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A novel stability study on volterra integral equations with delay (VIE-D) using the fuzzy minimum optimal controller in matrix-valued fuzzy Banach spaces

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Abstract

Our main goal in this article is to investigate the Hyers–Ulam–Rassias stability (HURS) for a type of integral equation called Volterra integral equation with delay (VIE-D). First, by considering special functions such as the Wright function (WR), Mittag–Leffler function (ML), Gauss hypergeometric function (GH), $\mathscr{H}$-Fox function ($\mathscr{H}$-F), and also by introducing the aggregation function, we select the best control function by performing numerical calculations to investigate the stability of the desired equation. In the following, using the selected optimal function, i.e., the minimum function, we prove the existence of a unique solution and the HURS of the VI-D equation in the matrix-valued fuzzy space (MVFS) with two different intervals. At the end of each section, we provide a numerical example of the obtained results.

Keywords Mittag–Leffler function · Gauss hypergeometric function · Wright function · $\mathscr{H}$-Fox function · HU stability · HUR stability · Aggregation function (AF) · Optimal control function · Minimum function · Volterra integral equation with delay (VIE-D) · MVFB-spaces

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Abbreviations

HURS	Hyers–Ulam–Rassias stability
VIE-D	Volterra integral equation with delay
WR	Wright function
M-L	Mittag–Leffler function
Gauss hypergeometric function	Gauss hypergeometric function
$\mathscr{H}$-F	$\mathscr{H}$-Fox function
CGTN	Continuous generalized t-norm
HUS	Hyers–Ulam stability
MVFS	Matrix-valued fuzzy space
FPM	Fixed point method
S-BI-AM	Stochastic bi-additive mapping
S-O	Stochastic operator
$\mathbb{C}$-BI-SO	$\mathbb{C}$-bilinear stochastic operator

0 Introduction

For the first time, the concept of stability was used for functional equations (Hyers et al. 1998). The question is if we convert the functional equation into a functional inequality, what is the difference between the solutions obtained from the functional equation and the solutions obtained from the functional inequality? This subject has always been of interest to many researchers, and they have done much research to prove stability, such as Hyers–Ulam stability (HUS) and HURS for functional equations. The beginning of the stability proof was done by Hyers in 1941, which resulted in the following result (Hyers 1941; Ulam 1974, 1960). On two Banach spaces $\mathcal{J}_1$ and $\mathcal{J}_2$, for $\upsilon > 0$ we consider the function V such that $\|\mathrm{V}(\phi + \psi) - \mathrm{V}(\phi) - \mathrm{V}(\psi)\| \leq \upsilon$, for any $\phi, \psi \in \mathcal{J}_1$. Therefore, there is a unique additive function (UAF) $\mathrm{W} : \mathcal{I}_1 \to \mathcal{I}_2$ such that for V and W and for $\phi \in \mathcal{I}_1$, we have $\|\mathrm{W}(\phi) - \mathrm{V}(\phi)\| \leq \upsilon$. He also proved that for any $\phi \in \mathcal{I}_1$, $\mathrm{W}(\phi) = \lim_{\mathrm{m}\to+\infty} \frac{\mathrm{V}(2^{\mathrm{m}}(\phi)}{2^{\mathrm{m}}}$. Recently, much research has been conducted regarding the stability of Hyers–Ulam (HU) type for Cauchy's additive equation (CAE) $\Xi(\phi + \psi) = \Xi(\phi) + \Xi(\psi)$. Following these researches, many researchers expanded the topic of stability in different directions. Th. M. Rassias was among those who expanded this topic and considered functions with x and y variables for stability (Rassias 1978). Following Rassias's research, other researchers have also addressed this issue, see (Eidinejad et al. 2021; Eidinejad and Saadati 2022, 2021, 2022; Dong and Son 2022; Eidinejad et al. 2022; Moi et al. 2022). Despite the much research that has been done in this field, there are fewer studies on the stability of integral equations (Castro and Ramos 2009; Jung 2007). Therefore, in this work, we introduce a type of integral equation and prove the Hyers–Ulam–Rassias and Hyers–Ulam stability for this integral equation. Considering the continuous function $\Xi : [\imath, \jmath] \times [\imath, \jmath] \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ and the delayed continuous function $u : [\imath, \imath] \to [\imath, \jmath]$, where $\theta(\phi) \leq \phi$, $\imath$, $\jmath$ and ℓ are real constants, where $\imath < \jmath$ and $\ell \in (\imath, \jmath)$, we define Volterra integral equation with delay (VIE-D) as follows (Burton 2005; Corduneanu 1988; Gripenberg et al. 1990; Lakshmikantham and Rama 1995)

$$\psi(\phi) = \int_{\ell}^{\phi} \Xi(\phi, \zeta, \psi(\zeta), \psi(\theta(\zeta)))\mathrm{d}\zeta, \qquad (-\infty < \imath \leq \phi \leq \jmath < +\infty). \tag{0.1}$$

After Rassias's research, attention to specific functions to construct control functions found a special role (Eidinejad et al. 2022). Performing numerical calculations, we select the best control function from aggregation functions (AGs). This optimal function, which is the minimum function, is selected by the introduced special functions. In fact, since we are working with matrix-valued fuzzy spaces, our chosen optimal control function is also matrix-valued fuzzy type.

The remainder of this article is organized as follows: In the second section, all the required concepts, including the definition of special functions and examples of these functions, introduction of fuzzy spaces, calculations related to the selection of the optimal control function, lemmas, and basic theorems are presented. The minimum-Aggregation-Rassias-Stability for Volterra's integral equation introduced in the introduction section is investigated on a compact interval. In the fourth section, we prove the minimum-Aggregation-Rassias-Stability of this equation on the infinite interval using all the concepts in the previous section. At the end of each section, we provide a numerical example of all the obtained results to get a better understanding of the application of these results.

1 Preliminaries

As we mentioned in the introduction, in this section we present all the concepts needed for the main proofs. Since Banach's fixed point theorem (alternative fixed point theorem (AFPT)) is our main tool for investigating the VIE-D (0.1), we present this theorem at the beginning of the section. Before bringing AFPT, we state the details used in the theorem.

On the non-empty set $\mathscr{X}$, we consider the function $\Delta_{\mathscr{X}} : \mathscr{X} \times \mathscr{X} \to [0, +\infty]$. $\Delta_{\mathscr{X}}$ is a generalized metric space (GMS) if and only if the following conditions hold

(1) $\Delta_{\mathscr{X}}(\phi, \psi) = 0$ if and only if $\phi = \psi$;
(2) $\Delta_{\mathscr{X}}(\phi, \psi) = \Delta_{\mathscr{X}}(\psi, \phi)$ for all $\phi, \psi \in \mathscr{X}$;
(3) $\Delta_{\mathscr{X}}(\phi, \delta) \leq \Delta_{\mathscr{X}}(\phi, \psi) + \Delta_{\mathscr{X}}(\psi, \delta)$ for all $\phi, \psi, \delta \in \mathscr{X}$.

If we assume that $(\mathscr{X}, \Delta_{\mathscr{X}})$ is a generalized complete metric space (GCMS), we define a set as follows

$$CON(\mathscr{X}) := \Big\{ \mathrm{H} : \mathscr{X} \to \mathscr{X} | \Delta_{\mathscr{X}}(\mathrm{H}\mathscr{X}_1, \mathrm{H}\mathscr{X}_2) \leq \alpha_{\mathrm{H}} \Delta_{\mathscr{X}}(\psi_1, \psi_2), \\ \forall \psi_1, \psi_2 \in \mathscr{X}, \quad \alpha_{\mathrm{H}} \in [0, 1) \Big\}, \tag{1.1}$$

which is the set of all contraction mappings. Next, we present the Diaz–Margolis theorem (FTP) ([Eidinejad et al. 2021; Eidinejad and Saadati 2022]).

Theorem 1.1 *We consider GCMS* $(\mathscr{X}, \Delta_{\mathscr{X}})$ *and assume that* $\phi, \psi \in \mathscr{X}$, *and also* $\Lambda \in CON(\mathscr{X})$ *such that* $\alpha_\Lambda < 1$. *With these assumptions, we assume that for every* $\mathbf{s}, \mathbf{s}_0 \in \mathbb{N}$ $(\mathbf{s} \geq \mathbf{s}_0)$ *and for* $\phi \in \mathscr{X}$, $\Delta_{\mathscr{X}}\left(\Lambda^{\mathbf{s}}\psi, \Lambda^{\mathbf{s}+1}\psi\right) < +\infty$,. *If this condition holds, we have*

(1) The fixed point k *of* Λ *is the convergence point of the sequence* $\{\Lambda^{\mathbf{s}}\phi\}$;
(2) In the set $\mathcal{K} = \{\psi \in \mathscr{X} \mid \Delta_{\mathscr{X}}(\Lambda^{\mathbf{s}_0}\phi, \psi) < +\infty\}$, k *is the unique fixed point of* Λ;
(3) $(1 - \alpha_\Lambda)\Delta_{\mathscr{X}}(\psi, \mathrm{k}) \leq \Delta_{\mathscr{X}}(\psi, \Lambda\psi)$ *for every* $\psi \in \mathscr{X}$.

We denote the set of all m × m diagonal matrices by $\mathrm{D}_{\mathrm{E_m}} = \mathrm{diag}\,\mathrm{E_m}([0,1])$, and we consider this set as follows

$$\mathrm{D}_{\mathrm{E_m}} = \mathrm{diag}\,\mathrm{E_m}([0,1]) = \left\{ \begin{bmatrix} \mathrm{d}_1 & & \\ & \ddots & \\ & & \mathrm{d_m} \end{bmatrix} = \mathrm{diag}[\mathrm{d}_1, \ldots, \mathrm{d_m}],\ \mathrm{d}_1, \ldots, \mathrm{d_m} \in [0,1] \right\}.$$

For the above set, we have

- If $\mathbf{d}, \mathbf{f} \in \mathrm{D}_{\mathrm{E_m}}$, then $\mathbf{d} = \mathrm{diag}[\mathrm{d}_1, \ldots, \mathrm{d_m}]$ and $\mathbf{f} = \mathrm{diag}[\mathrm{f}_1, \ldots, \mathrm{f_m}]$;
- $\mathbf{d} \preceq \mathbf{f}$ means that $\mathrm{d}_\iota \le \mathrm{m}_\iota$ for every $\iota = 1, \ldots, \mathrm{m}$;
- $\mathbf{d} \prec \mathbf{f}$ denotes that $\mathbf{d} \preceq \mathbf{f}$ and $\mathbf{d} \neq \mathbf{f}$;
- $\mathrm{diag}[1, \ldots, 1] = \mathbf{1}$ and $\mathrm{diag}[0, \ldots, 0] = \mathbf{0}$.

Definition 1.2 (Eidinejad et al. (2021)) A mapping $\circledast : \mathrm{D}_{\mathrm{E_m}} \times \mathrm{D}_{\mathrm{E_m}} \to \mathrm{D}_{\mathrm{E_m}}$ is called a GTN if the boundary condition, commutativity condition, associativity condition and monotonicity condition are established as follows:

(I) $\mathbf{f} \circledast \mathbf{1} = \mathbf{d}$ for all $\mathbf{d} \in \mathrm{D}_{\mathrm{E_m}}$;
(II) $\mathbf{d} \circledast \mathbf{f} = \mathbf{f} \circledast \mathbf{d}$ for all $\mathbf{d}, \mathbf{f} \in \mathrm{D}_{\mathrm{E_m}}$;
(III) $\mathbf{d} \circledast (\mathbf{f} \circledast \mathbf{g}) = (\mathbf{d} \circledast \mathbf{f}) \circledast \mathbf{g}$ for all $\mathbf{d}, \mathbf{f}, \mathbf{g} \in \mathrm{D}_{\mathrm{E_m}}$;
(IV) $\mathbf{d} \preceq \mathbf{f}$ and $\mathbf{g} \preceq \mathbf{e}$ implies that $\mathbf{d} \circledast \mathbf{g} \preceq \mathbf{f} \circledast \mathbf{e}$, for all $\mathbf{d}, \mathbf{g}, \mathbf{f}, \mathbf{e} \in \mathrm{D}_{\mathrm{E_m}}$. For convergent sequences $\{\mathbf{d}_\mathrm{m}\}$ and $\{\mathbf{f}_\mathrm{m}\}$ with convergence points $\mathbf{d}$ and $\mathbf{f}$, if we have
(V) $\lim_\mathrm{m}(\mathbf{d}_\mathrm{m} \circledast \mathbf{f}_\mathrm{m}) = \mathbf{d} \circledast \mathbf{f}$,

then the GTN $\circledast$ is a continuous mapping (CGTN).

In the following, we will give examples of CGTN:

Example 1.3 $\circledast_M : \mathrm{D}_{\mathrm{E_m}} \times \mathrm{D}_{\mathrm{E_m}} \to \mathrm{D}_{\mathrm{E_m}}$ is called minimum CGTN (MIN-CGTN), which is defined as follows

$$\mathbf{d} \circledast_M \mathbf{f} = \mathrm{diag}[\mathrm{d}_1, \cdots, \mathrm{d_m}] \circledast_\mathrm{M} \mathrm{diag}[\mathrm{f}_1, \cdots, \mathrm{f_m}] = \mathrm{diag}[\min\{\mathrm{d}_1, \mathrm{f}_1\}, \cdots, \min\{\mathrm{d_m}, \mathrm{f_m}\}].$$

Example 1.4 $\circledast_P : \mathrm{D}_{\mathrm{E_m}} \times \mathrm{D}_{\mathrm{E_m}} \to \mathrm{D}_{\mathrm{E_m}}$ is called product CGTN (P-CGTN), which is defined as follows

$$\mathbf{d} \circledast_P \mathbf{f} = \mathrm{diag}[\mathrm{d}_1, \cdots, \mathrm{d_m}] \circledast_\mathrm{P} \mathrm{diag}[\mathrm{f}_1, \cdots, \mathrm{f_m}] = \mathrm{diag}[\mathrm{d}_1.\mathrm{f}_1, \cdots, \mathrm{d_m}.\mathrm{f_m}].$$

Example 1.5 $\circledast_L : \mathrm{D}_{\mathrm{E_m}} \times \mathrm{D}_{\mathrm{E_m}} \to \mathrm{D}_{\mathrm{E_m}}$ is called Lukasiewicz CGTN(L-CGTN), which is defined as follows

$$\begin{aligned} \mathbf{d} \circledast_L \mathbf{f} &= \mathrm{diag}[\mathrm{d}_1, \cdots, \mathrm{d_m}] \circledast_\mathrm{L} \mathrm{diag}[\mathrm{f}_1, \cdots, \mathrm{f_m}] \\ &= \mathrm{diag}[\max\{\mathrm{d}_1 + \mathrm{f}_1 - 1, 0\}, \cdots, \max\{\mathrm{d_m} + \mathrm{f_m} - 1, 0\}]. \end{aligned}$$

For MIN-CGTN, P-CGTN and L-CGTN introduced in the above examples, the following inequality always holds (we have considered the dimension of the matrix as 3)

$$\begin{aligned} \mathrm{diag}\left[\mathrm{d}, \mathrm{f}, \mathrm{g}\right] \circledast_\mathrm{M} \mathrm{diag}\,[\mathrm{e}, \mathrm{h}, \mathrm{k}] &\succeq \mathrm{diag}\left[\mathrm{d}, \mathrm{f}, \mathrm{g}\right] \circledast_\mathrm{P} \mathrm{diag}\,[\mathrm{e}, \mathrm{h}, \mathrm{k}] \\ &\succeq \mathrm{diag}\left[\mathrm{d}, \mathrm{f}, \mathrm{g}\right] \circledast_\mathrm{L} \mathrm{diag}\,[\mathrm{e}, \mathrm{h}, \mathrm{k}]. \end{aligned}$$

Refer to Eidinejad et al. (2021); Eidinejad and Saadati (2022); Eidinejad et al. (2022) to see more numerical examples of the introduced CGTNs.

Definition 1.6 (Eidinejad et al. (2021)) Consider the matrix-valued fuzzy function (MVFF) $\mathcal{Z} : [0, \jmath]\times(0, +\infty) \to \mathrm{D}_{\mathrm{E_m}}$, then we have,

(MVFF1) MVFF is increasing and continuous;
(MVFF2) $\lim_{\omega\to+\infty} \mathcal{Z}(\phi, \omega) = \mathbf{1}$ for any $\phi \in [0, \jmath]$ and $\omega \in (0, +\infty)$.
(MVFF3) If $\mathcal{S}$ is another MVFF, the relationship of " $\preceq$" for these functions is defined $\mathcal{Z} \precsim \mathcal{S} \iff \mathcal{Z}(\phi, \omega) \preceq \mathcal{S}(\phi, \omega)$, for all $\omega \in (0, +\infty)$ and $\phi\in[0, \jmath]$.

Definition 1.7 (Eidinejad et al. (2021, 2022)) Let $\circledast$ be a CGTN, $\mathscr{X}$ be a vector space and $\mathscr{N}_{\mathscr{X}} : \mathscr{X}\times(0, +\infty) \to \mathrm{D}_{\mathrm{E_m}}$ be a matrix valued fuzzy set (MVFS). Triple $(\mathscr{X}, \mathscr{N}_{\mathscr{X}}, \circledast)$ is called a matrix-valued fuzzy normed space (MVFN-space) if

(NORM1) $\mathscr{N}_{\mathscr{X}}(\phi, \omega) = \mathbf{1}$ if and only if $\phi = 0$ for $\omega \in (0, +\infty)$;
(NORM2) $\mathscr{N}_{\mathscr{X}}(\gamma\phi, \omega) = \mathscr{N}_{\mathscr{X}}(\phi, \frac{\omega}{|\gamma|})$ for all $\phi \in \mathscr{X}$ and $\gamma \in \mathbb{C}$ with $\gamma \neq 0$;
(NORM3) $\mathscr{N}_{\mathscr{X}}(\phi + \psi, \omega + \nu) \succeq \mathscr{N}_{\mathscr{X}}(\phi, \omega) \circledast \mathscr{N}_{\mathscr{X}}(\psi, \nu)$ for all $\phi \in \mathscr{X}$ and any $\omega, \nu \in (0, +\infty)$;
(NORM4) $\lim_{\omega\to+\infty} \mathscr{N}_{\mathscr{X}}(\phi, \omega) = \mathbf{1}$ for any $\omega \in (0, +\infty)$.

There are many examples of the MVFN-space, which we can mention in the following two

$$\mathscr{N}_{\mathscr{X}}(\phi, \omega) = \mathrm{diag}\left[\frac{!}{! + \|\text{Œ}\|}, \frac{!}{! + \|\text{Œ}\|}\right];$$
$$\mathscr{N}_{\mathscr{X}}(\phi, \omega) = \mathrm{diag}\left[\sum_{\mathrm{m}=0}^{+\infty}\frac{(\frac{\|\text{Œ}\|}{!})^{\mathrm{m}}}{\text{0(mˇ + ¯)}}, \sum_{\mathrm{m}=0}^{+\infty}\frac{(\frac{\|\text{Œ}\|}{!})^{\mathrm{m}}}{\text{0(mˇ + ¯)}}\right]. \tag{1.2}$$

When a MVFN-space is complete we denote it by MVFB-space (Eidinejad et al. 2021; Eidinejad and Saadati 2021, 2022; Eidinejad et al. 2022). In the following, we introduce special functions that we need to select the optimal control function. We also perform the necessary calculations on these functions and show the obtained information in the form of graphs and tables.

Definition 1.8 (Eidinejad et al. 2022) Let ϕ be a real number and consider the generic parameters $\mathfrak{p}, \mathfrak{q}, \mathfrak{r} > 0$. We define the Gauss hypergeometric function ${}_2F_1 : \mathbb{R}^3 \times \mathcal{T} \longrightarrow (0, +\infty)$ by the infinite sum (that is convergent)

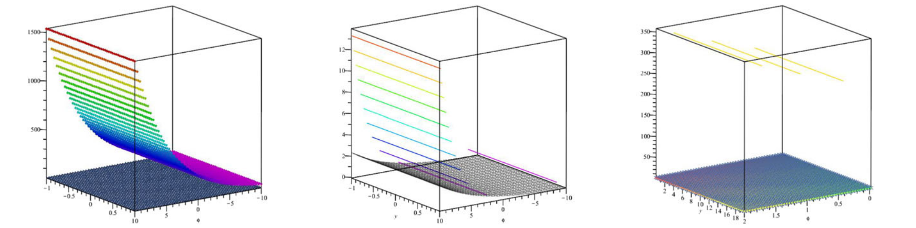
$${}_2\mathscr{F}_1(\mathrm{a}, \mathrm{b}, \mathrm{c}; \phi) = \sum_{\mathrm{m}=0}^{+\infty}\frac{(\mathrm{a})_{\mathrm{m}}(\mathrm{b})_{\mathrm{m}}}{(\mathrm{c})_{\mathrm{m}}}\frac{\phi^{\mathrm{m}}}{\mathrm{m}!} = \frac{\Gamma(\mathrm{c})}{\Gamma(\mathrm{a})\Gamma(\mathrm{b})}\sum_{\mathrm{m}=0}^{+\infty}\frac{\Gamma(\mathrm{a}+\mathrm{m})\Gamma(\mathrm{b}+\mathrm{m})}{\Gamma(\mathrm{c}+\mathrm{m})}\frac{\phi^{\mathrm{m}}}{\mathrm{m}!}.$$

Definition 1.9 (Eidinejad and Saadati 2022; Eidinejad et al. 2022) One-parameter and two-parameter Mittag–Leffler functions (M-LF) are defined as follows, respectively,

$$\mathscr{E}_{\kappa}(\phi) = \sum_{\mathrm{m}=0}^{+\infty}\frac{\phi^{\mathrm{m}}}{\Gamma(\mathrm{m}\kappa + 1)},$$
$$\mathscr{E}_{\kappa,\mu}(\phi) = \sum_{\mathrm{m}=0}^{+\infty}\frac{\phi^{\mathrm{m}}}{\Gamma(\mathrm{m}\kappa + \mu)},$$

where $\kappa, \mu \in \mathbb{C},\ Re(\kappa), Re(\mu) > 0$ and $\Gamma(.)$ used in the above functions is the famous gamma function.

Definition 1.10 (Eidinejad et al. 2022) For $0 \le \mathrm{x} \le \mathrm{s}, 1 \le \mathrm{y} \le \mathrm{t}, \{\mathrm{p}_\iota, \mathrm{q}_\iota\} \in \mathbb{C}, \{\phi_\iota, \psi_\iota\} \in \mathbb{R}^+$, we define the following functions



(a) Graphs of exponential function and $\mathscr{H}$-Fox function for $\phi \in (-10, 10)$

(b) Graphs of Wright function and Mittag-Leffler function for $\phi \in (-10, 10)$

(c) Graphs of exponential function, $\mathscr{H}$-Fox function, Wright functions and Mittag-Leffler function for $\phi \in (-0.02, 2)$

Fig. 1 Graphs of exponential function, $\mathscr{H}$-Fox function, Wright function and Mittag–Leffler function for different values ϕ

- $\mathscr{F}(\mathrm{z}) = \prod_{\iota=1}^{\mathrm{x}} \Gamma(\mathrm{q}_\iota - \psi_\iota \mathrm{z})$,
- $\mathscr{G}(\mathrm{z}) = \prod_{\iota=1}^{\mathrm{m}} \Gamma(1 - \mathrm{p}_\iota + \phi_\iota \mathrm{z})$,
- $\mathscr{P}(\mathrm{z}) = \prod_{\iota=\mathrm{y}+1}^{\mathrm{t}} \Gamma(1 - \mathrm{q}_\iota + \psi_\iota \mathrm{z})$,
- $\mathscr{L}(\mathrm{z}) = \prod_{\iota=\mathrm{x}+1}^{\mathrm{s}} \Gamma(\mathrm{p}_\iota - \phi_\iota \mathrm{z})$.

In the introduced functions $\mathrm{x} = 0$ if and only if $\mathscr{G}(\mathrm{z}) = 1$, $m = q$ if and only if $\mathscr{P}(\mathrm{z}) = 1$ and $\mathrm{x} = \mathrm{s}$ if and only if $\mathscr{L}(\mathrm{z}) = 1$. According to the introduced functions, we consider $\mathscr{H}_{\mathrm{p,t}}^{\mathrm{y,x}}(\mathrm{z}) = \frac{\mathscr{F}(\mathrm{z})\mathscr{G}(\mathrm{z})}{\mathscr{P}(\mathrm{z})\mathscr{L}(e)}$. The Mellin–Barnes integral (M-BI) representation of $\mathscr{H}$-Fox function ($\mathscr{H}$-FF) is shown as below

$$\mathcal{H}_{\mathrm{s,t}}^{\mathrm{y,x}}(\phi) = \frac{1}{2\pi i} \int_{\mathcal{T}} \mathscr{H}_{\mathrm{s,t}}^{\mathrm{y,x}}(\mathrm{z}) \phi^{\mathrm{z}} d\mathrm{z}, \tag{1.3}$$

where $\phi^{\mathrm{z}} = \exp\{\mathrm{z}(\log|\phi| + i \arg \phi)\}$. Also, the symbol $\mathcal{H}_{\mathrm{s,t}}^{\mathrm{y,x}}(\phi) = \mathcal{H}_{\mathrm{s,t}}^{\mathrm{y,x}}\left[\phi \left| \begin{matrix} (\mathrm{p}_\iota, \epsilon_\iota)_{\iota=1,\ldots,\mathrm{s}} \\ (\mathrm{q}_\iota, \lambda_\iota)_{\iota=1,\ldots,ps} \end{matrix} \right.\right]$ is considered for this integral.

Definition 1.11 (Eidinejad and Saadati 2022, ?; Eidinejad et al. 2022) The generalized Bessel Maitland function (GBMF) or Wright function (WF) of order $1/(1+\sigma)$ is represented using the series as

$$\mathscr{W}_{\kappa,\mu}(\phi) = \sum_{\mathrm{m}=0}^{+\infty} \frac{\phi^{\mathrm{m}}}{\mathrm{m}!\Gamma(\kappa\mathrm{m} + \mu)},$$

for $\kappa > -1, \mu > 0, \phi \in \mathbb{R}$,

Definition 1.12 (Eidinejad et al. 2022) an m-ary ($\mathrm{m} \in \mathbb{N}$) generalized aggregation function (m-AGAF) $\mathscr{A}^{(\mathrm{m})} : \mathbb{R}^{\mathrm{m}} \longrightarrow \mathbb{R}$ has the following property

- $\mathrm{x}_\iota \leq \mathrm{y}_\iota$ results that $\mathscr{A}^{(\mathrm{m})}(\mathrm{x}_1, \cdots, \mathrm{x}_\mathrm{m}) \leq \mathscr{A}^{(\mathrm{m})}(\mathrm{y}_1, \cdots, \mathrm{y}_\mathrm{m})$,

for all $\iota \in \{1, \ldots, \mathrm{m}\}$, and for $(\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{m})$, $(\mathrm{y}_1, \cdots, \mathrm{y}_\mathrm{m}) \in \mathbb{R}^{\mathrm{m}}$. For the sake of simplicity, we can remove the m number, which represents the number of variables of the aggregation function, and denote this function as $\mathscr{A}$. Also, when $\mathrm{m} = 1$, the aggregation function is shown as $\mathscr{A}^{(1)}(\phi) = \phi$ for all $\phi \in \mathbb{R}$.

What is shown in Fig. 1 is a graphical representation of the functions that we have introduced to define the control function. Figure 1 shows these functions for different values. The following are examples of generalized aggregation functions:

Example 1.13 The arithmetic mean function (AMF) $AM : \mathbb{R}^{\mathrm{m}} \longrightarrow \mathbb{R}$ is defined by

$$AM(\phi) = \frac{1}{\phi}\sum_{\iota=1}^{\mathrm{m}} \phi_\iota.$$

Example 1.14 The geometric mean function (GMF) $GM : \mathbb{R}^{\mathrm{m}} \longrightarrow \mathbb{R}$ is defined by

$$GM(\phi) = \left(\prod_{\iota=1}^{\mathrm{m}} \phi_\iota\right)^{\frac{1}{\mathrm{m}}}.$$

Example 1.15 The projection function (PF) $\mathscr{P}_n : \mathbb{R}^{\mathrm{m}} \longrightarrow \mathbb{R}$ for $n \in [\mathrm{m}]$ and nth argument is defined by

$$\mathscr{P}_n(\phi) = \phi_n,$$

where $\phi_{(n)}$ is the nth lowest coordinate of ϕ, i.e., $\phi_{(1)} \leq \cdots \leq \phi_{(n)} \leq \cdots \phi_{(\mathrm{m})}$. Also, the following functions show the PF in the first and last coordinates

$$\begin{aligned} \mathscr{P}_F(\phi) &= \mathscr{P}_1(\phi) = \phi_1, \\ \mathscr{P}_L(\phi) &= \mathscr{P}_{\mathrm{m}}(\phi) = \phi_{\mathrm{m}}. \end{aligned} \tag{1.4}$$

Example 1.16 The order statistic function (OSF) $OS_n : \mathbb{R}^{\mathrm{m}} \longrightarrow \mathbb{R}$ with the nth argument and nth lowest coordinate is defined by

$$OS_n(\phi) = \phi_{(n)},$$

for any $n \in [\mathrm{m}]$.

Example 1.17 The minimum function (MIN-F) and maximum function (MAX-F) are defined as follows, respectively,

$$\begin{aligned} MIN(\phi) &= OS_1(\phi) = \min\{\phi_1, \cdots, \phi_{\mathrm{m}}\} = \bigwedge_{\iota=1}^{\mathrm{m}} \phi_\iota, \\ MAX(\phi) &= OS_{\mathrm{m}}(\phi) = \max\{\phi_1, \cdots, \phi_{\mathrm{m}}\} = \bigvee_{\iota=1}^{\mathrm{m}} \phi_\iota. \end{aligned} \tag{1.5}$$

Example 1.18 The median function (MF) is defined as follows for odd and even values of $(\phi_1, \ldots, \phi_{2n-1})$ and $(\phi_1, \cdots, \phi_{2n})$, respectively,

$$\begin{aligned} MED(\phi_1, \ldots, \phi_{2n-1}) &= \phi_{(n)}, \\ MED(\phi_1, \ldots, \phi_{2n}) &= AM(\phi_{(n)}, \phi_{(n+1)}) = \frac{\phi_{(n)} + \phi_{(n+1)}}{2}. \end{aligned} \tag{1.6}$$

We are looking for the optimal control function to investigate VIE-D (0.1), that is, to prove the existence of a unique solution and the stability of this equation. Tables below gives us good information about the introduced functions so that can select the appropriate control function. By studying the information in these tables, we choose the minimum function as the optimal control function.

Consider the function

$$\Pi(\Phi,\omega)=\mathrm{diag}\Bigg[{}_2\mathscr{F}_1\left(\mathrm{a},\mathrm{b},\mathrm{c},\frac{-\|Œ\|}{!}\right),\mathscr{E}_{,\text{-}}\left(\frac{-\|Œ\|}{!}\right),\mathscr{W}_{,\text{-}}\left(\frac{-\|Œ\|}{!}\right),\\ \mathcal{H}^{\mathrm{y,x}}_{\mathrm{s,t}}\left(\frac{-\|\phi\|}{\omega}\right),\exp\left(\frac{-\|\phi\|}{\omega}\right)\Bigg]. \tag{1.7}$$

The table below shows the different values of the aggregation functions for Π.

By comparing the values obtained in the Table 1, we consider the minimum aggregate function

$$\boldsymbol{MIN}\Big(\boldsymbol{\Pi(\phi,\omega)}\Big)=\mathrm{diag}\Big[\bigwedge\Big(\Pi(\phi,\omega)\Big),\bigwedge\Big(\Pi(\phi,\omega)\Big),\bigwedge\Big(\Pi(\phi,\omega)\Big),\bigwedge\Big(\Pi(\phi,\omega)\Big)\Big], \tag{1.8}$$

as a control function. Figure 2 shows the graphical representation of aggregation functions for different values.

Definition 1.19 Let function $\boldsymbol{MIN}\Big(\boldsymbol{\Pi(\phi,\omega)}\Big)$ be a MVF function. The equation (0.1) is said to be Minimum-Aggregation-stable, if ψ is a given continuous function, satisfying

$$\mathscr{N}_{\mathscr{X}}\left(\psi(\phi)-\int_0^{\phi}\Xi(\phi,\zeta,\psi(\zeta),\psi(\theta(\zeta)))\mathrm{d}\zeta,\omega\right)\succeq\boldsymbol{MIN}\Big(\boldsymbol{\Pi(\phi,\omega)}\Big), \tag{1.9}$$

for $\phi\in[0,\jmath]$, and we can find a solution ψ_0 of equation (0.1) such that for some $\varepsilon>0$,

$$\mathscr{N}_{\mathscr{X}}\Big(\psi(\phi)-\psi_0(\phi)),\omega\Big)\succeq\boldsymbol{MIN}\Big(\boldsymbol{\Pi(\phi,\frac{\omega}{\gamma})}\Big).$$

In Definition 1.9, if we consider the minimum control function to be constant, then VIE-D (0.1) has the Hyers–Ulam stability in the MVFBS, which we will prove in the fifth section.

2 Minimum-aggregation-stability of the VIE-D in the compact interval case

Now, we use the fixed point method based on the Theorem 1.1 to show (0.1) is Minimum-Aggregation-stable (Eidinejad and Saadati 2021) in MVFB-space $(\mathscr{X},\mathscr{N}_{\mathscr{X}},\circledast)$ with optimal MVFF $\boldsymbol{MIN}\Big(\boldsymbol{\Pi(\phi,\omega)}\Big)$. First, we prove the following theorem, which is a prerequisite of the main theorem

Theorem 2.1 *We consider the set of all continuous functions as* $\mathcal{M}=\{\psi:[\imath,\jmath]\to\mathbb{C},\psi$ *is continuous*$\}$. *We define the mapping* $\Delta_{\mathcal{M}}:\mathcal{M}\times\mathcal{M}\to[0,+\infty]$ *on this set, for all* $\psi,\mathcal{G}\in\mathcal{M},\quad\phi\in[\imath,\jmath],\quad\omega\in(0,+\infty)$ *as follows*

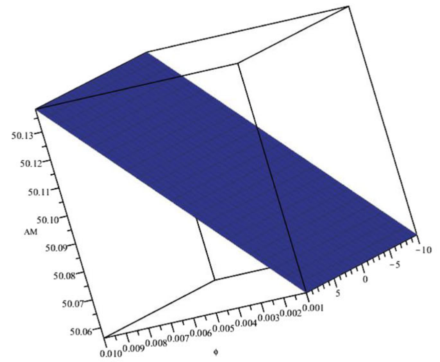
$$\Delta_{\mathcal{M}}(\psi,\mathcal{G})=\inf\left\{\beta\in[0,+\infty):\ \mathscr{N}_{\mathscr{X}}\Big(\psi(\phi)-\mathcal{G}(\phi),\omega\Big)\succeq\boldsymbol{MIN}\Big(\boldsymbol{\Pi(\phi,\frac{\omega}{\beta})}\Big)\right\}.$$

We prove that $(\mathcal{M},\Delta_{\mathcal{M}})$ *is a CGMS.*

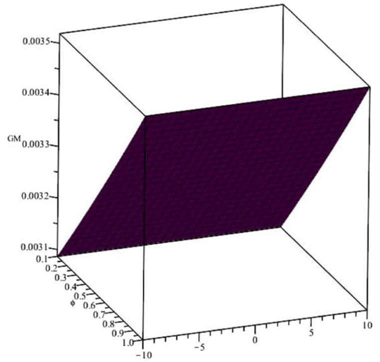
Proof To prove this theorem, we investigate all conditions of GCMS for $(\mathcal{M},\Delta_{\mathcal{M}})$.

Table 1 Aggregation functions for values between [0, 1]

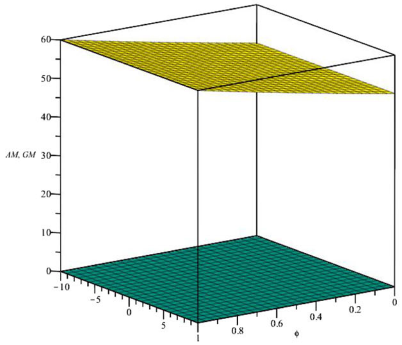
ϕ	$AM(\Pi)$	$GM(\Pi)$	$MAX(\Pi)$	$MIN(\Pi)$	$MED(\Pi)$
0.01	0.8082272830	0.001212411587	2.489115201	$1.376998069 \times 10^{(-14)}$	0.2766355873
0.02	2.517784924	0.001808433979	11.03650901	$1.378146039 \times 10^{(-14)}$	0.2774577317
0.03	11.31176500	0.002256733251	55.00600897	$1.379294982 \times 10^{(-14)}$	0.2782828628
0.04	23.33881550	0.002618921411	115.1408550	$1.380444898 \times 10^{(-14)}$	0.2791109936
0.05	41.02507214	0.002938314792	203.5717257	$1.381595787 \times 10^{(-14)}$	0.2799421374
0.06	63.75695342	0.003214459871	317.2307134	$1.382747652 \times 10^{(-14)}$	0.2807763072
0.07	40.72074192	0.002940372940	202.0492313	$1.383900493 \times 10^{(-14)}$	0.2816135167
0.08	46.53587898	0.003023834825	231.1244858	$1.385054311 \times 10^{(-14)}$	0.2824537789



(a) The aggregation AM function for$\phi \in (0.001, 0.01)$



(b) The aggregation GM function for $\phi \in (0.1, 1)$



(c) AM and GM function for $\phi \in (0, 1)$

Fig. 2 Graph of aggregate functions AM and GM for $\omega = 8$ and different values ϕ

(*1*) Assume that $\Delta_{\mathcal{M}}(\psi, \mathcal{G}) = 0$, then for all $\psi, \mathcal{G} \in \mathcal{M}$, $\phi \in [\imath, \jmath]$, $\omega \in (0, +\infty)$, we have

$$\inf \left\{ \beta \in [0, +\infty) : \mathscr{N}_{\mathscr{X}} \left(\psi(\phi) - \mathcal{G}(\phi), \omega \right) \succeq \boldsymbol{MIN} \left(\boldsymbol{\Pi}(\boldsymbol{\phi}, \frac{\boldsymbol{\omega}}{\boldsymbol{\beta}}) \right) \right\} = 0,$$

and this means $\mathscr{N}_{\mathscr{X}} \left(\psi(\phi) - \mathcal{G}(\phi), \omega \right) \succeq \boldsymbol{MIN} \left(\boldsymbol{\Pi}(\boldsymbol{\phi}, \frac{\boldsymbol{\omega}}{\boldsymbol{\beta}}) \right)$, for all $\beta \in [0, +\infty)$. Tend β to zero in this inequality, we get $\mathscr{N}_{\mathscr{X}} \left(\psi(\phi) - \mathcal{G}(\phi), \omega \right) = \mathbf{1}$. Thus, $\psi(\phi) = \mathcal{G}(\phi)$ for every $\phi \in [\imath, \jmath]$, and vice versa.

(*2*) It is easy to see that $\Delta_{\mathcal{M}}(\psi, \mathcal{G}) = \Delta_{\mathcal{M}}(\mathcal{G}, \psi)$ for every $\psi, \mathcal{G} \in \mathcal{M}$.

(*3*) Let $\Delta_{\mathcal{M}}(\psi, \mathcal{G}) = \eta_1 \in (0, +\infty)$ and $\mathscr{N}_{\mathscr{X}}(\mathcal{G}, \mathcal{R}) = \eta_2 \in (0, +\infty)$. Then, for every $\omega \in (0, +\infty)$ we have $\mathscr{N}_{\mathscr{X}} \left(\psi(\phi) - \mathcal{G}(\phi), \omega \right) \succeq \boldsymbol{MIN} \left(\boldsymbol{\Pi}(\boldsymbol{\phi}, \frac{\boldsymbol{\omega}}{\boldsymbol{\eta_1}}) \right)$, and $\mathscr{N}_{\mathscr{X}} \left(\mathcal{G}(\phi) - \mathcal{R}(\phi), \omega \right) \succeq \boldsymbol{MIN} \left(\boldsymbol{\Pi}(\boldsymbol{\phi}, \frac{\boldsymbol{\omega}}{\boldsymbol{\eta_2}}) \right)$. We get

$$\mathscr{N}_{\mathscr{X}} \left((\psi(\phi) - \mathcal{R}(\phi), (\eta_1 + \eta_2)\omega \right)$$

$$\succeq\Big[\mathscr{N}_{\mathscr{X}}\Big((\psi(\phi)-\mathcal{G}(\phi),(\eta_1)\omega\Big)\circledast\mathscr{N}_{\mathscr{X}}\Big((\mathcal{G}(\phi)-\mathcal{R}(\phi),(\eta_2)\omega\Big)\Big]$$
$$\succeq \boldsymbol{MIN}\Big(\boldsymbol{\Pi}(\boldsymbol{\phi},\boldsymbol{\omega})\Big)\circledast \boldsymbol{MIN}\Big(\boldsymbol{\Pi}(\boldsymbol{\phi},\boldsymbol{\omega})\Big)$$
$$=\boldsymbol{MIN}\Big(\boldsymbol{\Pi}(\boldsymbol{\phi},\boldsymbol{\omega})\Big).$$

Therefore, $\Delta_{\mathcal{M}}(\mathcal{G},\mathcal{R})\leq\eta_1+\eta_2$ and $\Delta_{\mathcal{M}}(\mathcal{G},\mathcal{R})\leq\Delta_{\mathcal{M}}(\psi,\mathcal{G})+\Delta_{\mathcal{M}}(\mathcal{G},\mathcal{R})$.

(*4*) To show the completeness of $(\mathcal{M},\Delta_{\mathcal{M}})$, we suppose that $\{\mathcal{G}_{\mathrm{m}}\}_{\mathrm{m}}$ is a Cauchy sequence in $(\mathcal{M},\Delta_{\mathcal{M}})$. Assume that $\phi\in[\imath,\jmath]$ and $\omega\in(0,+\infty)$, $\varkappa\in[0,1]^{\circ}$ are arbitrary such that $\boldsymbol{MIN}\Big(\boldsymbol{\Pi}(\boldsymbol{\phi},\boldsymbol{\omega})\Big)\succ\mathbf{1}-\varkappa$. For $\rho\omega<\vartheta$ choose $\mathrm{r}_0\in\mathbb{N}$ such that for all $\mathrm{r}_1,\mathrm{r}_2\geq\mathrm{r}_0$, we have $\Delta_{\mathcal{M}}(\mathcal{G}_{\mathrm{r}_1},\mathcal{G}_{\mathrm{r}_2})<\rho$. Then,

$$\mathscr{N}_{\mathscr{X}}\Big(\mathcal{G}_{\mathrm{r}_1}(\phi)-\mathcal{G}_{\mathrm{r}_2}(\phi),\vartheta\Big)\succeq\mathscr{N}_{\mathscr{X}}\Big(\mathcal{G}_{\mathrm{r}_1}(\phi)-\mathcal{G}_{\mathrm{r}_2}(\phi),\rho\omega\Big)\succeq\boldsymbol{MIN}\Big(\boldsymbol{\Pi}(\boldsymbol{\phi},\boldsymbol{\omega})\Big)\succ\mathbf{1}-\varkappa,$$

and this means $\mathscr{N}_{\mathscr{X}}\Big(\mathcal{G}_{\mathrm{r}}(\phi)-\mathcal{G}_{\mathrm{r}}(\phi),\omega\Big)\succ\mathbf{1}-\varkappa$, which implies that, the sequence $\{\mathcal{G}_{\mathrm{r}}(\phi)\}_{\mathrm{m}}$ is Cauchy in complete space $(\mathscr{X},\mathscr{N}_{\mathscr{X}},\circledast)$ on compact set $[\imath,\jmath]$. Then, it is uniformly convergent to the mapping $\mathcal{G}:[\imath,\jmath]\rightarrow\mathscr{X}$. By uniform convergence property, we conclude that $\mathcal{G}$ is continuous, i.e., an element of $\mathcal{M}$ and then $(\mathcal{M},\Delta_{\mathcal{M}})$ is complete. □

Now, we can investigate Minimum-Aggregation-stability and get an approximation for the VIE-D (0.1). First, we consider the following assumptions, then continue to prove the theorem.

(A1) For the continuous function $\Xi:[\imath,\jmath]\times[\imath,\jmath]\times\mathbb{C}\times\mathbb{C}\rightarrow\mathbb{C}$, we consider the following inequality

$$\mathscr{N}_{\mathscr{X}}\Big(\Xi(\phi,\zeta,\psi_1(\zeta),\psi_1(\theta(\zeta)))-\Xi(\phi,\zeta,\psi_2(\zeta),\psi_2(\theta(\zeta))),\omega\Big)\succeq\mathscr{N}_{\mathscr{X}}\Big(\psi_1-\psi_2,\frac{\omega}{\mathcal{Q}}\Big). \tag{2.1}$$

(A2) By considering the MVFF $\boldsymbol{MIN}:[0,1]\longrightarrow[0,1]$ as the control function, we have

$$\inf_{\xi\in[\imath,\jmath]}\boldsymbol{MIN}\Big(\boldsymbol{\Pi}(\boldsymbol{\xi},\boldsymbol{\omega})\Big)\succeq\boldsymbol{MIN}\Big(\boldsymbol{\Pi}(\boldsymbol{\phi},\frac{\boldsymbol{\omega}}{\boldsymbol{\mathcal{PF}}})\Big). \tag{2.2}$$

(A3) For the continuous function $\theta:[\imath,\jmath]\rightarrow[\imath,\jmath]$, we have

$$\mathscr{N}_{\mathscr{X}}\Big(\theta(\phi),\omega\Big)\succeq\mathscr{N}_{\mathscr{X}}\Big(\phi,\omega\Big). \tag{2.3}$$

(A4) Let $\psi:[\imath,\jmath]\rightarrow\mathscr{X}$ be a continuous function satisfying

$$\mathscr{N}_{\mathscr{X}}\Big(\psi(\phi)-\int_{\ell}^{\phi}\Xi(\phi,\zeta,\psi(\zeta),\psi(\theta(\zeta)))\mathrm{d}\zeta,\omega\Big)\succeq\boldsymbol{MIN}\Big(\boldsymbol{\Pi}(\boldsymbol{\phi},\boldsymbol{\omega})\Big). \tag{2.4}$$

Theorem 2.2 *We consider the MVFBS* $(\mathscr{X}, \mathscr{N}_{\mathscr{X}}, \circledast)$ *and assume that there are* $\mathcal{F}$ *and* $\mathcal{Q}$ *constants such that* $0 < \mathcal{QF} < 1$. *With these conditions and considering Assumptions (A1)–(A4), there exists a function* $\psi_0 : [\imath, \jmath] \to \mathscr{X}$ *in the form*

$$\psi_0(\phi) = \int_{\ell}^{\phi} \Xi(\phi, \zeta, \psi_0(\zeta), \psi_0(\theta(\zeta)))\mathrm{d}\zeta, \tag{2.5}$$

which is a unique solution of VIE-D (0.1), such that

$$\mathscr{N}_{\mathscr{X}}\Big(\psi(\phi) - \psi_0(\phi), \omega\Big) \succeq \boldsymbol{MIN}\Big(\boldsymbol{\Upsilon}(\boldsymbol{\phi}, \frac{\boldsymbol{\omega}}{\boldsymbol{\mathcal{A}}})\Big), \tag{2.6}$$

where in $\phi \in [\imath, \jmath]$, $\omega \in (0, +\infty)$ *and* $\mathcal{A} = \frac{1}{1-\mathcal{QF}}$.

Proof We consider the set $\mathcal{M} := \{\psi : [\imath, \jmath] \to \mathscr{X}, \psi \text{ is continuous}\}$, and the function $\Delta_{\mathscr{M}}$ defined in Theorem 2.1. As we proved in Theorem 2.1, $(\mathscr{M}, \Delta_{\mathscr{M}})$ is a GCMS. Now, we define the mapping $\Omega^\star : \mathscr{M} \to \mathscr{M}$, as follows

$$\Omega^\star(\psi(\phi)) = \int_{\ell}^{\phi} \Xi(\phi, \zeta, \phi(\zeta), \psi(\theta(\zeta)))\mathrm{d}\zeta, \tag{2.7}$$

for $\phi \in [\imath, \jmath]$ and we show that $\Omega^\star \in CON(\mathscr{X})$. we conclude that the continuity of $\Omega^\star\psi$ mapping. It means that

$$\begin{aligned}
&\mathscr{N}_{\mathscr{X}}\Big(\Omega^\star\psi(\phi) - \Omega^\star\psi(\phi_0), \omega + \tau\Big) = \\
&\mathscr{N}_{\mathscr{X}}\Big(\int_{\ell}^{\phi} \Xi(\phi, \zeta, \psi(\zeta), \psi(\theta(\zeta)))\mathrm{d}\zeta - \int_{\ell}^{\phi_0} \Xi(\phi_0, \zeta, \psi(\zeta), \psi(\theta(\zeta)))\mathrm{d}\zeta, \omega + \tau\Big) \\
&= \mathscr{N}_{\mathscr{X}}\Big(\int_{\ell}^{\phi} \Xi(\phi, \zeta, \psi(\zeta), \psi(\theta(\zeta)))\mathrm{d}\zeta - \int_{\ell}^{\phi} \Xi(\phi_0, \zeta, \psi(\zeta), \psi(\theta(\zeta)))\mathrm{d}\zeta \\
&\quad + \int_{\ell}^{\phi} \Xi(\phi_0, \zeta, \psi(\zeta), \psi(\theta(\zeta)))\mathrm{d}\zeta - \int_{\ell}^{\phi_0} \Xi(\phi_0, \zeta, \psi(\zeta), \psi(\theta(\zeta)))\mathrm{d}\zeta, \omega + \tau\Big) \\
&\succeq \mathscr{N}_{\mathscr{X}}\Big(\int_{\ell}^{\phi} \Xi(\phi, \zeta, \psi(\zeta), \psi(\theta(\zeta)))\mathrm{d}\zeta - \Big(\int_{\ell}^{\phi} \Xi(\phi_0, \zeta, \psi(\zeta), \psi(\theta(\zeta)))\mathrm{d}\zeta, \omega\Big) \circledast \\
&\mathscr{N}_{\mathscr{X}}\Big(\int_{\ell}^{\phi} \Xi(\phi_0, \zeta, \psi(\zeta), \psi(\theta(\zeta)))\mathrm{d}\zeta - \int_{\ell}^{\phi_0} \Xi(\phi_0, \zeta, \psi(\zeta), \psi(\theta(\zeta)))\mathrm{d}\zeta, \tau\Big) \\
&\succeq \mathscr{N}_{\mathscr{X}}\Big(\int_{\ell}^{\phi} \Xi(\phi, \zeta, \psi(\zeta), \psi(\theta(\zeta))) - \Xi(\phi_0, \zeta, \psi(\zeta), \psi(\theta(\zeta)))\mathrm{d}\zeta, \omega\Big) \\
&\circledast\, \mathscr{N}_{\mathscr{X}}\Big(\int_{\phi_0}^{\phi} \Xi(\phi_0, \zeta, \psi(\zeta), \psi(\theta(\zeta)))\mathrm{d}\zeta, \tau\Big) = \mathbf{1}.
\end{aligned} \tag{2.8}$$

Since ϕ tends to ϕ_0, the last line of (2.8) is the result. Let $\psi, \mathcal{S} \in \mathcal{M}$ and consider the coefficient $\mathcal{O}_{\psi\mathcal{S}} \in [0, +\infty]$ with $\Delta_{\mathcal{M}}(\psi, \mathcal{S}) \leq \mathcal{O}_{\psi\mathcal{S}}$, thus $\mathscr{N}_{\mathscr{X}}\Big(\psi(\phi) - \mathcal{S}(\phi), \mathcal{O}_{\psi\mathcal{S}}\omega\Big) \succeq \boldsymbol{MIN}\Big(\boldsymbol{\Pi}(\boldsymbol{\phi}, \boldsymbol{\omega})\Big)$ for all $\psi, \mathcal{S} \in \mathcal{M}$, $\phi \in [\imath, \jmath]$ and $\omega \in (\imath, \jmath)$. We consider

$$\delta\bar{\mathrm{h}}_k = \mathrm{h}_k - \mathrm{h}_{k-1} = \frac{Jk}{\mathrm{m}},$$

such that $\imath = \overline{\mathrm{h}}_1 < \overline{\mathrm{h}}_2 < \cdots < \overline{\mathrm{h}}_{\mathrm{m}} = \jmath_k$ and for all $k = 1, \cdots, \mathrm{m}$, also $\|\delta\overline{\mathrm{h}}_k\| = \max_{1\le k\le \mathrm{m}}(\delta\overline{\mathrm{h}}_k)$. Applying (NORM2) and (NORM3), we have

$$\begin{aligned}
&\mathscr{N}_{\mathscr{X}}\Big(\Omega^\star(\psi(\phi) - \Omega^\star(\mathcal{S}(\phi)), \mathcal{O}_{\psi\mathcal{S}}\omega\Big)\\
&= \mathscr{N}_{\mathscr{X}}\Big(\int_\ell^\phi [\Xi(\phi,\zeta,\psi(\zeta),\psi(\theta(\zeta))) - \Xi(\phi,\zeta,\mathcal{S}(\zeta),\mathcal{S}(\theta(\zeta)))]\mathrm{d}\zeta, \mathcal{O}_{\psi\mathcal{S}}\omega\Big)\\
&= \mathscr{N}_{\mathscr{X}}\Big(\lim_{\|\delta\overline{\mathrm{h}}\|\to 0}\sum_{\mathrm{n}=1}^{\mathrm{m}}[\Xi(\phi,\overline{\mathrm{h}}_{\mathrm{n}},\psi(\overline{\mathrm{h}}_{\mathrm{n}}),\psi(\theta(\overline{\mathrm{h}}_{\mathrm{n}}))) - \Xi(\phi,\overline{\mathrm{h}}_{\mathrm{n}},\mathcal{S}(\overline{\mathrm{h}}_{\mathrm{n}}),\mathcal{S}(\theta(\overline{\mathrm{h}}_{\mathrm{n}})))]\delta\mathrm{h}_k, \mathcal{O}_{\psi\mathcal{S}}\omega\Big)\\
&= \lim_{\|\delta\overline{\mathrm{h}}\|\to 0}\mathscr{N}_{\mathscr{X}}\Big(\sum_{\mathrm{n}=1}^{\mathrm{m}}[\Xi(\phi,\overline{\mathrm{h}}_{\mathrm{n}},\psi(\overline{\mathrm{h}}_{\mathrm{n}}),\psi(\theta(\overline{\mathrm{h}}_{\mathrm{n}}))) - \Xi(\phi,\overline{\mathrm{h}}_{\mathrm{n}},\mathcal{S}(\overline{\mathrm{h}}_{\mathrm{n}}),\mathcal{S}(\theta(\overline{\mathrm{h}}_{\mathrm{n}})))]\delta\mathrm{h}_k, \mathcal{O}_{\psi\mathcal{S}}\omega\Big)\\
&\succeq \lim_{\|\delta\overline{\mathrm{h}}\|\to 0}\circledast_M\mathscr{N}_{\mathscr{X}}\Big([\Xi(\phi,\overline{\mathrm{h}}_{\mathrm{n}},\psi(\overline{\mathrm{h}}_{\mathrm{n}}),\psi(\theta(\overline{\mathrm{h}}_{\mathrm{n}}))) - \Xi(\phi,\overline{\mathrm{h}}_{\mathrm{n}},\mathcal{S}(\overline{\mathrm{h}}_{\mathrm{n}}),\mathcal{S}(\theta(\overline{\mathrm{h}}_{\mathrm{n}})))]\delta\mathrm{h}_k, \frac{\mathcal{O}_{\psi\mathcal{S}}\omega}{\mathrm{m}}\Big)\\
&\succeq \inf_{\xi\in[\imath,\jmath]}\mathscr{N}_{\mathscr{X}}\Big([\Xi(\phi,\xi,\psi(\xi),\psi(\theta(\xi))) - \Xi(\phi,\xi,\mathcal{S}(\xi),\mathcal{S}(\theta(\xi)))], \frac{\mathcal{O}_{\psi\mathcal{S}}\omega}{\mathrm{m}\mathcal{P}}\Big)\\
&\succeq \inf_{\xi\in[\imath,\jmath]}\mathscr{N}_{\mathscr{X}}\Big(\psi(\xi) - \mathcal{S}(\xi), \frac{\mathrm{m}\mathcal{O}_{\psi\mathcal{S}}\mathcal{F}\omega}{\mathrm{m}\mathcal{P}}\Big). \qquad (2.9)
\end{aligned}$$

From (2.1), we get

$$\begin{aligned}
&\mathscr{N}_{\mathscr{X}}\Big([\Xi(\phi,\zeta,\psi(\zeta),\psi(\theta(\zeta))) - \Xi(\phi,\zeta,\mathcal{S}(\zeta),\mathcal{S}(\theta(\zeta)))], \mathcal{O}_{\psi\mathcal{S}}\omega\Big)\\
&\succeq \mathscr{N}_{\mathscr{X}}\Big(\psi(\zeta) - \mathcal{S}(\zeta), \mathcal{O}_{\psi\mathcal{S}}\omega\Big) \succeq \boldsymbol{MIN}\Big(\boldsymbol{\Pi}(\boldsymbol{\phi}, \frac{\boldsymbol{\omega}}{\boldsymbol{\mathcal{F}}})\Big), \qquad (2.10)
\end{aligned}$$

then

$$\begin{aligned}
&\mathscr{N}_{\mathscr{X}}\Big(\Omega^\star(\psi(\phi)) - \Omega^\star(\mathcal{S}(\phi)), \mathcal{O}_{\psi\mathcal{S}}\omega\Big) \succeq \inf_{\xi\in[\imath,\jmath]}\boldsymbol{MIN}\Big(\boldsymbol{\Pi}(\boldsymbol{\phi}, \frac{\boldsymbol{\mathcal{P}\omega}}{\boldsymbol{\mathcal{F}}})\Big)\\
&\succeq \boldsymbol{MIN}\Big(\boldsymbol{\Pi}(\boldsymbol{\phi}, \frac{\boldsymbol{\omega}}{\boldsymbol{\mathcal{QF}}})\Big), \qquad (2.11)
\end{aligned}$$

which implies that $\Delta_{\mathcal{M}}(\Omega^\star(\psi(\phi)), \Omega^\star(\mathcal{S}(\phi))) \le \mathcal{O}_{\psi\mathcal{S}}CL\mathcal{QF}$, therefore,

$$\Delta_{\mathcal{M}}(\Omega^\star(\psi(\phi)), \Omega^\star(\mathcal{S}(\phi))) \le \mathcal{QF}\Delta_{\mathcal{M}}(\psi, \mathcal{S}),$$

where $0 < \mathcal{QF} < 1$. It means that $\Omega^\star \in CON(\mathscr{X})$. In the sequel, we will show that $\Delta_{\mathcal{M}}(\Omega^\star(\psi(\phi)), \psi(\phi)) < +\infty$. Then, for $\mathcal{S} \in \mathcal{M}$ and for every $\omega \in (0, +\infty)$, we have

$$\begin{aligned}
&\mathscr{N}_{\mathscr{X}}\Big(\Omega^\star(\psi(\phi)) - \psi(\phi), \omega\Big)\\
&= \mathscr{N}_{\mathscr{X}}\Big(\int_\ell^\phi \Xi(\phi,\zeta,\psi(\zeta),\psi(\theta(\zeta)))\mathrm{d}\zeta - \psi(\zeta), \omega\Big) \succeq \boldsymbol{MIN}\Big(\boldsymbol{\Pi}(\boldsymbol{\phi}, \boldsymbol{\omega})\Big), \qquad (2.12)
\end{aligned}$$

and this means $\Delta_{\mathcal{M}}(\Omega^\star(\psi(\phi)), \psi(\phi)) < +\infty$. Since our main method of proof is based on Theorem 1.1, so far we see that all the assumptions of this theorem are established. Therefore, the sequence $\{\Omega^{\star\mathbf{s}}\psi(\phi)\}$ converges to a fixed point such as ψ. The unique element ψ is in the set $\mathcal{K} = \{\mathcal{G} \in \mathcal{M} : \Delta_{\mathcal{M}}(\Omega^\star\mathcal{G}, \mathcal{G}) < +\infty\}$ and is the unique fixed point of $\Omega^\star$, it means

$\Omega^\star\psi(\phi)=\psi(\phi)$ or equivalently

$$\psi(\phi)=\int_\ell^\phi \Xi(\phi,\zeta,\psi(\zeta),\psi(\theta(\zeta)))\mathrm{d}\zeta. \tag{2.13}$$

Using Inequality (2.12), we get

$$\Delta_{\mathcal{M}}(\psi,\psi)\leq\frac{1}{1-\mathcal{QF}}\Delta_{\mathcal{M}}(\Omega^\star\psi,\psi)\leq\frac{1}{1-\mathcal{QF}}.$$

it shows that the Minimum-Aggregation-stability for VIE-D 0.1 is established. Now, we go to the proof of uniqueness. First, we consider $\aleph=\frac{1}{1-\mathcal{QF}}$. Next, we consider another continuous function ψ_0 and assume that this continuous function applies to (2.13), i.e.,

$$\psi_0(\zeta)=\int_\ell^\phi \Xi(\phi,\zeta,\psi_0(\zeta),\psi_0(\theta(\zeta)))\mathrm{d}\zeta. \tag{2.14}$$

We are ready to prove that ψ_0 is a fixed point of $\mathcal{K}$ and $\psi_0\in\mathcal{K}$. Using Equation (2.14), we get $\Omega^\star\psi_0=\psi_0$. Now, we show that $\Delta_{\mathcal{M}}(\Omega^\star\psi(\phi),\psi_0)<+\infty$. Let $\psi(\phi)\in\mathcal{K}$, $\Delta_{\mathcal{M}}(\psi(\phi),\psi_0(\phi))<\aleph$, i.e.,

$$\mathscr{N}_{\mathscr{X}}\Big(\psi(\phi)-\psi_0(\phi),\omega\Big)\succeq \boldsymbol{MIN}\Big(\boldsymbol{\Pi}(\boldsymbol{\phi},\frac{\boldsymbol{\omega}}{\boldsymbol{\aleph}})\Big), \tag{2.15}$$

From (2.1), (2.15), we have

$$\begin{aligned}&\mathscr{N}_{\mathscr{X}}\Big(\Xi(\phi,\zeta,\psi(\zeta),\psi(\theta(\zeta)))-\Xi(\phi,\zeta,\psi_0(\zeta),\psi_0(\theta(\zeta))),\omega\Big)\\ \succeq&\mathscr{N}_{\mathscr{X}}\Big(\psi(\zeta)-\psi_0(\zeta),\frac{\omega}{\mathcal{F}}\Big)\\ \succeq&\boldsymbol{MIN}\Big(\boldsymbol{\Pi}(\boldsymbol{\phi},\frac{\boldsymbol{\omega}}{\boldsymbol{\mathcal{F}\aleph}})\Big),\end{aligned} \tag{2.16}$$

and

$$\begin{aligned}&\mathscr{N}_{\mathscr{X}}\Big(\int_\ell^\phi \Xi(\phi,\zeta,\psi(\zeta),\psi(\theta(\zeta)))-\xi(\phi,\zeta,\psi_0(\zeta),\psi_0(\theta(\zeta))),\omega\Big)\\ \succeq&\boldsymbol{MIN}\Big(\boldsymbol{\Pi}(\boldsymbol{\phi},\frac{\boldsymbol{\omega}}{\boldsymbol{\mathcal{FQ}}})\Big)\\ \succeq&\boldsymbol{MIN}\Big(\boldsymbol{\Pi}(\boldsymbol{\phi},\frac{\boldsymbol{\omega}}{\boldsymbol{\mathcal{FQ}\aleph}})\Big),\end{aligned} \tag{2.17}$$

and use Equation (2.14), we get

$$\begin{aligned}&\mathscr{N}_{\mathscr{X}}\Big(\Omega^\star(\psi(\phi))-\psi_0(\phi),\omega\Big)\\ &=\mathscr{N}_{\mathscr{X}}\Big(\int_\ell^\phi \Xi(\phi,\zeta,\psi(\zeta),\psi(\theta(\zeta)))\mathrm{d}\zeta-\int_\ell^\phi \Xi(\phi,\zeta,\phi_0(\phi),\psi_0(\theta(\zeta)))\mathrm{d}\zeta,\omega\Big)\\ &=\mathscr{N}_{\mathscr{X}}\Big(\int_\ell^\phi \Xi(\phi,\zeta,\psi(\zeta),\psi(\theta(\zeta)))-\Xi(\phi,\zeta,\psi_0(\zeta),\psi_0(\theta(\zeta)))\mathrm{d}\zeta,\omega\Big)\\ &\succeq\boldsymbol{MIN}\Big(\boldsymbol{\Pi}(\boldsymbol{\phi},\frac{\boldsymbol{\omega}}{\boldsymbol{\mathcal{FQ}\aleph}})\Big),\end{aligned}$$

and this means $\Delta_{\mathcal{M}}(\Omega^{\star}(\psi(\phi)), \psi_0(\zeta)) \leq \mathcal{F}\mathcal{Q}\aleph < +\infty$, which shows the uniqueness of the solution, and this is the end of the proof. □

Example 2.3 We consider a VIE-D as follows

$$\psi(\phi) = \int_{\ell}^{\phi} \frac{1}{16}(2 - 3\phi^2 - 2\cos(\zeta) + 2\zeta\sin(\zeta)) + \frac{1}{2000}\sin(\psi(\zeta)) + \frac{1}{2000}\sin(\psi(sin(\zeta))) + \frac{1}{2}(\zeta - \exp(\phi) - \sin(\phi))\mathrm{d}\zeta, \tag{2.18}$$

where $\ell = \frac{3}{4}$, $\theta(\phi) = \sin(\phi)$ and for all $\phi \in [0, 1]$. Considering the constant coefficients $\mathcal{Q} = \frac{4}{27}$ and $\mathcal{F} = \frac{1}{1000}$. Conditions (A1)–(A4) are established as follows

(A1)

$$\mathcal{N}_{\mathscr{X}}\Bigg(\frac{1}{16}(2-3)\phi^2 - 2\cos(\zeta) + 2\zeta\sin(\zeta)) + \frac{1}{2000}\sin(\psi_1(\zeta)) + \frac{1}{2000}\sin(\psi_1(sin(\zeta))) + \frac{1}{2}(\zeta - \exp(\phi) - \sin(\phi)) - \frac{1}{16}(2-3)\phi^2 - 2\cos(\zeta) + 2\zeta\sin(\zeta)) + \frac{1}{2000}\sin(\psi_2(\zeta)) + \frac{1}{2000}\sin(\psi_2(sin(\zeta))) + \frac{1}{2}(\zeta - \exp(\phi) - \sin(\phi)), \omega\Bigg) \succeq \mathcal{N}_{\mathscr{X}}\Big(\psi_1 - \psi_2, 1000\omega\Big). \tag{2.19}$$

(A2) By considering the MVFF $\boldsymbol{MIN} : [0, 1] \longrightarrow [0, 1]$ as the control function, we have

$$\inf_{\xi\in[\imath,\jmath]} \boldsymbol{MIN}\Big(\boldsymbol{\Pi}(\boldsymbol{\xi}, \boldsymbol{\omega})\Big) \succeq \boldsymbol{MIN}\Big(\boldsymbol{\Pi}(\boldsymbol{\phi}, \frac{\mathbf{72}\boldsymbol{\omega}}{\mathbf{4}})\Big), \tag{2.20}$$

where $\mathcal{P} = \frac{3}{8}$.

(A3) For the continuous function $\theta : [0, 1] \to [0, 1]$, we have

$$\mathcal{N}_{\mathscr{X}}\Big(\sin(\phi), \omega\Big) \succeq \mathcal{N}_{\mathscr{X}}\Big(\phi, \omega\Big). \tag{2.21}$$

(A4) Let $\psi : [\imath, \jmath] \to \mathscr{X}$ be a continuous function satisfying

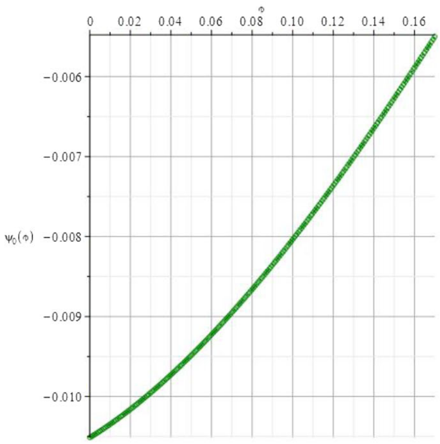
$$\mathcal{N}_{\mathscr{X}}\Bigg(\psi(\phi) - \int_{\ell}^{\phi} \frac{1}{16}(2-3)\phi^2 - 2\cos(\zeta) + 2\zeta\sin(\zeta)) + \frac{1}{2000}\sin(\psi(\zeta)) + \frac{1}{2000}\sin(\psi(sin(\zeta))) + \frac{1}{2}(\zeta - \exp(\phi) - \sin(\phi))\mathrm{d}\zeta, \omega\Bigg) \succeq \boldsymbol{MIN}\Big(\boldsymbol{\Pi}(\boldsymbol{\phi}, \boldsymbol{\omega})\Big). \tag{2.22}$$

Therefore, there exists a unique function ψ_0 such that

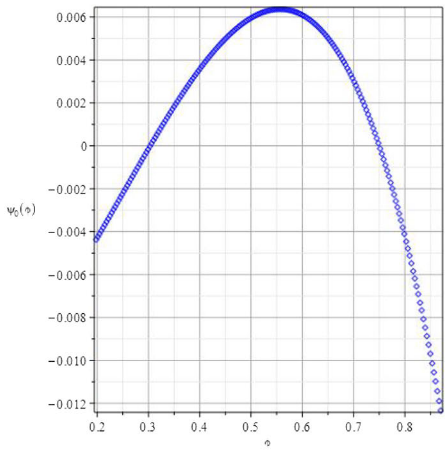
$$\psi_0(\phi) = \int_{\ell}^{\phi} \frac{1}{16}(2-3)\phi^2 - 2\cos(\zeta) + 2\zeta\sin(\zeta)) + \frac{1}{2000}\sin(\psi(\zeta)) + \frac{1}{2000}\sin(\psi(sin(\zeta))) + \frac{1}{2}(\zeta - \exp(\phi) - \sin(\phi))\mathrm{d}\zeta, \tag{2.23}$$

and the following inequality holds

$$\mathcal{N}_{\mathscr{X}} = \Big(\psi(\phi) - \psi_0(\phi), \omega\Big) \succeq \boldsymbol{MIN}\Big(\boldsymbol{\Pi}(\boldsymbol{\phi}, \frac{\mathbf{27000}\boldsymbol{\omega}}{\mathbf{26996}})\Big), \tag{2.24}$$



(a) Graphic representation of the exact solution of the VIE-D (2.18) for $\phi \in (0, 0.17)$



(b) Graphic representation of the exact solution of the VIE-D (2.18) for $\phi \in (0.198, 0.87)$

Fig. 3 2D graph of the exact solution of the VIE-D (2.18) for different values ϕ

where $\mathcal{A} = \frac{1}{1-\mathcal{QF}} = \frac{26996}{27000}$.

In addition to the calculations performed and obtaining an approximation for VIE-D (2.18), in the following, we show the graphic representation of the exact solutions of this equation for different values in Fig. 3.

3 Minimum-aggregation-stability of the VIE-D in the infinite interval case

In this section, we will once again examine VIE-D (0.1) with the difference that here we consider $\mathbb{R}$ instead of $[\imath, \jmath]$ interval. In fact, by converting the compact interval into an infinite interval, we prove the Minimum-Aggregation-Stability for VIE-D (0.1). Therefore, we assume that conditions (A1)–(A4) for functions $\theta : \mathbb{R} \to \mathbb{R}$ and $\Xi : \mathbb{R} \times \mathbb{R} \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ are satisfied. Before proceeding to the proof of the theorem, we remind you that we can also consider $\phi \in [\imath, +\infty)$ and $(+\infty, \jmath]$ modes.

Theorem 3.1 *We consider the MVFBS* $(\mathscr{X}, \mathscr{N}_{\mathscr{X}}, \circledast)$ *and assume that there are* $\mathcal{F}$ *and* $\mathcal{Q}$ *constants such that* $0 < \mathcal{QF} < 1$. *With these conditions and considering Assumptions (A1)–(A4), there exists a function* $\psi_0 : \mathbb{R} \to \mathscr{X}$ *in the form*

$$\psi_0(\phi) = \int_{\ell}^{\phi} \Xi(\phi, \zeta, \psi_0(\zeta), \psi_0(\theta(\zeta)))\mathrm{d}\zeta, \tag{3.1}$$

which is a unique solution of VIE-D (0.1)*, such that*

$$\mathscr{N}_{\mathscr{X}}\Big(\psi(\phi) - \psi_0(\phi), \omega\Big) \succeq \boldsymbol{MIN}\Big(\boldsymbol{\Pi}(\boldsymbol{\phi}, \frac{\boldsymbol{\omega}}{\boldsymbol{\mathcal{A}}})\Big), \tag{3.2}$$

where in $\phi \in \mathbb{R}$, $\omega \in (0, +\infty)$ *and* $\mathcal{A} = \frac{1}{1-\mathcal{QF}}$.

Proof We first prove that ψ_0 is a continuous function. For this purpose, we consider Theorem 2.2. According to this theorem, by considering $\mathrm{U_m} = [\ell - \mathrm{m}, \ell + \mathrm{m}]$ and every θ on $\mathrm{U_m}$, there is a unique continuous function like $\psi_{0,\mathrm{m}} : \mathrm{U_m} \to \mathbb{C}$ as follows

$$\psi_{0,\mathrm{m}}(\phi) = \int_\ell^\phi \Xi(\phi, \zeta, \psi_{0,\mathrm{m}}(\zeta), \psi_{0,\mathrm{m}}(\theta(\zeta)))\mathrm{d}\zeta, \tag{3.3}$$

for every $\mathrm{m} \in \mathbb{N}$ and $\phi \in \mathrm{U_m}$. Then, according to ψ and $\psi_{0,\mathrm{m}}$ the following inequality is also holds

$$\mathscr{N}_{\mathscr{X}}\left(\psi(\phi) - \psi_{0,\mathrm{m}}(\phi), \omega\right) \succeq \boldsymbol{MIN}\left(\boldsymbol{\Pi}(\boldsymbol{\phi}, \frac{\boldsymbol{\omega}}{\boldsymbol{\mathcal{A}}})\right). \tag{3.4}$$

We know that $\psi_{0,\mathrm{m}}$ is a unique function. Therefore, we have $\psi_{0,\mathrm{m}}(\phi) = \psi_{0,\mathrm{m}+1}(\phi) = \psi_{0,\mathrm{m}+2}(\phi) = \cdots$. Now, we define $\mathrm{m}(\phi) \in \mathbb{N}$ as $\mathrm{m}(\phi) = \min\{\mathrm{m} \in \mathbb{N} | \phi \in \mathrm{U_m}\}$. If we define the function $\psi_0(\phi) : \mathbb{R} \to \mathbb{C}$ as $\psi_0(\phi) = \psi_{0,\mathrm{m}}(\phi)$, then ψ_0 is a continuous function. Because if we assume $\mathrm{m}_1 = \mathrm{m}(\phi_1)$ for every $\phi \in \mathbb{R}$, then $\phi_1 \in \mathrm{U}_{\mathrm{m}_1+1}$. Therefore, there exists $\mathrm{g} > 0$, such that for every $\phi \in (\phi_1 - \mathrm{g}, \phi_1 + \mathrm{g})$, we have $\psi_0(\phi) = \psi_{0,\mathrm{m}_1+1}(\phi)$. Now, according to Theorem 2.2, we conclude that ψ_{0,m_1+1} is continuous and this means that ψ_0 is continuous. In the second part of the proof, we show that for $\phi \in \mathbb{R}$, (3.1) and (3.2) hold for ψ_0. First, we show that ψ_0 satisfies (3.1). Then, for $\phi \in \mathbb{R}$, we consider $\mathrm{m}(\phi)$ such that $\phi \in \mathrm{U}_{\mathrm{m}(\phi)}$. With these conditions and using (3.3) and due to the fact that for each $\zeta \in \mathrm{U}_{\mathrm{m}(\phi)}$, $\mathrm{m}(\zeta) \le \mathrm{m}(\phi)$ and $\mathrm{m}(\theta(\zeta)) \le \mathrm{m}(\theta(\phi))$, we have

$$\begin{aligned}\psi_0(\phi) = \psi_{0,\mathrm{m}(\phi)}(\phi) &= \int_\ell^\phi \Xi(\phi, \zeta, \psi_{0,\mathrm{m}}(\zeta), \psi_{0,\mathrm{m}(\phi)}(\theta(\zeta)))\mathrm{d}\zeta \\ &= \int_{ell}^\phi \Xi(\phi, \zeta, \psi_0(\zeta), \psi_0(\theta(\zeta)))\mathrm{d}\zeta.\end{aligned} \tag{3.5}$$

Now, we show that (3.2) holds for ψ_0. Considering that $\psi_{0,\mathrm{m}}(\phi) = \psi_{0,\mathrm{m}+1}(\phi) = \psi_{0,\mathrm{m}+2}(\phi) = \cdots$, we have $\psi_0(\zeta) = \psi_{0,\mathrm{m}(\zeta)}(\zeta) = \psi_{0,\mathrm{m}(\phi)}(\zeta)$ and $\psi_0(\theta(\zeta)) = \psi_{0,\mathrm{m}(\zeta)}(\theta(\zeta)) = \psi_{0,\mathrm{m}(\phi)}(\theta(\zeta))$. Then, also by (3.2), we get

$$\begin{aligned}&\mathscr{N}_{\mathscr{X}}\left(\psi(\phi) - \psi_0(\phi), \omega\right) \\ &= \mathscr{N}_{\mathscr{X}}\left(\psi(\phi) - \psi_{0,\mathrm{m}(\phi)}(\phi), \omega\right) \\ &\succeq \boldsymbol{MIN}\left(\boldsymbol{\Pi}(\boldsymbol{\phi}, (\boldsymbol{1} - \boldsymbol{\mathcal{Q}\mathcal{F}})\boldsymbol{\omega})\right),\end{aligned} \tag{3.6}$$

for all $\phi \in \mathbb{R}$. What remains to be proved is uniqueness. Then, according to the uniqueness proof procedure, we consider another function like ψ_1, such that for all $\phi \in \mathbb{R}$ (3.1 and (3.2) are also valid for this function. Considering $y_{0|I_n(x)} = y_{0,n(x)})$ and $\psi_{1|\mathrm{U}_{\mathrm{m}(\phi)}}$, which both apply to (3.1) and (3.2), and from the uniqueness of $\psi_{0|\mathrm{U}_{\mathrm{m}(\phi)}} = \psi_{0,\mathrm{m}(\phi)}$, we have $\psi_0(\phi) = \psi_{0|\mathrm{U}_{\mathrm{m}(\phi)}}(\phi) = \psi_{1|\mathrm{U}_{\mathrm{m}(\phi)}}(\phi) = \psi_1(\phi)$. □

We provide a numerical example according to the results obtained.

Example 3.2 For functions $\Xi : \mathbb{R} \times \mathbb{R} \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ and $\theta : \mathbb{R} \to \mathbb{R}$, we Consider the following VIE-D

$$\psi(\phi) = \int_\ell^\phi 4 + \exp(\frac{3\phi}{8}) + \exp(\frac{\zeta}{9})(8 + \frac{3\phi}{4}) + \frac{1}{10}\psi(\zeta)$$

$$+ \frac{3}{10}\psi(\sin(\frac{1}{\zeta^2+1})) + \phi^2 \mathrm{d}\zeta, \tag{3.7}$$

for $\ell = \frac{8}{10}$ and $\theta(\phi) = \sin(\frac{1}{\phi^2+1})$, $\zeta \in \mathbb{R}$. For the introduced functions, we have

(A1)

$$\mathcal{N}_{\mathcal{X}}\Bigg(4 + \exp(\frac{3\phi}{8}) + \exp(\frac{\zeta}{9})(8 + \frac{3\phi}{4}) + \frac{1}{10}\psi_1(\zeta) + \frac{3}{10}\psi_1(\sin(\frac{1}{\zeta^2+1})) + \phi^2 - 4 + \exp(\frac{3\phi}{8}) + \exp(\frac{\zeta}{9})(8 + \frac{3\phi}{4}) + \frac{1}{10}\psi_2(\zeta) + \frac{3}{10}\psi_2(\sin(\frac{1}{\zeta^2+1})) + \phi^2, \omega\Bigg) \succeq \mathcal{N}_{\mathcal{X}}\left(\psi_1 - \psi_2, \frac{10\omega}{4}\right). \tag{3.8}$$

(A2) By considering the MVFF $\boldsymbol{MIN} : [0, 1] \longrightarrow [0, 1]$ as the control function, we have

$$\inf_{\xi \in [\iota, \jmath]} \boldsymbol{MIN}\Big(\boldsymbol{\Pi(\xi, \omega)}\Big) \succeq \boldsymbol{MIN}\Big(\boldsymbol{\Pi(\phi, \frac{1000\omega}{2})}\Big), \tag{3.9}$$

where $\mathcal{P} = \frac{2}{10}$ (Fig. 4).

(A3) For the continuous function $\theta : \mathbb{R} \to \mathbb{R}$, we have

$$\mathcal{N}_{\mathcal{X}}\left(\sin(\frac{1}{\phi^2+1}), \omega\right) \succeq \mathcal{N}_{\mathcal{X}}\left(\frac{1}{\phi^2+1}, \omega\right). \tag{3.10}$$

(A4) Let $\psi : \mathbb{R} \to \mathcal{X}$ be a continuous function satisfying

$$\mathcal{N}_{\mathcal{X}}\Bigg(\psi(\phi) - \int_{\ell}^{\phi} 4 + \exp(\frac{3\phi}{8}) + \exp(\frac{\zeta}{9})(8 + \frac{3\phi}{4}) + \frac{1}{10}\psi(\zeta) + \frac{3}{10}\psi(\sin(\frac{1}{\zeta^2+1})) + \phi^2 \mathrm{d}\zeta, \omega\Bigg) \succeq \boldsymbol{MIN}\Big(\boldsymbol{\Pi(\phi, \omega)}\Big). \tag{3.11}$$

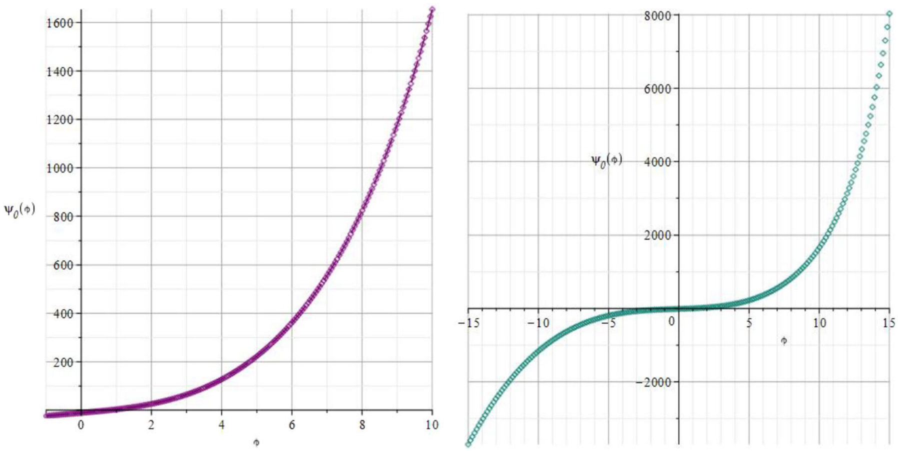
Therefore, there exists a unique function ψ_0 such that

$$\psi_0(\phi) = \int_{\ell}^{\phi} 4 + \exp(\frac{3\phi}{8}) + \exp(\frac{\zeta}{9})(8 + \frac{3\phi}{4}) + \frac{1}{10}\psi_0(\zeta) + \frac{3}{10}\psi_0(\sin(\frac{1}{\zeta^2+1})) + \phi^2 \mathrm{d}\zeta, \tag{3.12}$$

and the following inequality holds

$$\mathcal{N}_{\mathcal{X}} = \Big(\psi(\phi) - \psi_0(\phi), \omega\Big) \succeq \boldsymbol{MIN}\Big(\boldsymbol{\Pi(\phi, \frac{1000\omega}{996})}\Big), \tag{3.13}$$

where $\mathcal{A} = \frac{1}{1-\mathcal{QF}} = \frac{996}{1000}$.



(a) Graphic representation of the exact solution of the VIE-D (3.7) for $\phi \in (-1, 10)$

(b) Graphic representation of the exact solution of the VIE-D (2.18) for $\phi \in (-15, 15)$

Fig. 4 2D graph of the exact solution of the VIE-D (3.7) for different values ϕ

4 Conclusion

In this article, we have considered a VIE-D and we have tried to obtain the best approximation for this equation in the fuzzy space of matrix type. For this purpose, we have investigated several types of functions, and with the help of these functions, we have introduced a special controller, which is the main factor in obtaining a suitable approximation. We have considered two intervals and performed all these checks on two intervals. In each part, we have provided a practical example to investigate the accuracy of the obtained results.

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Declarations

Conflict of interest All authors declare that they have no competing interests.

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