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RESEARCH ARTICLE

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A nonlinear fractional partial integro-differential equation with nonlocal initial value conditions

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In this work, we study a new nonlinear partial integro-differential equation with nonlocal initial value conditions and investigate the solutions of this equation. By considering an equivalent implicit integral equation via series, we prove the uniqueness of solutions of the equation by Babenko's approach, Banach's contraction principle, and the multivariable Mittag–Leffler function. We also demonstrate the application of our key theorem with an illustrative example.

KEYWORDS

Babenko's approach, Banach's contractive principle, multivariate Mittag–Leffler function, nonlinear partial integro-differential equation

MSC CLASSIFICATION

35A02, 35C15, 45E10, 26A33

1 | INTRODUCTION AND PRELIMINARIES

Let $b > 0$ and $c_0, c_1, c_2 \in C[0, b]$. We consider the uniqueness of solutions for the following NPDIE for $2 < \varsigma \leq 3$, $0 < \varepsilon \leq 1$, and $\upsilon > 0$:

$$\begin{cases} \frac{{}_c\partial^{\varsigma}}{\partial\theta^{\varsigma}}\Phi(\theta,\sigma) + c_0(\sigma)\frac{{}_c\partial^{\varepsilon}}{\partial\theta^{\varepsilon}}\Phi(\theta,\sigma) + c_1(\sigma)\Phi(\theta,\sigma) + c_2(\sigma)I_{\sigma}^{\upsilon}\Phi(\theta,\sigma) = (\theta,\sigma,\Phi(\theta,\sigma)), \\ \Phi(0,\sigma) = \eta\int_0^1 \Phi(\theta,\sigma)d\theta, \ \ \frac{\partial}{\partial\theta}\Phi(0,\sigma) = \int_0^1 \psi(\theta)\Phi(\theta,\sigma)d\theta, \ \Phi_{\theta}''(0,\sigma) = 0, \end{cases} \tag{1.1}$$

where $(\theta, \sigma) \in [0, 1] \times [0, b]$, η is a constant, $\psi \in C[0, 1]$, and $: \ [0, 1] \times [0, b] \times R \rightarrow R$ satisfies certain conditions to be given later. Assume that $\upsilon \in R^{+}$ and $\sigma \in [0, b]$, the operator I_{σ}^{υ} is the partial Riemann–Liouville fractional integral of order

υ w.r.t σ with initial point zero (see [1]),

$$(I_\sigma^\upsilon \Phi)(\theta,\sigma) = \frac{1}{\Gamma(\upsilon)}\int_0^\sigma (\sigma-\zeta)^{\upsilon-1}\Phi(\theta,\zeta)d\zeta,$$

and $\frac{{}_c\partial^\varsigma}{\partial\theta^\varsigma}$ is the partial Liouville–Caputo fractional derivative of order ς with respect to $\theta \in [0,1]$

$$\left(\frac{{}_c\partial^\varsigma}{\partial\theta^\varsigma}\Phi\right)(\theta,\sigma) = \frac{1}{\Gamma(3-\varsigma)}\int_0^\theta (\theta-s)^{2-\varsigma}\Phi_s^{(3)}(s,\sigma)ds, \ \ 2<\varsigma\leq 3,$$

and similarly,

$$\left(\frac{{}_c\partial^\varepsilon}{\partial\theta^\varepsilon}\Phi\right)(\theta,\sigma) = \frac{1}{\Gamma(1-\varepsilon)}\int_0^\theta (\theta-s)^{-\varepsilon}\Phi_s'(s,\sigma)ds, \quad 0<\varepsilon\leq 1.$$

A function Φ is said to be a solution of Equation (1.1) if it satisfies the equation and its three initial conditions.

If $\psi(\theta)=\eta$ for all $\theta\in[0,1]$, then Equation (1.1) is

$$\begin{cases} \frac{{}_c\partial^\varsigma}{\partial\theta^\varsigma}\Phi(\theta,\sigma) + c_0(\sigma)\frac{{}_c\partial^\varepsilon}{\partial\theta^\varepsilon}\Phi(\theta,\sigma) + c_1(\sigma)\Phi(\theta,\sigma) + c_2(\sigma)I_\sigma^\upsilon\Phi(\theta,\sigma) = F(\theta,\sigma,\Phi(\theta,\sigma)), \\ \Phi(0,\sigma) = \frac{\partial}{\partial\theta}\Phi(0,\sigma) = \eta\int_0^1 \Phi(\theta,\sigma)d\theta, \ \Phi_t''(0,\sigma)=0. \end{cases}$$

It follows from [2] that

$$\left(I_\sigma^0\Phi\right)(\theta,\sigma) = \Phi(\theta,\sigma).$$

It is well known that fractional partial differential equations have been used in many fields such as quantum mechanics, fluid dynamics, and plasma physics [3]. In 2016, Petitta [4] considered the following boundary value problem and obtained solutions for this equation and investigated the uniqueness and regularity of the solutions:

$$\begin{cases} \mathcal{L}^s\Phi = \mu \text{ in } \Omega,\ 0<s<1, \\ \Phi(\sigma)=0, \ \text{ on } \ R^n\setminus\Omega, \end{cases}$$

where Ω is a bounded domain of R^n, μ is a bounded Radon measure on Ω, and $\mathcal{L}^s$ is given as

$$\mathcal{L}^s\Phi(\sigma) = PV\int_{R^n}(\Phi(\sigma)-\Phi(\sigma+y))K(y)dy,$$

and K is a non-negative kernel satisfying

$$\frac{\lambda}{|y|^{n+2s}} \leq K(y) \leq \frac{\Lambda}{|y|^{n+2s}}, \ \ 0<\lambda\leq\Lambda.$$

This class of kernels contains the fractional Laplacian $(-\triangle)^s$ as a particular case, that is,

$$(-\triangle)^s\Phi(\sigma) := c_{n,s}PV\int_{R^n}\frac{\Phi(\sigma)-\Phi(\sigma+y)}{|y|^{n+2s}}dy,$$

where

$$c_{n,s} = \frac{4^s s(s-1)\Gamma\left(\frac{n+2s}{2}\right)}{\pi^{n/2}\Gamma(2-s)}.$$

In [5], the author extended the fractional Laplacian $(-\triangle)^s\Phi$ over a space to all values of $s>0$ and $s\neq 1,2,\cdots$, based on the normalization in distribution theory (in the sense of Schwartz), Pizzetti's formula, and surface integrals in R^n. Recently, Mahor et al. [6] investigated several fractional equations and obtained analytical solutions using the fractional

Fourier transform and MLF for these equations. The fractional heat-diffusion equation is one of these equations, which is considered as follows:

$$\begin{cases} \frac{\partial^\varsigma}{\partial\theta^\varsigma}\phi(\sigma,\theta) = h\frac{\partial^{2\varsigma}}{\partial\theta^{2\varsigma}}\phi(\sigma,\theta),\ \sigma\in R,\ \theta\in R^+,\ 0<\varsigma<1, \\ \phi(\sigma,0)=\psi(\sigma), \end{cases}$$

where $\phi(\sigma,\theta)$ is the temperature function and $h\in R^+$ is the thermal conductivity.

In the following, we introduce the multivariable Mittag–Leffler function (MMLF) and Babenko's approach (B-A); we solve the fractional kinetic equation in detail. In this paper, we consider the Banach space $C([0,1]\times[0,b])$ with the norm

$$\|\Phi\| = \sup_{\theta\in[0,1],\ \sigma\in[0,b]} |\Phi(\theta,\sigma)| \ \text{ for } \ \Phi\in C([0,1]\times[0,b]).$$

Definition 1 ([7]). The two-parameter Mittag–Leffler (M-L) function is represented by the following series:

$$E_{\varsigma,\varepsilon}(z) = \sum_{h=0}^{\infty}\frac{z^h}{\Gamma(\varsigma h+\varepsilon)},$$

where $\varsigma,\varepsilon>0$ and $z\in C$ (the complex plane). The MMLF is defined as follows:

$$E_{(\varsigma_1,\dots,\varsigma_m),\varepsilon}(z_1,\cdots,z_m) = \sum_{h=0}^{\infty}\sum_{\substack{h_1+\dots+h_m=h\\ h_1\geq 0,\dots,h_m\geq 0}}\binom{h}{h_1,\cdots,h_m}\frac{z_1^{h_1}\cdots z_m^{h_m}}{\Gamma(\varsigma_1h_1+\cdots+\varsigma_mh_m+\varepsilon)},$$

where $\varsigma_i,\varepsilon>0$, for $i=1,2,\cdots,m$ and

$$\binom{h}{h_1,\cdots,h_m} = \frac{h!}{h_1!\cdots h_m!}.$$

One of the useful tools for dealing with differential equations with initial conditions as well as integral equations is B-A. This method is generally the same as the Laplace transform, while working on equations with constant coefficients. However, this method is widely used for differential and integral equations with continuous variable coefficients and boundary value problems [8]. In the following, to show some details of this approach, we will deduce the solution to the following fractional kinetic equation investigated by Saxena and Kalla in [9]

$$N(\theta) - N_0\psi(\theta) = -c^\nu I_t^\nu N(\theta),\ \ \nu>0, \tag{1.2}$$

where $N(\theta)$ denotes the number density of a given species at time θ, $N_0 = N(0)$ is the number density of that species at time $\theta=0$ and c is a positive constant.

Clearly, Equation (1.2) can be written as

$$\left(1+c^\nu I_\theta^\nu\right)N(\theta) = N_0\psi(\theta).$$

Treating the factor $\left(1+c^\nu I_\theta^\nu\right)$ as a variable, we derive from B-A

$$\begin{aligned} N(\theta) &= N_0\left(1+c^\nu I_\theta^\nu\right)^{-1}\psi(\theta) = N_0\sum_{h=0}^{\infty}(-1)^h(c^\nu I_\theta^\nu)^h\psi(\theta) \\ &= N_0\sum_{h=0}^{\infty}(-1)^h c^{h\nu} I_\theta^{h\nu}\psi(\theta) \\ &= N_0\psi(\theta) + N_0\sum_{h=1}^{\infty}(-1)^h c^{h\nu}\frac{1}{\Gamma(h\nu)}\int_0^\theta(\theta-\zeta)^{h\nu-1}\psi(\zeta)d\zeta \end{aligned}$$

$$= N_0\psi(\theta) + N_0 \sum_{h=0}^{\infty} (-1)^{h+1} c^{h\nu+\nu} \frac{1}{\Gamma(h\nu+\nu)} \int_0^{\theta} (\theta-\zeta)^{h\nu+\nu-1}\psi(\zeta)d\zeta$$
$$= N_0\psi(\theta) - N_0 c^{\nu} \int_0^{\theta} (\theta-\zeta)^{\nu-1} \sum_{h=0}^{\infty} (-1)^h c^{h\nu} \frac{(\theta-\zeta)^{h\nu}}{\Gamma(h\nu+\nu)}\psi(\zeta)d\zeta$$
$$= N_0\psi(\theta) - N_0 c^{\nu} \int_0^{\theta} (\theta-\zeta)^{\nu-1} E_{\nu,\nu}\left(-c^{\nu}(\theta-\zeta)^{\nu}\right)\psi(\zeta)d\zeta,$$

which is well defined. We should note that this is indeed Abel's equation of the second kind. There are some recent studies on several general classes of fractional-order kinetic equations using various approaches [6, 10–12]. In addition, researchers have investigated the existence and uniqueness for various types of fractional integro-differential equations, with boundary conditions and the Hilfer derivative, via different methods [13–16].

The main goal of this paper is to transform Equation (1.1) into an equivalent implicit integral equation in the series using B-A, which is derived in Section 2. Also, with the help of Banach's contraction principle (BCP), the uniqueness of the solutions in $C([0,1]\times[0,b])$ is studied. In Section 3, we present the application of proven results with an example, and finally, we summarize the entire work in Section 4.

2 | MAIN RESULTS

Theorem 2. *Let $b > 0$, $2 < \varsigma \le 3$, $0 < \varepsilon \le 1$, $\upsilon > 0$, η be a constant and $c_0, c_1, c_2, \psi \in C[0,b]$. We assume $F : [0,1]\times[0,b]\times R \to R$ is a bounded and continuous function, and*

$$2\left(|\eta| + m_0|\eta| + M\right) E_{(\varsigma-\varepsilon,\varsigma,\varsigma),1}\left(m_0, m_1, m_2 b^{\upsilon}\right) < 1,$$

where

$$m_i = \max_{\sigma\in[0,b]} |c_i(\sigma)|, \text{ and } M = \max_{\sigma\in[0,b]} |\psi(\sigma)|,$$

for $i = 0, 1, 2$. Then Φ is a solution of Equation (1.1) if and only if it satisfies the following implicit integral equation in the space $C([0,1]\times[0,b])$:

$$\begin{aligned}
\Phi(\theta,\sigma) = & \sum_{h=0}^{\infty} (-1)^h \sum_{\substack{h_1+h_2+h_3=h \\ h_1\ge 0,\, h_2\ge 0,\, h_3\ge 0}} \binom{h}{h_1, h_2, h_3} c_0^{h_1}(\sigma) c_1^{h_2}(\sigma) \\
& \cdot I_t^{(\varsigma-\varepsilon)h_1+\varsigma h_2+\varsigma h_3+\varsigma} (c_2(\sigma) I_\sigma^{\upsilon})^{h_3} F(\theta,\sigma,\Phi(\theta,\sigma)) \\
& + \sum_{h=0}^{\infty} (-1)^h \sum_{\substack{h_1+h_2+h_3=h \\ h_1\ge 0,\, h_2\ge 0,\, h_3\ge 0}} \binom{h}{h_1, h_2, h_3} c_0^{h_1}(\sigma) c_1^{h_2}(\sigma) \\
& \cdot \frac{\theta^{(\varsigma-\varepsilon)h_1+\varsigma h_2+\varsigma h_3}}{\Gamma((\varsigma-\varepsilon)h_1+\varsigma h_2+\varsigma h_3+1)} \eta \int_0^1 (c_2(\sigma) I_\sigma^{\upsilon})^{h_3} \Phi(\theta,\sigma) d\theta \\
& + \sum_{h=0}^{\infty} (-1)^h \sum_{\substack{h_1+h_2+h_3=h \\ h_1\ge 0,\, h_2\ge 0,\, h_3\ge 0}} \binom{h}{h_1, h_2, h_3} c_0^{h_1}(\sigma) c_1^{h_2}(\sigma) \\
& \cdot \frac{\theta^{(\varsigma-\varepsilon)h_1+\varsigma h_2+\varsigma h_3+\varsigma-\varepsilon}}{\Gamma((\varsigma-\beta)h_1+\varsigma h_2+\varsigma h_3+\varsigma-\beta+1)} \eta \int_0^1 (c_2(\sigma) I_\sigma^{\upsilon})^{h_3} c_0(\sigma)\Phi(\theta,\sigma) d\theta \\
& + \sum_{h=0}^{\infty} (-1)^h \sum_{\substack{h_1+h_2+h_3=h \\ h_1\ge 0,\, h_2\ge 0,\, h_3\ge 0}} \binom{h}{h_1, h_2, h_3} c_0^{h_1}(\sigma) c_1^{h_2}(\sigma) \\
& \cdot \frac{\theta^{(\varsigma-\varepsilon)h_1+\varsigma h_2+\varsigma h_3+1}}{\Gamma((\varsigma-\varepsilon)h_1+\varsigma h_2+\varsigma h_3+2)} \int_0^1 \psi(\theta)(c_2(\sigma) I_\sigma^{\upsilon})^{h_3} \Phi(\theta,\sigma) d\theta.
\end{aligned} \tag{2.1}$$

Proof. It can be clearly seen that

$$I_\theta^\varsigma \frac{{}_c\partial^\varsigma}{\partial\theta^\varsigma}\Phi(\theta,\sigma) = \Phi(\theta,\sigma) - \Phi(0,\sigma) - \Phi_t'(0,\sigma)\theta - \frac{\Phi_t''(0,\sigma)}{2!}\theta^2$$
$$= \Phi(\theta,\sigma) - \eta\int_0^1 \Phi(\theta,\sigma)d\theta - \int_0^1 \psi(\theta)\Phi(\theta,\sigma)d\theta\ \theta,$$

and since $0 < \varepsilon \leq 1$

$$I_\theta^\varsigma\ c_0(\sigma)\frac{{}_c\partial^\varepsilon}{\partial\theta^\varepsilon}\Phi(\theta,\sigma) = c_0(\sigma)I_\theta^{\varsigma-\beta}I_\theta^\varepsilon\frac{{}_c\partial^\varepsilon}{\partial\theta^\beta}\Phi(\theta,\sigma)$$
$$= c_0(\sigma)I_\theta^{\varsigma-\varepsilon}\left(\Phi(\theta,\sigma) - \Phi(0,\sigma)\right)$$
$$= c_0(\sigma)I_\theta^{\varsigma-\varepsilon}\Phi(\theta,\sigma) - \frac{c_0(\sigma)\theta^{\varsigma-\beta}\eta}{\Gamma(\varsigma-\beta+1)}\int_0^1 \Phi(\theta,\sigma)d\theta.$$

Applying the operator I_θ^ς to the equation

$$\frac{{}_c\partial^\varsigma}{\partial\theta^\varsigma}\Phi(\theta,\sigma) + c_0(\sigma)\frac{{}_c\partial^\varepsilon}{\partial\theta^\varepsilon}\Phi(\theta,\sigma) + c_1(\sigma)\Phi(\theta,\sigma) + c_2(\sigma)I_\sigma^\upsilon\Phi(\theta,\sigma) = F(\theta,\sigma,\Phi(\theta,\sigma)),$$

we get

$$\left(1 + c_0(\sigma)I_\theta^{\varsigma-\varepsilon} + c_1(\sigma)I_\theta^\varsigma + c_2(\sigma)I_\theta^\varsigma I_\sigma^\upsilon\right)\Phi(\theta,\sigma)$$
$$= I_\theta^\varsigma F(\theta,\sigma,\Phi(\theta,\sigma)) + \left(1 + \frac{c_0(\sigma)\theta^{\varsigma-\varepsilon}}{\Gamma(\varsigma-\varepsilon+1)}\right)\eta\int_0^1 \Phi(\theta,\sigma)d\theta + \int_0^1 \psi(\theta)\Phi(\theta,\sigma)d\theta\ \theta.$$

Using B-A, we obtain

$$\Phi(\theta,\sigma) = \left(1 + c_0(\sigma)I_\theta^{\varsigma-\varepsilon} + c_1(\sigma)I_\theta^\varsigma + c_2(\sigma)I_\theta^\varsigma I_\sigma^\upsilon\right)^{-1}$$
$$\cdot\left[I_\theta^\varsigma F(\theta,\sigma,\Phi(\theta,\sigma)) + \left(1 + \frac{c_0(\sigma)\theta^{\varsigma-\varepsilon}}{\Gamma(\varsigma-\varepsilon+1)}\right)\eta\int_0^1 \Phi(\theta,\sigma)d\theta + \int_0^1 \psi(\theta)\Phi(\theta,\sigma)d\theta\ \theta\right]$$
$$= \sum_{h=0}^\infty (-1)^h\left(c_0(\sigma)I_\theta^{\varsigma-\varepsilon} + c_1(\sigma)I_\theta^\varsigma + c_2(\sigma)I_\theta^\varsigma I_\sigma^\upsilon\right)^h$$
$$\cdot\left[I_\theta^\varsigma F(\theta,\sigma,\Phi(\theta,\sigma)) + \left(1 + \frac{c_0(\sigma)\theta^{\varsigma-\varepsilon}}{\Gamma(\varsigma-\varepsilon+1)}\right)\eta\int_0^1 \Phi(\theta,\sigma)d\theta + \int_0^1 \psi(\theta)\Phi(\theta,\sigma)d\theta\ \theta\right]$$
$$= \sum_{h=0}^\infty (-1)^h \sum_{\substack{h_1+h_2+h_3=h\\ h_1\geq 0,\, h_2\geq 0,\, h_3\geq 0}} \binom{h}{h_1,h_2,h_3}\left(c_0(\sigma)I_\theta^{\varsigma-\varepsilon}\right)^{h_1}\left(c_1(\sigma)I_\theta^\varsigma\right)^{h_2}\left(c_2(\sigma)I_\theta^\varsigma I_\sigma^\upsilon\right)^{h_3}$$
$$\cdot\left[I_\theta^\varsigma F(\theta,\sigma,\Phi(\theta,\sigma)) + \left(1 + \frac{c_0(\sigma)\theta^{\varsigma-\varepsilon}}{\Gamma(\varsigma-\varepsilon+1)}\right)\eta\int_0^1 \Phi(\theta,\sigma)d\theta + \int_0^1 \psi(\theta)\Phi(\theta,\sigma)d\theta\ \theta\right].$$

Clearly,

$$\left(c_0(\sigma)I_\theta^{\varsigma-\varepsilon}\right)^{h_1} = c_0(\sigma)I_\theta^{\varsigma-\varepsilon}c_0(\sigma)I_\theta^{\varsigma-\varepsilon}\cdots c_0(\sigma)I_\theta^{\varsigma-\varepsilon} = c_0^{h_1}(\sigma)I_\theta^{(\varsigma-\varepsilon)h_1}$$

and $\left(c_1(\sigma)I_\theta^\varsigma\right)^{h_2} = c_1^{h_2}(\sigma)I_\theta^{\varsigma h_2}$. As for the term $\left(c_2(\sigma)I_\theta^\varsigma I_\sigma^\upsilon\right)^{h_3}$, we have

$$\left(c_2(\sigma)I_\theta^\varsigma I_\sigma^\upsilon\right)^{h_3} = \left(c_2(\sigma)I_\theta^\varsigma I_\sigma^\upsilon\right)\left(c_2(\sigma)I_\theta^\varsigma I_\sigma^\upsilon\right)\cdots\left(c_2(\sigma)I_\theta^\varsigma I_\sigma^\upsilon\right) = I_\theta^{\varsigma h_3}(c_2(\sigma)I_\sigma^\upsilon)^{h_3}.$$

Therefore,

$$\Phi(\theta,\sigma) = \sum_{h=0}^\infty (-1)^h \sum_{\substack{h_1+h_2+h_3=h\\ h_1\geq 0,\, h_2\geq 0,\, h_3\geq 0}} \binom{h}{h_1,h_2,h_3} c_0^{h_1}(\sigma)c_1^{h_2}(\sigma)I_\theta^{(\varsigma-\varepsilon)h_1+\varsigma h_2+\varsigma h_3}(c_2(\sigma)I_\sigma^\upsilon)^{h_3}$$
$$\cdot\left[I_\theta^\varsigma F(\theta,\sigma,\Phi(\theta,\sigma)) + \left(1 + \frac{c_0(\sigma)\theta^{\varsigma-\varepsilon}}{\Gamma(\varsigma-\varepsilon+1)}\right)\eta\int_0^1 \Phi(\theta,\sigma)d\theta + \int_0^1 \psi(\theta)\Phi(\theta,\sigma)d\theta\ \theta\right]$$

$$
\begin{aligned}
&= \sum_{h=0}^{\infty} (-1)^h \sum_{\substack{h_1+h_2+h_3=h \\ h_1\geq 0,\, h_2\geq 0,\, h_3\geq 0}} \binom{h}{h_1, h_2, h_3} c_0^{h_1}(\sigma) c_1^{h_2}(\sigma) \\
&\quad \cdot I_\theta^{(\varsigma-\varepsilon)h_1+\varsigma h_2+\varsigma h_3+\varsigma} (c_2(\sigma) I_\sigma^{\upsilon})^{h_3} F(\theta,\sigma,\Phi(\theta,\sigma)) \\
&\quad + \sum_{h=0}^{\infty} (-1)^h \sum_{\substack{h_1+h_2+h_3=h \\ h_1\geq 0,\, h_2\geq 0,\, h_3\geq 0}} \binom{h}{h_1, h_2, h_3} c_0^{h_1}(\sigma) c_1^{h_2}(\sigma) \\
&\quad \cdot \frac{\theta^{(\varsigma-\varepsilon)h_1+\varsigma h_2+\varsigma h_3}}{\Gamma((\varsigma-\varepsilon)h_1+\varsigma h_2+\varsigma h_3+1)} \eta \int_0^1 (c_2(\sigma) I_\sigma^{\upsilon})^{h_3} \Phi(\theta,\sigma) d\theta \\
&\quad + \sum_{h=0}^{\infty} (-1)^h \sum_{\substack{h_1+h_2+h_3=h \\ h_1\geq 0,\, h_2\geq 0,\, h_3\geq 0}} \binom{h}{h_1, h_2, h_3} c_0^{h_1}(\sigma) c_1^{h_2}(\sigma) \\
&\quad \cdot \frac{I_\theta^{(\varsigma-\varepsilon)h_1+\varsigma h_2+\varsigma h_3} \theta^{\varsigma-\varepsilon}}{\Gamma(\varsigma-\beta+1)} \eta \int_0^1 (c_2(\sigma) I_\sigma^{\upsilon})^{h_3} c_0(\sigma) \Phi(\theta,\sigma) d\theta \\
&\quad + \sum_{h=0}^{\infty} (-1)^h \sum_{\substack{h_1+h_2+h_3=h \\ h_1\geq 0,\, h_2\geq 0,\, h_3\geq 0}} \binom{h}{h_1, h_2, h_3} c_0^{h_1}(\sigma) c_1^{h_2}(\sigma) \\
&\quad \cdot I_\theta^{(\varsigma-\varepsilon)h_1+\varsigma h_2+\varsigma h_3} \theta \int_0^1 \psi(\theta) (c_2(\sigma) I_\sigma^{\upsilon})^{h_3} \Phi(\theta,\sigma) d\theta.
\end{aligned}
$$

Using

$$
\begin{aligned}
I_\theta^{(\varsigma-\varepsilon)h_1+\varsigma h_2+\varsigma h_3} \theta^{\varsigma-\varepsilon} &= \frac{\Gamma(\varsigma-\varepsilon+1)}{\Gamma((\varsigma-\varepsilon)h_1+\varsigma h_2+\varsigma h_3+\varsigma-\beta+1)} \theta^{(\varsigma-\beta)h_1+\varsigma h_2+\varsigma h_3+\varsigma-\beta}, \\
I_\theta^{(\varsigma-\varepsilon)h_1+\varsigma h_2+\varsigma h_3} \theta &= \frac{\theta^{(\varsigma-\varepsilon)h_1+\varsigma h_2+\varsigma h_3+1}}{\Gamma((\varsigma-\beta)h_1+\varsigma h_2+\varsigma h_3+2)},
\end{aligned}
$$

we have

$$
\begin{aligned}
\Phi(\theta,\sigma) &= \sum_{h=0}^{\infty} (-1)^h \sum_{\substack{h_1+h_2+h_3=h \\ h_1\geq 0,\, h_2\geq 0,\, h_3\geq 0}} \binom{h}{h_1, h_2, h_3} c_0^{h_1}(\sigma) c_1^{h_2}(\sigma) \\
&\quad \cdot I_\theta^{(\varsigma-\varepsilon)h_1+\varsigma h_2+\varsigma h_3+\varsigma} (c_2(\sigma) I_\sigma^{\upsilon})^{h_3} F(\theta,\sigma,\Phi(\theta,\sigma)) \\
&\quad + \sum_{h=0}^{\infty} (-1)^h \sum_{\substack{h_1+h_2+h_3=h \\ h_1\geq 0,\, h_2\geq 0,\, h_3\geq 0}} \binom{h}{h_1, h_2, h_3} c_0^{h_1}(\sigma) c_1^{h_2}(\sigma) \\
&\quad \cdot \frac{\theta^{(\varsigma-\varepsilon)h_1+\varsigma h_2+\varsigma h_3}}{\Gamma((\varsigma-\varepsilon)h_1+\varsigma h_2+\varsigma h_3+1)} \eta \int_0^1 (c_2(\sigma) I_\sigma^{\upsilon})^{h_3} \Phi(\theta,\sigma) d\theta \\
&\quad + \sum_{h=0}^{\infty} (-1)^h \sum_{\substack{h_1+h_2+h_3=h \\ h_1\geq 0,\, h_2\geq 0,\, h_3\geq 0}} \binom{h}{h_1, h_2, h_3} c_0^{h_1}(\sigma) c_1^{h_2}(\sigma) \\
&\quad \cdot \frac{\theta^{(\varsigma-\varepsilon)h_1+\varsigma h_2+\varsigma h_3+\varsigma-\varepsilon}}{\Gamma((\varsigma-\beta)h_1+\varsigma h_2+\varsigma h_3+\varsigma-\beta+1)} \eta \int_0^1 (c_2(\sigma) I_\sigma^{\upsilon})^{h_3} c_0(\sigma) \Phi(\theta,\sigma) d\theta \\
&\quad + \sum_{h=0}^{\infty} (-1)^h \sum_{\substack{h_1+h_2+h_3=h \\ h_1\geq 0,\, h_2\geq 0,\, h_3\geq 0}} \binom{h}{h_1, h_2, h_3} c_0^{h_1}(\sigma) c_1^{h_2}(\sigma) \\
&\quad \cdot \frac{\theta^{(\varsigma-\varepsilon)h_1+\varsigma h_2+\varsigma h_3+1}}{\Gamma((\varsigma-\varepsilon)h_1+\varsigma h_2+\varsigma h_3+2)} \int_0^1 \psi(\theta) (c_2(\sigma) I_\sigma^{\upsilon})^{h_3} \Phi(\theta,\sigma) d\theta.
\end{aligned}
$$

The function $\Phi(\theta,\sigma)$ satisfies the initial conditions: First $\Phi(0,\sigma)=\eta\int_0^1\Phi(\theta,\sigma)d\sigma$, which is the term

$$(-1)^0\sum_{\substack{h_1+h_2+h_3=0\\h_1\geq 0,\,h_2\geq 0,\,h_3\geq 0}}\binom{0}{h_1,h_2,h_3}c_0^{h_1}(\sigma)c_1^{h_2}(\sigma)\cdot\frac{\theta^{(\varsigma-\varepsilon)h_1+\varsigma h_2+\varsigma h_3}}{\Gamma((\varsigma-\varepsilon)h_1+\varsigma h_2+\varsigma h_3+1)}\eta\int_0^1(c_2(\sigma)I_\sigma^{\upsilon})^{h_3}\Phi(\theta,\sigma)d\theta,$$

and the rest being zero by setting $\theta=0$. Similarly, $\frac{\partial}{\partial\theta}\Phi(0,\sigma)=\int_0^1\psi(\theta)\Phi(\theta,\sigma)d\theta$, $\Phi''_\theta(0,\sigma)=0$, by noting that $\varsigma-\varepsilon>1$. Hence Φ is a solution of Equation (1.1) if and only if it satisfies Equation (2.1). Next, in $C([0,1]\times[0,b])$, we show the convergence of the right side of Equation (2.1), which is a series. Since $c_0,\,c_1,\,c_2\in C[0,b]$, we let $m_i=\max_{\sigma\in[0,b]}|c_i(\sigma)|$, for $i=0,1,2$. Then,

$$\begin{aligned}\|\Phi\|\leq&\sum_{h=0}^{\infty}\sum_{\substack{h_1+h_2+h_3=h\\h_1\geq 0,\,h_2\geq 0,\,h_3\geq 0}}\binom{h}{h_1,h_2,h_3}m_0^{h_1}m_1^{h_2}m_2^{h_3}\\
&\cdot\frac{b^{\upsilon h_3}}{\Gamma((\varsigma-\varepsilon)h_1+\varsigma h_2+\varsigma h_3+\varsigma+1)\Gamma(\upsilon h_3+1)}\sup_{(\theta,\sigma)\in[0,1]\times[0,b]}|F(\theta,\sigma,\Phi(\theta,\sigma))|\\
&+\sum_{h=0}^{\infty}\sum_{\substack{h_1+h_2+h_3=h\\h_1\geq 0,\,h_2\geq 0,\,h_3\geq 0}}\binom{h}{h_1,h_2,h_3}m_0^{h_1}m_1^{h_2}m_2^{h_3}\\
&\cdot\frac{b^{\upsilon h_3}|\eta|}{\Gamma((\varsigma-\varepsilon)h_1+\varsigma h_2+\varsigma h_3+1)\Gamma(\upsilon h_3+1)}\|\Phi\|\\
&+\sum_{h=0}^{\infty}\sum_{\substack{h_1+h_2+h_3=h\\h_1\geq 0,\,h_2\geq 0,\,h_3\geq 0}}\binom{h}{h_1,h_2,h_3}m_0^{h_1+1}m_1^{h_2}m_2^{h_3}\\
&\cdot\frac{b^{\upsilon h_3}|\eta|}{\Gamma((\varsigma-\varepsilon)h_1+\varsigma h_2+\varsigma h_3+\varsigma-\varepsilon+1)\Gamma(\upsilon h_3+1)}\|\Phi\|\\
&+\sum_{h=0}^{\infty}\sum_{\substack{h_1+h_2+h_3=h\\h_1\geq 0,\,h_2\geq 0,\,h_3\geq 0}}\binom{h}{h_1,h_2,h_3}m_0^{h_1}m_1^{h_2}m_2^{h_3}\\
&\cdot\frac{M\,b^{\upsilon h_3}}{\Gamma((\varsigma-\varepsilon)h_1+\varsigma h_2+\varsigma h_3+2)\Gamma(\upsilon h_3+1)}\|\Phi\|,\end{aligned}$$

where $M=\max_{\sigma\in[0,b]}|\psi(\sigma)|$. Applying the property $\Gamma(\upsilon h_3+1)\geq 1/2$, we deduce

$$\begin{aligned}\|\Phi\|\leq&\,2E_{(\varsigma-\varepsilon,\varsigma,\varsigma),\varsigma+1}(m_0,m_1,m_2b^{\upsilon})\sup_{(\theta,\sigma)\in[0,1]\times[0,b]}|F(\theta,\sigma,\Phi(\theta,\sigma))|\\
&+2|\eta|E_{(\varsigma-\varepsilon,\varsigma,\varsigma),1}(m_0,m_1,m_2b^{\upsilon})\|\Phi\|\\
&+2m_0|\eta|E_{(\varsigma-\varepsilon,\varsigma,\varsigma),\varsigma-\varepsilon+1}(m_0,m_1,m_2b^{\upsilon})\|\Phi\|\\
&+2ME_{(\varsigma-\varepsilon,\varsigma,\varsigma),2}(m_0,m_1,m_2b^{\upsilon})\|\Phi\|\\
\leq&\,2E_{(\varsigma-\varepsilon,\varsigma,\varsigma),\varsigma+1}(m_0,m_1,m_2b^{\upsilon})\sup_{(\theta,\sigma)\in[0,1]\times[0,b]}|F(\theta,\sigma,\Phi(\theta,\sigma))|\\
&+2\left(|\eta|+m_0|\eta|+M\right)E_{(\varsigma-\varepsilon,\varsigma,\varsigma),1}(m_0,m_1,m_2b^{\upsilon})\|\Phi\|.\end{aligned}$$

From $q=1-2\left(|\eta|+m_0|\eta|+M\right)E_{(\varsigma-\varepsilon,\varsigma,\varsigma),1}(m_0,m_1,m_2b^{\upsilon})>0$, we have

$$\|\Phi\|\leq\frac{2}{q}E_{(\varsigma-\varepsilon,\varsigma,\varsigma),\varsigma+1}(m_0,m_1,m_2b^{\upsilon})\sup_{(\theta,\sigma)\in[0,1]\times[0,b]}|F(\theta,\sigma,\Phi(\theta,\sigma))|<+\infty,$$

since F is a bounded function. This implies that the series is well defined in the space $C([0,1]\times[0,b])$. This completes the proof. □

In the following, we present and prove the uniqueness of the solutions of Equation (1.1) in the form of a theorem.

Theorem 3. *Let $b > 0$, $2 < \varsigma \leq 3$, $0 < \varepsilon \leq 1$, $\upsilon > 0$, η be a constant and $c_0, c_1, c_2, f \in C[0,b]$. We assume the continuous and bounded function $F : [0,1] \times [0,b] \times R \to R$ satisfies the following Lipschitz condition with a constant $L \geq 0$:*

$$|F(\theta,\sigma,y_1) - F(\theta,\sigma,y_2)| \leq L|y_1 - y_2|, \ (\theta,\sigma) \in [0,1]\times[0,b], \ y_1, y_2 \in R.$$

In addition, we suppose

$$2\,(L + |\eta| + m_0|\eta| + M)\,E_{(\varsigma-\varepsilon,\varsigma,\varsigma),1}\,(m_0, m_1, m_2 b^{\upsilon}) < 1,$$

where

$$m_i = \max_{\sigma\in[0,b]} |c_i(\sigma)|, \ \textit{and}\ M = \max_{\sigma\in[0,b]} |\psi(\sigma)|,$$

for $i = 0, 1, 2$. Then Equation (1.1) has a unique solution in the space $C([0,1]\times[0,b])$.

Proof. We consider the nonlinear mapping T over the space $C([0,1]\times[0,b])$ and define it as follows:

$$\begin{aligned}
(T\Phi)(\theta,\sigma) &= \sum_{h=0}^{\infty}(-1)^h \sum_{\substack{h_1+h_2+h_3=h \\ h_1\geq 0,\, h_2\geq 0,\, h_3\geq 0}} \binom{h}{h_1, h_2, h_3} c_0^{h_1}(\sigma) c_1^{h_2}(\sigma) \\
&\cdot I_t^{(\varsigma-\varepsilon)h_1+\varsigma h_2+\varsigma h_3+\varsigma}(c_2(\sigma) I_\sigma^{\upsilon})^{h_3} F(\theta,\sigma,\Phi(\theta,\sigma)) \\
&+ \sum_{h=0}^{\infty}(-1)^h \sum_{\substack{h_1+h_2+h_3=h \\ h_1\geq 0,\, h_2\geq 0,\, h_3\geq 0}} \binom{h}{h_1, h_2, h_3} c_0^{h_1}(\sigma) c_1^{h_2}(\sigma) \\
&\cdot \frac{\theta^{(\varsigma-\varepsilon)h_1+\varsigma h_2+\varsigma h_3}}{\Gamma((\varsigma-\varepsilon)h_1+\varsigma h_2+\varsigma h_3+1)}\eta \int_0^1 (c_2(\sigma) I_\sigma^{\upsilon})^{h_3}\Phi(\theta,\sigma)d\theta \\
&+ \sum_{h=0}^{\infty}(-1)^h \sum_{\substack{h_1+h_2+h_3=h \\ h_1\geq 0,\, h_2\geq 0,\, h_3\geq 0}} \binom{h}{h_1, h_2, h_3} c_0^{h_1}(\sigma) c_1^{h_2}(\sigma) \\
&\cdot \frac{\theta^{(\varsigma-\varepsilon)h_1+\varsigma h_2+\varsigma h_3+\varsigma-\varepsilon}}{\Gamma((\varsigma-\beta)h_1+\varsigma h_2+\varsigma h_3+\varsigma-\beta+1)}\eta \int_0^1 (c_2(\sigma) I_\sigma^{\upsilon})^{h_3} c_0(\sigma)\Phi(\theta,\sigma)d\theta \\
&+ \sum_{h=0}^{\infty}(-1)^h \sum_{\substack{h_1+h_2+h_3=h \\ h_1\geq 0,\, h_2\geq 0,\, h_3\geq 0}} \binom{h}{h_1, h_2, h_3} c_0^{h_1}(\sigma) c_1^{h_2}(\sigma) \\
&\cdot \frac{\theta^{(\varsigma-\varepsilon)h_1+\varsigma h_2+\varsigma h_3+1}}{\Gamma((\varsigma-\varepsilon)h_1+\varsigma h_2+\varsigma h_3+2)} \int_0^1 \psi(\theta)(c_2(\sigma) I_\sigma^{\upsilon})^{h_3}\Phi(\theta,\sigma)d\theta.
\end{aligned}$$

It follows from the proof of Theorem 2 that $T\Phi \in C([0,1]\times[0,b])$. We now show that T is contractive. Indeed, similar to that in the proof of Theorem 2,

$$\begin{aligned}
\|T\Phi - T\vartheta\| &\leq 2E_{(\varsigma-\varepsilon,\varsigma,\varsigma),\varsigma+1}\,(m_0,m_1,m_2b^{\upsilon}) \sup_{(\theta,\sigma)\in[0,1]\times[0,b]} |F(\theta,\sigma,\Phi(\theta,\sigma)) - F(\theta,\sigma,\vartheta(\theta,\sigma))| \\
&\quad + 2|\eta| E_{(\varsigma-\varepsilon,\varsigma,\varsigma),1}\,(m_0,m_1,m_2b^{\upsilon})\,\|\Phi-\vartheta\| \\
&\quad + 2m_0|\eta|\,E_{(\varsigma-\varepsilon,\varsigma,\varsigma),\varsigma-\varepsilon+1}\,(m_0,m_1,m_2b^{\upsilon})\,\|\Phi-\vartheta\| \\
&\quad + 2M\,E_{(\varsigma-\varepsilon,\varsigma,\varsigma),2}\,(m_0,m_1,m_2b^{\upsilon})\,\|\Phi-\vartheta\| \\
&\leq 2LE_{(\varsigma-\varepsilon,\varsigma,\varsigma),\varsigma+1}\,(m_0,m_1,m_2b^{\upsilon})\,\|\Phi-\vartheta\| \\
&\quad + 2\,(|\eta| + m_0|\eta| + M)\,E_{(\varsigma-\varepsilon,\varsigma,\varsigma),1}\,(m_0,m_1,m_2b^{\upsilon})\,\|\Phi-\vartheta\| \\
&\leq 2\,(L + |\eta| + m_0|\eta| + M)\,E_{(\varsigma-\varepsilon,\varsigma,\varsigma),1}\,(m_0,m_1,m_2b^{\upsilon})\,\|\Phi-\vartheta\|.
\end{aligned}$$

Since $2\,(L + |\eta| + m_0|\eta| + M)\,E_{(\varsigma-\varepsilon,\varsigma,\varsigma),1}\,(m_0,m_1,m_2b^{\upsilon}) < 1$, Equation (1.1) has a unique solution in the space $C([0,1]\times[0,b])$ by BCP. □

3 | EXAMPLE

Example 4. We consider an NPDIE-NIVC as follows:

$$\begin{cases} \frac{{}^{c}\partial^{2.5}}{\partial\theta^{2.5}}\Phi(\theta,\sigma)+\sin\sigma\,\frac{{}^{c}\partial^{0.5}}{\partial\theta^{0.5}}\Phi(\theta,\sigma)+\frac{\sigma^2}{1+\sigma^2}\,\Phi(\theta,\sigma)+\cos\sigma\, I_{\sigma}\Phi(\theta,\sigma)=\frac{1}{21}\sin|\Phi(\theta,\sigma)|,\\ \Phi(0,\sigma)=\frac{1}{40}\int_0^1\Phi(\theta,\sigma)d\theta,\quad \frac{\partial}{\partial\theta}\Phi(0,\sigma)=\frac{1}{22}\int_0^1\theta\Phi(\theta,\sigma)d\theta,\ \Phi''_{\theta}(0,\sigma)=0.\end{cases}$$

Then, it has a unique solution in the space $C([0,1]\times[0,1])$.

Proof. Clearly, $m_0=m_1=m_2=1$, and $F(\theta,\sigma,y)=\frac{1}{21}\sin|y|$, is a continuous and bounded function satisfying

$$|F(\theta,\sigma,y_1)-F(\theta,\sigma,y_2)|\le\frac{1}{21}|y_1-y_2|.$$

Thus, $L=1/21$. Moreover, $M=1/22$ and $\eta=1/40$. Finally, we note

$$\begin{aligned}&2\,(L+|\eta|+m_0|\eta|+M)\,E_{(\varsigma-\epsilon,\varsigma,\varsigma),1}\,(m_0,m_1,m_2b^{\upsilon})\\ &=2\,(1/21+1/40+1/40+1/22)\,E_{(2,2.5,2.5),1}(1,1,1)\\ &\le\frac{3}{10}E_{(2,2.5,2.5),1}(1,1,1),\end{aligned}$$

and

$$\begin{aligned}E_{(2,2.5,2.5),1}(1,1,1)&=\sum_{h=0}^{\infty}\sum_{\substack{h_1+h_2+h_3=h\\ h_1\ge0,\,h_2\ge0,\,h_3\ge0}}\binom{h}{h_1,h_2,h_3}\frac{1}{\Gamma(2h_1+2.5h_2+2.5h_3+1)}\\ &\le\sum_{h=0}^{\infty}\frac{3^h}{(2h)!}=\cosh\left(\sqrt{3}\right)\approx2.9146,\end{aligned}$$

since

$$\sum_{\substack{h_1+h_2+h_3=h\\ h_1\ge0,\,h_2\ge0,\,h_3\ge0}}\binom{h}{h_1,h_2,h_3}=3^h,$$

$$\frac{1}{\Gamma(2h_1+2.5h_2+2.5h_3+1)}\le\frac{1}{\Gamma(2h+1)}=\frac{1}{(2h)!}.$$

Hence,

$$2\,(L+|\eta|+m_0|\eta|+M)\,E_{(\varsigma-\epsilon,\varsigma,\varsigma),1}\,(m_0,m_1,m_2b^{\upsilon})<1.$$

By Theorem 3, the NPDIE-NIVC has a unique solution in the space $C([0,1]\times[0,1])$. □

4 | CONCLUSION

In this article, we have considered the NPDIE with nonlocal initial value conditions. Our main work was to investigate the uniqueness of solutions for this equation using an implicit integral equation, B-A, MMLF, and BCP. In the final part, we have presented the numerical results of our key theorem by giving an illustrative example.

AUTHOR CONTRIBUTIONS

All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

CONFLICT OF INTEREST STATEMENT

The authors declare that they have no competing interests.

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