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ORIGINAL RESEARCH

Existence and regularity results for a system of Λ-Hilfer fractional differential equations by the generalized Lax–Milgram theorem

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Abstract We study the existence of weak solutions for a system and a coupled system of Λ-Hilfer fractional differential equations on compact domains using the Lax–Milgram and Minty–Browder theorems. Furthermore, we provide an illustrative example, and a regularity result to imply that the obtained solution is classical.

Keywords Λ-Hilfer fractional differential equation · Lax–Milgram theorem · Minty–Browder theorem · Weak solution · Regularity

Mathematics Subject Classification 34A08 · 26A33 · 34B08

1 Introduction

Fractional differential equations (in short FDEs) are useful in a variety of domains, including physics, biology, and engineering, (see [1–7] and references therein). The variational approach to FDEs articulated in boundary value issues, p-Laplacian problems, critical point theory (CPT) problems, and so on is gaining popularity [8–10]. For example, Jiao and Zhou [11] used CPT to prove the existence and uniqueness of solutions for FBV equations on a variational structure. In addition, [12] applying the same theory proved the existence and uniqueness results for fractional p-Laplacian in the Caputo sense, under the Dirichlet boundary condition with mixed derivatives and integral boundary constraints. Fattahi and Alimohammady [13] studied the existing solutions for an FBV problem utilizing non-smooth CPT and variational approaches in 2017. For more results, one can see [14, 15] and references therein.

Motivated by the research described above and to further investigate in the field, the main goals of our study focus on the following:

1. Applying the Lax–Milgram theorem we prove the existence and uniqueness of solutions to a class of FDEs. Moreover, we present a regularity result for the solution and demonstrate an example using our main theorem.
2. Applying the Minty- Browder theorem to extend our result to a system of coupled of FDEs.

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2 Preliminaries

We begin by recalling some fundamental definitions of Λ-Riemann-Liouville fractional integral, Λ-Hilfer fractional derivative and variational definitions and related results. We refer readers to [14,15] and references therein for more details.

Definition 1 Let $\kappa \in (0, 1]$ and $\theta \in [0, 1]$. The Λ-fractional derivative space(in short Λ-FDS) $\mathcal{H}_2^{\kappa,\theta,\Lambda} := \mathcal{H}_2^{\kappa,\theta,\Lambda}([0, \mathcal{T}], \mathbb{R}))$ is defined as the space $\overline{C_0^{\infty}([0, \mathcal{T}], \mathbb{R})}$. That is,

$$\begin{aligned}\mathcal{H}_2^{\kappa,\theta,\Lambda} &= \left\{\vartheta \in \mathcal{L}^2([0, \mathcal{T}], \mathbb{R}) : \ {}^{H}\mathfrak{D}_{0^+}^{\kappa,\theta;\Lambda}\vartheta \in \mathcal{L}^2([0, \mathcal{T}], \mathbb{R}) \text{ and } I_{0^+}^{\theta(\theta-1)}\vartheta(0) = I_{0^+}^{\theta(\theta-1)}\vartheta(\mathcal{T}) = 0\right\} \\ &= \overline{C_0^{\infty}([0, \mathcal{T}], \mathbb{R})},\end{aligned}$$

with the norm

$$||\vartheta||_{\mathcal{H}_2^{\kappa,\theta,\Lambda}} = \left(||\vartheta||_{\mathcal{L}_2}^2 + ||{}^{H}\mathfrak{D}_{0^+}^{\kappa,\theta;\Lambda}\vartheta||_{\mathcal{L}_2}^2\right)^{1/2}.$$

The Λ-FDS $\mathcal{H}_2^{\kappa,\theta,\Lambda}$ is a reflexive and separable Banach space, for $0 < \kappa \leq 1$ and $0 \leq \theta \leq 1$.

The following Theorems 1 and 2, as well as Remark 1 can be found from [14,15].

Theorem 1 *Let $\kappa > 1/2$ and $\theta \in [0, 1]$. If the sequence $\{\varrho_n\}$ converges weakly to ϱ in $\mathcal{H}_2^{\kappa,\theta,\Lambda}$, then $\varrho_n \longrightarrow \varrho$ in $C[0, \mathcal{T}]$.*

Remark 1 For all $\mathfrak{w} \in \mathcal{H}_2^{\kappa,\theta,\Lambda}$ we have $||\mathfrak{w}||_{\infty} \leq \dfrac{(\Lambda(\mathcal{T}) - \Lambda(0))^{\kappa-1/2}}{\Gamma(\kappa)(2\kappa-1)^{1/2}}||\mathfrak{w}||_{\mathcal{H}_2^{\kappa,\theta,\Lambda}}$.

We recall the generalizations of the Lax–Milgram theorem [14,15] and Browder–Minty theorem [16], which will play an important role in presenting existence results of our FDEs and coupled system of FDEs respectively below.

The Generalization of the Lax–Milgram Theorem

Theorem 2 *Let $\mathcal{H}$ be a Hilbert space, $\mathcal{B}(\omega_1, \omega_2) : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be a continuous coercive bilinear form, and $\mathcal{F} : \mathcal{H} \to \mathcal{H}^*$ satisfy:*

- *(K1) For some positive constant C we have $||\mathcal{F}(\omega)||_{\mathcal{H}} \leq C$ for all $\omega \in N_1(0)$, where $N_1(0)$ stands for unit ball in $\mathcal{H}$.*
- *(K2) If $\{\omega_n\}$ is a sequence in $\mathcal{H}$ so that $\omega_n \rightharpoonup \omega$ weakly in $\mathcal{H}$, then the sequence $\{\mathcal{F}(\omega_n)\}$ has a subsequence $\{\mathcal{F}(\omega_{n_k})\}$ such that $\mathcal{F}(\omega_{n_k}) \rightharpoonup \mathcal{F}(\omega)$ weakly in $\mathcal{H}$.*

Then for some constant $C_{\mathcal{H}} > 0$, there exists an element $\tilde{\omega} \in \mathcal{H}$ such that $\mathcal{B}(\tilde{\omega}, \omega) = \lambda \prec \mathcal{F}(\tilde{\omega}), \omega \succ$, for all $\omega \in \mathcal{H}$ and $|\lambda| \leq 1$.

The Browder–Minty Theorem

Theorem 3 ([16]) *Assume that Ξ is a separable, reflexive Banach space, and $\mathfrak{A} : \Xi \longrightarrow \Xi^*$ is a monotone and continuous mapping on finite dimensional subspace, and assume that $\mathfrak{A}$ is coercive in the sense that*

$$\frac{\prec \mathfrak{A}\varphi, \varphi \succ}{||\varphi||_{\Xi}} \longrightarrow \infty \quad as \quad ||\varphi||_{\Xi} \longrightarrow \infty.$$

Then for all $\mathfrak{F} \in \Xi^*$, there exists $\varphi \in \Xi$ such that $\mathfrak{A}\varphi = \mathfrak{F}$.

3 Existence results

We first prove that there exists at least a weak solution for the following FDE, then we state some conditions so that the weak solution can be classical. Let $\kappa \in (1/2, 1)$, $\theta \in [0, 1]$. Assume ${}^{H}\mathfrak{D}^{\kappa,\theta;\Lambda}_{\mathcal{T}^-}(.)$ and ${}^{H}\mathfrak{D}^{\kappa,\theta;\Lambda}_{0^+}(.)$ are the Λ-Hilfer fractional derivatives left-sided and right-sided of order κ and type θ, and consider

$$\begin{cases} {}^{H}\mathfrak{D}^{\kappa,\theta;\Lambda}_{\mathcal{T}^-}\left({}^{H}\mathfrak{D}^{\kappa,\theta;\Lambda}_{0^+}\mathfrak{w}(\eta)\right) - \mathcal{A}(\mathfrak{w}(\eta)) = \lambda\left[\mathfrak{g}(\eta, \mathfrak{w}(\eta)) + \int_0^\eta \mathfrak{h}(s, \mathfrak{w}(s))ds\right], \\ I^{\theta(\theta-1);\Lambda}_{0^+}\mathfrak{w}(0) = I^{\theta(\theta-1);\Lambda}_{0^+}\mathfrak{w}(\mathcal{T}^-) = 0, \end{cases} \tag{1}$$

where the boundary conditions are given by the Λ -Riemann-Liouville left and right sided fractional integrals. Moreover, λ is a parameter, the operator $\mathfrak{g} : [0, \mathcal{T}] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous mapping, $\mathfrak{h} : [0, \mathcal{T}] \times \mathbb{R} \longrightarrow \mathbb{R}$ is an integrable function, and $\mathcal{A}$ is a bounded linear operator.

Definition 2 A function $\mathfrak{w} \in \mathcal{H}^{\kappa,\theta,\Lambda}_2[0, \mathcal{T}]$ is a **weak solution** of (1) if

$$\int_0^{\mathcal{T}} \left({}^{H}\mathfrak{D}^{\kappa,\theta;\Lambda}_{0^+}\mathfrak{w}(\eta)\,{}^{H}\mathfrak{D}^{\kappa,\theta;\Lambda}_{0^+}\phi(\eta) - \mathcal{A}(\mathfrak{w}(\eta))\phi(\eta)\right) d\eta = \lambda \int_0^{\mathcal{T}} \left[\mathfrak{g}(\eta, \mathfrak{w}(\eta)) + \mu \int_0^\eta \mathfrak{h}(s, \mathfrak{w}(s))ds\right]\phi(\eta)d\eta,$$

for all $\phi \in \mathcal{H}^{\kappa,\theta,\Lambda}_2[0, \mathcal{T}]$.

Also, a function $\mathfrak{w} \in \mathcal{C}[0, \mathcal{T}]$ is a **classical solution** of (1) if it satisfies Equation (1) and its boundary conditions.

Theorem 4 *Assume that* $\kappa \in (\frac{1}{2}, 1]$, $\theta \in [0, 1]$, $\mathfrak{g} \in C([0, \mathcal{T}] \times \mathbb{R}, \mathbb{R})$ *is a continuous mapping,* $\mathfrak{h} : [0, \mathcal{T}] \times \mathbb{R} \longrightarrow \mathbb{R}$ *is an integrable function, and* $\mathcal{A}$ *is a bounded linear operator. Also take*

$$\zeta := \max\left\{\mathfrak{g}(\eta, \nu) + \int_0^\eta \mathfrak{h}(s, \nu)ds \quad : \quad (\eta, \nu) \in [0, \mathcal{T}] \times \left[\frac{-(\Lambda(\mathcal{T}) - \Lambda(0))^{\kappa-1/2}}{\Gamma(\kappa)(2\kappa-1)^{1/2}}, \frac{(\Lambda(\mathcal{T}) - \Lambda(0))^{\kappa-1/2}}{\Gamma(\kappa)(2\kappa-1)^{1/2}}\right]\right\}. \tag{2}$$

Then for every $|\lambda| < \frac{1}{\zeta(\Lambda(\mathcal{T})-\Lambda(0))^{1/2}}$ *FDE (1) has at least one weak solution.*

Proof To prove this theorem, we consider the following bilinear form $\mathcal{B}(\mathfrak{w}, \phi)$and show that it satisfies the condition of the Lax–Milgram theorem:

$$\mathcal{B}(\mathfrak{w}, \phi) := \int_0^{\mathcal{T}} \left({}^{H}\mathfrak{D}^{\kappa,\theta;\Lambda}_{0^+}\mathfrak{w}(\eta)\,{}^{H}\mathfrak{D}^{\kappa,\theta;\Lambda}_{0^+}\phi(\eta) - \mathcal{A}(\mathfrak{w}(\eta))\phi(\eta)\right) d\eta.$$

Using Holder's inequality and boundedness of the operator $\mathcal{A}$ we obtain that

$$|\mathcal{B}(\mathfrak{w}, \phi)| \le ||{}^{H}\mathfrak{D}^{\kappa,\theta;\Lambda}_{0^+}\mathfrak{w}(\eta)||_{\mathcal{L}_2}||{}^{H}\mathfrak{D}^{\kappa,\theta;\Lambda}_{0^+}\phi(\eta)||_{\mathcal{L}_2} + ||\mathcal{A}||||\mathfrak{w}||_{\mathcal{L}_2}||\phi||_{\mathcal{L}_2} \le (1 + ||\mathcal{A}||)||\mathfrak{w}||_{\mathcal{H}^{\kappa,\theta,\Lambda}_2}||\phi||_{\mathcal{H}^{\kappa,\theta,\Lambda}_2} \tag{3}$$

and $|\mathcal{B}(\mathfrak{w}, \mathfrak{w})| \ge ||\mathfrak{w}||^2_{\mathcal{H}^{\kappa,\theta,\Lambda}_2}$. Therefore, $\mathcal{B}$ is a continuous, bounded and coercive bilinear form on $\mathcal{H}^{\kappa,\theta,\Lambda}_2[0, \mathcal{T}]$.

Now, we set the operator $\Theta : \mathcal{H}^{\kappa,\theta,\Lambda}_2[0, \mathcal{T}] \to (\mathcal{H}^{\kappa,\theta,\Lambda}_2[0, \mathcal{T}])^*$ as follows:

$$\prec \Theta(\mathfrak{w}), \phi \succ = \int_0^{\mathcal{T}} (\mathfrak{g}(\eta, \mathfrak{w}(\eta)) + \mathcal{F}(\eta, \mathfrak{w}(\eta)))\,\phi(\eta)d\eta,$$

where $\mathcal{F}(\eta, \mathfrak{w}(\eta)) = \int_0^\eta \mathfrak{h}(s, \mathfrak{w}(s))ds$.

Assume that $\mathfrak{w} \in N_1(0) \subset \mathcal{H}^{\kappa,\theta,\Lambda}_2[0, \mathcal{T}]$, so $||\mathfrak{w}||_{\mathcal{H}^{\kappa,\theta,\Lambda}_2} \le 1$, and by Remark 1 we have $\Gamma(\kappa)(2\kappa - 1)^{1/2}||\mathfrak{w}||_\infty \le (\Lambda(\mathcal{T}) - \Lambda(0))^{\kappa-1/2}$, for all $\eta \in [0, \mathcal{T}]$. So, we deduce that $|\mathfrak{g}(\eta, \mathfrak{w}(\eta)) + \int_0^\eta \mathfrak{h}(s, \mathfrak{w}(s))ds| \le \zeta$, where ζ is defined in (2). For any arbitrary $\phi \in \mathcal{H}^{\kappa,\theta,\Lambda}_2$ with $||\phi||_{\mathcal{H}^{\kappa,\theta,\Lambda}_2} = 1$ we have

$$\begin{aligned} &| \prec \Theta(\mathfrak{w}), \phi \succ | \\ &\quad \le \int_0^{\mathcal{T}} (\mathfrak{g}(\eta, \mathfrak{w}(\eta)) + \mathcal{F}(\eta, \mathfrak{w}(\eta)))\,\phi(\eta)d\eta \end{aligned}$$

$$\leq \left(\int_0^{\mathcal{T}} |\left(\mathfrak{g}(\eta, \mathfrak{w}(\eta)) + \mathcal{F}(\eta, \mathfrak{w}(\eta)) \right)|^2 d\eta \right)^{1/2} ||\phi(\eta)||_{\mathcal{L}^2}$$
$$\leq \mathcal{T}\sqrt{\zeta},$$

where $\mathcal{F}(\eta, \mathfrak{w}(\eta)) = \int_0^\eta \mathfrak{h}(s, \mathfrak{w}(s))ds$. Choosing $C_{\mathcal{F}} = \mathcal{T}\sqrt{\zeta}$ we show that hypothesis $(H1)$ in the Lax–Milgram theorem holds.

Next, we assume that $\{\omega_n\}$ is an arbitrary sequence in $\mathcal{H}_2^{\kappa,\theta,\Lambda}[0, \mathcal{T}]$ which is weakly convergent in the space. Using Theorem 1 we have $\omega_n(\eta) \to \omega(\eta)$ for all $\eta \in [0, \mathcal{T}]$. Applying continuity of $\mathfrak{g}(\eta, w(\eta)) + \int_0^\eta \mathfrak{h}(s, w(s))ds$, we come to

$$\mathfrak{g}(\eta, \omega_n(\eta)) + \int_0^\eta \mathfrak{h}(s, \omega_n(s))ds \to \mathfrak{g}(\eta, \omega(\eta)) + \int_0^\eta \mathfrak{h}(s, \omega(s))ds, \tag{4}$$

whenever n tends to infinity, and for all $\eta \in [0, \mathcal{T}]$. Since $\{\omega_n\}$ is bounded, there exists a positive constant C_ω such that $||\omega_n|| \leq C_\omega$, for every $n \in \mathbb{N}$. In addition, from Remark 1 we have $\Gamma(\kappa)(2\kappa - 1)^{1/2}||\omega_n||_\infty \leq C_\omega(\Lambda(\mathcal{T}) - \Lambda(0))^{\kappa-1/2}$, for all $\eta \in [0, \mathcal{T}]$. Hence we can conclude that there exists a positive constant ζ' such that

$$|\mathfrak{g}(\eta, \omega_n(\eta)) + \int_0^\eta \mathfrak{h}(s, \omega_n(s))ds| \leq \zeta', \tag{5}$$

for all $\eta \in [0, \mathcal{T}]$, and $n \in \mathbb{N}$. Therefore, from (4), (5) and the Lebesgue dominated theorem, we obtain that

$$\int_0^{\mathcal{T}} |\mathfrak{g}(\eta, \omega_n(\eta)) - \mathfrak{g}(\eta, \omega(\eta)) + \int_0^\eta \mathfrak{h}(s, \omega_n(s))ds - \int_0^\eta \mathfrak{h}(s, \omega_n(s))ds|d\eta \leq$$
$$\int_0^{\mathcal{T}} |\left(\mathfrak{g}(\eta, \omega_n(\eta)) - \mathfrak{g}(\eta, \omega(\eta))\right) d\eta + \int_0^{\mathcal{T}} \int_0^\eta |\mathfrak{h}(s, \omega_n(s)) - \mathfrak{h}(s, \omega_n(s))ds|d\eta \to 0.$$

It follows, for an arbitrary $\phi \in \mathcal{H}_2^{\kappa,\theta,\Lambda}$ with $||\phi||_{\mathcal{H}_2^{\kappa,\theta,\Lambda}} = 1$, that

$$\prec \Theta(\omega_n) - \Theta(\omega), \phi \succ = |\int_0^{\mathcal{T}} (\mathcal{F}_1(\eta, \omega_n(\eta) - \mathcal{F}_1(\eta, \omega(\eta))\phi(\eta)d\eta| \leq ||\mathcal{F}_1(\eta, \omega_n(\eta) - \mathcal{F}_1(\eta, \omega(\eta)||_{\mathcal{L}^2} \to 0,$$

where $\mathcal{F}_1(\eta, \mathfrak{w}(\eta)) = (\mathfrak{g}(\eta, \mathfrak{w}(\eta)) + \mathcal{F}(\eta, \mathfrak{w}(\eta)))$. This implies condition $(K2)$ of the Lax–Milgram theorem also holds. By Theorem 2 we get the desired result. □

Remark 2 Clearly, we have the following from the above result:

- Choosing $\Lambda(\eta) = \eta$ in (1), then there exists at least a solution in the Caputo fractional derivative sense for FDE (1), whenever $\theta \longrightarrow 1$ and $|\lambda| < \dfrac{1}{\zeta\sqrt{\eta}}$.
- Taking $\Lambda(\eta) = \eta^\kappa, (\kappa > 0)$ in (1), then there exists at least a solution in the Katugampola fractional derivative sense for FDE (1), whenever $\theta \longrightarrow 0$ and $|\lambda| < \dfrac{1}{\zeta\sqrt{\eta^\kappa}}$.
- Similarly for $\Lambda(\eta) = \eta^\kappa$, $(\kappa > 0)$ in (1), then there exists at least a solution in the Caputo–Katugampola fractional derivative sense for FDE (1), whenever $\theta \longrightarrow 1$ and $|\lambda| < \dfrac{1}{\zeta\sqrt{\eta^\kappa}}$.

3.1 **Regularity result**

To prove our regularity result, we first recall some preliminaries.

Definition 3 Let $u, v, w \in L^2[0, \mathcal{T}]$. Then for all $\phi \in C_0^\infty[0, \mathcal{T}]$ we define

$$\int_0^{\mathcal{T}} u(\eta)\, {}^H\mathfrak{D}_{\mathcal{T}^-}^{\kappa,\theta;\Lambda}\phi(\eta)d\eta = \int_0^{\mathcal{T}} v(\eta)\phi(\eta)d\eta,$$

and

$$\int_0^{\mathcal{T}} u(\eta)^H\mathfrak{D}_{0^+}^{\kappa,\theta;\Lambda}\phi(\eta)d\eta = \int_0^{\mathcal{T}} w(\eta)\phi(\eta)d\eta,$$

where ${}^H\bar{\mathfrak{D}}_{\mathcal{T}^-}^{\kappa,\theta;\Lambda}u(\eta) = v(\eta)$ and ${}^H\bar{\mathfrak{D}}_{0^+}^{\kappa,\theta;\Lambda}u(\eta) = w(\eta)$. The functions v and w are called the weak left and the weak right fractional derivative of $u(\eta)$, of order $\kappa \in (0, 1]$ and type $\theta \in [0, 1]$, respectively.

Lemma 1 ([14,15]) *Let $\Omega = [0, \mathcal{T}]$ and $w(\eta) \in \mathcal{L}^2(\Omega)$, if*

- *${}^H\bar{\mathfrak{D}}_{0^+}^{\kappa,\theta;\Lambda}w(\eta)$ exists and it is almost everywhere equal to a function in $C(\Omega)$, then $w(\eta)$ is a.e. equal to a function $\tilde{w}(\eta) \in C(\Omega)$. Moreover, ${}^H\mathfrak{D}_{0^+}^{\kappa,\theta;\Lambda}w(\eta)$ exists for every $\eta \in \Omega$, and $w(\eta)$ belongs to $C(\Omega)$.*
- *${}^H\bar{\mathfrak{D}}_{\mathcal{T}^-}^{\kappa,\theta;\Lambda}w(\eta)$ exists and it is almost everywhere equal to a function in $C(\Omega)$, then $w(\eta)$ is a.e. equal to a function $\tilde{w}(\eta) \in C(\Omega)$. Furthermore, ${}^H\mathfrak{D}_{\mathcal{T}^-}^{\kappa,\theta;\Lambda}w(\eta)$ exists for every $\eta \in (\Omega)$, and $w(\eta)$ belongs to $C(\Omega)$.*

Now, we state our regularity result with the similar proof of Theorem 10 in [14] and Theorem 3.1 in [8].

Theorem 5 *Assume that $\kappa \in (\frac{1}{2}, 1]$, $\theta \in [0, 1]$, $\mathfrak{g} \in C([0, \mathcal{T}] \times \mathbb{R}, \mathbb{R})$ is a continuous mapping, $\mathfrak{h} : [0, \mathcal{T}] \times \mathbb{R} \longrightarrow \mathbb{R}$ is an integrable function, and $\mathcal{A}$ is a bounded linear operator. Also, let ζ be defined in (2), and $|\lambda| < \frac{1}{\zeta(\Lambda(\mathcal{T})-\Lambda(0))^{1/2}}$. Then every weak solution of FDE (1) is classical.*

Proof Let $v(\eta)$ be a weak solution of equation (1) and take $L(\eta) := \mathcal{A}v(\eta) + \lambda(\mathfrak{g}(\eta, v(\eta)) + \int_0^\eta \mathfrak{h}(s, v(s))ds)$. From the definition of weak solution, we have

$$\int_0^{\mathcal{T}} \left({}^H\mathfrak{D}_{0^+}^{\kappa,\theta;\Lambda}v(\eta)^H\mathfrak{D}_{0^+}^{\kappa,\theta;\Lambda}\phi(\eta)\right)d\eta = \int_0^{\mathcal{T}} L(\eta)\phi(\eta)d\eta,$$

for all $\phi \in \mathcal{H}_2^{\kappa,\theta,\Lambda}[0, \mathcal{T}]$. From Definition 3 we claim $L(\eta) = {}^H\bar{\mathfrak{D}}_{\mathcal{T}^-}^{\kappa,\theta;\Lambda}\,{}^H\bar{\mathfrak{D}}_{0^+}^{\kappa,\theta;\Lambda}v(\eta)$. Moreover, from Remark 1 and Theorem 1, we imply $L(\eta) = {}^H\bar{\mathfrak{D}}_{\mathcal{T}^-}^{\kappa,\theta;\Lambda}\,{}^H\bar{\mathfrak{D}}_{0^+}^{\kappa,\theta;\Lambda}v(\eta)$ belongs to $C[0, \mathcal{T}]$. It follows from Lemma 1 that ${}^H\mathfrak{D}_{\mathcal{T}^-}^{\kappa,\theta;\Lambda}\,{}^H\bar{\mathfrak{D}}_{0^+}^{\kappa,\theta;\Lambda}v(\eta)$ exists for all $\eta \in [0, \mathcal{T}]$. In addition, it belongs to $C[0, \mathcal{T}]$ and ${}^H\bar{\mathfrak{D}}_{0^+}^{\kappa,\theta;\Lambda}v(\eta)$ is a.e equal to an element of $C[0, \mathcal{T}]$. Moreover, the first part of the Lamma 1 shows that there exists ${}^H\bar{\mathfrak{D}}_{0^+}^{\kappa,\theta;\Lambda}v(\eta)$ for all $[0, \mathcal{T}]$, and by Remark 1, we derive that ${}^H\bar{\mathfrak{D}}_{0^+}^{\kappa,\theta;\Lambda}v(\eta) = {}^H\mathfrak{D}_{0^+}^{\kappa,\theta;\Lambda}v(\eta)$ a.e. on $[0, \mathcal{T}]$. Therefore, ${}^H\bar{\mathfrak{D}}_{\mathcal{T}^-}^{\kappa,\theta;\Lambda}\,{}^H\mathfrak{D}_{0^+}^{\kappa,\theta;\Lambda}v(\eta)$ exists for any $\eta \in [0, \mathcal{T}]$. Since $L(\eta) = {}^H\bar{\mathfrak{D}}_{\mathcal{T}^-}^{\kappa,\theta;\Lambda}\,{}^H\mathfrak{D}_{0^+}^{\kappa,\theta;\Lambda}v(\eta)$ is a.e equal and both are continuous, hence we conclude that $L(\eta) = {}^H\bar{\mathfrak{D}}_{\mathcal{T}^-}^{\kappa,\theta;\Lambda}\,{}^H\mathfrak{D}_{0^+}^{\kappa,\theta;\Lambda}v(\eta)$ for all $\eta \in [0, \mathcal{T}]$, which is our desired result. □ □

We are ready to demonstrate an application of our main result by the following example:

Example 1 Let

$$\begin{cases} {}^H\mathfrak{D}_{\mathcal{T}^-}^{3/4,1/4;e^\eta}\left({}^H\mathfrak{D}_{0^+}^{3/4,1/4;e^\eta}\mathfrak{w}(\eta)\right) - \mathfrak{w}(\eta) = \lambda\left[\frac{1}{2}e^{-\eta}\mathfrak{w}(\eta)) + \frac{1}{3}\int_0^\eta sin(s\mathfrak{w}(s))ds\right], \\ I_{0^+}^{3/16;e^\eta}\mathfrak{w}(0) = I_{1^-}^{3/16;e^\eta}\mathfrak{w}(1) = 0, \end{cases} \tag{6}$$

where $\eta \in [0, 1]$, ${}^H\mathfrak{D}_{\mathcal{T}^-}^{3/4,1/4;e^\eta}(.)$ and ${}^H\mathfrak{D}_{0^+}^{3/4,1/4;e^\eta}(.)$ are the Λ-Hilfer fractional derivatives left-sided and right-sided of order $3/4$ and type $1/4$. The function $\frac{1}{2}e^{-\eta}\mathfrak{w}(\eta) + \frac{1}{3}\int_0^\eta sin(s\mathfrak{w}(s))ds$ is clearly bounded for $\eta \in [0, 1]$, and let

$$\mathcal{M} := \max\left\{\frac{1}{2}e^{-\eta}\nu + \frac{1}{3}\int_0^\eta sin(s\nu)ds \;\;:\;\; (\eta, \nu) \in [0, 1] \times \left[\frac{-(e-1)^{0.25}}{\Gamma(0.75)\sqrt{0.5}}, \frac{(e-1)^{0.25}}{\Gamma(0.75)\sqrt{0.5}}\right]\right\} \simeq 2.17.$$

So Theorem 4 implies that (6) has at least one weak solution whenever $|\lambda| < \frac{1}{\mathcal{M}\sqrt{e-1}} \simeq 0.36$. Also, by Theorem 5 every weak solution of (6) is a classical solution.

4 System of coupled fractional equations

We are going to extend the existence result discussed in the previous section to the following coupled system of FDEs

$$\begin{cases} {}^H\mathfrak{D}_{\mathcal{T}^-}^{\kappa_1,\theta_1;\Lambda}\left({}^H\mathfrak{D}_{0^+}^{\kappa_1,\theta_1;\Lambda}\mathfrak{w}(\eta)\right) - \mathcal{A}_1(\mathfrak{w}(\eta)) = \lambda\left(\mathfrak{g}_1(\eta,\mathfrak{w}(\eta),\mathfrak{v}(\eta)) + \int_0^\eta \mathfrak{h}_1(s,\mathfrak{w}(s),\mathfrak{v}(s))ds\right), \\ {}^H\mathfrak{D}_{\mathcal{T}^-}^{\kappa_2,\theta_2;\Lambda}\left({}^H\mathfrak{D}_{0^+}^{\kappa_2,\theta_2;\Lambda}\mathfrak{v}(\eta)\right) - \mathcal{A}_2(\mathfrak{v}(\eta)) = \lambda\left(\mathfrak{g}_2(\eta,\mathfrak{w}(\eta),\mathfrak{v}(\eta)) + \int_0^\eta \mathfrak{h}_2(s,\mathfrak{w}(s),\mathfrak{v}(s))ds\right), \\ I_{0^+}^{\theta_1(\theta_1-1);\Lambda}\mathfrak{w}(0) = I_{0^+}^{\theta_1(\theta_1-1);\Lambda}\mathfrak{w}(\mathcal{T}^-) = 0, \\ I_{0^+}^{\theta_2(\theta_2-1);\Lambda}\mathfrak{v}(0) = I_{0^+}^{\theta_2(\theta_2-1);\Lambda}\mathfrak{v}(\mathcal{T}^-) = 0 \end{cases} \tag{7}$$

where $\eta \in [0,\mathcal{T}]$, ${}^H\mathfrak{D}_{\mathcal{T}^-}^{\kappa_i,\theta_i;\Lambda}(.)$ and ${}^H\mathfrak{D}_{0^+}^{\kappa_i,\theta_i;\Lambda}(.)$, for $i = 1, 2$, are the Λ-Hilfer left and right sided fractional derivatives of order $\frac{1}{2} < \kappa_i < 1$ type $0 \le \theta_i \le 1$, and the boundary conditions are given by the Λ -Riemann-Liouville left and right sided fractional integrals. Moreover, λ is a parameter, the operator $\mathfrak{g}_i, \mathfrak{h}_i : [0,\mathcal{T}]\times\mathbb{R}^2 \longrightarrow \mathbb{R}$, where $\mathfrak{g}_i(\eta,\mathfrak{w}(\eta),\mathfrak{v}(\eta))$ are continuous mappings, and $\mathfrak{h}_i(s,\mathfrak{w}(s),\mathfrak{v}(\eta))$, for $i = 1, 2$, are integrable functions, and $\mathcal{A}_i$, for $i = 1, 2$, are bounded linear operators.

Definition 4 A pair of functions $\mathfrak{w}, \mathfrak{v} \in \mathcal{H}_2^{\kappa,\theta,\Lambda}[0,\mathcal{T}]$ are **weak solutions** of coupled system of FDEs (7), if

$$\begin{aligned} &\int_0^{\mathcal{T}} {}^H\mathfrak{D}_{0^+}^{\kappa_1,\theta_1;\Lambda}\mathfrak{w}(\eta)\,{}^H\mathfrak{D}_{0^+}^{\kappa_1,\theta_1;\Lambda}\phi_1(\eta)d\eta \\ &- \int_0^{\mathcal{T}}\left[\mathcal{A}_1(\mathfrak{w}(\eta)) + \lambda\left(\mathfrak{g}_1(\eta,\mathfrak{w}(\eta),\mathfrak{v}(\eta)) + \int_0^\eta \mathfrak{h}_1(s,\mathfrak{w}(s),\mathfrak{v}(s))ds\right)\right]\phi_1(\eta)d\eta \\ &+ \int_0^{\mathcal{T}} {}^H\mathfrak{D}_{0^+}^{\kappa_2,\theta_2;\Lambda}\mathfrak{v}(\eta)\,{}^H\mathfrak{D}_{0^+}^{\kappa_2,\theta_2;\Lambda}\phi_2(\eta)d\eta \\ &- \int_0^{\mathcal{T}}\left[\mathcal{A}_2(\mathfrak{v}(\eta)) + \lambda\left(\mathfrak{g}_2(\eta,\mathfrak{w}(\eta),\mathfrak{v}(\eta)) + \int_0^\eta \mathfrak{h}_2(s,\mathfrak{w}(s),\mathfrak{v}(s))ds\right)\right]\phi_2(\eta)d\eta = 0 \end{aligned}$$

for all $\phi_1, \phi_2 \in \mathcal{H}_2^{\kappa,\theta,\Lambda}[0,\mathcal{T}]$.

To prove our main existence result, we first state the following growth conditions:

(i) There exist constants c_i, d_i, $i = 1, 2$, and $p \in [2, 2^*)$ so that for all $(\eta, \nu_i, \xi_i) \in ([0,\mathcal{T}], \mathbb{R}^2)$, $i = 1, 2$, we have

$$|\mathfrak{g}_i(\eta,\nu,\xi)| \le \sigma_i(\eta) + c_i|\nu|^{p-1} + d_i|\xi|^{p-1},$$

where $\sigma_i \in \mathcal{L}^q[0,\mathcal{T}]$ and $q \in (2^*, 2]$. Moreover, $\mathfrak{g}_i(\eta, 0, \xi)$ and $\mathfrak{g}_i(\eta, \nu, 0)$, for $i = 1, 2$, belong to $\mathcal{L}^q[0,\mathcal{T}]$ as functions of η.

(ii) There exist constants e_i, f_i, $i = 1, 2$, and $p \in [2, 2^*)$ so that for all $(\eta, \nu_i, \xi_i) \in ([0,\mathcal{T}], \mathbb{R}^2)$, $i = 1, 2$, we have

$$|\mathfrak{h}_i(\eta,\nu,\xi)| \le \delta_i(\eta) + e_i|\nu|^{p-1} + f_i|\xi|^{p-1},$$

where $\delta_i \in \mathcal{L}^q[0,\mathcal{T}]$ and $q \in (2^*, 2]$. Moreover, $\mathfrak{h}_i(\eta, 0, \xi)$ and $\mathfrak{h}_i(\eta, \nu, 0)$, for $i = 1, 2$, belong to $\mathcal{L}^q[0,\mathcal{T}]$ as functions of η.

(iii) For all $\eta \in [0,\mathcal{T}]$, and $\nu_i, \xi_i \in \mathbb{R}$ such that $\nu_i \ne \xi_i$, for $i = 1, 2$, we have

$$\frac{\mathfrak{g}_1(\eta,\nu_1,\xi_1) - \mathfrak{g}_1(\eta,\nu_2,\xi_2)}{\nu_1 - \nu_2} \ge \rho \quad \text{and} \quad \frac{\mathfrak{g}_2(\eta,\nu_1,\xi_1) - \mathfrak{g}_2(\eta,\nu_2,\xi_2)}{\xi_1 - \xi_2} \ge \rho,$$

(iv) For all $\eta \in [0,\mathcal{T}]$, and $\nu_i, \xi_i \in \mathbb{R}$ such that $\nu_i \ne \xi_i$, for $i = 1, 2$, we have

$$\frac{\int_0^\eta[\mathfrak{h}_1(s,\nu_1,\xi_1) - \mathfrak{h}_1(s,\nu_2,\xi_2)]ds}{\nu_1 - \nu_2} \ge \rho^* \quad \text{and} \quad \frac{\int_0^\eta[\mathfrak{h}_2(s,\nu_1,\xi_1) - \mathfrak{h}_2(s,\nu_2,\xi_2)]ds}{\xi_1 - \xi_2} \ge \rho^*.$$

Theorem 6 *Assume that* $\kappa \in (\frac{1}{2}, 1]$, $\theta \in [0, 1]$, $\mathfrak{g} \in C([0, \mathcal{T}] \times \mathbb{R}, \mathbb{R})$ *is a continuous mapping,* $\mathfrak{h} : [0, \mathcal{T}] \times \mathbb{R} \longrightarrow \mathbb{R}$ *is an integrable function, and* $\mathcal{A}$ *is a bounded linear operator. Also suppose that the hypotheses* (i) *-*(iv) *are satisfied and* $||\mathcal{A}_1|| + ||\mathcal{A}_2|| \leq \frac{\Gamma(\kappa+1)}{(\psi((T)-\psi(0))^{\kappa}}$. *Then there exists a unique pair of functions* $(\mathfrak{w}, \mathfrak{v})$ *satisfying the coupled differential equation* (7) *in the weak sense.*

Proof Take the operator $\mathcal{S} : \mathcal{H}_2^{\kappa,\theta,\Lambda}[0, \mathcal{T}] \to (\mathcal{H}_2^{\kappa,\theta,\Lambda}[0, \mathcal{T}])^*$ as follows:

$$\prec \mathcal{S}(\mathfrak{w}, \mathfrak{v}), (\phi_1, \phi_2) \succ := \int_0^{\mathcal{T}} {}^H\mathfrak{D}_{0^+}^{\kappa_1,\theta_1;\Lambda}\mathfrak{w}(\eta)\,{}^H\mathfrak{D}_{0^+}^{\kappa_1,\theta_1;\Lambda}\phi_1(\eta)d\eta + \int_0^{\mathcal{T}} {}^H\mathfrak{D}_{0^+}^{\kappa_2,\theta_2;\Lambda}\mathfrak{v}(\eta)\,{}^H\mathfrak{D}_{0^+}^{\kappa_2,\theta_2;\Lambda}\phi_2(\eta)d\eta$$

$$-\int_0^{\mathcal{T}} \left[\mathcal{A}_1(\mathfrak{w}(\eta)) + \lambda\left(\mathfrak{g}_1(\eta, \mathfrak{w}(\eta), \mathfrak{v}(\eta)) + \int_0^{\eta} \mathfrak{h}_1(s, \mathfrak{w}(s), \mathfrak{v}(s))ds\right)\right]\phi_1(\eta)d\eta$$

$$-\int_0^{\mathcal{T}} \left[\mathcal{A}_2(\mathfrak{v}(\eta)) + \lambda\left(\mathfrak{g}_2(\eta, \mathfrak{w}(\eta), \mathfrak{v}(\eta)) + \int_0^{\eta} \mathfrak{h}_2(s, \mathfrak{w}(s), \mathfrak{v}(s))ds\right)\right]\phi_2(\eta)d\eta,$$

for all $\mathfrak{w}, \mathfrak{v}, \phi_1, \phi_2 \in \mathcal{H}_2^{\kappa,\theta,\Lambda}[0, \mathcal{T}]$.
Note that from hypothesis (i) we derive that

$$\left|\int_0^{\mathcal{T}} \mathfrak{g}_i(\eta, \mathfrak{w}(\eta), \mathfrak{v}(\eta))\phi_i d\eta\right| \leq ||\sigma_i||_q||\phi_i||_p + c_i||\phi_i||_p||\mathfrak{w}||_p^{p-1} + d_i||\phi_i||_p||\mathfrak{v}||_p^{p-1} < \infty,$$

and from (ii) we come to

$$\left|\int_0^{\mathcal{T}} \left[\int_0^{\eta} \mathfrak{h}_2(s, \mathfrak{w}(s), \mathfrak{v}(s))ds\right]\phi_2(\eta)d\eta\right| \leq ||\delta_i||_q||\phi_i||_p + e_i||\phi_i||_p||\mathfrak{w}||_p^{p-1} + f_i||\phi_i||_p||\mathfrak{v}||_p^{p-1} < \infty,$$

for all $\mathfrak{w}, \mathfrak{v}, \phi_1, \phi_2 \in \mathcal{H}_2^{\kappa,\theta,\Lambda}[0, T]$. Moreover, due to boundedness of the operator $\mathcal{A}_i$, for $i = 1, 2$, we deduce

$$\left|\int_0^{\mathcal{T}} \mathcal{A}_1(\mathfrak{w}(\eta))\phi_1 d\eta\right| \leq ||\mathcal{A}||(||\mathfrak{w}||_2||\phi_1||_2) < \infty,$$

and

$$\left|\int_0^{\mathcal{T}} \mathcal{A}_2(\mathfrak{v}(\eta))\phi_2 d\eta\right| \leq ||\mathcal{A}||(||\mathfrak{v}||_2||\phi_2||_2) < \infty.$$

Therefore, $\prec \mathcal{S}(\mathfrak{w}, \mathfrak{v}), (\phi_1, \phi_2) \succ \in \mathcal{H}_2^{\kappa,\theta,\Lambda}[0, T]^*$ for all $\mathfrak{w}, \mathfrak{v}, \phi_1, \phi_2 \in \mathcal{H}_2^{\kappa,\theta,\Lambda}[0, T]$. So the operator $\mathcal{S}$ is well defined.

Taking $\mathfrak{w}_j, \mathfrak{v}_j \in \mathcal{H}_2^{\kappa,\theta,\Lambda}[0, T]$, $i = 1, 2$, we obtain that

$$\begin{aligned}
\prec \mathcal{S}(\mathfrak{w}_1, \mathfrak{v}_1) - \mathcal{S}(\mathfrak{w}_2, \mathfrak{v}_2), (\mathfrak{w}_1 - \mathfrak{w}_2, \mathfrak{v}_1 - \mathfrak{v}_2) \succ &= \prec \mathcal{S}(\mathfrak{w}_1, \mathfrak{v}_1), (\mathfrak{w}_1 - \mathfrak{w}_2, \mathfrak{v}_1 - \mathfrak{v}_2) \succ \\
&\quad - \prec \mathcal{S}(\mathfrak{w}_2, \mathfrak{v}_2), (\mathfrak{w}_1 - \mathfrak{w}_2, \mathfrak{v}_1 - \mathfrak{v}_2) \succ \\
= \int_0^{\mathcal{T}} {}^H\mathfrak{D}_{0^+}^{\kappa_1,\theta_1;\Lambda}\mathfrak{w}_1(\eta)\,{}^H\mathfrak{D}_{0^+}^{\kappa_1,\theta_1;\Lambda}(\mathfrak{w}_1(\eta) - \mathfrak{w}_2(\eta))d\eta \\
-\int_0^{\mathcal{T}} \left[\mathcal{A}_1(\mathfrak{w}_1(\eta)) + \lambda\left(\mathfrak{g}_1(\eta, \mathfrak{w}_1(\eta), \mathfrak{v}_1(\eta)) + \int_0^{\eta} \mathfrak{h}_1(s, \mathfrak{w}_1(s), \mathfrak{v}_1(s))ds\right)\right](\mathfrak{w}_1(\eta) - \mathfrak{w}_2(\eta))d\eta \\
+\int_0^{\mathcal{T}} {}^H\mathfrak{D}_{0^+}^{\kappa_2,\theta_2;\Lambda}\mathfrak{v}_1(\eta)\,{}^H\mathfrak{D}_{0^+}^{\kappa_2,\theta_2;\Lambda}(\mathfrak{v}_1(\eta) - \mathfrak{v}_2(\eta))d\eta \\
-\int_0^{\mathcal{T}} \left[\mathcal{A}_2(\mathfrak{v}_1(\eta)) + \lambda\left(\mathfrak{g}_2(\eta, \mathfrak{w}_1(\eta), \mathfrak{v}_1(\eta)) + \int_0^{\eta} \mathfrak{h}_2(s, \mathfrak{w}_1(s), \mathfrak{v}_1(s))ds\right)\right](\mathfrak{v}_1(\eta) - \mathfrak{v}_2(\eta))d\eta
\end{aligned}$$

$$
\begin{aligned}
&-\int_0^{\mathcal{T}} {}^H\mathfrak{D}_{0^+}^{\kappa_1,\theta_1;\Lambda}\mathfrak{w}_2(\eta)\,{}^H\mathfrak{D}_{0^+}^{\kappa_1,\theta_1;\Lambda}(\mathfrak{w}_1(\eta)-\mathfrak{w}_2(\eta))d\eta\\
&+\int_0^{\mathcal{T}}\left[\mathcal{A}_1(\mathfrak{w}_2(\eta))+\lambda\left(\mathfrak{g}_1(\eta,\mathfrak{w}_2(\eta),\mathfrak{v}_2(\eta))+\int_0^{\eta}\mathfrak{h}_1(s,\mathfrak{w}_2(s),\mathfrak{v}_2(s))ds\right)\right](\mathfrak{w}_1(\eta)-\mathfrak{w}_2(\eta))d\eta\\
&-\int_0^{\mathcal{T}} {}^H\mathfrak{D}_{0^+}^{\kappa_2,\theta_2;\Lambda}\mathfrak{v}_2(\eta)\,{}^H\mathfrak{D}_{0^+}^{\kappa_2,\theta_2;\Lambda}(\mathfrak{v}_1(\eta)-\mathfrak{v}_2(\eta))d\eta\\
&+\int_0^{\mathcal{T}}\left[\mathcal{A}_2(\mathfrak{v}_2(\eta))+\lambda\left(\mathfrak{g}_2(\eta,\mathfrak{w}_2(\eta),\mathfrak{v}_2(\eta))+\int_0^{\eta}\mathfrak{h}_2(s,\mathfrak{w}_2(s),\mathfrak{v}_2(s))ds\right)\right](\mathfrak{v}_1(\eta)-\mathfrak{v}_2(\eta))d\eta\\
=&\int_0^{\mathcal{T}}|{}^H\mathfrak{D}_{0^+}^{\kappa_1,\theta_1;\Lambda}(\mathfrak{w}_1(\eta)-\mathfrak{w}_2(\eta))|^2+\int_0^{\mathcal{T}}|{}^H\mathfrak{D}_{0^+}^{\kappa_2,\theta_2;\Lambda}(\mathfrak{v}_1(\eta)-\mathfrak{v}_2(\eta))|^2\\
&+\int_0^{\mathcal{T}}(\mathcal{A}_1(\mathfrak{w}_2(\eta)-\mathfrak{w}_1(\eta))+\lambda[\mathfrak{g}_1(\eta,\mathfrak{w}_2(\eta),\mathfrak{v}_2(\eta))-\mathfrak{g}_1(\eta,\mathfrak{w}_1(\eta),\mathfrak{v}_1(\eta))])(\mathfrak{w}_1(\eta)-\mathfrak{w}_2(\eta))d\eta\\
&+\int_0^{\mathcal{T}}\left(\lambda\int_0^{\eta}[\mathfrak{h}_1(s,\mathfrak{w}_2(s),\mathfrak{v}_2(s))-\mathfrak{h}_1(s,\mathfrak{w}_1(s),\mathfrak{v}_1(s)]ds\right)(\mathfrak{w}_1(\eta)-\mathfrak{w}_2(\eta))d\eta\\
&+\int_0^{\mathcal{T}}[\mathcal{A}_2(\mathfrak{v}_2(\eta)-\mathfrak{v}_1(\eta))+\lambda(\mathfrak{g}_2(\eta,\mathfrak{w}_2(\eta),\mathfrak{v}_2(\eta))-\mathfrak{g}_2(\eta,\mathfrak{w}_1(\eta),\mathfrak{v}_1(\eta)))](\mathfrak{v}_1(\eta)-\mathfrak{v}_2(\eta))d\eta\\
&+\int_0^{\mathcal{T}}\left(\lambda\int_0^{\eta}[\mathfrak{h}_2(s,\mathfrak{w}_2(s),\mathfrak{v}_2(s))ds-\mathfrak{h}_2(s,\mathfrak{w}_1(s),\mathfrak{v}_1(s))]ds\right)(\mathfrak{v}_1(\eta)-\mathfrak{v}_2(\eta))d\eta.
\end{aligned}
$$

Applying hypotheses (iii) and (iv) we infer

$$\prec \mathcal{S}(u_1,v_1)-\mathcal{S}(u_2,v_2),(u_1-u_2,v_1-v_2)\succ>0.$$

Thus $\mathcal{S}$ is a monotone operator. To complete our proof, it suffices to show that $\mathcal{S}$ is a coercive mapping. To do so, we take $(\phi_1,\phi_2)=(\mathfrak{w},\mathfrak{v})$ in the definition of the operator $\mathcal{S}$, then we get

$$
\begin{aligned}
&\prec \mathcal{S}(\mathfrak{w},\mathfrak{v}),(\mathfrak{w},\mathfrak{v})\succ=\int_0^{\mathcal{T}}|{}^H\mathfrak{D}_{0^+}^{\kappa_1,\theta_1;\Lambda}\mathfrak{w}(\eta)|^2d\eta+\int_0^{\mathcal{T}}|{}^H\mathfrak{D}_{0^+}^{\kappa_2,\theta_2;\Lambda}\mathfrak{v}(\eta)|^2d\eta\\
&-\int_0^{\mathcal{T}}\left[\mathcal{A}_1(\mathfrak{w}(\eta))+\lambda\left(\mathfrak{g}_1(\eta,\mathfrak{w}(\eta),\mathfrak{v}(\eta))+\int_0^{\eta}\mathfrak{h}_1(s,\mathfrak{w}(s),\mathfrak{v}(s))ds\right)\right]\mathfrak{w}(\eta)d\eta\\
&-\int_0^{\mathcal{T}}\left[\mathcal{A}_2(\mathfrak{v}(\eta))+\lambda\left(\mathfrak{g}_2(\eta,\mathfrak{w}(\eta),\mathfrak{v}(\eta))+\int_0^{\eta}\mathfrak{h}_2(s,\mathfrak{w}(s),\mathfrak{v}(s))ds\right)\right]\mathfrak{v}(\eta)d\eta\\
&\geq||\mathfrak{w}||^2+||\mathfrak{v}||^2-||\mathcal{A}_1|||\mathfrak{w}|^2_{\mathcal{L}^2}-||\mathcal{A}_2|||\mathfrak{v}|^2_{\mathcal{L}^2}\\
&-\lambda\left(|\mathfrak{w}|_{\mathcal{L}^p}|\mathfrak{g}_1(\eta,0,\mathfrak{v}(\eta))|_{\mathcal{L}^q}-\rho|\mathfrak{w}|^2_{\mathcal{L}^2}-|\mathfrak{v}|_{\mathcal{L}^p}|\mathfrak{g}_2(\eta,\mathfrak{w}(\eta),0)|_{\mathcal{L}^q}+\rho|\mathfrak{v}|^2_{\mathcal{L}^2}\right)\\
&-\lambda\left(|\mathfrak{w}|_{\mathcal{L}^p}|\int_0^{\eta}\mathfrak{h}_1(s,0,\mathfrak{v}(s))ds|_{\mathcal{L}^q}-\rho^*|\mathfrak{w}|^2_{\mathcal{L}^2}-|\mathfrak{v}|_{\mathcal{L}^p}|\int_0^{\eta}\mathfrak{h}_2(s,\mathfrak{w}(s),0)ds|_{\mathcal{L}^q}+\rho^*|\mathfrak{v}|^2_{\mathcal{L}^2}\right).
\end{aligned}
$$

From Proposition 4.6 in [15], we imply that $||\mathfrak{w}||_{\mathcal{L}^p}\leq C_\psi||{}^H\mathfrak{D}_{0^+}^{\kappa,\theta;\Lambda}\mathfrak{w}(\eta)||_{\mathcal{L}^p}$, where $C_\psi=\frac{(\psi((T)-\psi(0))^\kappa}{\Gamma(\kappa+1)}$. Thus for $p=2$ in the last inequality we have

$$
\begin{aligned}
&\prec \mathcal{S}(\mathfrak{w},\mathfrak{v}),(\mathfrak{w},\mathfrak{v})\succ\geq[1-C_\psi(||A_1||-\lambda\rho-\lambda\rho^*)]||\mathfrak{w}||^2+[1-C_\psi(||A_2||+\lambda\rho+\lambda\rho^*)]||\mathfrak{v}||^2\\
&-\lambda\left(|\mathfrak{w}|_{\mathcal{L}^p}|\mathfrak{g}_1(\eta,0,\mathfrak{v}(\eta))|_{\mathcal{L}^q}-|\mathfrak{v}|_{\mathcal{L}^p}|\mathfrak{g}_2(\eta,\mathfrak{w}(\eta),0)|_{\mathcal{L}^q}\right)\\
&-\lambda\left(|\mathfrak{w}|_{\mathcal{L}^p}|\int_0^{\eta}\mathfrak{h}_1(s,0,\mathfrak{v}(s))ds|_{\mathcal{L}^q}-|\mathfrak{v}|_{\mathcal{L}^p}|\int_0^{\eta}\mathfrak{h}_2(s,\mathfrak{w}(s),0)ds|_{\mathcal{L}^q}\right)\\
&\geq[1-C_\psi(||A_1||-\lambda\rho-\lambda\rho^*)]||\mathfrak{w}||^2+[1-C_\psi(||A_2||+\lambda\rho+\lambda\rho^*)]||\mathfrak{v}||^2\\
&-C_p\lambda||\mathfrak{w}||\left(|\mathfrak{g}_1(\eta,0,\mathfrak{v}(\eta))|_{\mathcal{L}^q}+|\int_0^{\eta}\mathfrak{h}_1(s,0,\mathfrak{v}(s))ds|_{\mathcal{L}^q}\right)\\
&-C'_p\lambda||\mathfrak{v}||\left(|\mathfrak{g}_2(\eta,\mathfrak{w}(\eta),0)|_{\mathcal{L}^q}+|\int_0^{\eta}\mathfrak{h}_2(s,\mathfrak{w}(s),0)ds|_{\mathcal{L}^q}\right),
\end{aligned}
$$

where C_p, C'_p are the Sobolev constants corresponding to $\mathfrak{w}, \mathfrak{v}$. We now choose the following norm which is equivalent to the standard product norm:

$$||(\mathfrak{w}, \mathfrak{v})|| := \max\{||\mathfrak{w}||, ||\mathfrak{v}||\}.$$

Hence we get

$$\begin{aligned}&\prec \mathcal{S}(\mathfrak{w}, \mathfrak{v}), (\mathfrak{w}, \mathfrak{v}) \succ \geq \left(1 - C_\psi(||A_1|| + ||A_2||)\right) ||(\mathfrak{w}, \mathfrak{v})||^2\\ &-C'_\psi \lambda ||(\mathfrak{w}, \mathfrak{v})|| \left(|\mathfrak{g}_1(\eta, 0, \mathfrak{v}(\eta))|_{\mathcal{L}^q} - |\mathfrak{g}_2(\eta, \mathfrak{w}(\eta), 0)|_{\mathcal{L}^q} + |\int_0^\eta \mathfrak{h}_1(s, 0, \mathfrak{v}(s)) ds|_{\mathcal{L}^q}\right.\\ &\left. -|\int_0^\eta \mathfrak{h}_2(s, \mathfrak{w}(s), 0) ds|_{\mathcal{L}^q} \right),\end{aligned}$$

where $C'_\psi := \max\{C_p, C'_p\}$. Thus, by the above inequality we have

$$\lim_{||(\mathfrak{w}, \mathfrak{v})|| \to \infty} \frac{\prec \mathcal{S}(\mathfrak{w}, \mathfrak{v}), (\mathfrak{w}, \mathfrak{v}) \succ}{||(\mathfrak{w}, \mathfrak{v})||} \to \infty.$$

Therefore the operator $\mathcal{S}$ is coercive. The desired result immediately follows from the Minty–Browder Theorem. □

5 Conclusion

Applying the generalized Lax–Milgram Theorem, we have studied the existence and regularity of weak solutions to the nonlinear Λ-Hilfer fractional boundary value problem on certain function spaces, with an applicable example. Additionally, we extended our obtained results to the system of coupled FDEs based on the Minty-Browder FPT and some growth requirements.

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Data availability No data set is used to support this study.

Declarations

Conflict of interest The authors declare that they have no competing interests.

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