \nu)a_0\right] + \left\{3(2)a_3 - \left[2(1+\gamma) - (\nu^2 + 2\gamma\nu + \nu)\right]a_1\right\}s$$
$$+ \sum_{n=2}^{\infty}\left\{(n+2)(n+1)a_{n+2} - \left[n(n-1) + 2(1+\gamma)n - (\nu^2 + 2\gamma\nu + \nu)\right]a_n\right\}s^n = 0.$$

Since $s^n \neq 0$ for all $n \geq 0$, it follows that

$$2a_2 + (\nu^2 + 2\gamma\nu + \nu)a_0 = 0, \quad 6a_3 - \left[2(1+\gamma) - (\nu^2 + 2\gamma\nu + \nu)\right]a_1 = 0,$$
$$(n+2)(n+1)a_{n+2} - \left[n(n-1) + 2(1+\gamma)n - (\nu^2 + 2\gamma\nu + \nu)\right]a_n = 0, \quad n \geq 2,$$

which can be grouped into a recursion formula

$$a_{n+2} = -\frac{(\nu - n)(\nu + n + 1 + 2\gamma)}{(n+2)(n+1)} a_n, \quad n \geq 0. \tag{36}$$

Calculation of the coefficients from the recurrence relation in (36) leads to the following:

$$\begin{aligned}
a_2 &= \frac{-(\nu)(\nu + 1 + 2\gamma)}{2!} a_0, \\
a_4 &= \frac{(-1)^2(\nu - 2)(\nu)(\nu + 1 + 2\gamma)(\nu + 3 + 2\gamma)}{4!} a_0, \\
&\vdots \\
a_{2n} &= \frac{(-1)^n(\nu - 2n + 2)\cdots(\nu - 2)(\nu)(\nu + 1 + 2\gamma)(\nu + 3 + 2\gamma)\cdots(\nu + 2n - 1 + 2\gamma)}{(2n)!} a_0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
a_3 &= \frac{-(\nu - 1)(\nu + 2 + 2\gamma)}{3!} a_1, \\
a_5 &= \frac{(-1)^2(\nu - 3)(\nu - 1)(\nu + 2 + 2\gamma)(\nu + 4 + 2\gamma)}{5!} a_1, \\
&\vdots \\
a_{2n+1} &= \frac{(-1)^n(\nu - 2n + 1)\cdots(\nu - 3)(\nu - 1)(\nu + 2 + 2\gamma)(\nu + 4 + 2\gamma)\cdots(\nu + 2n + 2\gamma)}{(2n+1)!} a_1.
\end{aligned}$$

With $a_0 = 1, a_1 = 0$ in one case and $a_0 = 0, a_1 = 1$ in the others, we obtain two linearly independent solutions of Equation (35) as

$$Y_e(s) = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{(\nu - 2n + 2)\cdots(\nu - 2)(\nu)(\nu + 1 + 2\gamma)(\nu + 3 + 2\gamma)\cdots(\nu + 2n - 1 + 2\gamma)}{(2n)!} s^{2n}$$

and

$$Y_o(s) = s + \sum_{n=1}^{\infty} (-1)^n \frac{(\nu - 2n + 1)\cdots(\nu - 3)(\nu - 1)(\nu + 2 + 2\gamma)(\nu + 4 + 2\gamma)\cdots(\nu + 2n + 2\gamma)}{(2n+1)!} s^{2n+1},$$

respectively. Now consider the case when ν is an even number. Writing $\nu = 2m$ for a non-negative integer, m, we have

$$(\nu - 2n + 2) \cdots (\nu - 2)(\nu) = \begin{cases} 0, & \text{if } m = 0; \\ \dfrac{2^n m!}{(m-n)!}, & \text{if } m \geq n \geq 1; \\ 0, & \text{if } n > m \geq 1. \end{cases}$$

and

$$(\nu + 1 + 2\gamma)(\nu + 3 + 2\gamma) \cdots (\nu + 2n - 1 + 2\gamma) = \frac{2^n \Gamma(m + \lambda + n)}{\Gamma(m + \lambda)},$$

where $\lambda = \gamma + 1/2 > 0$ and Γ is the gamma function. Hence, $Y_e(s)$ reduces to a finite series

$$\begin{aligned} Y_e(s) &= 1 + m! \sum_{n=1}^{m} \frac{(-1)^n 2^{2n}}{(m-n)!(2n)!} \frac{\Gamma(m+\lambda+n)}{\Gamma(m+\lambda)} s^{2n} \\ &= m! \sum_{n=0}^{m} \frac{(-1)^n 2^{2n}}{(m-n)!(2n)!} \frac{\Gamma(m+\lambda+n)}{\Gamma(m+\lambda)} s^{2n} \\ &= \frac{(-1)^m m! \Gamma(\lambda)}{\Gamma(m+\lambda)} \sum_{k=0}^{m} (-1)^k \frac{2^{2m-2k} \Gamma(2m+\lambda-k)}{k!(2m-2k)!\Gamma(\lambda)} s^{2m-2k} \\ &= \frac{(-1)^m m! \Gamma(\lambda)}{\Gamma(m+\lambda)} C_{2m}^{\gamma}(s), \end{aligned} \tag{37}$$

where $C_{2m}^{\gamma}(s)$ is the Gegenbauer polynomial of s with degree $2m$.

Next, consider the case when ν is an odd number. Writing $\nu = 2m + 1$ for a non-negative integer, m, we have

$$(\nu - 2n + 1) \cdots (\nu - 3)(\nu - 1) = \begin{cases} 0, & \text{if } m = 0; \\ \dfrac{2^n m!}{(m-n)!}, & \text{if } m \geq n \geq 1; \\ 0, & \text{if } n > m \geq 1. \end{cases}$$

and

$$(\nu + 2 + 2\gamma)(\nu + 4 + 2\gamma) \cdots (\nu + 2n + 2\gamma) = \frac{2^n \Gamma(m + \lambda + n + 1)}{\Gamma(m + \lambda + 1)},$$

where $\lambda = \gamma + 1/2 > 0$.

Such as in the case of an even integer, $Y_o(s)$ reduces to a finite series

$$\begin{aligned} Y_o(s) &= s + m! \sum_{n=1}^{m} \frac{(-1)^n 2^{2n}}{(m-n)!(2n+1)!} \frac{\Gamma(m+\lambda+n+1)}{\Gamma(m+\lambda+1)} s^{2n+1} \\ &= m! \sum_{n=0}^{m} \frac{(-1)^n 2^{2n}}{(m-n)!(2n+1)!} \frac{\Gamma(m+\lambda+n+1)}{\Gamma(m+\lambda+1)} s^{2n+1} \\ &= \frac{(-1)^m m! \Gamma(\lambda)}{2\Gamma(m+\lambda+1)} \sum_{k=0}^{m} (-1)^k \frac{2^{2m+1-2k} \Gamma(2m+1+\lambda-k)}{k!(2m+1-2k)!\Gamma(\lambda)} s^{2m+1-2k} \\ &= \frac{(-1)^m m! \Gamma(\lambda)}{2\Gamma(m+\lambda+1)} C_{2m+1}^{\gamma}(s), \end{aligned} \tag{38}$$

where $C_{2m+1}^{\gamma}(s)$ is the Gegenbauer polynomial of s with degree $2m+1$. As Equation (35) is linear and homogeneous, C_{ν}^{γ}, in the forms of (37) and (38), is its solution for $\nu = 0, 1, 2, \ldots$.

Appealing to (ii) and (iii) in Lemma 1, the inverse Laplace transform of $C_\nu^\gamma(s)$ gives us the distributional solutions of Equation (31) in the form

$$\begin{aligned} y(t) &= \sum_{k=0}^{\lfloor \nu/2 \rfloor} (-1)^k \frac{2^{\nu-2k}\Gamma(\nu+\lambda-k)}{k!(\nu-2k)!\Gamma(\lambda)} D^{\nu-2k}\delta(t) \\ &= C_\nu^\gamma(D)\delta(t), \end{aligned} \tag{39}$$

where $C_\nu^\gamma(D)$ is the Gegenbauer polynomial in terms of the distributional derivative operator, $D = d/dt$. $\square$

To shorten our notation, from now on we shall refer to the distributional derivative operator as D.

Remark 1.

(*i*) *If* $\gamma = 0$ *then Equation* (31) *reduces to*

$$t^2y''(t) + 2ty'(t) - [t^2 + \nu(\nu+1)]y(t) = 0, \tag{40}$$

whose distribution solution, from Formula (32), *is*

$$y(t) = P_\nu(D)\delta(t), \tag{41}$$

where

$$P_\nu(D) = \frac{1}{2^\nu} \sum_{k=0}^{\lfloor \nu/2 \rfloor} (-1)^k \frac{(2\nu-2k)!}{k!(\nu-k)!(\nu-2k)!} D^{\nu-2k}$$

is the Legendre polynomial in D (see [3]*);*

(*ii*) *If* $\gamma = -1/2$ *then Equation* (31) *reduces to*

$$t^2y''(t) + 3ty'(t) - (t^2 + \nu^2 - 1)y(t) = 0, \tag{42}$$

whose distributional solution is

$$y(t) = T_\nu(D)\delta(t), \tag{43}$$

where

$$T_\nu(D) = \frac{\nu}{2} \sum_{k=0}^{\lfloor \nu/2 \rfloor} (-1)^k \frac{2^{\nu-2k}(\nu-k-1)!}{k!(\nu-2k)!} D^{\nu-2k}$$

is the Chebyshev polynomial, in D, of the first kind (see [15]*);*

(*iii*) *If* $\gamma = 1/2$ *then Equation* (31) *reduces to Equation* (18),

$$t^2y''(t) + ty'(t) - [t^2 + (\nu+1)^2]y(t) = 0, \tag{44}$$

whose distribution solution, from Formula (32), *is*

$$y(t) = U_\nu(D)\delta(t), \tag{45}$$

where

$$U_\nu(D) = \sum_{k=0}^{\lfloor \nu/2 \rfloor} (-1)^k \frac{2^{\nu-2k}(\nu-k)!}{k!(\nu-2k)!} D^{\nu-2k}$$

is the Chebyshev polynomial, in D, of the second kind, which appears in Theorem 1.

Example 4. *For* $\gamma = 5$ *and* $\nu = 2$, *Equation* (31) *appears as*

$$t^2y''(t) - 8ty'(t) - (t^2 + 36)y(t) = 0, \tag{46}$$

whose solution, according to Theorem 2, is

$$y(t) = \frac{143}{2}\delta''(t) - \frac{11}{2}\delta(t). \tag{47}$$

For $\gamma = 0$ *and* $\nu = 3$, *Equation* (31) *appears as*

$$t^2y''(t) + 2ty'(t) - (t^2 + 12)y(t) = 0, \tag{48}$$

whose solution, according to Theorem 2, is

$$y(t) = \frac{5}{2}\delta'''(t) - \frac{3}{2}\delta'(t). \tag{49}$$

For $\gamma = 1/2$ *and* $\nu = 4$, *Equation* (31) *appears as*

$$t^2y''(t) + ty'(t) - (t^2 + 25)y(t) = 0, \tag{50}$$

whose solution, according to Theorem 2, is

$$y(t) = 16\delta^{(4)}(t) - 12\delta''(t) + \delta(t), \tag{51}$$

which coincides with the solution in (30) *in Example 3. With the help of Formula* (17)*, it is straight forward to check that distributions* (47)*,* (49)*, and* (51) *satisfy Equations* (46)*,* (48)*, and* (50)*, respectively.*

4. Conclusions

Within the space of right-sided distributions, we derived the distributional solutions of the modified Bessel equation, Equation (18), and its related equation, Equation (31), by employing the Laplace transform and the power series method. Relying on the values of ν and γ, we found that our distributional solutions of Equations (18) and (31) could be perceived as the Chebyshev polynomial, in D, of the second kind and the Gegenbauer polynomial, in D, acting on the Dirac delta functions, respectively. Evidently there are classical solutions of both equations that are not stated here, but they can be found in mainstream textbooks.

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