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On the Uniqueness of the Bounded Solution for the Fractional Nonlinear Partial Integro-Differential Equation with Approximations

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Abstract: This paper studies the uniqueness of the bounded solution to a new Cauchy problem of the fractional nonlinear partial integro-differential equation based on the multivariate Mittag–Leffler function as well as Banach's contractive principle in a complete function space. Applying Babenko's approach, we convert the fractional nonlinear equation with variable coefficients to an implicit integral equation, which is a powerful method of studying the uniqueness of solutions. Furthermore, we construct algorithms for finding analytic and approximate solutions using Adomian's decomposition method and recurrence relation with the order convergence analysis. Finally, an illustrative example is presented to demonstrate constructions for the numerical solution using MATHEMATICA.

Keywords: adomian's decomposition method; banach's contractive principle; multivariate Mittag–Leffler function; babenko's approach; analytic and approximate solution

MSC: 35A02; 35C10; 35C15; 26A33

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1. Introduction and Preliminaries

The operator I_θ^β is the partial Riemann–Liouville fractional integral of order $\beta > 0$ with respect to θ with the initial point zero [1],

$$(I_\theta^\beta W)(\theta,\sigma) = \frac{1}{\Gamma(\beta)} \int_0^\theta (\theta - \zeta)^{\beta-1} W(\zeta,\sigma)d\zeta, \ \theta > 0,$$

and $\frac{{}_c\partial^\alpha}{\partial\theta^\alpha}$ is the partial Caputo fractional derivative of order α with respect to θ

$$\left(\frac{{}_c\partial^\alpha}{\partial\theta^\alpha} W\right)(\theta,\sigma) = \frac{1}{\Gamma(1-\alpha)} \int_0^\theta (\theta - s)^{-\alpha} W_s'(s,\sigma)ds, \quad 0 < \alpha \leq 1.$$

It follows from [2] that, for $0 < \alpha \leq 1$,

$$I_\theta^\alpha \left(\frac{{}_c\partial^\alpha}{\partial\theta^\alpha} W\right)(\theta,\sigma) = W(\theta,\sigma) - W(0,\sigma).$$

We define the Banach space $C([0,1] \times [0,b])$ with the norm for $b > 0$ as

$$\|W\| = \sup_{\theta\in[0,1],\, \sigma\in[0,b]} |W(\theta,\sigma)| \ \text{ for } \ W \in C([0,1] \times [0,b]).$$

In this work, we consider the uniqueness of the bounded solution to the following fractional nonlinear partial integro-differential equation with the initial condition in the Caputo sense for $0 < \alpha_2 < \alpha_1 \leq 1$ in the space $C([0,1] \times [0,b])$:

$$\begin{cases} \frac{{}_c\partial^{\alpha_1}}{\partial\theta^{\alpha_1}} W(\theta,\sigma) + C\frac{{}_c\partial^{\alpha_2}}{\partial\theta^{\alpha_2}} W(\theta,\sigma) + \sum_{i=1}^{m} \lambda_i(\sigma) I_\theta^{\gamma_i} I_\sigma^{\beta_i} W(\theta,\sigma) \\ \qquad = f(\theta,\sigma) + g(W(\theta,\sigma)), \\ W(0,\sigma) = \phi(\sigma), \quad (\theta,\sigma) \in [0,1] \times [0,b], \end{cases} \tag{1}$$

where $\lambda_i(\sigma)$, $f(\theta,\sigma)$, g, $\phi(\sigma)$ are functions satisfying certain conditions and C is a constant. In addition, we avoid computational difficulties in the series and construct the algorithm used to find analytic and approximate solutions to (1) using Adomian's decomposition method. To the best of the authors' knowledge, Equation (1) is new and has never been studied before. Furthermore, any existing integral transform seems hard to deal with using this equation due to the variable coefficients and double integral operators involved.

A function W is said to be a solution of Equation (1) if it satisfies the equation over $C([0,1] \times [0,b])$ and its initial condition.

Definition 1. *The two-parameter Mittag–Leffler function is represented by the following series*

$$E_{\varsigma,\varepsilon}(z) = \sum_{l=0}^{\infty} \frac{z^l}{\Gamma(\varsigma l + \varepsilon)},$$

where $\varsigma, \varepsilon > 0$ *and* $z \in \mathcal{C}$ *(the complex plane).*

The multivariate Mittag–Leffler function [3] is defined as follows

$$E_{(\varsigma_1,\cdots,\varsigma_m),\varepsilon}(z_1,\cdots,z_m) = \sum_{l=0}^{\infty} \sum_{\substack{l_1+\cdots+l_m=l \\ l_1\geq 0,\cdots,l_m\geq 0}} \binom{l}{l_1,\cdots,l_m} \frac{z_1^{l_1}\cdots z_m^{l_m}}{\Gamma(\varsigma_1 l_1 + \cdots + \varsigma_m l_m + \varepsilon)},$$

where $\varsigma_i, \varepsilon > 0$, $z_i \in \mathcal{C}$ *for* $i = 1,2,\cdots,m$ *and*

$$\binom{l}{l_1,\cdots,l_m} = \frac{l!}{l_1!\cdots l_m!}.$$

Babenko's approach [4] is an efficient tool used to solve differential equations (including partial differential equations) with initial conditions or boundary value problems, as well as integral equations [5,6]. In the following, to show the details of this technique, we will deduce the solution to the following fractional integro-differential equation with the initial condition in the space $C[0,b]$ for a constant λ:

$$\begin{cases} {}_cD_0^\alpha y(x) + \lambda I_0^\beta y(x) = F(x), \quad F \in C[0,b], \quad 0 < \alpha \leq 1, \quad \beta > 0, \\ y(0) = 0, \end{cases}$$

where

$${}_cD_0^\alpha y(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-s)^{-\alpha} y'(s) ds,$$

and

$$I_0^\beta y(x) = \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} y(s) ds.$$

Clearly,

$$I_0^\alpha({}_cD_0^\alpha y(x)) = y(x) - y(0) = y(x).$$

Applying the operator I_0^α to both sides of equation, we obtain

$$\left(1+\lambda I_0^{\alpha+\beta}\right)y(x) = I_0^\alpha F(x).$$

Treating the operator $\left(1+\lambda I_0^{\alpha+\beta}\right)$ as a variable, we informally derive that, using Babenko's approach

$$\begin{aligned} y(x) &= \left(1+\lambda I_0^{\alpha+\beta}\right)^{-1} I_0^\alpha F(x) = \sum_{l=0}^{\infty}(-1)^l\left(\lambda I_0^{\alpha+\beta}\right)^l I_0^\alpha F(x) \\ &= \sum_{l=0}^{\infty}(-1)^l\lambda^l I_0^{l(\alpha+\beta)} I_0^\alpha F(x) = \sum_{l=0}^{\infty}(-1)^l\lambda^l I_0^{l(\alpha+\beta)+\alpha} F(x). \end{aligned}$$

Obviously,

$$\begin{aligned} \left\| I_0^{l(\alpha+\beta)+\alpha} \right\| &= \sup_{x\in[0,b],\ \|\phi\|\le 1} \left| I_0^{l(\alpha+\beta)+\alpha}\phi \right| \\ &= \sup_{x\in[0,b],\ \|\phi\|\le 1} \frac{1}{\Gamma(l(\alpha+\beta)+\alpha)} \left| \int_0^x (x-s)^{l(\alpha+\beta)+\alpha-1}\phi(s) \right| ds \\ &\le \sup_{x\in[0,b],\ \|\phi\|\le 1} \|\phi\| \frac{x^{l(\alpha+\beta)+\alpha}}{\Gamma(l(\alpha+\beta)+\alpha+1)} \\ &\le \frac{b^{l(\alpha+\beta)+\alpha}}{\Gamma(l(\alpha+\beta)+\alpha+1)}, \end{aligned}$$

due to the fact that

$$\Gamma(l(\alpha+\beta)+\alpha+1) = \Gamma(l(\alpha+\beta)+\alpha)(l(\alpha+\beta)+\alpha).$$

These imply that

$$\begin{aligned} \|y\| &\le \sum_{l=0}^{\infty}|\lambda|^l \left\| I_0^{l(\alpha+\beta)+\alpha} \right\| \|F\| \le b^\alpha \|F\| \sum_{l=0}^{\infty} \frac{\left(|\lambda|\, b^{\alpha+\beta}\right)^l}{\Gamma(l(\alpha+\beta)+\alpha+1)} \\ &= b^\alpha \|F\| E_{\alpha+\beta,\alpha+1}\left(|\lambda|\, b^{\alpha+\beta}\right) < +\infty, \end{aligned}$$

which infers that the series solution

$$y(x) = \sum_{l=0}^{\infty}(-1)^l\lambda^l I_0^{l(\alpha+\beta)+\alpha} F(x),$$

is uniformly convergent in $C[0,b]$, and thus well defined.

In particular, for $F(x) = x$,

$$y(x) = x^{\alpha+1}\sum_{l=0}^{\infty}(-1)^l\lambda^l \frac{x^{l(\alpha+\beta)}}{\Gamma(l(\alpha+\beta)+\alpha+2)}.$$

Nonlinear partial differential equations have been used to describe a wide range of phenomena and dynamical processes in many scientific areas, such as physics, fluid mechanics, geophysics, plasma physics and optical fibres [7]. Atangana [8] introduced new fractal–fractional differential and integral operators to work on more non-local problems using analytic and numerical methods. Researchers have made a great deal of effort to find approximate, stable numerical and analytical methods to solve fractional partial differential equations of physical interest, which include the finite difference method [9], Adomian's decomposition method [10], variational iteration method [11], and homotopy perturbation method [12]. On the other hand, the uniqueness and existence of solutions are among the most important and interesting topics for fractional nonlinear partial differential or integral equations with initial value or boundary conditions [13]. Very recently, Li et al. [6] applied Babenko's approach, the multivariate Mittag–Leffler function, and Krasnoselskii's fixed point theorem, and investigated the existence of solutions to a Liouville–Caputo nonlinear integro-differential equations with variable coefficients and initial conditions in a Banach space.

The rest of this paper is organized as follows. Section 2 deals with the uniqueness of the bounded solution to Equation (1) using the multivariate Mittag–Leffler function and Banach's contractive principle. Section 3 finds the recurrence algorithms for analytic and approximate solutions based on Adomian's decomposition method with an order analysis. We further demonstrate the applications of the main results with an example in Section 4. In Section 5, we provide a summary of the work.

2. Uniqueness of Bounded Solution

Theorem 1. *Let $\phi(\sigma)$, $\lambda_i(\sigma) \in C[0,b]$ for all $i = 1, 2, \cdots, m$, $f(\theta,\sigma) \in C([0,1] \times [0,b])$ and g be a continuous and bounded function on $\mathcal{R}$ (the set of all real numbers). In addition, we assume that $0 < \alpha_2 < \alpha_1 \leq 1$, γ_i, $\beta_i \geq 0$ for all $i = 1, 2, \cdots, m$ and C is a constant in R. Then $W(\theta,\sigma)$ is a solution of Equation (1) if and only if it is bounded and satisfies the following implicit integral equation in the space $C([0,1] \times [0,b])$:*

$$\begin{aligned}
&W(\theta,\sigma) \\
&= \sum_{l=0}^{\infty}(-1)^l \sum_{\substack{l_1+\cdots+l_{m+1}=l \\ l_1\geq 0,\cdots,l_{m+1}\geq 0}} \binom{l}{l_1,\cdots,l_{m+1}} C^{l_{m+1}} I_\theta^{l_1(\alpha_1+\gamma_1)+\cdots+l_m(\alpha_1+\gamma_m)+l_{m+1}(\alpha_1-\alpha_2)+\alpha_1} \\
&\quad \cdot \left(\lambda_1(\sigma) I_\sigma^{\beta_1}\right)^{l_1} \cdots \left(\lambda_m(\sigma) I_\sigma^{\beta_m}\right)^{l_m} f(\theta,\sigma) \\
&\quad + \sum_{l=0}^{\infty}(-1)^l \sum_{\substack{l_1+\cdots+l_{m+1}=l \\ l_1\geq 0,\cdots,l_{m+1}\geq 0}} \binom{l}{l_1,\cdots,l_{m+1}} C^{l_{m+1}} I_\theta^{l_1(\alpha_1+\gamma_1)+\cdots+l_{m+1}(\alpha_1-\alpha_2)+\alpha_1} \\
&\quad \cdot \left(\lambda_1(\sigma) I_\sigma^{\beta_1}\right)^{l_1} \cdots \left(\lambda_m(\sigma) I_\sigma^{\beta_m}\right)^{l_m} g(W(\theta,\sigma)) + \sum_{l=0}^{\infty}(-1)^l \\
&\quad \cdot \sum_{\substack{l_1+\cdots+l_{m+1}=l \\ l_1\geq 0,\cdots,l_{m+1}\geq 0}} \binom{l}{l_1,\cdots,l_{m+1}} \frac{C^{l_{m+1}}\, \theta^{l_1(\alpha_1+\gamma_1)+\cdots+l_{m+1}(\alpha_1-\alpha_2)}}{\Gamma(l_1(\alpha_1+\gamma_1)+\cdots+l_{m+1}(\alpha_1-\alpha_2)+1)} \\
&\quad \cdot \left(\lambda_1(\sigma) I_\sigma^{\beta_1}\right)^{l_1} \cdots \left(\lambda_m(\sigma) I_\sigma^{\beta_m}\right)^{l_m} \phi(\sigma) + C\sum_{l=0}^{\infty}(-1)^l \\
&\quad \cdot \sum_{\substack{l_1+\cdots+l_{m+1}=l \\ l_1\geq 0,\cdots,l_{m+1}\geq 0}} \binom{l}{l_1,\cdots,l_{m+1}} \frac{C^{l_{m+1}}\, \theta^{l_1(\alpha_1+\gamma_1)+\cdots+l_{m+1}(\alpha_1-\alpha_2)+\alpha_1-\alpha_2}}{\Gamma(l_1(\alpha_1+\gamma_1)+\cdots+l_{m+1}(\alpha_1-\alpha_2)+\alpha_1-\alpha_2+1)} \\
&\quad \cdot \left(\lambda_1(\sigma) I_\sigma^{\beta_1}\right)^{l_1} \cdots \left(\lambda_m(\sigma) I_\sigma^{\beta_m}\right)^{l_m}.
\end{aligned} \tag{2}$$

furthermore,

$$
\begin{aligned}
\|W\| \leq & 2E_{(\alpha_1+\gamma_1,\cdots,\alpha_1+\gamma_m,\alpha_1-\alpha_2),\,\alpha_1+1}\left(Bb^{\beta_1},\cdots,Bb^{\beta_m},|C|\right) \\
& \cdot\left(\max_{(\theta,\sigma)\in[0,1]\times[0,b]}|f(\theta,\sigma)| + \sup_{x\in R}|g(x)|\right) \\
& + 2E_{(\alpha_1+\gamma_1,\cdots,\alpha_1+\gamma_m,\alpha_1-\alpha_2),\,1}\left(Bb^{\beta_1},\cdots,Bb^{\beta_m},|C|\right)\max_{\sigma\in[0,b]}|\phi(\sigma)| \\
& + 2E_{(\alpha_1+\gamma_1,\cdots,\alpha_1+\gamma_m,\alpha_1-\alpha_2),\,\alpha_1-\alpha_2+1}\left(Bb^{\beta_1},\cdots,Bb^{\beta_m},|C|\right) < +\infty,
\end{aligned}
$$

where B is a positive constant satisfying

$$\max_{\sigma\in[0,b]}|\lambda_i(\sigma)| \leq B, \quad \textit{for } i = 1,2,\cdots,m.$$

Proof. Applying the operator $I_\theta^{\alpha_1}$ to both sides of Equation (1), we obtain

$$
\begin{aligned}
& I_\theta^{\alpha_1}\frac{{}^c\partial^{\alpha_1}}{\partial\theta^{\alpha_1}}W(\theta,\sigma) + CI_\theta^{\alpha_1-\alpha_2}I_\theta^{\alpha_2}\frac{{}^c\partial^{\alpha_2}}{\partial\theta^{\alpha_2}}W(\theta,\sigma) + \sum_{i=1}^{m}\lambda_i(\sigma)I_\theta^{\alpha_1+\gamma_i}I_\sigma^{\beta_i}W(\theta,\sigma) \\
& \quad = I_\theta^{\alpha_1}f(\theta,\sigma) + I_\theta^{\alpha_1}g(W(\theta,\sigma)).
\end{aligned}
$$

This implies that

$$
\begin{aligned}
& W(\theta,\sigma) - \phi(\sigma) + CI_\theta^{\alpha_1-\alpha_2}(W(\theta,\sigma)-\phi(\sigma)) + \sum_{i=1}^{m}\lambda_i(\sigma)I_\theta^{\alpha_1+\gamma_i}I_\sigma^{\beta_i}W(\theta,\sigma) \\
& \quad = I_\theta^{\alpha_1}f(\theta,\sigma) + I_\theta^{\alpha_1}g(W(\theta,\sigma)).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
& \left(1 + CI_\theta^{\alpha_1-\alpha_2} + \sum_{i=1}^{m}\lambda_i(\sigma)I_\theta^{\alpha_1+\gamma_i}I_\sigma^{\beta_i}\right)W(\theta,\sigma) \\
& \quad = I_\theta^{\alpha_1}f(\theta,\sigma) + I_\theta^{\alpha_1}g(W(\theta,\sigma)) + \phi(\sigma)\left(1 + C\frac{\theta^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\right).
\end{aligned}
$$

Using Babenko's approach and the multinomial theorem,

$$
\begin{aligned}
&W(\theta,\sigma)\\
&=\left(1+CI_\theta^{\alpha_1-\alpha_2}+\sum_{i=1}^{m}\lambda_i(\sigma)I_\theta^{\alpha_1+\gamma_i}I_\sigma^{\beta_i}\right)^{-1}\\
&\quad\cdot\left(I_\theta^{\alpha_1}f(\theta,\sigma)+I_\theta^{\alpha_1}g(W(\theta,\sigma))+\phi(\sigma)\left(1+C\frac{\theta^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\right)\right)\\
&=\sum_{l=0}^{\infty}(-1)^l\left(CI_\theta^{\alpha_1-\alpha_2}+\sum_{i=1}^{m}\lambda_i(\sigma)I_\theta^{\alpha_1+\gamma_i}I_\sigma^{\beta_i}\right)^{l}\\
&\quad\cdot\left(I_\theta^{\alpha_1}f(\theta,\sigma)+I_\theta^{\alpha_1}g(W(\theta,\sigma))+\phi(\sigma)\left(1+C\frac{\theta^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\right)\right)\\
&=\sum_{l=0}^{\infty}(-1)^l\sum_{\substack{l_1+\cdots+l_{m+1}=l\\ l_1\geq 0,\cdots,l_{m+1}\geq 0}}\binom{l}{l_1,\cdots,l_{m+1}}\left(\lambda_1(\sigma)I_\theta^{\alpha_1+\gamma_1}I_\sigma^{\beta_1}\right)^{l_1}\cdots\\
&\quad\cdot\left(\lambda_m(\sigma)I_\theta^{\alpha_1+\gamma_m}I_\sigma^{\beta_m}\right)^{l_m}\left(CI_\theta^{\alpha_1-\alpha_2}\right)^{l_{m+1}}\\
&\quad\cdot\left(I_\theta^{\alpha_1}f(\theta,\sigma)+I_\theta^{\alpha_1}g(W(\theta,\sigma))+\phi(\sigma)\left(1+C\frac{\theta^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\right)\right)\\
&=\sum_{l=0}^{\infty}(-1)^l\sum_{\substack{l_1+\cdots+l_{m+1}=l\\ l_1\geq 0,\cdots,l_{m+1}\geq 0}}\binom{l}{l_1,\cdots,l_{m+1}}C^{l_{m+1}}I_\theta^{l_1(\alpha_1+\gamma_1)+\cdots+l_m(\alpha_1+\gamma_m)+l_{m+1}(\alpha_1-\alpha_2)+\alpha_1}\\
&\quad\cdot\left(\lambda_1(\sigma)I_\sigma^{\beta_1}\right)^{l_1}\cdots\left(\lambda_m(\sigma)I_\sigma^{\beta_m}\right)^{l_m}f(\theta,\sigma)\\
&\quad+\sum_{l=0}^{\infty}(-1)^l\sum_{\substack{l_1+\cdots+l_{m+1}=l\\ l_1\geq 0,\cdots,l_{m+1}\geq 0}}\binom{l}{l_1,\cdots,l_{m+1}}C^{l_{m+1}}I_\theta^{l_1(\alpha_1+\gamma_1)+\cdots+l_{m+1}(\alpha_1-\alpha_2)+\alpha_1}\\
&\quad\cdot\left(\lambda_1(\sigma)I_\sigma^{\beta_1}\right)^{l_1}\cdots\left(\lambda_m(\sigma)I_\sigma^{\beta_m}\right)^{l_m}g(W(\theta,\sigma))+\sum_{l=0}^{\infty}(-1)^l\\
&\quad\cdot\sum_{\substack{l_1+\cdots+l_{m+1}=l\\ l_1\geq 0,\cdots,l_{m+1}\geq 0}}\binom{l}{l_1,\cdots,l_{m+1}}\frac{C^{l_{m+1}}\,\theta^{l_1(\alpha_1+\gamma_1)+\cdots+l_{m+1}(\alpha_1-\alpha_2)}}{\Gamma(l_1(\alpha_1+\gamma_1)+\cdots+l_{m+1}(\alpha_1-\alpha_2)+1)}\\
&\quad\cdot\left(\lambda_1(\sigma)I_\sigma^{\beta_1}\right)^{l_1}\cdots\left(\lambda_m(\sigma)I_\sigma^{\beta_m}\right)^{l_m}\phi(\sigma)+C\sum_{l=0}^{\infty}(-1)^l\\
&\quad\cdot\sum_{\substack{l_1+\cdots+l_{m+1}=l\\ l_1\geq 0,\cdots,l_{m+1}\geq 0}}\binom{l}{l_1,\cdots,l_{m+1}}\frac{C^{l_{m+1}}\,\theta^{l_1(\alpha_1+\gamma_1)+\cdots+l_{m+1}(\alpha_1-\alpha_2)+\alpha_1-\alpha_2}}{\Gamma(l_1(\alpha_1+\gamma_1)+\cdots+l_{m+1}(\alpha_1-\alpha_2)+\alpha_1-\alpha_2+1)}\\
&\quad\cdot\left(\lambda_1(\sigma)I_\sigma^{\beta_1}\right)^{l_1}\cdots\left(\lambda_m(\sigma)I_\sigma^{\beta_m}\right)^{l_m}.
\end{aligned}
$$

Clearly, all the above steps are reversible. It remains to be shown that $W \in C([0,1] \times [0,b])$. Let B be a positive constant such that

$$\max_{\sigma\in[0,b]}|\lambda_i(\sigma)| \leq B, \quad \text{for } i = 1,2,\cdots,m.$$

then,

$$
\begin{aligned}
&\|W\| \\
&\leq \sum_{l=0}^{\infty} \sum_{\substack{l_1+\cdots+l_{m+1}=l \\ l_1\geq 0,\cdots,l_{m+1}\geq 0}} \binom{l}{l_1,\cdots,l_{m+1}} \frac{|C|^{l_{m+1}} B^{l_1}\cdots B^{l_m}}{\Gamma(l_1(\alpha_1+\gamma_1)+\cdots+l_{m+1}(\alpha_1-\alpha_2)+\alpha_1+1)} \\
&\quad \cdot \frac{b^{l_1\beta_1+\cdots+l_m\beta_m}}{\Gamma(l_1\beta_1+\cdots+l_m\beta_m+1)} \max_{(\theta,\sigma)\in[0,1]\times[0,b]} |f(\theta,\sigma)| \\
&\quad + \sum_{l=0}^{\infty} \sum_{\substack{l_1+\cdots+l_{m+1}=l \\ l_1\geq 0,\cdots,l_{m+1}\geq 0}} \binom{l}{l_1,\cdots,l_{m+1}} \frac{|C|^{l_{m+1}} B^{l_1}\cdots B^{l_m}}{\Gamma(l_1(\alpha_1+\gamma_1)+\cdots+l_{m+1}(\alpha_1-\alpha_2)+\alpha_1+1)} \\
&\quad \cdot \frac{b^{l_1\beta_1+\cdots+l_m\beta_m}}{\Gamma(l_1\beta_1+\cdots+l_m\beta_m+1)} \sup_{x\in R}|g(x)| + \sum_{l=0}^{\infty} \sum_{\substack{l_1+\cdots+l_{m+1}=l \\ l_1\geq 0,\cdots,l_{m+1}\geq 0}} \binom{l}{l_1,\cdots,l_{m+1}} \\
&\quad \frac{|C|^{l_{m+1}}\, B^{l_1}\cdots B^{l_m}\, b^{l_1\beta_1+\cdots+l_m\beta_m}}{\Gamma(l_1(\alpha_1+\gamma_1)+\cdots+l_{m+1}(\alpha_1-\alpha_2)+1)\,\Gamma(l_1\beta_1+\cdots+l_m\beta_m+1)} \max_{\sigma\in[0,b]} |\phi(\sigma)| \\
&\quad + |C| \sum_{l=0}^{\infty} \sum_{\substack{l_1+\cdots+l_{m+1}=l \\ l_1\geq 0,\cdots,l_{m+1}\geq 0}} \binom{l}{l_1,\cdots,l_{m+1}} \\
&\quad \cdot \frac{|C|^{l_{m+1}}\, (Bb^{\beta_1})^{l_1}\cdots(Bb^{\beta_m})^{l_m}}{\Gamma(l_1(\alpha_1+\gamma_1)+\cdots+l_{m+1}(\alpha_1-\alpha_2)+\alpha_1-\alpha_2+1)\,\Gamma(l_1\beta_1+\cdots+l_m\beta_m+1)}.
\end{aligned}
$$

Clearly,

$$\Gamma(l_1\beta_1+\cdots+l_m\beta_m+1) \geq 1/2.$$

Therefore,

$$
\begin{aligned}
\|W\| \leq & 2E_{(\alpha_1+\gamma_1,\cdots,\alpha_1+\gamma_m,\alpha_1-\alpha_2),\,\alpha_1+1}\left(Bb^{\beta_1},\cdots,Bb^{\beta_m},|C|\right) \\
& \left(\max_{(\theta,\sigma)\in[0,1]\times[0,b]} |f(\theta,\sigma)| + \sup_{x\in R}|g(x)|\right) \\
& + 2E_{(\alpha_1+\gamma_1,\cdots,\alpha_1+\gamma_m,\alpha_1-\alpha_2),\,1}\left(Bb^{\beta_1},\cdots,Bb^{\beta_m},|C|\right) \max_{\sigma\in[0,b]}|\phi(\sigma)| \\
& + 2E_{(\alpha_1+\gamma_1,\cdots,\alpha_1+\gamma_m,\alpha_1-\alpha_2),\,\alpha_1-\alpha_2+1}\left(Bb^{\beta_1},\cdots,Bb^{\beta_m},|C|\right) < +\infty.
\end{aligned}
$$

This completes the proof of Theorem 1. □

Theorem 2. *Let $\phi(\sigma)$, $\lambda_i(\sigma) \in C[0,b]$ for all $i = 1,2,\cdots,m$, $f(\theta,\sigma) \in C([0,1]\times[0,b])$ and g be a continuous and bounded function on R, satisfying the following Lipschitz condition for a positive constant $\mathcal{L}$:*

$$|g(x)-g(y)| \leq \mathcal{L}|x-y|, \quad x, y \in R.$$

In addition, we assume that $0 < \alpha_2 < \alpha_1 \leq 1$, γ_i, $\beta_i \geq 0$ for all $i = 1,2,\cdots,m$, C is a constant in R and

$$q = 2\mathcal{L}E_{(\alpha_1+\gamma_1,\cdots,\alpha_1+\gamma_m,\alpha_1-\alpha_2),\,\alpha_1+1}\left(Bb^{\beta_1},\cdots,Bb^{\beta_m},|C|\right) < 1,$$

where B is defined in Theorem 1. Then Equation (1) has a unique solution in the space $C([0,1]\times[0,b])$.

Proof. We define a nonlinear mapping $\mathcal{T}$ over the space $C([0,1]\times[0,b])$ by

$$\begin{aligned}
&(\mathcal{T}W)(\theta,\sigma)\\
&=\sum_{l=0}^{\infty}(-1)^l \sum_{\substack{l_1+\cdots+l_{m+1}=l\\ l_1\geq 0,\cdots,l_{m+1}\geq 0}} \binom{l}{l_1,\cdots,l_{m+1}} C^{l_{m+1}} I_\theta^{l_1(\alpha_1+\gamma_1)+\cdots+l_m(\alpha_1+\gamma_m)+l_{m+1}(\alpha_1-\alpha_2)+\alpha_1}\\
&\quad\cdot\left(\lambda_1(\sigma)I_\sigma^{\beta_1}\right)^{l_1}\cdots\left(\lambda_m(\sigma)I_\sigma^{\beta_m}\right)^{l_m} f(\theta,\sigma)\\
&\quad+\sum_{l=0}^{\infty}(-1)^l \sum_{\substack{l_1+\cdots+l_{m+1}=l\\ l_1\geq 0,\cdots,l_{m+1}\geq 0}} \binom{l}{l_1,\cdots,l_{m+1}} C^{l_{m+1}} I_\theta^{l_1(\alpha_1+\gamma_1)+\cdots+l_{m+1}(\alpha_1-\alpha_2)+\alpha_1}\\
&\quad\cdot\left(\lambda_1(\sigma)I_\sigma^{\beta_1}\right)^{l_1}\cdots\left(\lambda_m(\sigma)I_\sigma^{\beta_m}\right)^{l_m} g(W(\theta,\sigma))+\sum_{l=0}^{\infty}(-1)^l\\
&\quad\cdot\sum_{\substack{l_1+\cdots+l_{m+1}=l\\ l_1\geq 0,\cdots,l_{m+1}\geq 0}} \binom{l}{l_1,\cdots,l_{m+1}} \frac{C^{l_{m+1}}\,\theta^{l_1(\alpha_1+\gamma_1)+\cdots+l_{m+1}(\alpha_1-\alpha_2)}}{\Gamma(l_1(\alpha_1+\gamma_1)+\cdots+l_{m+1}(\alpha_1-\alpha_2)+1)}\\
&\quad\cdot\left(\lambda_1(\sigma)I_\sigma^{\beta_1}\right)^{l_1}\cdots\left(\lambda_m(\sigma)I_\sigma^{\beta_m}\right)^{l_m} \phi(\sigma)+C\sum_{l=0}^{\infty}(-1)^l\\
&\quad\cdot\sum_{\substack{l_1+\cdots+l_{m+1}=l\\ l_1\geq 0,\cdots,l_{m+1}\geq 0}} \binom{l}{l_1,\cdots,l_{m+1}} \frac{C^{l_{m+1}}\,\theta^{l_1(\alpha_1+\gamma_1)+\cdots+l_{m+1}(\alpha_1-\alpha_2)+\alpha_1-\alpha_2}}{\Gamma(l_1(\alpha_1+\gamma_1)+\cdots+l_{m+1}(\alpha_1-\alpha_2)+\alpha_1-\alpha_2+1)}\\
&\quad\cdot\left(\lambda_1(\sigma)I_\sigma^{\beta_1}\right)^{l_1}\cdots\left(\lambda_m(\sigma)I_\sigma^{\beta_m}\right)^{l_m}.
\end{aligned}$$

It follows from Theorem 1 that $\mathcal{T}W\in C([0,1]\times[0,b])$. We are going to show that $\mathcal{T}$ is contractive. Indeed, for $W_1,W_2\in C([0,1]\times[0,b])$, we come to

$$\begin{aligned}
&\mathcal{T}W_1-\mathcal{T}W_2\\
&=\sum_{l=0}^{\infty}(-1)^l \sum_{\substack{l_1+\cdots+l_{m+1}=l\\ l_1\geq 0,\cdots,l_{m+1}\geq 0}} \binom{l}{l_1,\cdots,l_{m+1}} C^{l_{m+1}} I_\theta^{l_1(\alpha_1+\gamma_1)+\cdots+l_{m+1}(\alpha_1-\alpha_2)+\alpha_1}\\
&\quad\cdot\left(\lambda_1(\sigma)I_\sigma^{\beta_1}\right)^{l_1}\cdots\left(\lambda_m(\sigma)I_\sigma^{\beta_m}\right)^{l_m} g(W_1(\theta,\sigma))\\
&\quad-\sum_{l=0}^{\infty}(-1)^l \sum_{\substack{l_1+\cdots+l_{m+1}=l\\ l_1\geq 0,\cdots,l_{m+1}\geq 0}} \binom{l}{l_1,\cdots,l_{m+1}} C^{l_{m+1}} I_\theta^{l_1(\alpha_1+\gamma_1)+\cdots+l_{m+1}(\alpha_1-\alpha_2)+\alpha_1}\\
&\quad\cdot\left(\lambda_1(\sigma)I_\sigma^{\beta_1}\right)^{l_1}\cdots\left(\lambda_m(\sigma)I_\sigma^{\beta_m}\right)^{l_m} g(W_2(\theta,\sigma)),
\end{aligned}$$

and

$$\begin{aligned}
&\|\mathcal{T}W_1-\mathcal{T}W_2\|\\
&\leq 2\mathcal{L}E_{(\alpha_1+\gamma_1,\cdots,\alpha_1+\gamma_m,\alpha_1-\alpha_2),\alpha_1+1}\left(Bb^{\beta_1},\cdots,Bb^{\beta_m},|C|\right)\|W_1-W_2\|\\
&=q\|W_1-W_2\|,
\end{aligned}$$

from the proof of Theorem 1 and noting that

$$|g(W_1)-g(W_2)|\leq\mathcal{L}|W_1-W_2|.$$

Since $q<1$, Equation (1) has a unique solution in the space $C([0,1]\times[0,b])$ by Banach's contractive principle. This completes the proof of Theorem 2. □

3. Analytic and Approximate Solutions

We begin to derive analytic and approximate solutions to Equation (1) from the implicit integral Equation (2) obtained in Section 2 using an initial value $W_0 = 0$ and the following recurrence (iterative method):

$$\begin{aligned}
&W_{n+1}(\theta,\sigma) \\
&= \sum_{l=0}^{\infty}(-1)^l \sum_{\substack{l_1+\cdots+l_{m+1}=l \\ l_1\geq 0,\cdots,l_{m+1}\geq 0}} \binom{l}{l_1,\cdots,l_{m+1}} C^{l_{m+1}} I_\theta^{l_1(\alpha_1+\gamma_1)+\cdots+l_m(\alpha_1+\gamma_m)+l_{m+1}(\alpha_1-\alpha_2)+\alpha_1} \\
&\quad \cdot \left(\lambda_1(\sigma) I_\sigma^{\beta_1}\right)^{l_1} \cdots \left(\lambda_m(\sigma) I_\sigma^{\beta_m}\right)^{l_m} f(\theta,\sigma) \\
&\quad + \sum_{l=0}^{\infty}(-1)^l \sum_{\substack{l_1+\cdots+l_{m+1}=l \\ l_1\geq 0,\cdots,l_{m+1}\geq 0}} \binom{l}{l_1,\cdots,l_{m+1}} C^{l_{m+1}} I_\theta^{l_1(\alpha_1+\gamma_1)+\cdots+l_{m+1}(\alpha_1-\alpha_2)+\alpha_1} \\
&\quad \cdot \left(\lambda_1(\sigma) I_\sigma^{\beta_1}\right)^{l_1} \cdots \left(\lambda_m(\sigma) I_\sigma^{\beta_m}\right)^{l_m} g(W_n(\theta,\sigma)) + \sum_{l=0}^{\infty}(-1)^l \\
&\quad \cdot \sum_{\substack{l_1+\cdots+l_{m+1}=l \\ l_1\geq 0,\cdots,l_{m+1}\geq 0}} \binom{l}{l_1,\cdots,l_{m+1}} \frac{C^{l_{m+1}}\, \theta^{l_1(\alpha_1+\gamma_1)+\cdots+l_{m+1}(\alpha_1-\alpha_2)}}{\Gamma(l_1(\alpha_1+\gamma_1)+\cdots+l_{m+1}(\alpha_1-\alpha_2)+1)} \\
&\quad \cdot \left(\lambda_1(\sigma) I_\sigma^{\beta_1}\right)^{l_1} \cdots \left(\lambda_m(\sigma) I_\sigma^{\beta_m}\right)^{l_m} \phi(\sigma) + C\sum_{l=0}^{\infty}(-1)^l \\
&\quad \cdot \sum_{\substack{l_1+\cdots+l_{m+1}=l \\ l_1\geq 0,\cdots,l_{m+1}\geq 0}} \binom{l}{l_1,\cdots,l_{m+1}} \frac{C^{l_{m+1}}\, \theta^{l_1(\alpha_1+\gamma_1)+\cdots+l_{m+1}(\alpha_1-\alpha_2)+\alpha_1-\alpha_2}}{\Gamma(l_1(\alpha_1+\gamma_1)+\cdots+l_{m+1}(\alpha_1-\alpha_2)+\alpha_1-\alpha_2+1)} \\
&\quad \cdot \left(\lambda_1(\sigma) I_\sigma^{\beta_1}\right)^{l_1} \cdots \left(\lambda_m(\sigma) I_\sigma^{\beta_m}\right)^{l_m},
\end{aligned}$$

for $n = 0, 1, \cdots$.

Let W_{exact} be an exact solution to Equation (1). Then, it follows from Theorem 2 that

$$\begin{aligned}
&\|W_{n+1} - W_{exact}\| \leq q\|W_n - W_{exact}\| \leq q^2\|W_{n-1} - W_{exact}\| \leq \cdots \leq q^n\|W_1 - W_{exact}\| \\
&\leq 2q^n E_{(\alpha_1+\gamma_1,\cdots,\alpha_1+\gamma_m,\alpha_1-\alpha_2),\alpha_1+1}\left(Bb^{\beta_1},\cdots,Bb^{\beta_m},|C|\right)\|g(W_0) - g(W_{exact})\| \\
&\leq 4q^n E_{(\alpha_1+\gamma_1,\cdots,\alpha_1+\gamma_m,\alpha_1-\alpha_2),\alpha_1+1}\left(Bb^{\beta_1},\cdots,Bb^{\beta_m},|C|\right)\sup_{x\in R}|g(x)|,
\end{aligned}$$

which is the absolute error between W_{n+1} and W_{exact}.

We can apply Adomian's decomposition method to construct an algorithm for finding analytic and approximate solutions to Equation (1).

It follows from the proof of Theorem 1 that

$$\begin{aligned}
W(\theta,\sigma) \quad = \quad & \phi(\sigma)\left(1 + \frac{C\theta^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\right) - CI_\theta^{\alpha_1-\alpha_2}W(\theta,\sigma) - \sum_{i=1}^{m}\lambda_i(\sigma)I_\theta^{\alpha_1+\gamma_i}I_\sigma^{\beta_i}W(\theta,\sigma) \\
& + I_\theta^{\alpha_1}f(\theta,\sigma) + I_\theta^{\alpha_1}g(W(\theta,\sigma)).
\end{aligned} \tag{3}$$

let

$$W(\theta,\sigma) = W_1(\theta,\sigma) + W_2(\theta,\sigma) + W_3(\theta,\sigma) + \cdots, \tag{4}$$

by noting that $W_0 = 0$, and the nonlinear term

$$g(W(\theta,\sigma)) = A_1(W_1) + A_2(W_1,W_2) + \cdots. \tag{5}$$

Substituting (4) and (5) into (3), we have

$$\begin{aligned}
&W_1 + W_2 + W_3 + \cdots \\
&= \phi(\sigma)\left(1 + \frac{C\theta^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\right) + I_\theta^{\alpha_1} f(\theta,\sigma) - C I_\theta^{\alpha_1-\alpha_2}(W_1 + W_2 + W_3 + \cdots) \\
&\quad - \sum_{i=1}^{m} \lambda_i(\sigma) I_\theta^{\alpha_1+\gamma_i} I_\sigma^{\beta_i}(W_1 + W_2 + W_3 + \cdots) + I_\theta^{\alpha_1}(A_1 + A_2 + A_3 + \cdots),
\end{aligned}$$

which implies, using Adomian's decomposition method, that

$$\begin{aligned}
W_1 &= \phi(\sigma)\left(1 + \frac{C\theta^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\right) + I_\theta^{\alpha_1} f(\theta,\sigma), \\
W_{k+1} &= -C I_\theta^{\alpha_1-\alpha_2} W_k - \sum_{i=1}^{m} \lambda_i(\sigma) I_\theta^{\alpha_1+\gamma_i} I_\sigma^{\beta_i} W_k + I_\theta^{\alpha_1} A_k,
\end{aligned} \tag{6}$$

for $k = 1, 2, \cdots$.

Here, a couple of examples to establish A_k from a given nonlinear function $g(W)$ are presented. First, we assume that $g(W) = W\bar{W} = |W|^2$. Then,

$$\begin{aligned}
&(W_1 + W_2 + \cdots)(\bar{W}_1 + \bar{W}_2 + \cdots) = A_1 + A_2 + \cdots, \\
&= |W_1|^2 + 2W_1\bar{W}_2 + \cdots + \sum_{i=0}^{n-1} W_{i+1}\bar{W}_{n-i} + \cdots
\end{aligned}$$

which derives that

$$\begin{aligned}
A_1 &= |W_1|^2, \\
A_2 &= W_1\bar{W}_2 + \bar{W}_2 W_1 = 2W_1\bar{W}_2, \\
&\vdots \\
A_n &= \sum_{i=0}^{n-1} W_{i+1}\bar{W}_{n-i},
\end{aligned}$$

for $n = 1, 2, \cdots$ from [14].

Secondly, we consider $g(W) = \sin W$. Using Taylor's series, we obtain

$$g(W) = \sin W \approx W - \frac{W^3}{3!} + \frac{W^5}{5!}.$$

Thus,

$$\begin{aligned}
&(W_1 + W_2 + \cdots) - \frac{(W_1 + W_2 + \cdots)^3}{3!} + \frac{(W_1 + W_2 + \cdots)^5}{5!} \\
&= A_1 + A_2 + \cdots.
\end{aligned}$$

This infers that

$$\begin{aligned}
A_1 &= W_1 - \frac{W_1^3}{3!} + \frac{W_1^5}{5!}, \\
A_2 &= W_2 - \frac{W_1^2 W_2}{2!} + \frac{W_1^4 W_2}{4!}, \\
&\vdots
\end{aligned}$$

In summary, we are able to find analytic and approximate solutions to Equation (1) as

$$W = W_1 + W_2 + W_3 + \cdots + W_n,$$

from (6) and the recurrence of A_n for certain nonlinear functions $g(W)$.

4. Example

Example 1. *Consider the following fractional nonlinear partial integro-differential equation with the initial condition:*

$$\begin{cases} \frac{\partial}{\partial\theta}W(\theta,\sigma) + 2\frac{{}^{c}\partial^{0.5}}{\partial\theta^{0.5}}W(\theta,\sigma) + (\sigma^2+1)I_\theta I_\sigma^{0.5}W(\theta,\sigma) \\ \qquad = \theta + \frac{1}{22216010}\cos(W(\theta,\sigma)), \\ W(0,\sigma) = \sigma, \quad (\theta,\sigma) \in [0,1]\times[0,1]. \end{cases} \tag{7}$$

Then Equation (7) has a unique solution in the space $C([0,1]\times[0,1])$.

Proof. Clearly, $g(W) = \frac{1}{22216010}\cos(W(\theta,\sigma))$ is a continuous and bounded function satisfying the following Lipschitz condition for $W \in C([0,1]\times[0,1])$:

$$|g(x) - g(y)| \leq \frac{1}{22216010}|x-y|, \quad x, y \in R.$$

Furthermore,

$$\max_{\sigma\in[0,1]} |\sigma^2+1| \leq B = 2, \quad C = 2,$$

and

$$\begin{aligned} q &= 2\mathcal{L}E_{(\alpha_1+\gamma_1,\cdots,\alpha_1+\gamma_m,\alpha_1-\alpha_2),\alpha_1+1}\left(Bb^{\beta_1},\cdots,Bb^{\beta_m},|C|\right) \\ &= \frac{2}{22216010}E_{(2,\,0.5),2}(2,2) = \frac{2}{22216010}\sum_{l=0}^{\infty}\sum_{l_1+l_2=l}\binom{l}{l_1,l_2}\frac{2^{l_1}2^{l_2}}{\Gamma(2l_1+0.5l_2+2)}. \end{aligned}$$

Using

$$\sum_{l_1+l_2=l}\binom{l}{l_1,l_2} = 2^l,$$
$$\Gamma(2l_1+0.5l_2+2) = \Gamma(1.5l_1+0.5l+2) \geq \Gamma(0.5l+2),$$

we deduce that

$$q \leq \frac{2}{22216010}\sum_{l=0}^{\infty}\frac{4^l}{\Gamma(0.5l+2)} \approx \frac{2*1.1108*10^6}{22216010} \approx 0.1 < 1,$$

using online calculators from the site https://www.wolframalpha.com/ (accessed on 2 March 2023). By Theorem 2, Equation (7) has a unique solution in the space $C([0,1]\times[0,1])$. □

Finally, we are going to find an analytic and approximate solution to (7) using Recurrence (6). Let

$$g(W) = \frac{1}{22216010}\cos W \approx \frac{1}{22216010}\left(1 - \frac{W^2}{2!} + \frac{W^4}{4!}\right) = A_1 + A_2 + \cdots,$$

and

$$W = W_1 + W_2 + \cdots.$$

This infers that

$$A_1 = \frac{1}{22216010}\left(1 - \frac{W_1^2}{2!} + \frac{W_1^4}{4!}\right),$$
$$A_2 = \frac{1}{22216010}\left(-\frac{W_1 W_2}{1!} + \frac{W_1^3 W_2}{3!}\right),$$
$$\vdots$$

Moreover,

$$W_1 = \sigma\left(1 + \frac{2\,\theta^{0.5}}{\Gamma(0.5+1)}\right) + I_\theta \theta = \sigma\left(1 + \frac{4\,\theta^{0.5}}{\sqrt{\pi}}\right) + \frac{\theta^2}{2!},$$

and

$$W_2 = -2I_\theta^{0.5} W_1 - (\sigma^2+1) I_\theta^2 I_\sigma^{0.5} W_1 + I_\theta A_1.$$

Using MATHEMATICA and manual simplification, we finally obtain

$$\begin{aligned}
W_{appro} \approx & W_1 + W_2 \\
= & \sigma + \frac{\theta(24 - 2132736960\sigma - 12\sigma^2 + \sigma^4)}{533184240} + \frac{2\theta^{3/2}\sigma^2(-6+\sigma^2)}{99972045\sqrt{\pi}} \\
& + \theta^2\left(\frac{1}{2} + \frac{\sigma^2(-2+\sigma^2)}{11108005\pi} - \frac{2\sigma^{3/2}(1+\sigma^2)}{3\sqrt{\pi}}\right) \\
& + \theta^{5/2}\left(-\frac{16}{15\sqrt{\pi}} + \frac{32\sigma^4}{166620075\pi^{3/2}} - \frac{64\sigma^{3/2}(1+\sigma^2)}{45\pi}\right) \\
& + \theta^3\left(\frac{16\sigma^4}{99972045\pi^2} + \frac{\sigma(-6+\sigma^2)}{799776360}\right) + \frac{\theta^{7/2}\sigma(-2+\sigma^2)}{77756035\sqrt{\pi}} \\
& + \theta^4\left(\frac{\sigma^3}{22216010\pi} - \frac{\sqrt{\sigma}(1+\sigma^2)}{12\sqrt{\pi}}\right) + \frac{16\theta^{9/2}\sigma^3}{299916135\pi^{3/2}} + \frac{\theta^5(-2+\sigma^2)}{1777280800} \\
& + \frac{\theta^{11/2}\sigma^2}{244376110\sqrt{\pi}} + \frac{\theta^6\sigma^2}{133296060\pi} + \frac{\theta^7\sigma}{7464579360} + \frac{\theta^{15/2}\sigma}{1999440900\sqrt{\pi}} + \frac{\theta^9}{76778530560},
\end{aligned}$$

which is an approximation in a separate form of θ and σ, arranged in an increasing power of θ from 0 to 9. Clearly,

$$W_{appro}(0,\sigma) = \sigma,$$

which demonstrates that the initial condition is satisfied.

Clearly, the absolute error is

$$\left\|W_{appro} - W_{exact}\right\| \leq 4 * 0.1 * \frac{1.1168 * 10^6}{22216010} \approx 0.02. \quad (8)$$

Remark 1. *Since the exact solution to Equation (7) is unknown, we were unable to set up a table showing absolute errors between the approximate and exact solutions at particular values. In addition, there is no existing available research for comparison as the equation is new. However, Inequality (8) has plainly indicated the efficiency of our proposed method for the first two-term approximation only.*

5. Conclusions

We obtained sufficient conditions for the uniqueness of the bounded solution to the new Cauchy problem of the fractional nonlinear partial integro-differential Equation (1) using the multivariate Mittag–Leffler function as well as Banach's contractive principle. In addition, we constructed the algorithm for finding analytic and approximate solutions based on Adomian's decomposition method and presented an approximation order analysis, with one example showing the applications of the main results. Clearly, the technique used in this work can be applied to other nonlinear differential equations with initial or boundary conditions as well as integral equations with variable coefficients.

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