Received: 18 January 2023 | Revised: 12 April 2023 | Accepted: 24 April 2023
DOI: 10.1002/jnm.3121

RESEARCH ARTICLE

WILEY

Numerical solutions of 2D stochastic time-fractional Sine–Gordon equation in the Caputo sense

Zahra Eidinejad[1] | Reza Saadati[1] | Javad Vahidi[1] | Chenkuan Li[2]

[1]School of Mathematics, Iran University of Science and Technology, Narmak, Tehran, Iran
[2]Department of Mathematics and Computer Science, Brandon University, Brandon, Manitoba, Canada

Correspondence
Reza Saadati, School of Mathematics, Iran University of Science and Technology, Narmak, Tehran 13114-16846, Iran.
Email: rsaadati@eml.cc; rsaadati@iust.ac.ir

Abstract
We study the two-dimensional stochastic time-fractional Sine–Gordon equation (2D ST-FS-G) and provide a solution for it. We use the new clique polynomial method to obtain a numerical solution to the 2D TFS-G equation. In this technique, the clique polynomial is considered as a basic function for operational matrices. By converting the 2D ST-FS-G equation to algebraic equations, a solution is obtained for the desired equation, which clearly shows that this approach is suitable and accurate for dealing with the equation. We further present the error boundary for the obtained approximation of the desired three-variable function based on the clique polynomial. Finally, we compare some numerical results obtained with the exact ones by a few practical examples.

KEYWORDS
Brownian motion process, clique polynomial, operational matrices, stochastic partial differential equations, two-dimensional stochastic time-fractional Sine–Gordon equation (2D ST-FS-G)

1 | INTRODUCTION

In engineering and various sciences, partial differential equations are used to model many phenomena. Therefore, numerical solutions to these equations have always been of interest to scientists and they have presented many numerical schemes to find approximate solutions since it seems impossible to get exact solutions in general.[1] Since fractional differential operators are generalizations of integer order operators, they are used in many engineering problems including mathematical modeling. Viscoelastic materials, bioengineering applications, and viscous fluid flows can be mentioned among these issues. Likewise, numerical methods such as finite element methods, spectral methods, and difference schemes are used to solve fractional partial differential equations.[2–5] Nonlinear equations are among the topics required in various sciences such as physics, which have been studied by many researchers because of their importance.[6] The Sine–Gordon nonlinear partial differential equation (S-GN PDE) first introduced by Edmond Bour (1862) is one of the most important equations in the modern nonlinear wave hypothesis, which was initially proposed in the study of the differential geometry of surfaces with Gaussian constant curvature.[7] The S-GN PDE is also used in other subjects such as nonlinear optics solid-state physics, metal displacement and so on. For this reason, researchers have conducted extensive studies to investigate these equations to find accurate solutions.[8–15] For the nonlinear Sine–Gordon equation and generalized Sine–Gordon equation, elements Galerkin method is used by Li et al.[16] The spectral collocation method based on Legendre Wavelets is presented by Yin et al.[17] to solve the nonlinear Klein–Sine–Gordon equations. Shan et al.[18] investigated the two-dimensional generalized space–time Legendre–Gauss–Lobatto collocation method. In reference 19, a finite difference scheme and a meshless method were used in time and place variables, respectively. The meshless procedures with the Neumann boundary conditions have been applied in reference 20. In reference 21, a combination of the local discontinuous Galerkin method and the finite difference has been used to obtain solutions of the time–space fractional Sine–Gordon equation. Soliton structured methods represent special wave-like solutions for nonlinear dynamic equations, which have attracted the attention of mathematicians. Various nonlinear differential equations such as S-GN PDE and Kortewegde Vries equation (KVE) have been investigated using methods with soliton structures. Many analytical methods have been proposed to numerically solve the S-GN PDE at higher dimensions. For example, Zagrodzinsky, Balcklund, and Painleve developed solutions for the S-GN PDE using the Hirota method. Bratsos, Guo et al. have proposed a numerical solution for the third-order S-GN PDE and the two-dimensional S-GN PDE, respectively.[22–25] Since the occurrence of many real phenomena depends on random factors and there is no certainty about them, stochastic partial differential equations (SPDEs) are used to investigate and explain such problems. SPDEs are applied in various scientific fields, such as nanotechnology, physics, chemistry, biology, economics, etc., which have made researchers to pay attention to solving these equations. One of the most important partial differential equations (SPDEs) is the S-GN PDE. Since there is no exact solution to this equation in most cases, numerical solutions to these equations have always been of interest to researchers. Various methods have been proposed for the numerical solutions. For example, there are the boundary elements method, Taylor expansion method, finite element methods, meshless method, etc.[26–28] Among the meshless methods for solving high-order equations, we can name the element free Galerkin methods, the moving least squares, reproducing kernel particle, and so on.[29] Another well-known method of solving time-independent and time-dependent SPDEs is the meshless method based on the radial basis functions (RBFs). In the present work, we study the generalized form of two-dimensional stochastic time-fractional Sine–Gordon equation (2D ST-FS-G) as follows:

$$\begin{aligned}&\frac{\partial^{\kappa}\Psi(u,w,\tau)}{\partial\tau^{\kappa}}+\mu_1\frac{\partial^{\kappa-1}\Psi(u,w,\tau)}{\partial\tau^{\kappa-1}}=\mu_2\frac{\partial^2\Psi(u,w,\tau)}{\partial u^2}+\mu_3\frac{\partial^2\Psi(u,w,\tau)}{\partial w^2}\\&+s(u,w)\sin(\Psi(u,w,\tau))+\phi(u,w,\tau)+\mu_4\frac{dM(\tau)}{d\tau},\quad (u,w)\in\mathcal{Y},\tau\in[0,\mathrm{R}],\end{aligned}\tag{1.1}$$

with the initial and boundary conditions:

$$\begin{cases}\Psi(u,w,0)=\Psi_0(u,w), & (u,w)\in\mathcal{Y},\\ \frac{\partial\Psi(u,w,\tau)}{\partial\tau}\Big|_{\tau=0}=z(u,w), & (u,w)\in\mathcal{Y},\end{cases}$$

and

$$\Psi(u,w,\tau)=k(\tau),\quad 0\le\tau\le R,(u,w)\in\partial\mathcal{Y},$$

where $\mu_J, J=1,2,3,4$ is known positive constants, the functions and $\phi(u,w,\tau), s(u,w), \Psi_0(u,w), z(u,w)$ are known functions, and $\Psi(u,w,\tau)$ is an unknown function. Also, $M(\tau)$ is the standard Brownian motion process defined on the probability space $(\Xi,\mathscr{H},\mathcal{J})$, which is given below, and $\frac{\partial^{\kappa}\Psi(u,w,\tau)}{\partial\tau^{\kappa}}$ is the Caputo fractional derivative of order κ, and $1<\kappa<2$.

Definition 1.1. The[30] Brownian motion process $\mathrm{M}(\tau)$ satisfies the following properties:

1. $M(0)=0$ (with probability 1).
2. For $0\le\sigma<\tau\le R$, the increment $M(\tau)-M(\sigma)$ is a random variable with mean zero and variance $\tau-\sigma$. Consequently, $M(\tau)-M(\sigma)\sim\sqrt{\tau-\sigma}\mathcal{N}(0,1)$, where $\mathcal{N}(0,1)$ denotes the normal distribution with zero mean and unit variance.
3. For $0\le\sigma<\tau<\lambda<\zeta\le T$, the increments $M(\tau)-M(\sigma)$ and $B(\zeta)-B(\lambda)$ are independent.

Our goal is to obtain the numerical solution to Equation (1.1) using the method based on clique polynomials. One of the common and practical numerical methods for solving differential, fractional differential, integral, and fractional integral equations is the operational matrix method using different polynomials. Among the applications of the operational matrix method, we can mention the poly-Bernoulli operational matrix, Jacobi wavelet operational matrix of fractional integration, etc., which are used for fractional delay differential equations and fractional integrodifferential equations, respectively. One of the main advantages of using operational matrix methods compared to other methods is its easy and applicable use in different subsidy systems.[31,32] In mathematics, there is a branch called graph theory that deals with the subject of vertices and edges. Considering a set of $\mathfrak{e}$ vertices called $V(\mathcal{Q})$ and a set of edges of $\mathcal{Q}$ called $E(\mathcal{Q})$, we have a graph $\mathcal{Q}=\mathcal{Q}(V(\mathcal{Q}),E(\mathcal{Q}))$ which is pairwise ordered. Our goal is to obtain the numerical solutions of Equation (1.1) using the method based on clique polynomials. The clique polynomial related to the maximal clique problem was first introduced by Hoede and Li which is defined as $\Omega(\mathcal{Q},\tau)=\sum_{\iota=0}^{J}\mathfrak{m}_{\mathfrak{e}}(\mathcal{Q})\tau^{\mathfrak{e}}$ where $\mathfrak{m}_0(\mathcal{Q})=1$ and $\mathfrak{m}_{\mathfrak{e}}(\mathcal{Q})$ is the number of $\mathfrak{e}$-cliques of $\mathcal{E}$.[33,34] The maximal clique problem, which is a hybrid optimization problem, is used in various fields such as economics, information retrieval and biomedical engineering, graph coloring, classification theory, set packing, and optimal winner determination.[35] Polynomials are used to solve many equations, that is, they solve these equations by expanding the solution by polynomials.[36,37] There are different types of graph polynomials such as characteristic polynomials, Tutte polynomials, chromatic polynomials, matching polynomials,[38,39] and clique polynomials. Using the technique of the operational method, which is a new method, we obtain the operational matrix with respect to the polynomial of the complete category of the graph. This work is actually a link between graph theory and numerical solutions of differential equations. The rest of this article is organized as follows:

In Section 2, we first define some basic concepts needed. Then we will describe the batch polynomial technique based on the operation matrix to obtain approximate solutions. In this section, we also obtain the approximation error of the three-variable function using batch polynomials. In Section 3, by presenting an example of Equation (1.1), we implement the batch polynomial technique based on the operational matrix to obtain approximate solutions to this equation. At the end and in Section 4, we provide numerical examples and compare the results obtained with other methods by examining the errors of this technique.

2 | PRELIMINARIES

In this section, we will introduce the basic concepts and the main technique of the article using clique polynomials.

Definition 2.1. The[40] κ-order ($\kappa>0$) Riemann–Liouville fractional integral (RL-integral) is defined by

$$ {}^{RL}I_{\rho}^{\kappa}\Psi(\rho)=\frac{1}{\Gamma(\kappa)}\int_0^{\rho}(\rho-r)^{\kappa-1}\Psi(r)dr. $$

Definition 2.2. The[40] κ-order Caputo derivative for $\Psi\in C[p,q]$ is given by

$$ {}^{C}D_{\rho}^{\kappa}\Psi(\rho)=\begin{cases}\dfrac{1}{\Gamma(\ell-\kappa)}\displaystyle\int_p^{\rho}(\rho-r)^{\ell-\kappa-1}\Psi^{\ell}(r)dr, & \ell-1<\kappa<\ell\\ \dfrac{d^{\ell}\Psi(\rho)}{d\rho^{\ell}}, & \kappa=\ell\in\mathbb{N}.\end{cases} $$

The clique polynomial of a graph $\mathcal{Q}$, which denotes by $\Omega(\mathcal{Q},\tau)$, is given by[41]

$$ \Omega(\mathcal{Q},\tau)=\sum_{\mathfrak{e}=0}^{J}\mathfrak{m}_{\mathfrak{e}}(\mathcal{Q})\tau^{\mathfrak{e}},\quad \mathfrak{m}_0(\mathcal{Q})=1 $$

where $\mathfrak{m}_{\mathfrak{e}}(\mathcal{Q})$ is the number of $\mathfrak{e}$-cliques of $\mathcal{Q}$. The CP of a complete graph $\mathcal{D}_J$ with J vertices is given by[42]

$$ \Omega(\mathcal{D}_J,\tau)=(1+\tau)^J. $$

We can approximate any function $\Psi(\tau)$ in $L^2(0,R)$ in terms of the CP as

$$\Psi(\tau) \simeq \widetilde{\Psi}(\tau) = \sum_{J=0}^{E-1} \Psi_J \Omega(\mathcal{D}_J, \tau) = \Delta^T \mathfrak{G}(\tau), \tag{2.1}$$

where $\Delta = [\Psi_0, \Psi_1, \cdots, \Psi_{E-1}]^T$, and

$$\mathfrak{G}(\tau) = [\Omega(\mathcal{D}_0,\tau), \Omega(\mathcal{D}_1,\tau), \cdots, \Omega(\mathcal{D}_{E-1},\tau)]^T. \tag{2.2}$$

From (2.1), we have $\Delta = A^{-1}\langle \Psi(\tau), \mathfrak{G}(\tau)\rangle$, where $A = \langle \mathfrak{G}(\tau), \mathfrak{G}(\tau)\rangle$ and $\langle \cdot, \cdot \rangle$ indicate the inner product of two arbitrary functions. Also, an arbitrary function $\Psi(u,\tau) \in L^2([p,q] \times [0,R])$ can be expanded in terms of the CP by the following infinite series

$$\Psi(u,\tau) = \sum_{J=0}^{\infty} \sum_{\iota=0}^{\infty} \Psi_{J\iota} \Omega(\mathcal{D}_J, u) \Omega(\mathcal{D}_j, \tau). \tag{2.3}$$

We consider the first E terms in (2.3), as an approximation of the function $\Psi(u,\tau)$. It is obtained as

$$\Psi(u,\tau) \simeq \sum_{J=0}^{E-1} \sum_{\iota=0}^{E-1} \Psi_{J\iota} \Omega(\mathcal{D}_J, u) \Omega(\mathcal{D}_\iota, \tau) = \mathfrak{G}^T(u) \Delta \mathfrak{G}(\tau),$$

where $\Delta = A^{-1}\langle \mathfrak{G}(p), \langle \Psi(u,\tau), \mathfrak{G}(\tau)\rangle\rangle A^{-1}$. Also, for function $\Psi(u,w,\tau) \in L^2([p,q] \times [p,q] \times [0,R])$, we have

$$\Psi(u,w,\tau) = \sum_{J=0}^{\infty} \sum_{i=0}^{\infty} \sum_{\epsilon=0}^{\infty} \Psi_{J\iota\epsilon} \Omega(\mathcal{D}_J, u) \Omega(\mathcal{D}_\epsilon, y) \Omega(\mathcal{D}_\iota, \tau).$$

Since in the process of solving differential equations using the clique polynomials technique, assume that $\mathfrak{G}(\tau)$ is the matrix defined in (2.2). This vector is written based on Taylor functions as follows

$$\mathfrak{G}(\tau) = A T_M(\tau), \tag{2.4}$$

where A is an invertible matrix defined as follows

$$A = [a_{ij}], \quad i,j = 0,1,...,M-1, \quad a_{ij} = \begin{cases} \binom{i}{j}, & i \geq j \\ 0, & i < j. \end{cases}$$

Also, $T_M(\tau)$ is the vector of Taylor's basic functions as follows

$$T_M(\tau) = \begin{bmatrix} 1 \\ \tau \\ \tau^2 \\ \vdots \\ \tau^{M-1} \end{bmatrix}. \tag{2.5}$$

By taking the integral from Equation (2.4), we have

$$\begin{aligned} \int_0^\tau \mathfrak{G}(t)dt &= \int_0^\tau AT_M(t)dt \\ &= A\int_0^u T_M(t)dt \\ &= B\overline{T}(\tau), \end{aligned} \tag{2.6}$$

where

$$B = A\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \frac{1}{2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \tag{2.7}$$

and

$$\overline{T}(\tau) = \begin{bmatrix} \tau \\ \tau^2 \\ \tau^3 \\ \vdots \\ \tau^M \end{bmatrix}. \tag{2.8}$$

Now we approximate the τ^k functions based on the clique base and using Equation (2.4) by

$$\tau^k \simeq A^{-1}_{[k+1]}\mathfrak{G}(\tau), k = 1, 2, \cdots, M-1,$$

where $A^{-1}_{[k+1]}$ is the $(k+1)$th row of the matrix A^{-1}. Furthermore, we estimate the function t^M based on the clique base as follows

$$\tau^M \simeq c_M^T \mathfrak{G}(\tau),$$

where c_M is defined as

$$c_M = Q^{-1}\int_0^1 \tau^M \mathfrak{G}(\tau)d\tau$$

and $Q = <\mathfrak{G}(\tau), \mathfrak{G}(\tau)>$. We suppose that

$$E = \begin{bmatrix} A^{-1}_{[2]} \\ A^{-1}_{[3]} \\ A^{-1}_{[4]} \\ \vdots \\ A^{-1}_{[M]} \\ c_M^T \end{bmatrix}, \tag{2.9}$$

then $\overline{T}(\tau) \simeq E\mathfrak{G}(\tau)$. Therefore, the integral operating matrix based on the clique base is derived as

$$\int_0^{\tau} \mathfrak{G}(\tau)d\tau \simeq P\mathfrak{G}(\tau), \tag{2.10}$$

and $P=BE$ is the operational matrix is called the clique base integral.

By taking twice the integral of Equation (2.2), we have

$$\begin{aligned} &\int_0^{\tau}\int_0^{\tau} \mathfrak{G}(t)dtdt = \int_0^{\tau} P\mathfrak{G}(t)dt \\ &= P\int_0^{\tau} \mathfrak{G}(t)dt \\ &= P^2\mathfrak{G}(\tau). \end{aligned} \tag{2.11}$$

Consider the following set

$$\mathfrak{X} = \text{span}\{\Omega(\mathcal{D}_0,u)\Omega(\mathcal{D}_0,w)\Omega(\mathcal{D}_0,\tau),...,\Omega(\mathcal{D}_0,u)\Omega(\mathcal{D}_0,w)\Omega(\mathcal{D}_{E-1},\tau),...,\Omega(\mathcal{D}_{E-1},u)\Omega(\mathcal{D}_{E-1},w)\Omega(\mathcal{D}_{E-1},\tau)\}, \tag{2.12}$$

which is the space of all three-variable polynomials. If is the degree of polynomials on u,w,τ, then $\leq E$. The best approximation for the function $\Psi(u,w,\tau)\in L^*$ to $\mathfrak{X}$ is the function $\overline{\Psi}(u,w,\tau)$.

In the next theorem, we obtain the error of approximation of the three-variable function using clique polynomials.

Theorem 2.3. Suppose $\Psi(u,w,\tau)$ is a sufficiently smooth function on $L=[p,q]\times[p,q]\times[0,R]$ and $\widetilde{\Psi}(u,w,\tau)$ is its approximation using clique polynomials. Then we have

$$\left\|\Psi(u,w,\tau)-\widetilde{\Psi}(u,w,\tau)\right\|_2 \leq \left(\frac{\aleph_1(q-p)^E}{E!2^{2E-1}}+\frac{\aleph_2(q-p)^E}{E!2^{2E-1}}+\frac{\aleph_3(\mathrm{R})^E}{E!2^{2E-1}}+\frac{\aleph_4(q-p)^{2E}R^E}{(E!)^3 2^{6E-3}}\right)(q-p)R. \tag{2.13}$$

Proof. Let $u_\jmath,w_\imath,\jmath,\imath=0,1,...,E-1$ and $\tau_\epsilon,\epsilon=0,1,...,E-1$ be the roots of E-degree shifted Chebyshev polynomials on $[0,1]$. By considering the interpolating polynomial Y_E to $\Psi(u,w,\tau)$ at points $(u_\jmath,w_\imath,\tau_\epsilon)$, we have

$$\begin{aligned} \Psi(u,w,\tau)-\Upsilon_E(u,w,\tau) = &\frac{\partial^E\Psi(\alpha,w,\tau)}{\partial u^E E!}\prod_{\jmath=0}^{E-1}(u-u_\jmath)+\frac{\partial^E\Psi(u,\beta,\tau)}{\partial w^E E!}\prod_{\imath=0}^{E-1}(w-w_\imath)+\frac{\partial^E\Psi(u,w,\nu)}{\partial\tau^E E!}\prod_{\epsilon=0}^{E-1}(\tau-\tau_\epsilon) \\ &-\frac{\partial^{3E}\Psi(\alpha',\beta',\nu')}{\partial u^E\partial w^E\partial\tau^E(E!)^3}\prod_{\jmath=0}^{E-1}(u-u_\jmath)\prod_{\imath=0}^{E-1}(w-w_\imath)\prod_{\epsilon=0}^{E-1}(\tau-\tau_\epsilon), \end{aligned} \tag{2.14}$$

where $\alpha,\alpha'\in[p,q]$, $\beta,\beta'\in[p,q]$, and $\nu,\nu'\in[0,R]$. Then we further imply

$$\begin{aligned} |\Psi(u,w,\tau)-\Upsilon_E(u,w,\tau)| \leqslant &\max_{(u,w,\tau)\in L}\left|\frac{\partial^E\Psi(\alpha,w,\tau)}{\partial u^E}\right|\frac{\prod_{\jmath=0}^{E-1}|u-u_\jmath|}{E!} \\ &+\max_{(u,w,\tau)\in L}\left|\frac{\partial^E\Psi(u,\beta,\tau)}{\partial w^E}\right|\frac{\prod_{\imath=0}^{E-1}|w-w_\imath|}{E!}+\max_{(u,w,\nu)\in L}\left|\frac{\partial\Psi(u,w,\nu)^E}{\partial\tau^E}\right|\frac{\prod_{\epsilon=0}^{E-1}|\tau-\tau_\epsilon|}{E!} \\ &+\max_{(u,w,\tau)\in L}\left|\frac{\partial^{3E}\Psi(\alpha',\beta',\nu')}{\partial u^E\partial w^E\partial\tau^E}\right|\frac{\prod_{\jmath=0}^{E-1}|u-u_\jmath|\prod_{\jmath=0}^{E-1}|w-w_\imath|\prod_{\tau^E}^{E-1}|\tau-\tau_\epsilon|}{(E!)^3}. \end{aligned} \tag{2.15}$$

According to the smooth function $\Psi(u,w,\tau)$ on L^*, there are constants $\aleph_1,\aleph_2,\aleph_3,\aleph_4$ such that

$$\begin{aligned}&\max_{(u,w,\tau)\in L}\left|\frac{\partial^{L}\Psi(u,w,\tau)}{\partial u^{E}}\right|\leq\aleph_1,\ \max_{(u,w,\tau)\in L}\left|\frac{\partial^{E}\Psi(u,w,\tau)}{\partial w^{E}}\right|\leq\aleph_2,\ \max_{(u,w,\tau)\in L}\left|\frac{\partial\Psi(u,w,\nu)^{E}}{\partial \tau^{E}}\right|\leq\aleph_3,\\&\max_{(u,w,\tau)\in L}\left|\frac{\partial^{3E}\Psi(\aleph',\beta',\nu')}{\partial u^{E}\partial w^{E}\partial\tau^{E}}\right|\leq\aleph_4.\end{aligned}\tag{2.16}$$

By substituting (2.16) into (2.15) and employing the estimates for Chebyshev interpolation nodes, we come to

$$|\Psi(u,w,\tau)-\Upsilon_E(u,w,\tau)|\leqslant\frac{\aleph_1(q-p)^E}{E!2^{2E-1}}+\frac{\aleph_2(q-p)^E}{E!2^{2E-1}}+\frac{\aleph_3(R)^E}{E!2^{2E-1}}+\frac{\aleph_4(q-p)^{2E}R^E}{(E!)^3 2^{6E-3}}.\tag{2.17}$$

Since $\widetilde{\Psi}(u,w,\tau)$ is the best approximation of $\Psi(u,w,\tau)$ in $\mathfrak{X}$ and using (2.17), we can write

$$\begin{aligned}\left\|\Psi(u,w,\tau)-\tilde{\Psi}(u,w,\tau)\right\|_2^2&=\int_0^{\mathrm{R}}\int_p^q\int_p^q\left|\Psi(u,w,\tau)-\tilde{\Psi}(u,w,\tau)\right|^2 dudwd\tau\\&\leqslant\int_0^{\mathrm{R}}\int_p^q\int_p^q|\Psi(u,w,\tau)-\Upsilon_E(u,w,\tau)|^2 dudwd\tau\\&=\int_0^{\mathrm{R}}\int_p^q\int_p^q\left(\frac{\aleph_1(q-p)^E}{E!2^{2E-1}}+\frac{\aleph_2(q-p)^E}{E!2^{2E-1}}+\frac{\aleph_3(\mathrm{R})^E}{E!2^{2E-1}}+\frac{\aleph_4(q-p)^{2E}\mathrm{R}^E}{(E!)^3 2^{6E-3}}\right)^2 dudwd\tau\\&=\left(\frac{\aleph_1(q-p)^E}{E!2^{2E-1}}+\frac{\aleph_2(q-p)^E}{E!2^{2E-1}}+\frac{\aleph_3(\mathrm{R})^E}{E!2^{2E-1}}+\frac{\aleph_4(q-p)^{2E}\mathrm{R}^E}{(E!)^3 2^{6E-3}}\right)^2(q-p)^2\mathrm{R},\end{aligned}\tag{2.18}$$

which gives the desired result. Note that for the special case, when $p=0$, $q=1$, and $R=1$, we have

$$\left\|\Psi(u,w,\tau)-\widetilde{\Psi}(u,w,\tau)\right\|_2\leqslant\frac{\aleph_1}{E!2^{2E-1}}+\frac{\aleph_2}{E!2^{2E-1}}+\frac{\aleph_3}{E!2^{2E-1}}+\frac{\aleph_4}{(E!)^3 2^{6E-3}},\tag{2.19}$$

therefore (Figure 1)

$$\left\|\Psi(u,w,\tau)-\widetilde{\psi}(u,w,\tau)\right\|_2=O\left(\frac{1}{E!2^{2E-1}}\right).$$

□

3 | THE PROPOSED TECHNIQUE

In this section, we use the clique polynomials technique to obtain the solution of an equation of type (1.1). For this purpose, we consider the following 2D ST-FS-G equation:

$$\begin{aligned}&\frac{\partial^2\Psi(u,w,\tau)}{\partial\tau^2}+\mu_1\frac{\partial\Psi(u,w,\tau)}{\partial\tau}=\mu_2\frac{\partial^2\Psi(u,w,\tau)}{\partial u^2}+\mu_3\frac{\partial^2\Psi(u,w,\tau)}{\partial w^2}\\&+s(u,w)\sin(\Psi(u,w,\tau))+\phi(u,w,\tau)+\mu_4\frac{dM(\tau)}{d\tau},\quad(u,w)\in\mathcal{Y},\tau\in[0,R],\end{aligned}\tag{3.1}$$

with the initial and boundary conditions in Equation (1.1). Assume that,

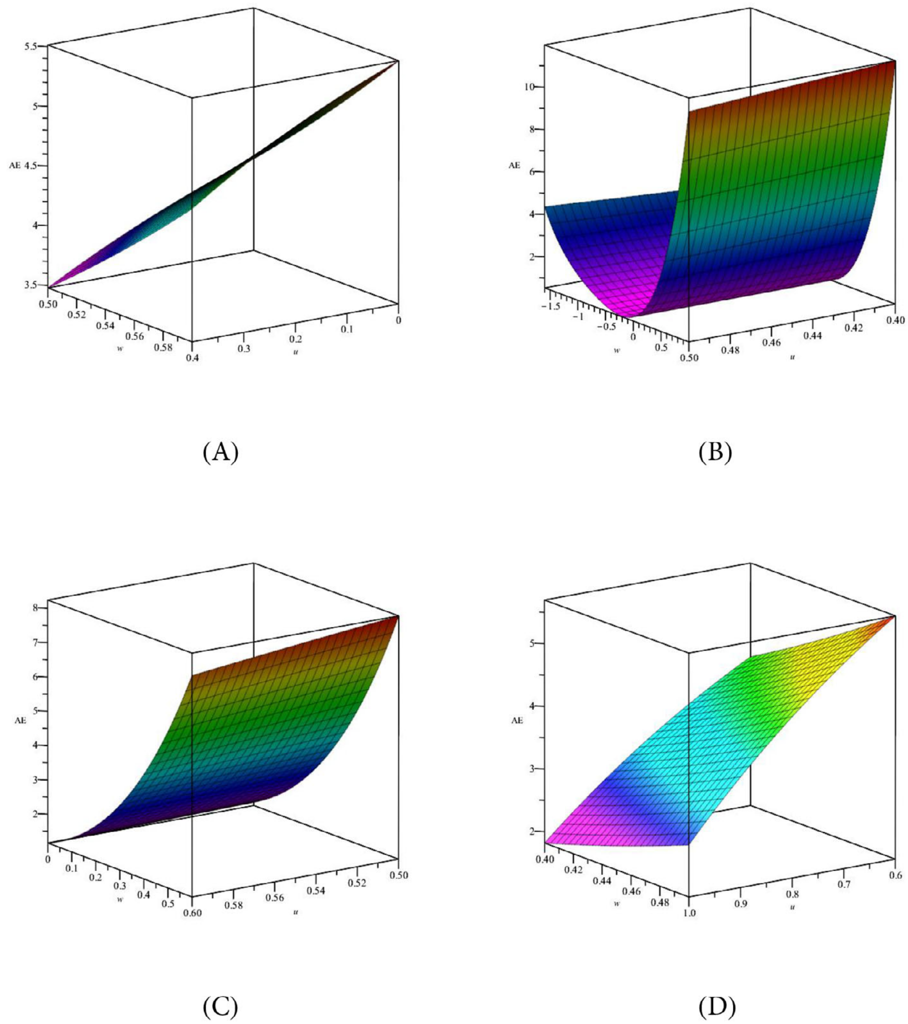

FIGURE 1 Absolute error of example (4.1). (A) $\tau = 0.02, u \in \left(0, \frac{2}{5}\right), w \in \left(\frac{1}{2}, \frac{3}{5}\right)$, (B) $\tau = 0.02, u \in \left(\frac{2}{5}, \frac{1}{2}\right), w \in \left(-\frac{8}{5}, 1\right)$, (C) $\tau = 0.18, u \in \left(\frac{1}{2}, \frac{3}{5}\right), w \in \left(0, \frac{3}{5}\right)$, and (D) $\tau = 0.14, u \in \left(\frac{3}{5}, 1\right), w \in \left(\frac{8}{20}, \frac{1}{2}\right)$.

$$\frac{\partial^5 \Psi(u,w,\tau)}{\partial u^2 \partial w^2 \partial \tau} = \sum_{J=0}^{\infty} \sum_{\iota=0}^{\infty} \sum_{\epsilon=0}^{\infty} \Psi_{J\iota\epsilon} \Omega\left(\mathcal{D}_J, u\right) \Omega(\mathcal{D}_\epsilon, w) \Omega(\mathcal{D}_\iota, \tau) = \mathfrak{G}(\tau)^T \Delta \mathfrak{G}(u) \mathfrak{G}(w), \tag{3.2}$$

and

$$\begin{aligned}
\mathfrak{G}(\tau)^T &= [\Omega(\mathcal{D}_0, \tau), \Omega(\mathcal{D}_1, \tau), \Omega(\mathcal{D}_2, \tau), \Omega(\mathcal{D}_3, \tau), \ldots, \Omega(\mathcal{D}_\ell, \tau)] \\
\mathfrak{G}(u) &= [\Omega(\mathcal{D}_0, u), \Omega(\mathcal{D}_1, u), \Omega(\mathcal{D}_2, u), \Omega(\mathcal{D}_3, u), \ldots, \Omega(\mathcal{D}_\ell, u)]^T \\
\mathfrak{G}(w) &= [\Omega(\mathcal{D}_0, u), \Omega(\mathcal{D}_1, u), \Omega(\mathcal{D}_2, u), \Omega(\mathcal{D}_3, u), \ldots, \Omega(\mathcal{D}_\ell, u)]^T.
\end{aligned}$$

Integrate Equation (3.2) regarding τ from 0 to τ,

$$\begin{aligned}
\frac{\partial^4 \Psi(u,w,\tau)}{\partial u^2 \partial w^2} &= \frac{\partial^4 \Psi(u,w,0)}{\partial u^2 \partial w^2} + \int_0^\tau \mathfrak{G}(\tau)^T \Delta \mathfrak{G}(u) \mathfrak{G}(w) d\tau, \\
\frac{\partial^4 \Psi(u,w,\tau)}{\partial u^2 \partial w^2} &= \frac{\partial^4 \Psi(u,w,0)}{\partial u^2 \partial w^2} + \Delta \mathfrak{G}(u) \mathfrak{G}(w) \int_0^\tau \mathfrak{G}(\tau)^T d\tau, \\
\frac{\partial^4 \Psi(u,w,\tau)}{\partial u^2 \partial w^2} &= \frac{\partial^4 \Psi(u,w,0)}{\partial u^2 \partial w^2} + [P \mathfrak{G}(\tau)]^T \Delta \mathfrak{G}(u) \mathfrak{G}(w).
\end{aligned} \tag{3.3}$$

Integrating Equation (3.3) regarding u from 0 to u,

$$\frac{\partial^3\Psi(u,w,\tau)}{\partial u\partial w^2}=\frac{\partial^3\Psi(0,w,\tau)}{\partial u\partial w^2}-\frac{\partial^3\Psi(0,w,0)}{\partial u\partial w^2}+\frac{\partial^3\Psi(u,w,0)}{\partial u\partial w2}+\int_u^0[P\mathfrak{G}(\tau)]^T\Delta\mathfrak{G}(u)\mathfrak{G}(w)du,$$
$$\frac{\partial^3\Psi(u,w,\tau)}{\partial u\partial w^2}=\frac{\partial^3\Psi(0,w,\tau)}{\partial u\partial w^2}-\frac{\partial^3\Psi(0,w,0)}{\partial u\partial w^2}+\frac{\partial^3\Psi(u,w,0)}{x\partial u\partial w^2}+\Delta\mathfrak{G}(w)[P\mathfrak{G}(\tau)]^T\int_u^0\mathfrak{G}(u)du, \tag{3.4}$$
$$\frac{\partial^3\Psi(u,w,\tau)}{\partial u\partial w^2}=\frac{\partial^3\Psi(0,w,\tau)}{\partial u\partial w^2}-\frac{\partial^3\Psi(0,w,0)}{\partial u\partial w^2}+\frac{\partial^3\Psi(u,w,0)}{\partial u\partial w^2}+\Delta\mathfrak{G}(w)[P\mathfrak{G}(\tau)]^T[P\mathfrak{G}(u)].$$

Again, by taking the integral from 0 to u relative to u

$$\begin{aligned}\frac{\partial^2\Psi(u,w,\tau)}{\partial w^2} &=\frac{\partial^2\Psi(0,w,\tau)}{\partial w^2}+u\left[-\frac{\partial^2\Psi(0,w,0)}{\partial w^2}+\frac{\partial^2\Psi(0,w,\tau)}{\partial w^2}\right]-\frac{\partial^2\Psi(0,w,0)}{\partial w^2}+\frac{\partial^2\Psi(u,w,0)}{\partial w^2}\\ &+\int_0^u\Delta\mathfrak{G}(w)[P\mathfrak{G}(\tau)]^T[P\mathfrak{G}(u)]du,\\ \frac{\partial^2\psi(u,w,\tau)}{\partial w^2} &=\frac{\partial^2\Psi(0,w,\tau)}{\partial w^2}+u\left[-\frac{\partial^2\Psi(0,w,0)}{\partial w^2}+\frac{\partial^2\Psi(0,w,\tau)}{\partial w^2}\right]-\frac{\partial^2\Psi(0,w,0)}{\partial w^2}+\frac{\partial^2\Psi(u,w,0)}{\partial w^2}\\ &+\Delta\mathfrak{G}(w)[P\mathfrak{G}(\tau)]^T\left[p^2\mathfrak{G}(u)\right].\end{aligned} \tag{3.5}$$

Now we take an integral from Equation (3.3) regarding w from 0 to w

$$\begin{aligned}\frac{\partial^3\Psi(u,w,\tau)}{\partial u^2\partial w} &=\frac{\partial^3\Psi(u,0,\tau)}{\partial u^2\partial w}-\frac{\partial^3\Psi(u,0,0)}{\partial u^2\partial w}+\frac{\partial^3\Psi(u,w,0)}{\partial u^2\partial w}\\ &+\Delta\mathfrak{G}(u)[P\mathfrak{G}(\tau)]^T[P\mathfrak{G}(w)],\end{aligned} \tag{3.6}$$

and again, by taking the integral from 0 to w relative to w

$$\begin{aligned}\frac{\partial^2\Psi(u,w,\tau)}{\partial u^2} &=\frac{\partial^2\Psi(u,0,\tau)}{\partial u^2}+w\left[-\frac{\partial^2\Psi(u,0,0)}{\partial u^2}+\frac{\partial^2\Psi(u,0,\tau)}{\partial u^2}\right]-\frac{\partial^2\Psi(u,0,0)}{\partial u^2}+\frac{\partial^2\Psi(u,w,0)}{\partial u^2}\\ &+\Delta\mathfrak{G}(u)[P\mathfrak{G}(\tau)]^T\left[p^2\mathfrak{G}(w)\right].\end{aligned} \tag{3.7}$$

Integrating twice from 0 to u relative to u from the above equation, we have

$$\begin{aligned}\Psi(u,w,\tau) &=\Psi(0,w,\tau)+\Psi(u,0,\tau)-\Psi(0,0,\tau)\\ &+w\left[\Psi(u,0,\tau)-\Psi(0,0,\tau)-\Psi(u,0,0)+\Psi(0,0,0)+u\left[-\frac{\partial\Psi(0,0,\tau)}{\partial u}+\frac{\partial\Psi(0,0,0)}{\partial u}\right]\right]\\ &-\Psi(u,0,0)+\Psi(0,0,0)+\Psi(u,w,0)-\Psi(0,w,0)\\ &+\Delta[P\mathfrak{G}(\tau)]^T\left[p^2\mathfrak{G}(w)\right]\left[P^2\mathfrak{G}(u)\right],\end{aligned} \tag{3.8}$$

by putting $u=c,w=d$ in the equation, we have obtained

$$\begin{aligned}\Psi(c,d,\tau) &=\Psi(0,d,\tau)+\Psi(c,0,\tau)-\Psi(0,0,\tau)\\ &+d\left[\Psi(c,0,\tau)-\Psi(0,0,\tau)-\Psi(c,0,0)+\Psi(0,0,0)+c\left[-\frac{\partial\Psi(0,0,\tau)}{\partial u}+\frac{\partial\Psi(0,0,0)}{\partial u}\right]\right]\\ &-\Psi(c,0,0)+\Psi(0,0,0)+\Psi(c,d,0)-\Psi(0,d,0)\\ &+\Delta[P\mathfrak{G}(\tau)]^T\left[p^2\mathfrak{G}(d)\right]\left[P^2\mathfrak{G}(c)\right].\end{aligned} \tag{3.9}$$

Then

$$\left[-\frac{\partial\Psi(0,0,\tau)}{\partial u}+\frac{\partial\Psi(0,0,0)}{\partial u}\right]=$$

$$\frac{1}{dc}[\Psi(c,d,\tau)-\Psi(0,d,\tau)-\Psi(c,0,\tau)+\Psi(0,0,\tau)-d\Psi(c,0,\tau)+d\Psi(0,0,\tau)+d\Psi(c,0,0)$$
$$-d\Psi(0,0,0)\qquad+\Psi(c,0,0)-\Psi(0,0,0)-\Psi(c,d,0)+\Psi(0,d,0)\qquad-\Delta[P\mathfrak{G}(\tau)]^T\left[p^2\mathfrak{G}(d)\right]\left[P^2\mathfrak{G}(c)\right]]. \tag{3.10}$$

Using Equation (3.8), we get

$$\begin{aligned}\Psi(u,w,\tau)\ &=\Psi(0,w,\tau)+\Psi(u,0,\tau)-\Psi(0,0,\tau)\\&+w[\Psi(u,0,\tau)-\Psi(0,0,\tau)-\Psi(u,0,0)\end{aligned}$$

$$+\Psi(0,0,0)+u\left[\frac{1}{dc}[\Psi(c,d,\tau)-\Psi(0,d,\tau)-\Psi(c,0,\tau)+\Psi(0,0,\tau)-d\Psi(c,0,\tau)+d\Psi(0,0,\tau)+d\Psi(c,0,0)-d\Psi(0,0,0)+\Psi(c,0,0)\right.$$
$$-\Psi(0,0,0)-\Psi(c,d,0)+\Psi(0,d,0)$$
$$-\Delta[P\mathfrak{G}(\tau)]^T\left[p^2\mathfrak{G}(d)\right]\left[P^2\mathfrak{G}(c)\right]]]]\ \ -\Psi(u,0,0)+\Psi(0,0,0)+\Psi(u,w,0)-\Psi(0,w,0)+\Delta[P\mathfrak{G}(\tau)]^T\left[p^2\mathfrak{G}(w)\right]\left[P^2\mathfrak{G}(u)\right]. \tag{3.11}$$

Continuing the process, we derive that

$$\begin{aligned}&\frac{\partial\Psi(u,w,\tau)}{\partial\tau}=\frac{\partial\Psi(0,w,\tau)}{\partial\tau}+\frac{\partial\Psi(u,0,\tau)}{\partial\tau}-\frac{\partial\Psi(0,0,\tau)}{\partial\tau}\\&+w\frac{d}{d\tau}\left[\Psi(u,0,\tau)-\Psi(0,0,\tau)-\Psi(u,0,0)+\Psi(0,0,0)+u\left[\frac{1}{dc}[\Psi(c,d,\tau)-\Psi(0,d,\tau)-\Psi(c,0,\tau)\right.\right.\\&+\Psi(0,0,\tau)-d\Psi(c,0,\tau)+d\Psi(0,0,\tau)+d\Psi(c,0,0)-d\Psi(0,0,0)\\&\left.\left.+\Psi(c,0,0)-\Psi(0,0,0)-\Psi(c,d,0)+\Psi(0,d,0)-\Delta[P\mathfrak{G}(\tau)]^T\left[p^2\mathfrak{G}(d)\right]\left[P^2\mathfrak{G}(c)\right]]\right]\right]\\&+\frac{d}{d\tau}\Delta[P\mathfrak{G}(\tau)]^T\left[P^2\mathfrak{G}(w)\right]\left[P^2\mathfrak{G}(u)\right],\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}&\frac{\partial^2\Psi(u,w,\tau)}{\partial\tau^2}=\frac{\partial^2\Psi(0,w,\tau)}{\partial\tau^2}+\frac{\partial^2\Psi(u,0,\tau)}{\partial\tau^2}-\frac{\partial^2\Psi(0,0,\tau)}{\partial\tau^2}\\&+w\frac{d^2}{d\tau^2}\left[\Psi(u,0,\tau)-\Psi(0,0,\tau)-\Psi(u,0,0)+\Psi(0,0,0)+u\left[\frac{1}{dc}[\Psi(c,d,\tau)-\Psi(0,d,\tau)-\Psi(c,0,\tau)\right.\right.\\&+\Psi(0,0,\tau)-d\Psi(c,0,\tau)+d\Psi(0,0,\tau)+d\Psi(c,0,0)-d\Psi(0,0,0)\\&\left.\left.+\Psi(c,0,0)-\Psi(0,0,0)-\Psi(c,d,0)+\Psi(0,d,0)-\Delta[P\mathfrak{G}(\tau)]^T\left[p^2\mathfrak{G}(d)\right]\left[P^2\mathfrak{G}(c)\right]]\right]\right]\\&+\frac{d^2}{d\tau^2}\Delta[P\mathfrak{G}(\tau)]^T\left[P^2\mathfrak{G}(w)\right]\left[P^2\mathfrak{G}(u)\right].\end{aligned} \tag{3.13}$$

To complete the algorithm, we put Equations (3.13), (3.12), (3.11), (3.7), and (3.5) in Equation (1.1)

$$\frac{\partial^2\Psi(0,w,\tau)}{\partial\tau^2}+\frac{\partial^2\Psi(u,0,\tau)}{\partial\tau^2}-\frac{\partial^2\Psi(0,0,\tau)}{\partial\tau^2}$$
$$+w\frac{d^2}{d\tau^2}[\Psi(u,0,\tau)-\Psi(0,0,\tau)-\Psi(u,0,0)+\Psi(0,0,0)$$

$$+u[\frac{1}{dc}[\Psi(c,d,\tau)-\Psi(0,d,\tau)-\Psi(c,0,\tau)+\Psi(0,0,\tau)-d\Psi(c,0,\tau)+d\Psi(0,0,\tau)+d\Psi(c,0,0)-d\Psi(0,0,0)$$

$$+\Psi(c,0,0)-\Psi(0,0,0)$$
$$-\Psi(c,d,0)+\Psi(0,d,0)-\Delta[P\mathfrak{G}(\tau)]^T\left[p^2\mathfrak{G}(d)\right]\left[P^2\mathfrak{G}(c)\right]]]]+\frac{d^2}{d\tau^2}\Delta[P\mathfrak{G}(\tau)]^T\left[P^2\mathfrak{G}(w)\right]\left[P^2\mathfrak{G}(u)\right],$$
$$+\mu_1\Bigg(\frac{\partial\Psi(0,w,\tau)}{\partial\tau}+\frac{\partial\Psi(u,0,\tau)}{\partial\tau}-\frac{\partial\Psi(0,0,\tau)}{\partial\tau}+w\frac{d}{d\tau}\Bigg[\Psi(u,0,\tau)-\Psi(0,0,\tau)-\Psi(u,0,0)+\Psi(0,0,0)+u\Bigg[\frac{1}{dc}[\Psi(c,d,\tau)-\Psi(0,d,\tau)$$
$$-\Psi(c,0,\tau)+\Psi(0,0,\tau)-d\Psi(c,0,\tau)+d\Psi(0,0,\tau)+d\Psi(c,0,0)$$
$$-d\Psi(0,0,0)+\Psi(c,0,0)-\Psi(0,0,0)-\Psi(c,d,0)$$
$$+\Psi(0,d,0)-\Delta[P\mathfrak{G}(\tau)]^T\left[p^2\mathfrak{G}(d)\right]\left[P^2\mathfrak{G}(c)\right]]]]$$
$$+\frac{d}{d\tau}\Delta[P\mathfrak{G}(\tau)]^T\left[P^2\mathfrak{G}(w)\right]\left[P^2\mathfrak{G}(u)\right]\Bigg)=\mu_2\Bigg(\frac{\partial^2\Psi(u,0,\tau)}{\partial u^2}+w[$$
$$-\frac{\partial^2\Psi(u,0,0)}{\partial u^2}+\frac{\partial^2\Psi(u,0,\tau)}{\partial u^2}]-\frac{\partial^2\Psi(u,0,0)}{\partial u^2}$$
$$+\frac{\partial^2\Psi(u,w,0)}{\partial u^2}+\Delta\mathfrak{G}(u)[P\mathfrak{G}(\tau)]^T\left[p^2\mathfrak{G}(w)\right]\Bigg)+\mu_3\Bigg(\frac{\partial^2\Psi(0,w,\tau)}{\partial w^2}+u\left[-\frac{\partial^2\Psi(0,w,0)}{\partial w^2}+\frac{\partial^2\Psi(0,w,\tau)}{\partial w^2}\right]-\frac{\partial^2\Psi(0,w,0)}{\partial w^2}+\frac{\partial^2\Psi(u,w,0)}{\partial w^2}$$
$$+\Delta\mathfrak{G}(w)[P\mathfrak{G}(\tau)]^T\left[p^2\mathfrak{G}(u)\right]\Bigg)+s(u,w)\sin(\Psi(0,w,\tau)+\Psi(u,0,\tau)-\Psi(0,0,\tau)$$
$$+w\Bigg[\Psi(u,0,\tau)-\Psi(0,0,\tau)-\Psi(u,0,0)+\Psi(0,0,0)+u\Bigg[\frac{1}{dc}[\Psi(c,d,\tau)-\Psi(0,d,\tau)-\Psi(c,0,\tau)+\Psi(0,0,\tau)-d\Psi(c,0,\tau)$$
$$+d\Psi(0,0,\tau)+d\Psi(c,0,0)$$
$$-d\Psi(0,0,0)+\Psi(c,0,0)-\Psi(0,0,0)-\Psi(c,d,0)+\Psi(0,d,0)$$
$$-\Delta[P\mathfrak{G}(\tau)]^T\left[p^2\mathfrak{G}(d)\right]\left[P^2\mathfrak{G}(c)\right]]]]-\Psi(u,0,0)+\Psi(0,0,0)+\Psi(u,w,0)$$
$$-\Psi(0,w,0)+\Delta[P\mathfrak{G}(\tau)]^T\left[p^2\mathfrak{G}(w)\right]\left[P^2\mathfrak{G}(u)\right]\Big)+\phi(u,w,\tau)+\mu_4\frac{dM(\tau)}{d\tau}. \tag{3.14}$$

By solving the above system at different points (local points $u_J, w_J, \tau_J=\frac{2J-1}{2E^2}$), the values of polynomial coefficients of the category are derived. Then by placing these coefficients in (3.11), we achieve the desired solution (Figure 2).

4 | NUMERICAL EXAMPLES

In this section, a performance of the clique polynomial method for a type of Equation (1.1) is demonstrated. We consider the domain Ξ and these results are calculated in this domain. We shall computer the errors, L_2-error, L_∞-error, and RMS-error. Therefore, in the following, we will first introduce these errors and then present them in the examples.

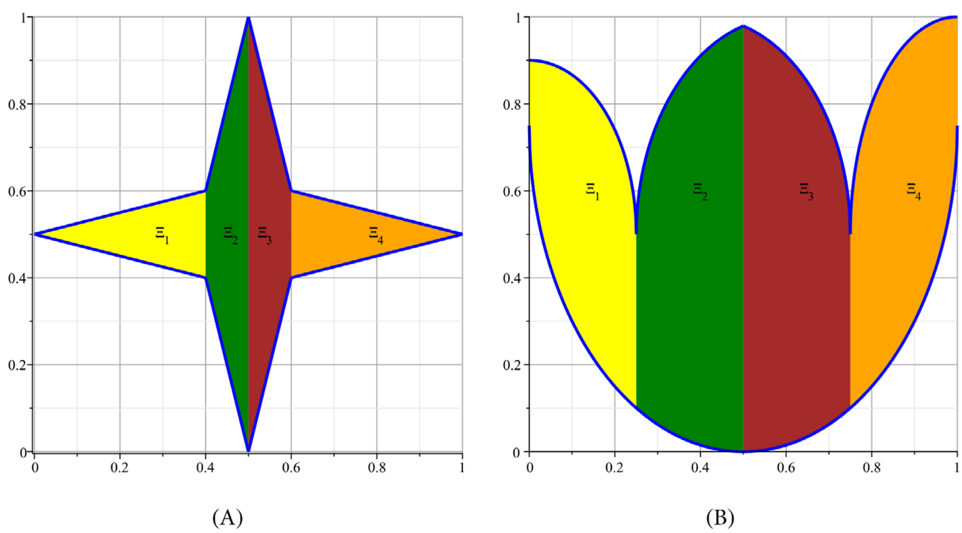

FIGURE 2 The intended domain of examples (4.1) and (2).

$$
\begin{aligned}
L^2\text{-error} &= \sqrt{\sum_{J=1}^{\ell} e_J^2} \\
L^\infty\text{-error} &= \max(e_J) \quad 1 \le J \le \ell - 1 \\
\text{RMS-error} &= \sqrt{\sum_{J=1}^{\ell} \frac{e_J^2}{\ell}},
\end{aligned} \tag{4.1}
$$

where e_J is an absolute error.

Example 4.1. Consider the following nonlinear 2D stochastic timefractional Sine–Gordon equation (2D ST-FS-G)

$$
\begin{aligned}
\frac{\partial^\kappa \Psi(u,w,\tau)}{\partial \tau^\kappa} + 3\frac{\partial^{\kappa-1}\Psi(u,w,\tau)}{\partial \tau^{\kappa-1}} &= 2\frac{\partial^2 \Psi(u,w,\tau)}{\partial u^2} + \frac{\partial^2 \Psi(u,w,\tau)}{\partial \tau^2} \\
&+ \frac{\pi}{3}\sin(\psi(u,w,\tau)) + \phi(u,w,\tau) + 4\frac{d\mathrm{M}(\tau)}{d\tau},
\end{aligned} \tag{4.2}
$$

by choosing $\kappa = 2$ and with the following initial and boundary conditions

$$
\begin{cases}
\Psi(u,w,0) = -\sin\left(\frac{\pi}{2}u\right)\sin\left(\frac{\pi}{2}w\right), & (u,w) \in \Xi, \\
\frac{\partial \Psi(u,w,\tau)}{\partial \tau}\Big|_{\tau=0} = -2\sin\left(\frac{\pi}{2}u\right)\sin\left(\frac{\pi}{2}w\right), & (u,w) \in \Xi,
\end{cases}
$$

and $\Psi(u,w,\tau) = -\exp(2\tau)\sin\left(\frac{\pi}{2}u\right)\sin\left(\frac{\pi}{2}w\right)$, where $(u,w) \in \partial\Xi, 0 \le \tau \le T$. We consider $\phi(u,w,\tau)$ such that the exact solution of Equation (4.3) is as follows (Table 1 and Figure 3)

$$
\Psi(u,w,\tau) = -\exp(2\tau)\sin\left(\frac{\pi}{2}u\right)\sin\left(\frac{\pi}{2}w\right),
$$

TABLE 1 Compare the errors of our proposed method and other methods for example (4.1).

R	L_2-error	RMS-error	L_∞-error
Numerical results using Gaussian RBF			
1	–	$4.2234e-3$	$5.1057e-3$
2	–	$2.6358e-3$	$3.3081e-3$
3	–	$8.9071e-4$	$9.0937e-4$
4	–	$5.9450e-4$	$5.9999e-4$
5	–	–	–
Numerical results using multiquadrics RBF			
1	–	$9.3207e-4$	$1.2396e-3$
2	–	$5.5302e-4$	$6.2397e-4$
3	–	$3.4501e-4$	$4.2139e-4$
4	–	$1.4540e-4$	$1.4999e-4$
5	–	–	–
Numerical results using our method			
1	$3.2e-5$	$8.1e-6$	$1.6e-5$
2	$4.6e-5$	$1.1e-5$	$2.3e-5$
3	$5.6e-5$	$1.4e-5$	$2.8e-5$
4	$6.5e-5$	$1.6e-5$	$3.2e-5$
5	$7.2e-5$	$1.8e-5$	$3.6e-5$

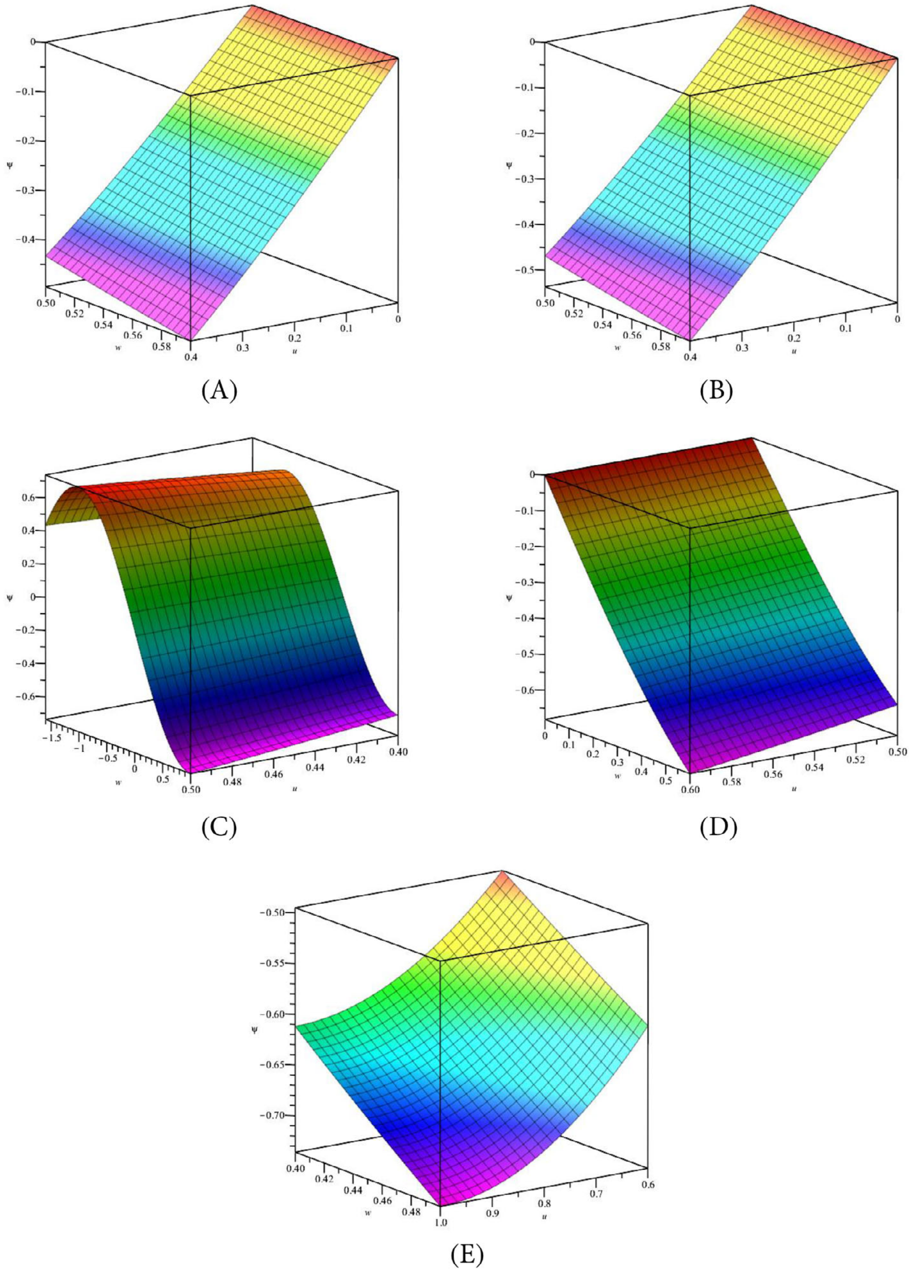

FIGURE 3 Exact solution for example (4.1). (A) $\tau = 0.02, u \in \left(0, \frac{2}{5}\right), w \in \left(\frac{1}{2}, \frac{3}{5}\right)$, (B) $\tau = 0.06, u \in \left(0, \frac{2}{5}\right), w \in \left(\frac{1}{2}, \frac{3}{5}\right)$, (C) $\tau = 0.02, u \in \left(\frac{2}{5}, \frac{1}{2}\right), w \in \left(-\frac{8}{5}, 1\right)$, (D) $\tau = 0.02, u \in \left(\frac{1}{2}, \frac{3}{5}\right), w \in \left(0, \frac{3}{5}\right)$, and (E) $\tau = 0.02, u \in \left(\frac{3}{5}, 1\right), w \in \left(\frac{8}{20}, \frac{1}{2}\right)$.

where $\Xi = \Xi_1 \cup \Xi_2 \cup \Xi_3 \cup \Xi_4$ and

$$\Xi_1 = \left\{ (u,w) \in \mathbb{R}^2 : \quad 0 < u < \frac{2}{5}, \quad -\frac{1}{4}u + \frac{1}{2} < w < \frac{1}{4}u + \frac{1}{2} \right\}$$

$$\Xi_2 = \left\{ (u,w) \in \mathbb{R}^2 : \quad \frac{2}{5} < u < \frac{1}{2}, \quad -4u + 2 < w < 4u - 1 \right\}$$

$$\Xi_3 = \left\{ (u,w) \in \mathbb{R}^2 : \quad \frac{1}{2} < u < \frac{3}{5}, \quad 4u - 2 < w < -4u + 3 \right\}$$

$$\Xi_4 = \left\{ (u,w) \in \mathbb{R}^2 : \quad \frac{3}{5} < u < 1, \quad \frac{1}{4}u + \frac{1}{4} < w < -\frac{1}{4}u + \frac{3}{4} \right\}.$$

Example 4.2. Consider the following nonlinear 2D stochastic timefractional Sine–Gordon equation (2D ST-FS-G)

$$\begin{aligned} \frac{\partial^{\alpha}\Psi(u,w,\tau)}{\partial\tau^{\kappa}} + \frac{\partial^{\kappa-1}\Psi(u,w,\tau)}{\partial\tau^{\kappa-1}} &= \pi\frac{\partial^2\Psi(u,w,\tau)}{\partial u^2} + \frac{\pi}{2}\frac{\partial^2\Psi(u,w,\tau)}{\partial w^2} \\ &+ \sin(\Psi(u,w,\tau)) + \phi(u,w,\tau) + \frac{\pi}{3}\frac{dM(\tau)}{d\tau}, \end{aligned} \tag{4.3}$$

for $\kappa = 2$ and with the following initial and boundary conditions:

$$\begin{cases} \Psi(u,w,0) = -w\exp(2u+w), & (u,w)\in\Xi, \\ \left.\frac{\partial\Psi(u,w,\tau)}{\partial\tau}\right|_{\tau=0} = -\exp(2u+w), & (u,w)\in\Xi, \end{cases}$$

and $\Psi(u,w,\tau) = (2\tau^2 - \tau - w)\exp(2u+w)$, where $(u,w)\in\partial\Xi, 0\le\tau\le T$. We consider $\phi(u,w,\tau)$ such that the exact solution of Equation (4.4) is as follows (Tables 2 and 3; Figures 4 and 5)

$$\Psi(u,w,\tau) = \left(2\tau^2 - \tau - w\right)\exp(2u+w),$$

where $\Xi = \Xi_1\cup\Xi_2\cup\Xi_3\cup\Xi_4$ and

$$\Xi_1 = \left\{(u,w)\in\mathbb{R}^2: \quad 0<u<0.25, \quad 0.75-0.75\sqrt{1-\frac{(u-0.5)^2}{0.25}}<w<0.5+0.4\sqrt{1-\frac{u^2}{0.0625}}\right\}$$

TABLE 2 Compare the errors of our proposed method and other methods for example (4.2).

R	L_2-error	RMS-error	L_∞-error
Numerical results using Gaussian RBF			
6	–	$5.0091e-4$	$5.8107e-4$
7	–	$2.5407e-4$	$3.1279e-4$
8	–	$9.8531e-5$	$1.0125e-4$
9	–	$7.5442e-5$	$7.5978e-5$
10	–	–	–
Numerical results using multiquadrics RBF			
6	–	$9.3728e-5$	$1.1215e-4$
7	–	$8.5703e-5$	$9.2271e-5$
8	–	$5.7301e-5$	$6.5317e-5$
9	–	$3.2362e-5$	$3.2832e-5$
10	–	–	–
Numerical results using our method			
6	$7.9e-5$	$1.9e-5$	$3.9e-5$
7	$8.6e-5$	$2.1e-5$	$4.3e-5$
8	$9.6e-5$	$2.3e-5$	$4.6e-5$
9	$9.5e-5$	$2.4e-5$	$4.8e-5$
10	–	$2.5e-5$	$5.1e-5$

TABLE 3 Exact and approximate solutions of examples (4.1) and (4.2).

R	ApproximateSolutions	ExactSolutions
Numerical results for example (4.1)		
1	−0.9797428447	−0.2445680150
2	1.210786774	−0.04126143009
3	3.288801672	0.6776125542
4	3.233305841	−0.4651960087
5	4.211685926	−0.5741286545
Numerical results for example (4.2)		
6	4.120551377	−1.480747753
7	6.731720887	0.04310902677
8	6.362513321	−1.141583089
9	−6.548391730	−14.59614450
10	−5.155084770	−14.01832209

$$\Xi_2 = \left\{ (u,w) \in \mathbb{R}^2 : \quad 0.25 < u < 0.5, \quad 0.75 - 0.75\sqrt{1 - \frac{(u-0.5)^2}{0.25}} < w < 0.5 + 0.5\sqrt{1 - \frac{(u-0.6)^2}{0.1225}} \right\}$$

$$\Xi_3 = \left\{ (u,w) \in \mathbb{R}^2 : \quad 0.5 < u < 0.75, \quad 0.75 - 0.75\sqrt{1 - \frac{(u-0.5)^2}{0.25}} < w < 0.5 + 0.5\sqrt{1 - \frac{(u-0.4)^2}{0.1225}} \right\}$$

$$\Xi_4 = \left\{ (u,w) \in \mathbb{R}^2 : \quad 0.75 < u < 1, \quad 0.75 - 0.75\sqrt{1 - \frac{(u-0.5)^2}{0.25}} < w < 0.5 + 0.4\sqrt{1 - \frac{(u-1)^2}{0.0625}} \right\}.$$

Example 4.3. Consider the following nonlinear 2D ST-FS-G

$$\begin{aligned} \frac{\partial^\alpha \Psi(u,w,\tau)}{\partial \tau^\kappa} + \frac{\partial^{\kappa-1}\Psi(u,w,\tau)}{\partial \tau^{\kappa-1}} &= \frac{\partial^2 \Psi(u,w,\tau)}{\partial u^2} + \frac{\partial^2 \Psi(u,w,\tau)}{\partial w^2} \\ &\quad - \sin(\Psi(u,w,\tau)) + \phi(u,w,\tau) + \frac{dM(\tau)}{d\tau}, \end{aligned} \tag{4.4}$$

where $\kappa = 2$. We consider the function $\phi(u,w,\tau)$ such that $\Psi(u,w,\tau) = -u^2 - w^2 + \tau^2 + (u - w - \tau)\cos(\tau)$, is the exact solution of the equation. We have the initial and boundary conditions as follows

$$\begin{cases} \Psi(u,w,0) = -u^2 - w^2 + u - w, & (u,w) \in \Xi, \\ \left.\frac{\partial \Psi(u,w,\tau)}{\partial \tau}\right|_{\tau=0} = -1, & (u,w) \in \Xi, \end{cases}$$

and where $(u,w) \in \partial\Xi, \;\; 0 \le \tau \le T$, $\Xi = \Xi_1 \cup \Xi_2 \cup \Xi_3 \cup \Xi_4$ and

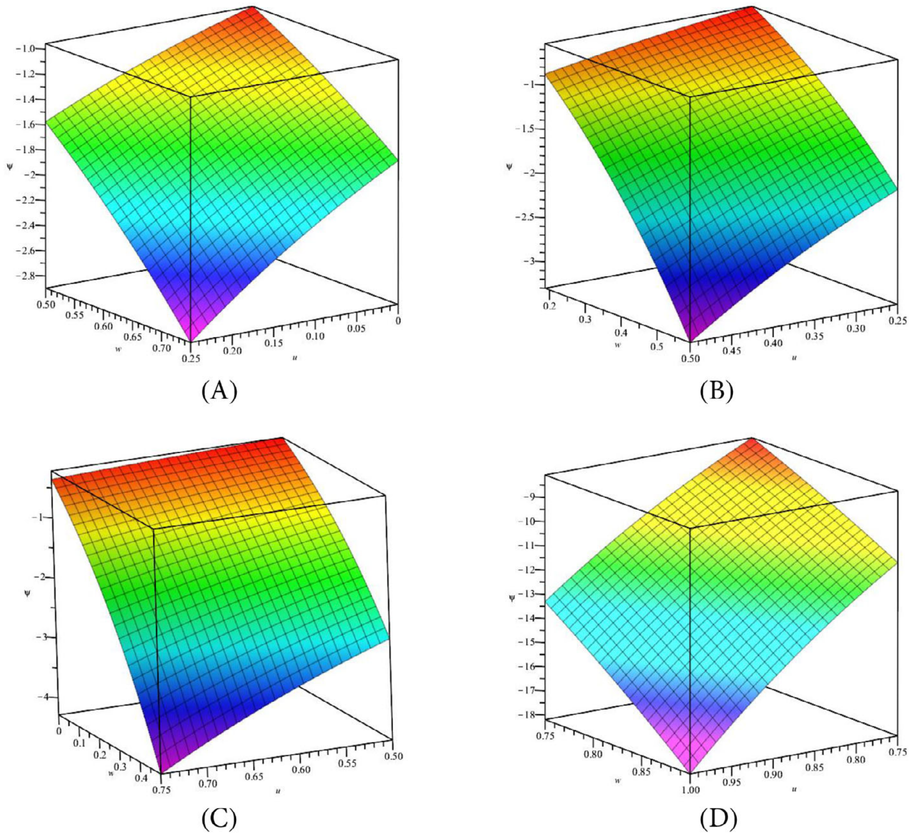

FIGURE 4 Exact solution for example (4.2). (A) $\tau = 0.1, u \in (0, 0.25), w \in (0.75, 0.5)$, (B) $\tau = 0.1, u \in (0.25, 0.5), w \in (-0.1875, 0.5918)$, (C) $\tau = 0.1, u \in (0.5, 0.75), w \in (0, 0.5)$, and (D) $\tau = 0.14, u \in (0.75, 1), w \in (0.75, 0.9)$.

$$\Xi_1 = \left\{ (u,w) \in \mathbb{R}^2 : \quad 0 < u < \frac{1}{6}, \quad -\frac{1}{3}u + \frac{1}{6} < w < \frac{1}{3}u + \frac{1}{6} \right\}$$
$$\Xi_2 = \left\{ (u,w) \in \mathbb{R}^2 : \quad \frac{1}{6} < u < \frac{1}{3}, \quad -3u + 6 < w < 3u + 2 \right\}$$
$$\Xi_3 = \left\{ (u,w) \in \mathbb{R}^2 : \quad \frac{1}{3} < u < \frac{1}{2}, \quad 3u - 6 < w < -3u + 4 \right\}$$
$$\Xi_4 = \left\{ (u,w) \in \mathbb{R}^2 : \quad \frac{1}{2} < u < 1, \quad \frac{1}{6}u + \frac{1}{6} < w < -\frac{1}{6}u + \frac{2}{3} \right\}.$$

In the following, we present the numerical results obtained for Equation (4.4) in Table 4. In this table, the exact solutions and approximate solutions obtained from the presented method are given. Also, the comparison of these solutions is shown graphically in Figure 6.

Example 4.4. Consider the following nonlinear 2D ST-FS-G

$$\frac{\partial^{\alpha}\Psi(u,w,\tau)}{\partial\tau^{\kappa}} + \frac{\partial^{\kappa-1}\Psi(u,w,\tau)}{\partial\tau^{\kappa-1}} = \frac{\partial^2\Psi(u,w,\tau)}{\partial u^2} - \frac{\partial^2\Psi(u,w,\tau)}{\partial w^2} - \sin(\Psi(u,w,\tau)) + \phi(u,w,\tau) + \pi\frac{dM(\tau)}{d\tau}, \tag{4.5}$$

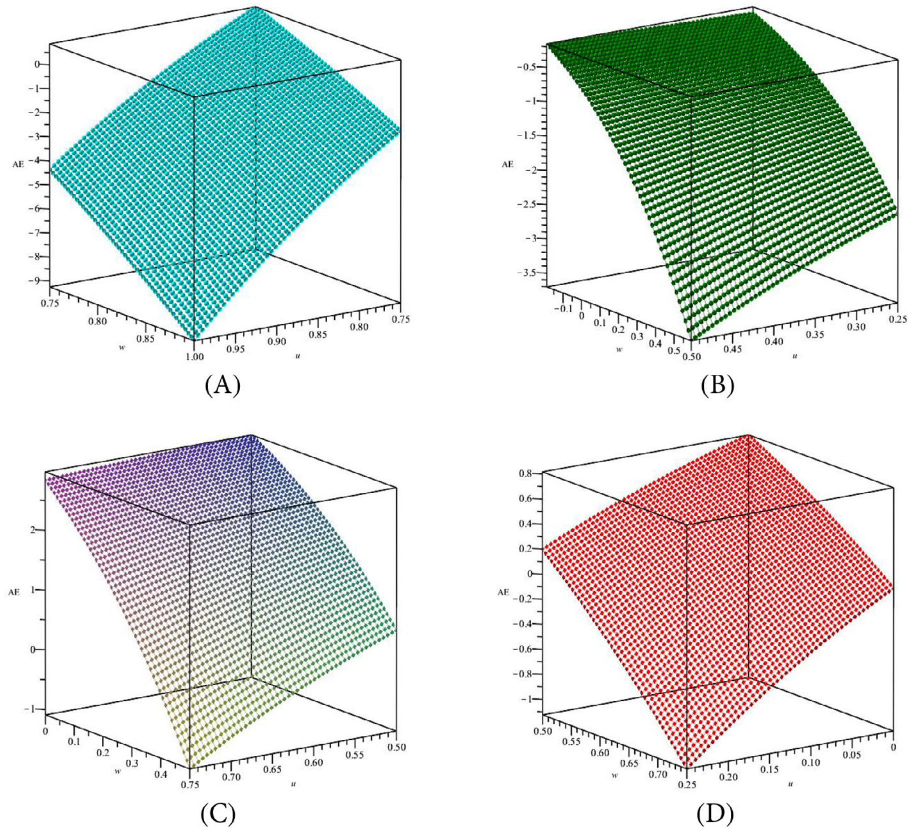

FIGURE 5 Absolute error for example (4.2). (A) $\tau = 0.1, u \in (0, 0.25), w \in (0.75, 0.5)$, (B) $\tau = 0.1, u \in (0.25, 0.5), w \in (-0.1875, 0.5918)$, (C) $\tau = 0.1, u \in (0.5, 0.75), w \in (0, 0.5)$, and (D) $\tau = 0.14, u \in (0.75, 1), w \in (0.75, 0.9)$.

TABLE 4 Numerical results for example (4.3).

R	ApproximateSolutions	ExactSolutions
1	−0.6056681223	−0.6136571156
2	−0.8800356855	−0.9700367267
3	−0.2913271607	−0.3110146471
4	−0.4623267413	−0.2627466525
5	−0.3354488719	−0.2312488719
6	−0.3927865112	−0.3475555741
7	−0.9455632147	−1.444063416
8	−2.874625172	−4.576032249
9	−1.126851245	−1.376880030
10	−1.864202633	−2.065604301

where $\kappa = 2$. We consider the function $\phi(u,w,\tau)$ such that $\Psi(u,w,\tau) = \frac{1}{4}\left(\sin\left(\frac{\pi}{2}(u+w+\tau)\right) + \sin\left(\frac{\pi}{2}(u+w-\tau)\right)\right)$, is the exact solution of the equation. We have the initial and boundary conditions as follows

$$\begin{cases} \Psi(u,w,0) = \dfrac{1}{2}\left(\sin\left(\dfrac{\pi}{2}(u+w)\right)\right), & (u,w) \in \Xi, \\ \left.\frac{\partial \Psi(u,w,\tau)}{\partial \tau}\right|_{\tau=0} = 0, & (u,w) \in \Xi, \end{cases}$$

and where $(u,w) \in \partial\Xi$, $0 \le \tau \le T$, $\Xi = \Xi_1 \cup \Xi_2 \cup \Xi_3 \cup \Xi_4$ and

$$\Xi_1 = \left\{(u,w) \in \mathbb{R}^2: \quad 0 < u < 0.15, \quad 0.35 - 0.35\left(0.15 - \sqrt{1-u^2}\right) < w < 0.75 + 0.55\left(1 - \sqrt{1-u^2}\right)\right\}$$
$$\Xi_2 = \left\{(u,w) \in \mathbb{R}^2: \quad 0.15 < u < 0.35, \quad 0.35 - 0.35\left(0.15 - \sqrt{1-u^2}\right) < w < 0.75 + 0.75\left(1 - \sqrt{u^2}\right)\right\}$$
$$\Xi_3 = \left\{(u,w) \in \mathbb{R}^2: \quad 0.35 < u < 0.55, \quad 0.35 - 0.35\left(0.15 - \sqrt{1-u^2}\right) < w < 0.75 + 0.75\left(1 - \sqrt{u^2}\right)\right\}$$
$$\Xi_4 = \left\{(u,w) \in \mathbb{R}^2: \quad 0.55 < u < 1, \quad 0.35 - 0.35\left(0.15 - \sqrt{1-u^2}\right) < w < 0.75 + 0.55\left(1 - \sqrt{1-u^2}\right)\right\}.$$

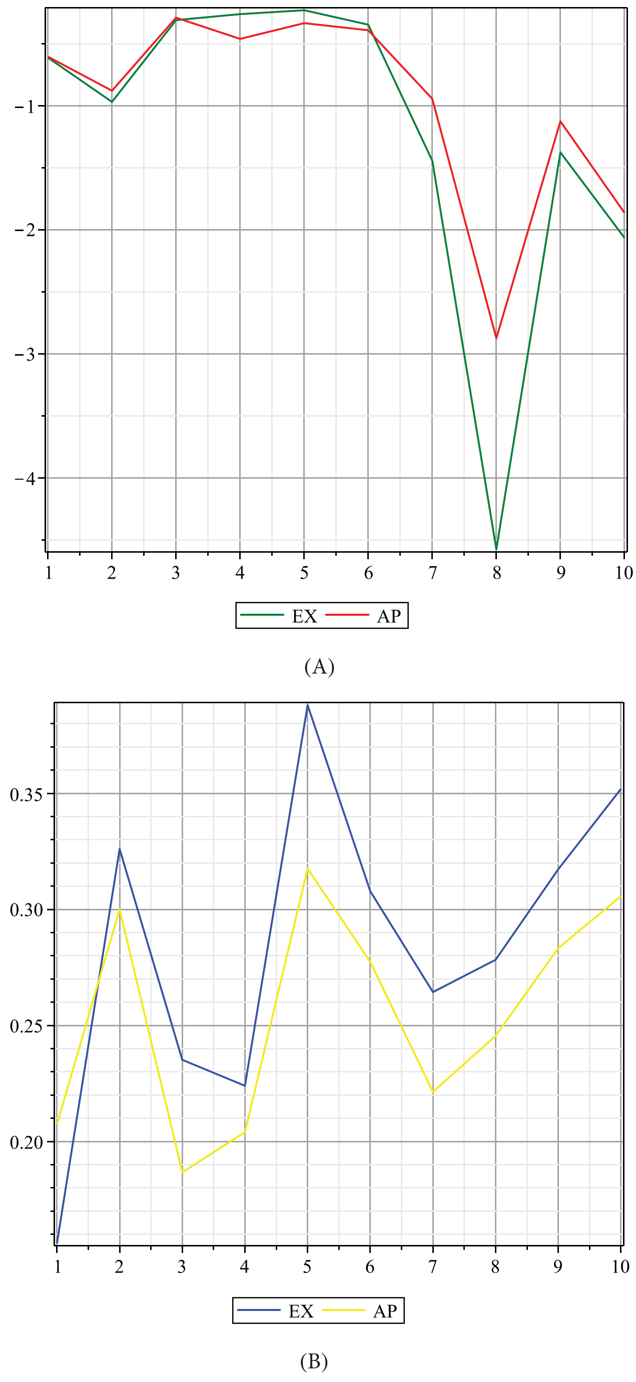

FIGURE 6 Comparison of exact solutions and approximate solutions for examples (4.3), (4.4) for different values of u, w in the given domains and for $\tau = 0.2, 0.3$.

TABLE 5 Numerical results for example (4.4).

R	ApproximateSolutions	ExactSolutions
1	0.15734821368	0.2461101360
2	0.3000472826	0.3260483918
3	0.1867232491	0.2351866150
4	0.2040013769	0.2240029381
5	0.3175440995	0.3880140310
6	0.2773576314	0.3078128528
7	0.2213675422	0.2643731790
8	0.2455000038	0.2783018098
9	0.2833147637	0.3173109120
10	0.3057133399	0.3519093811

In the following, we present the numerical results obtained for Equation (4.5) in Table 5. In this table, the exact solutions and approximate solutions obtained from the presented method are given. Also, the comparison of these solutions is shown graphically in Figure 6.

5 | CONCLUSION

Using the new clique polynomial method, we have studied the two-dimensional stochastic time-fractional Sine–Gordon equation (2D ST-FS-G) and provided a numerical algorithm for solving it. By converting the 2D ST-FS-G equation to algebraic equations, an approximate solution is derived for the desired equation. We also have presented the error boundary for the obtained approximation of the desired three-variable function based on the clique polynomial. At the end, we demonstrated some numerical examples using our techniques to indicate the efficiency of our algorithm.

AUTHOR CONTRIBUTIONS

Zahra Eidinejad, Reza Saadati, Javad Vahidi, and Chenkuan Li conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

FUNDING INFORMATION

Chenkuan Li was supported by the Natural Sciences and Engineering Research Council of Canada (Grant No. 2019-03907).

CONFLICT OF INTEREST STATEMENT

The authors declare that they have no competing interests.

DATA AVAILABILITY STATEMENT

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

ORCID

Zahra Eidinejad https://orcid.org/0000-0003-2517-3365
Reza Saadati https://orcid.org/0000-0002-6770-6951
Javad Vahidi https://orcid.org/0000-0001-9985-170X
Chenkuan Li https://orcid.org/0000-0001-7098-8059

REFERENCES

1. Wang L, Ma Y, Meng Z. Haar wavelet method for solving fractional partial differential equations numerically. *Appl Math Comput*. 2014; 227:66-76.

2. Mainardi F. *Fractional Calculus and Waves in Linear Viscoelasticity. An Introduction to Mathematical Models*. Imperial College Press; 2010.
3. Otarola E. A piecewise linear FEM for an optimal control problem of fractional operators: error analysis on curved domains. *ESAIM Math Model Numer Anal*. 2017;51(4):1473-1500.
4. Zayernouri M, Karniadakis GE. Discontinuous spectral element methods for time- and space-fractional advection equations. *SIAM J Sci Comput*. 2014;36(4):B684-B707.
5. Zeng F, Zhang Z, Karniadakis GE. A generalized spectral collocation method with tunable accuracy for variable-order fractional differential equations. *SIAM J Sci Comput*. 2015;37(6):A2710-A2732.
6. Meerschaert MM, Tadjeran C. Finite difference approximations for two-sided space-fractional partial differential equations. *Appl Numer Math*. 2006;56(1):80-90.
7. Dodd RK, Eilbeck JC, Gibbon JD, Morris HC. *Solitons and Nonlinear Wave Equations*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers]; 1982.
8. Mirzaee F, Rezaei S, Samadyar N. Numerical solution of two-dimensional stochastic time-fractional Sine–Gordon equation on non-rectangular domains using finite difference and meshfree methods. *Eng Anal Bound Elem*. 2021;127:53-63.
9. Kaya D. A numerical solution of the Sine-Gordon equation using the modified decomposition method. *Appl Math Comput*. 2003;143(2–3):309-317.
10. Wei GW. Discrete singular convolution for the Sine-Gordon equation. *Phys D*. 2000;137(3–4):247-259.
11. Wazwaz AM. Exact solutions for the generalized Sine-Gordon and the generalized sinh-Gordon equations. *Chaos, Solitons Fractals*. 2006;28(1):127-135.
12. Liu S, Fu Z, Liu S. Exact solutions to Sine-Gordon-type equations. *Phys Lett A*. 2006;351(1–2):59-63.
13. Isojima S, Murata M, Nobe A, Satsuma J. An ultradiscretization of the Sine-Gordon equation. *Phys Lett A*. 2004;331(6):378-386.
14. Argyris J, Haase M, Heinrich JC. Finite element approximation to two-dimensional Sine-Gordon solitons. *Comput Methods Appl Mech Eng*. 1991;86(1):1-26.
15. Bratsos AG. A third order numerical scheme for the two-dimensional Sine-Gordon equation. *Math Comput Simul*. 2007;76(4):271-282.
16. Li X, Zhang S, Wang Y, Chen H. Analysis and application of the element-free Galerkin method for nonlinear Sine-Gordon and generalized sinh-Gordon equations. *Comput Math Appl*. 2016;71(8):1655-1678.
17. Yin F, Tian T, Song J, Zhu M. Spectral methods using Legendre wavelets for nonlinear Klein\Sine-Gordon equations. *J Comput Appl Math*. 2015;275:321-334.
18. Shan Y, Liu W, Wu B. Space-time Legendre-gauss-Lobatto collocation method for two-dimensional generalized Sine-Gordon equation. *Appl Numer Math*. 2017;122:92-107.
19. Dehghan M, Abbaszadeh M, Mohebbi A. An implicit RBF meshless approach for solving the time fractional nonlinear Sine-Gordon and Klein-Gordon equations. *Eng Anal Bound Elem*. 2015;50:412-434.
20. Dehghan M, Abbaszadeh M. Two meshless procedures: moving kriging interpolation and element-free Galerkin for fractional PDEs. *Appl Anal*. 2017;96(6):936-969.
21. Ahmadinia M, Safari Z. Analysis of local discontinuous Galerkin method for time-space fractional Sine-Gordon equations. *Appl Numer Math*. 2020;148:1-17.
22. Robler A, Seaid M, Zahri M. Method of lines for stochastic boundary-value problems with additive noise. *Appl Math Comput*. 2008;199(1):301-314.
23. Cao Y, Yang H, Yin L. Finite element methods for semilinear elliptic stochastic partial differential equations. *Numer Math*. 2007;106(2):181-198.
24. Jentzen A, Kloeden PE. Taylor approximations for stochastic partial differential equations. *CBMS-NSF Regional Conference Series in Applied Mathematics* (Vol. 83). Society for Industrial and Applied Mathematics (SIAM); 2011.
25. Fasshauer GE, Ye Q. Kernel-based collocation methods versus Galerkin finite element methods for approximating elliptic stochastic partial differential equations. *Meshfree methods for partial differential equations VI, Lecture Notes in Computer Science*. Vol 89. Springer; 2013:155-170.
26. Cialenco I, Fasshauer GE, Ye Q. Approximation of stochastic partial differential equations by a kernel-based collocation method. *Int J Comput Math*. 2012;89(18):2543-2561.
27. Zagrodzinski J. Particular solutions of the Sine-Gordon equation in $2+1$ dimensions. *Phys Lett A*. 1979;72(4–5):284-286.
28. Hirota R. Exact three-soliton solution of the two-dimensional Sine-Gordon equation. *J Phys Soc Jpn*. 1973;35(5):1566.
29. Samadyar N, Mirzaee F. Numerical solution of two-dimensional weakly singular stochastic integral equations on non-rectangular domains via radial basis functions. *Eng Anal Bound Elem*. 2019;101:27-36.
30. Mirzaee F, Samadyar N. On the numerical solution of fractional stochastic integro-differential equations via meshless discrete collocation method based on radial basis functions. *Eng Anal Bound Elem*. 2019;100:246-255.
31. Md Nasrudin FS, Phang C. Numerical solution via operational matrix for solving Prabhakar fractional differential equations. *J Undergrad Math*. 2022;2022:7220433.
32. Owoyemi AE, Phang C, Toh YT. An efficient numerical scheme for solving multiorder tempered fractional differential equations via operational matrix. *J Undergrad Math*. 2022;2022:7628592.
33. Mirzaee F, Solhi E, Samadyar N. Moving least squares and spectral collocation method to approximate the solution of stochastic Volterra-Fredholm integral equations. *Appl Numer Math*. 2021;161:275-285.

34. Hoede C. Hard graphs for the maximum clique problem. Proceedings of the first Japan conference on graph theory and applications (Hakone, 1986). *Discret Math*. 1988;72(1–3):175-179.
35. Wu Q. *The maximum clique problems with applications to graph coloring* (PhD thesis). Universiti d'Angers. 2013.
36. Ganji RM, Jafari H, Adem AR. A numerical scheme to solve variable order diffusion-wave equations. *Therm Sci*. 2019;23(6):2063-2071.
37. Tuan NH, Ganji RM, Jafari H. A numerical study of fractional rheological models and fractional Newell–Whitehead–Segel equation with non-local and non-singular kernel. *Chin J Phys*. 2020;68(5):308-320.
38. Shah R, Hassan K, Kumam P, Muhammad A. An analytical technique to solve the system of nonlinear fractional partial differential equations. *Mathematics*. 2019;7(6):505.
39. Loh JR, Isah A, Phang C, Toh YT. On the new properties of Caputo-Fabrizio operator and its application in deriving shifted Legendre operational matrix. *Appl Numer Math*. 2018;132(4):138-153.
40. Podlubny I. Fractional differential equations. An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications. *Mathematics in Science and Engineering*. Academic Press, Inc.; 1999:198.
41. Hoede C, Li XL. Clique polynomials and independent set polynomials of graphs. 13th British combinatorial conference (Guildford, 1991). *Discret Math*. 1994;125(1–3):219-228.
42. Kumbinarasaiah S, Ramane HS, Pise KS, Hariharan G. Numerical-solution-for-nonlinear-Klein-Gordon equation via operational-matrix by clique polynomial of complete graphs. *Int J Appl Comput Math*. 2021;7(1):Paper No. 12.

How to cite this article: Eidinejad Z, Saadati R, Vahidi J, Li C. Numerical solutions of 2D stochastic time-fractional Sine–Gordon equation in the Caputo sense. *Int J Numer Model*. 2023;e3121. doi:10.1002/jnm.3121