fractal and fractional

Brief Report

A Matrix Mittag–Leffler Function and the Fractional Nonlinear Partial Integro-Differential Equation in $\mathbb{R}^n$

Chenkuan Li [1,*], Joshua Beaudin [1], Azedine Rahmoune [2] and Walid Remili [2]

[1] Department of Mathematics and Computer Science, Brandon University, Brandon, MB R7A 6A9, Canada; beaudinj@brandonu.ca
[2] Department of Mathematics, Faculty of Mathematics and Informatics, University Mohamed El Bachir El Ibrahimi of Bordj Bou Arreridj, El Anasser 34030, Algeria; a.rahmoune@univ-bba.dz (A.R.); walid.remili@univ-bba.dz (W.R.)
* Correspondence: lic@brandonu.ca; Tel.: +1-204-571-8549

Abstract: In this paper, we introduce the matrix Mittag–Leffler function, which is a generalization of the multivariate Mittag–Leffler function, in order to investigate the uniqueness of the solutions to a fractional nonlinear partial integro-differential equation in $\mathbb{R}^n$ with a boundary condition based on Banach's contractive principle and Babenko's approach. In addition, we present an example demonstrating applications of the key results derived using a Python code that computes the approximate value of our matrix Mittag–Leffler function.

Keywords: Banach's contractive principle; matrix Mittag–Leffler function; Babenko's approach; implicit integral equation

MSC: 35A02; 35C10; 35C15; 26A33

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1. Background

In this section, we introduce some basic notations, a new matrix Mittag–Leffler function as well as Babenko's approach with an illustrative example solving a fractional differential equation.

Let $y \in \Omega = [0,1]^n \subset \mathbb{R}^n$ $(n \geq 1)$ and $x \in [0,1]$. It follows from [1] for $\alpha_1 \geq 0, \cdots, \alpha_n \geq 0$ that:

$$I_1^{\alpha_1} \cdots I_n^{\alpha_n} Z(x,y) = \frac{1}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_n)} \int_0^{y_1} \cdots \int_0^{y_n} \prod_{k=1}^{n} (y_k - s_k)^{\alpha_k - 1} Z(x, s_1, \cdots, s_n) ds_n \cdots ds_1.$$

In particular, for $\alpha_2 = \cdots = \alpha_n = 0$, we have:

$$I_1^{\alpha_1} Z(x,y) = \frac{1}{\Gamma(\alpha_1)} \int_0^{y_1} (y_1 - s_1)^{\alpha_1 - 1} Z(x, s_1, y_2, \cdots, y_n) ds_1. \tag{1}$$

Let $1 < \alpha \leq 2$. Then, from [2]

$$I_x^{\alpha} \frac{{}^c\partial^{\alpha}}{\partial x^{\alpha}} Z(x,y) = Z(x,y) - Z(0,y) - Z_x'(0,y)x, \tag{2}$$

where the operator I_x^{α} is the partial Riemann–Liouville fractional integral of order $\alpha > 0$ with respect to x with initial point zero [3]:

$$(I_x^{\alpha} Z)(x,y) = \frac{1}{\Gamma(\alpha)} \int_0^x (x - \zeta)^{\alpha - 1} Z(\zeta, y) d\zeta, \tag{3}$$

and $\frac{{}_c\partial^\alpha}{\partial x^\alpha}$ is the partial Liouville–Caputo fractional derivative of order α with respect to x

$$\left(\frac{{}_c\partial^\alpha}{\partial x^\alpha} Z\right)(x,y) = \frac{1}{\Gamma(2-\alpha)} \int_0^x (x-s)^{-\alpha+1} W_x''(s,y) ds, \quad 1 < \alpha \le 2. \tag{4}$$

$\mathcal{S}([0,1]\times\Omega)$ is defined as the Banach space with the norm given as:

$$\|Z\|_{\mathcal{S}} = \sup_{x\in[0,1], y\in\Omega} |Z(x,y)| \text{ for } Z \in \mathcal{S}([0,1]\times\Omega).$$

In this paper, we study the uniqueness of the solutions to the following equation with the boundary condition for $1 < \alpha \le 2$ and $m \in \mathbb{N}$ in the space $\mathcal{S}([0,1]\times\Omega)$ for constants a_k:

$$\begin{cases} \frac{{}_c\partial^\alpha}{\partial x^\alpha} Z(x,y) + \sum_{k=1}^{m} a_k I_1^{\alpha_{1k}} \cdots I_n^{\alpha_{nk}} Z(x,y) = f(x,y,Z(x,y)), \\ Z(0,y) = Z(1,y) = 0, \quad (x,y) \in [0,1]\times\Omega, \end{cases} \tag{5}$$

where $\alpha_{ik} \ge 0$ for all $i = 1,\cdots,n$, $k = 1,\cdots,m$, and $f : [0,1]\times\Omega\times\mathbb{R} \to \mathbb{R}$ is a function satisfying certain conditions to be given. Equation (5), with its boundary condition, is new and, to the best of our knowledge, has never previously been investigated. This research has many potential applications since uniqueness is an important topic in many scientific areas, such as control theory, and the method used clearly opens up new directions for studying other types of equations with initial or boundary problems.

A function Z is a solution of the problem (5) if it satisfies the equation over $\mathcal{S}([0,1]\times\Omega)$ and its boundary condition $Z(0,y) = Z(1,y) = 0$.

Let $\alpha_{ij} \ge 0, \gamma_i > 0$ for all $i = 0,\cdots,n$, $j = 1,\cdots,m$, and

$$M = \begin{bmatrix} \alpha_{01} \cdots & \alpha_{0m} & \gamma_0 \\ \alpha_{11} \cdots & \alpha_{1m} & \gamma_1 \\ \cdots & & \\ \alpha_{n1} \cdots & \alpha_{nm} & \gamma_n \end{bmatrix}. \tag{6}$$

Definition 1. *A matrix Mittag–Leffler function is defined by the following series:*

$$E_M(z_1,\cdots,z_m) = \sum_{l=0}^{\infty} \sum_{\substack{l_1+\cdots+l_m=l \\ l_1\ge 0,\cdots,l_m\ge 0}} \binom{l}{l_1,\cdots,l_m} \frac{z_1^{l_1}\cdots z_m^{l_m}}{\Gamma(\alpha_{01}l_1+\cdots+\alpha_{0m}l_m+\gamma_0)}$$
$$\cdot \frac{1}{\Gamma(\alpha_{11}l_1+\cdots+\alpha_{1m}l_m+\gamma_1)\cdots\Gamma(\alpha_{n1}l_1+\cdots+\alpha_{nm}l_m+\gamma_n)}, \tag{7}$$

where $z_i \in \mathbb{C}$ for $i = 1,2,\cdots,m$ and

$$\binom{l}{l_1,\cdots,l_m} = \frac{l!}{l_1!\cdots l_m!}.$$

Clearly, the above series converges since there is a positive number δ such that

$$\Gamma(\alpha_{11}l_1+\cdots+\alpha_{1m}l_m+\gamma_1) \ge \delta,$$
$$\cdots,$$
$$\Gamma(\alpha_{n1}l_1+\cdots+\alpha_{nm}l_m+\gamma_n) \ge \delta.$$

In particular,

$$E_P(z_1,\cdots,z_m) = E_{(\alpha_{01},\cdots,\alpha_{0m}),\gamma_0}(z_1,\cdots,z_m) = \sum_{l=0}^{\infty}\sum_{\substack{l_1+\cdots+l_m=l \\ l_1\geq 0,\cdots,l_m\geq 0}} \binom{l}{l_1,\cdots,l_m} \frac{z_1^{l_1}\cdots z_m^{l_m}}{\Gamma(\alpha_{01}l_1+\cdots+\alpha_{0m}l_m+\gamma_0)}, \quad (8)$$

which is the multivariate Mittag–Leffler function given in [4], with

$$P = \begin{bmatrix} \alpha_{01}\cdots & \alpha_{0m} & \gamma_0 \\ 0\cdots & 0 & 1 \\ \cdots & & \\ 0\cdots & 0 & 1 \end{bmatrix}. \quad (9)$$

Moreover,

$$E_{P_0}(z) = E_{\alpha_{01},\gamma_0}(z) = \sum_{l=0}^{\infty} \frac{z^l}{\Gamma(\alpha_{01}l+\gamma_0)}, \ z\in\mathbb{C}, \quad (10)$$

which is the well known two-parameter Mittag–Leffler function, with

$$P_0 = \begin{bmatrix} \alpha_{01} & 0\cdots & 0 & \gamma_0 \\ 0 & 0\cdots & 0 & 1 \\ \cdots & & & \\ 0 & 0\cdots & 0 & 1 \end{bmatrix}. \quad (11)$$

Remark 1. *In 2018, Garrappa and Popolizio [5] also defined the matrix Mittag–Leffler function to study fractional calculus. They called it the Mittag–Leffler (ML) function with matrix arguments, which is based on the spectrum of the matrix with the Jordan canonical form, and is totally different from Definition 1.*

Babenko's approach [6] is an efficient tool for dealing with integral and differential equations with initial conditions or boundary value problems [1]. The method itself is similar to the Laplace transform while working on differential and integral equations with constant coefficients, but it can be applied to equations with continuous and bounded variable coefficients. To show this method in detail, we will derive the solution to the following equation with the initial conditions in the space $C[0,1]$ for constant λ:

$$\begin{cases} {}_cD_0^{\alpha}x(t) + \lambda\, {}_cD_0^{\beta}x(t) = t^2, \quad 0<\beta<\alpha, \quad 1<\alpha\leq 2, \\ x(0) = x'(0) = 0, \end{cases} \quad (12)$$

where

$${}_cD_0^{\alpha}x(t) = \frac{1}{\Gamma(2-\alpha)}\int_0^t (t-s)^{-\alpha+1}x''(s)ds, \quad 1<\alpha\leq 2,$$

and

$${}_cD_0^{\beta}x(t) = \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}x'(s)ds, \quad \text{if } 0<\beta\leq 1.$$

From (2), one has

$$I_0^{\alpha}({}_cD_0^{\alpha}x(t)) = x(t) - x(0) - x'(0)t = x(t). \quad (13)$$

Applying I_0^{α} to the first equation in (12), we have

$$\left(1+\lambda I_0^{\alpha-\beta}\right)x(t) = I_0^{\alpha}t^2 = \frac{2}{\Gamma(\alpha+3)}t^{\alpha+2}. \quad (14)$$

Treating the operator $\left(1+\lambda I_0^{\alpha-\beta}\right)$ as a variable, we informally derive using Babenko's approach

$$x(t) = \frac{2}{\Gamma(\alpha+3)}\left(1+\lambda I_0^{\alpha-\beta}\right)^{-1} t^{\alpha+2} = \frac{2}{\Gamma(\alpha+3)}\sum_{l=0}^{\infty}(-1)^l\lambda^l I_0^{l(\alpha-\beta)}\, t^{\alpha+2}$$
$$= 2t^{\alpha+2}\sum_{l=0}^{\infty}\frac{\left(-\lambda t^{\alpha-\beta}\right)^l}{\Gamma(l(\alpha-\beta)+\alpha+3)} = 2t^{\alpha+2}E_{\alpha-\beta,\,\alpha+3}\left(-\lambda t^{\alpha-\beta}\right), \tag{15}$$

using

$$I_0^{l(\alpha-\beta)}\, t^{\alpha+2} = \frac{\Gamma(\alpha+3)}{\Gamma(l(\alpha-\beta)+\alpha+3)}t^{l(\alpha-\beta)+\alpha+2}. \tag{16}$$

This infers that

$$\|x\| \leq 2\sum_{l=0}^{\infty}\frac{|\lambda|^l}{\Gamma(l(\alpha-\beta)+\alpha+3)} = 2E_{\alpha-\beta,\alpha+3}(|\lambda|) < +\infty, \tag{17}$$

which claims that the series solution

$$x(t) = 2t^{\alpha+2}\sum_{l=0}^{\infty}\frac{\left(-\lambda t^{\alpha-\beta}\right)^l}{\Gamma(l(\alpha-\beta)+\alpha+3)} = 2t^{\alpha+2}E_{\alpha-\beta,\,\alpha+3}\left(-\lambda t^{\alpha-\beta}\right), \tag{18}$$

is in $\mathcal{C}[0,1]$.

We can also readily show $x \in \mathcal{C}[0,1]$ from a different point of view: $E_{\alpha-\beta,\,\alpha+3}\left(-\lambda t^{\alpha-\beta}\right)$ is an entire function of t so it is continuous. Hence, $x(t)$ is continuous.

Fractional nonlinear PDEs have been used to describe many different physical systems, ranging from optical fibres to dynamical processes in various scientific fields [7,8]. There are many interesting studies on the uniqueness and existence of solutions, based on the theory of fixed points, for fractional nonlinear integro-differential equations with initial value or boundary condition problems, as well as for integral equations [9]. Very recently, Li [1] investigated the uniqueness of solutions for the following equation with the initial conditions for any positive integer m:

$$\begin{cases} \dfrac{\partial^m}{\partial x^m}Z(x,y) - \displaystyle\sum_{k=1}^{l}\lambda_k(y)I_1^{\beta_{1k}}\cdots I_n^{\beta_{nk}}Z(x,y) = f\left(x,y,Z(x,y),\cdots,\dfrac{\partial^{m-1}}{\partial x^{m-1}}Z(x,y)\right), \\ Z(0,y) = 0,\cdots,\dfrac{\partial^{m-1}}{\partial x^{m-1}}u(0,y) = 0, \end{cases} \tag{19}$$

in the Banach space

$$S_m([0,x_0]\times\Omega_1) = \left\{Z(x,y)\ :\frac{\partial^m}{\partial x^m}Z(x,y) \in \mathcal{S}([0,x_0]\times\Omega_1) \text{ and } \|Z\|_m < \infty\right\},$$

where $x_0 > 0$, $\Omega_1 = [0,\omega_1]\times\cdots\times[0,\omega_n] \subset \mathbb{R}^n$, and

$$\|Z\|_m = \max\left\{\|Z(x,y)\|_{\mathcal{S}}, \|\frac{\partial}{\partial x}Z(x,y)\|_{\mathcal{S}},\cdots,\|\frac{\partial^m}{\partial x^m}Z(x,y)\|_{\mathcal{S}}\right\}.$$

The remainder of this paper is structured as follows. Section 2 studies the uniqueness of the solutions to the problem (5) using the matrix Mittag–Leffler function given above, Babenko's approach and Banach's contractive principle. Section 3 presents an example to demonstrate the applications of the main results based on the value of a matrix Mittag–Leffler function evaluated using our Python code. In Section 4, we provide a summary of the work.

2. Uniqueness of Solutions

Theorem 1. *Let* $1 < \alpha \leq 2,\ \alpha_{ij} \geq 0$ *for* $i = 1, \cdots, n,\ j = 1, \cdots, m$, *and*

$$M_k = \begin{bmatrix} \alpha \cdots & \alpha & 2 \\ \alpha_{11} \cdots & \alpha_{1m} & \alpha_{1k}+1 \\ \cdots & & \\ \alpha_{n1} \cdots & \alpha_{nm} & \alpha_{nk}+1 \end{bmatrix}, \tag{20}$$

for $k = 1, \cdots, m$. *We assume that*

$$A = 1 - \frac{1}{\Gamma(\alpha+1)} \sum_{k=1}^{m} |a_k| E_{M_k}(|a_1|, \cdots, |a_m|) > 0.$$

Then, $Z(x, y)$ *is a solution to the problem (5), if and only if it is bounded and satisfies the following implicit integral equation in the space* $\mathcal{S}([0,1] \times \Omega)$:

$$\begin{aligned} Z(x,y) = &\sum_{l=0}^{\infty} (-1)^l \sum_{\substack{l_1+\cdots+l_m=l \\ l_1 \geq 0, \cdots, l_m \geq 0}} \binom{l}{l_1, \cdots, l_m} a_1^{l_1} \cdots a_m^{l_m} I_x^{l_1\alpha+\cdots+l_m\alpha+\alpha} \\ &\cdot I_1^{\alpha_{11}l_1+\cdots+\alpha_{1m}l_m} \cdots I_n^{\alpha_{n1}l_1+\cdots+\alpha_{nm}l_m} f(x,y,Z(x,y)) \\ &- \sum_{l=0}^{\infty} (-1)^l \sum_{\substack{l_1+\cdots+l_m=l \\ l_1 \geq 0, \cdots, l_m \geq 0}} \binom{l}{l_1, \cdots, l_m} a_1^{l_1} \cdots a_m^{l_m} \frac{x^{\alpha l_1+\cdots+\alpha l_m+1}}{\Gamma(\alpha l_1+\cdots+\alpha l_m+2)} \\ &\cdot I_1^{\alpha_{11}l_1+\cdots+\alpha_{1m}l_m} \cdots I_n^{\alpha_{n1}l_1+\cdots+\alpha_{nm}l_m} I_{x=1}^{\alpha} f(x,y,Z(x,y)) + \sum_{k=1}^{m} a_k \sum_{l=0}^{\infty} (-1)^l \\ &\cdot \sum_{\substack{l_1+\cdots+l_m=l \\ l_1 \geq 0, \cdots, l_m \geq 0}} \binom{l}{l_1, \cdots, l_m} a_1^{l_1} \cdots a_m^{l_m} \frac{x^{\alpha l_1+\cdots+\alpha l_m+1}}{\Gamma(\alpha l_1+\cdots+\alpha l_m+2)} \\ &\cdot I_1^{\alpha_{11}l_1+\cdots+\alpha_{1m}l_m+\alpha_{1k}} \cdots I_n^{\alpha_{n1}l_1+\cdots+\alpha_{nm}l_m+\alpha_{nk}} I_{x=1}^{\alpha} Z(x,y). \end{aligned} \tag{21}$$

Furthermore,

$$\begin{aligned} \|Z\| \leq \frac{1}{q} \bigg(E_{Q_\alpha}(|a_1|, \cdots, |a_m|) + \frac{1}{\Gamma(\alpha+1)} E_{Q_1}(|a_1|, \cdots, |a_m|) \bigg) \\ \cdot \sup_{(x,y) \in [0,1] \times \Omega,\, w \in R} |f(x,y,w)| < +\infty, \end{aligned} \tag{22}$$

where

$$Q_\alpha = \begin{bmatrix} \alpha \cdots & \alpha & \alpha+1 \\ \alpha_{11} \cdots & \alpha_{1m} & 1 \\ \cdots & & \\ \alpha_{n1} \cdots & \alpha_{nm} & 1 \end{bmatrix}, \tag{23}$$

and we define

$$Q_1 = \begin{bmatrix} \alpha \cdots & \alpha & 2 \\ \alpha_{11} \cdots & \alpha_{1m} & 1 \\ \cdots & & \\ \alpha_{n1} \cdots & \alpha_{nm} & 1 \end{bmatrix}. \tag{24}$$

Proof. Applying the operator I_x^{α} to the first equation in (5), we get

$$I_x^{\alpha} \frac{{}^c\partial^{\alpha}}{\partial x^{\alpha}} Z(x,y) + \sum_{k=1}^{m} a_k I_x^{\alpha} I_1^{\alpha_{1k}} \cdots I_n^{\alpha_{nk}} Z(x,y) = I_x^{\alpha} f(x,y,Z(x,y)). \tag{25}$$

According to (2), we get

$$Z(x,y) - Z(0,y) - Z'_x(0,y)x = I_x^{\alpha} f(x,y,Z(x,y)) - \sum_{k=1}^{m} a_k I_x^{\alpha} I_1^{\alpha_{1k}} \cdots I_n^{\alpha_{nk}} Z(x,y), \tag{26}$$

which implies

$$-Z'_x(0,y) = I_{x=1}^{\alpha} f(x,y,Z(x,y)) - \sum_{k=1}^{m} a_k I_1^{\alpha_{1k}} \cdots I_n^{\alpha_{nk}} I_{x=1}^{\alpha} Z(x,y), \tag{27}$$

using the boundary condition

$$Z(0,y) = Z(1,y) = 0.$$

Hence,

$$Z'_x(0,y) = -I_{x=1}^{\alpha} f(x,y,Z(x,y)) + \sum_{k=1}^{m} a_k I_1^{\alpha_{1k}} \cdots I_n^{\alpha_{nk}} I_{x=1}^{\alpha} Z(x,y), \tag{28}$$

which infers that

$$\begin{aligned} &Z(x,y) + \sum_{k=1}^{m} a_k I_x^{\alpha} I_1^{\alpha_{1k}} \cdots I_n^{\alpha_{nk}} Z(x,y) = \\ &\quad I_x^{\alpha} f(x,y,Z(x,y)) - x I_{x=1}^{\alpha} f(x,y,Z(x,y)) + \sum_{k=1}^{m} a_k I_1^{\alpha_{1k}} \cdots I_n^{\alpha_{nk}} x I_{x=1}^{\alpha} Z(x,y). \end{aligned} \tag{29}$$

Therefore,

$$\begin{aligned} &\left(1 + \sum_{k=1}^{m} a_k I_x^{\alpha} I_1^{\alpha_{1k}} \cdots I_n^{\alpha_{nk}}\right) Z(x,y) = \\ &\quad I_x^{\alpha} f(x,y,Z(x,y)) - x I_{x=1}^{\alpha} f(x,y,Z(x,y)) + \sum_{k=1}^{m} a_k I_1^{\alpha_{1k}} \cdots I_n^{\alpha_{nk}} x I_{x=1}^{\alpha} Z(x,y). \end{aligned} \tag{30}$$

Using Babenko's method and the multinomial theorem, we have

$$\begin{aligned} Z(x,y) =& \left(1 + \sum_{k=1}^{m} a_k I_x^{\alpha} I_1^{\alpha_{1k}} \cdots I_n^{\alpha_{nk}}\right)^{-1} \\ &\cdot \left(I_x^{\alpha} f(x,y,Z(x,y)) - x I_{x=1}^{\alpha} f(x,y,Z(x,y)) + \sum_{k=1}^{m} a_k I_1^{\alpha_{1k}} \cdots I_n^{\alpha_{nk}} x I_{x=1}^{\alpha} Z(x,y)\right) \\ =& \sum_{l=0}^{\infty} (-1)^l \left(\sum_{k=1}^{m} a_k I_x^{\alpha} I_1^{\alpha_{1k}} \cdots I_n^{\alpha_{nk}}\right)^l \\ &\cdot \left(I_x^{\alpha} f(x,y,Z(x,y)) - x I_{x=1}^{\alpha} f(x,y,Z(x,y)) + \sum_{k=1}^{m} a_k I_1^{\alpha_{1k}} \cdots I_n^{\alpha_{nk}} x I_{x=1}^{\alpha} Z(x,y)\right) \\ =& \sum_{l=0}^{\infty} (-1)^l \sum_{\substack{l_1 + \cdots + l_m = l \\ l_1 \geq 0, \cdots, l_m \geq 0}} \binom{l}{l_1, \cdots, l_m} a_1^{l_1} \cdots a_m^{l_m} \left(I_x^{\alpha} I_1^{\alpha_{11}} \cdots I_n^{\alpha_{n1}}\right)^{l_1} \cdots \left(I_x^{\alpha} I_1^{\alpha_{1m}} \cdots I_n^{\alpha_{nm}}\right)^{l_m} \\ &\cdot \left(I_x^{\alpha} f(x,y,Z(x,y)) - x I_{x=1}^{\alpha} f(x,y,Z(x,y)) + \sum_{k=1}^{m} a_k I_1^{\alpha_{1k}} \cdots I_n^{\alpha_{nk}} x I_{x=1}^{\alpha} Z(x,y)\right) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{l=0}^{\infty}(-1)^l \sum_{\substack{l_1+\cdots+l_m=l\\ l_1\geq 0,\cdots,l_m\geq 0}} \binom{l}{l_1,\cdots,l_m} a_1^{l_1}\cdots a_m^{l_m} I_x^{l_1\alpha+\cdots+l_m\alpha+\alpha} \\
&\cdot I_1^{\alpha_{11}l_1+\cdots+\alpha_{1m}l_m}\cdots I_n^{\alpha_{n1}l_1+\cdots+\alpha_{nm}l_m} f(x,y,Z(x,y)) \\
&- \sum_{l=0}^{\infty}(-1)^l \sum_{\substack{l_1+\cdots+l_m=l\\ l_1\geq 0,\cdots,l_m\geq 0}} \binom{l}{l_1,\cdots,l_m} a_1^{l_1}\cdots a_m^{l_m} \frac{x^{\alpha l_1+\cdots+\alpha l_m+1}}{\Gamma(\alpha l_1+\cdots+\alpha l_m+2)} \\
&\cdot I_1^{\alpha_{11}l_1+\cdots+\alpha_{1m}l_m}\cdots I_n^{\alpha_{n1}l_1+\cdots+\alpha_{nm}l_m} I_{x=1}^{\alpha} f(x,y,Z(x,y)) + \sum_{k=1}^{m} a_k \sum_{l=0}^{\infty}(-1)^l \\
&\cdot \sum_{\substack{l_1+\cdots+l_m=l\\ l_1\geq 0,\cdots,l_m\geq 0}} \binom{l}{l_1,\cdots,l_m} a_1^{l_1}\cdots a_m^{l_m} \frac{x^{\alpha l_1+\cdots+\alpha l_m+1}}{\Gamma(\alpha l_1+\cdots+\alpha l_m+2)} \\
&\cdot I_1^{\alpha_{11}l_1+\cdots+\alpha_{1m}l_m+\alpha_{1k}}\cdots I_n^{\alpha_{n1}l_1+\cdots+\alpha_{nm}l_m+\alpha_{nk}} I_{x=1}^{\alpha} Z(x,y).
\end{aligned}
$$

Clearly, all above steps are reversible. It remains to be shown that $Z \in \mathcal{S}([0,1]\times\Omega)$. In fact,

$$
\begin{aligned}
\|Z\| \leq &\sum_{l=0}^{\infty} \sum_{\substack{l_1+\cdots+l_m=l\\ l_1\geq 0,\cdots,l_m\geq 0}} \binom{l}{l_1,\cdots,l_m} \frac{|a_1|^{l_1}\cdots|a_m|^{l_m}}{\Gamma(l_1\alpha+\cdots+l_m\alpha+\alpha+1)} \\
&\cdot \frac{1}{\Gamma(\alpha_{11}l_1+\cdots+\alpha_{1m}l_m+1)}\cdots\frac{1}{\Gamma(\alpha_{n1}l_1+\cdots+\alpha_{nm}l_m+1)} \sup_{(x,y)\in[0,1]\times\Omega,\, w\in R} |f(x,y,w)| \\
&+ \frac{1}{\Gamma(\alpha+1)} \sum_{l=0}^{\infty} \sum_{\substack{l_1+\cdots+l_m=l\\ l_1\geq 0,\cdots,l_m\geq 0}} \binom{l}{l_1,\cdots,l_m} \frac{|a_1|^{l_1}\cdots|a_m|^{l_m}}{\Gamma(l_1\alpha+\cdots+l_m\alpha+2)} \\
&\cdot \frac{1}{\Gamma(\alpha_{11}l_1+\cdots+\alpha_{1m}l_m+1)}\cdots\frac{1}{\Gamma(\alpha_{n1}l_1+\cdots+\alpha_{nm}l_m+1)} \sup_{(x,y)\in[0,1]\times\Omega,\, w\in R} |f(x,y,w)| \\
&+ \frac{1}{\Gamma(\alpha+1)} \sum_{k=1}^{m}|a_k| \sum_{l=0}^{\infty} \sum_{\substack{l_1+\cdots+l_m=l\\ l_1\geq 0,\cdots,l_m\geq 0}} \binom{l}{l_1,\cdots,l_m} \frac{|a_1|^{l_1}\cdots|a_m|^{l_m}}{\Gamma(l_1\alpha+\cdots+l_m\alpha+2)} \\
&\cdot \frac{1}{\Gamma(\alpha_{11}l_1+\cdots+\alpha_{1m}l_m+\alpha_{1k}+1)}\cdots\frac{1}{\Gamma(\alpha_{n1}l_1+\cdots+\alpha_{nm}l_m+\alpha_{nk}+1)}\|Z\| \\
&= \left(E_{Q_\alpha}(|a_1|,\cdots,|a_m|) + \frac{1}{\Gamma(\alpha+1)} E_{Q_1}(|a_1|,\cdots,|a_m|)\right) \sup_{(x,y)\in[0,1]\times\Omega,\, w\in R} |f(x,y,w)| \\
&+ \frac{1}{\Gamma(\alpha+1)} \sum_{k=1}^{m} |a_k| E_{M_k}(|a_1|,\cdots,|a_m|)\|Z\|.
\end{aligned}
$$

Since,

$$
A = 1 - \frac{1}{\Gamma(\alpha+1)} \sum_{k=1}^{m} |a_k| E_{M_k}(|a_1|,\cdots,|a_m|) > 0,
$$

we come to

$$
\begin{aligned}
\|Z\| \leq \frac{1}{A}&\left(E_{Q_\alpha}(|a_1|,\cdots,|a_m|) + \frac{1}{\Gamma(\alpha+1)} E_{Q_1}(|a_1|,\cdots,|a_m|)\right) \\
&\cdot \sup_{(x,y)\in[0,1]\times\Omega,\, w\in R} |f(x,y,Z)| < +\infty,
\end{aligned}
$$

since f is bounded. Hence, $Z \in \mathcal{S}([0,1]\times\Omega)$. This completes the proof of Theorem 1. □

Theorem 2. *Let $f : [0,1] \times \Omega \times \mathbb{R} \to \mathbb{R}$ be a continuous and bounded function satisfying the following Lipschitz condition for a positive constant $\mathcal{B}$:*

$$|f(x,y,w_1) - f(x,y,w_2)| \leq \mathcal{B}|w_1 - w_2|, \quad w_1,\, w_2 \in \mathbb{R}. \tag{31}$$

In addition, we assume that $1 < \alpha \leq 2$ and

$$D = \mathcal{B}\bigg(E_{Q_\alpha}(|a_1|, \cdots, |a_m|) + \frac{1}{\Gamma(\alpha+1)}\, E_{Q_1}(|a_1|, \cdots, |a_m|) \bigg) + \frac{1}{\Gamma(\alpha+1)} \sum_{k=1}^{m} |a_k| E_{M_k}(|a_1|, \cdots, |a_m|) < 1. \tag{32}$$

Then, the problem (5) has a unique solution in the space $\mathcal{S}([0,1] \times \Omega)$.

Proof. The nonlinear mapping $\mathcal{T}$ is defined over the space $\mathcal{S}([0,1] \times \Omega)$ by

$$\begin{aligned}
(\mathcal{T}Z)(x,y) = &\sum_{l=0}^{\infty}(-1)^l \sum_{\substack{l_1+\cdots+l_m=l \\ l_1\geq 0,\cdots,l_m\geq 0}} \binom{l}{l_1,\cdots,l_m} a_1^{l_1}\cdots a_m^{l_m}\, I_x^{l_1\alpha+\cdots+l_m\alpha+\alpha} \\
&\cdot I_1^{\alpha_{11}l_1+\cdots+\alpha_{1m}l_m} \cdots I_n^{\alpha_{n1}l_1+\cdots+\alpha_{nm}l_m} f(x,y,Z(x,y)) \\
&- \sum_{l=0}^{\infty}(-1)^l \sum_{\substack{l_1+\cdots+l_m=l \\ l_1\geq 0,\cdots,l_m\geq 0}} \binom{l}{l_1,\cdots,l_m} a_1^{l_1}\cdots a_m^{l_m} \frac{x^{\alpha l_1+\cdots+\alpha l_m+1}}{\Gamma(\alpha l_1+\cdots+\alpha l_m+2)} \\
&\cdot I_1^{\alpha_{11}l_1+\cdots+\alpha_{1m}l_m} \cdots I_n^{\alpha_{n1}l_1+\cdots+\alpha_{nm}l_m} I_{x=1}^{\alpha} f(x,y,Z(x,y)) + \sum_{k=1}^{m} a_k \sum_{l=0}^{\infty}(-1)^l \\
&\cdot \sum_{\substack{l_1+\cdots+l_m=l \\ l_1\geq 0,\cdots,l_m\geq 0}} \binom{l}{l_1,\cdots,l_m} a_1^{l_1}\cdots a_m^{l_m} \frac{x^{\alpha l_1+\cdots+\alpha l_m+1}}{\Gamma(\alpha l_1+\cdots+\alpha l_m+2)} \\
&\cdot I_1^{\alpha_{11}l_1+\cdots+\alpha_{1m}l_m+\alpha_{1k}} \cdots I_n^{\alpha_{n1}l_1+\cdots+\alpha_{nm}l_m+\alpha_{nk}} I_{x=1}^{\alpha} Z(x,y).
\end{aligned}$$

It follows from Theorem 1 that $\mathcal{T}Z \in \mathcal{S}([0,1] \times \Omega)$. We shall show that $\mathcal{T}$ is contractive. Indeed, for $Z_1, Z_2 \in \mathcal{C}([0,1] \times \Omega)$, we get

$$\begin{aligned}
\|\mathcal{T}Z_1 - \mathcal{T}Z_2\| \leq\ & \mathcal{B}E_{Q_\alpha}(|a_1|, \cdots, |a_m|)\|Z_1 - Z_2\| \\
&+ \frac{\mathcal{B}}{\Gamma(\alpha+1)}\, E_{Q_1}(|a_1|, \cdots, |a_m|)\|Z_1 - Z_2\| \\
&+ \frac{1}{\Gamma(\alpha+1)} \sum_{k=1}^{m} |a_k| E_{M_k}(|a_1|, \cdots, |a_m|)\|Z_1 - Z_2\| \\
&= D\|Z_1 - Z_2\|,
\end{aligned} \tag{33}$$

from the proof of Theorem 1, noting that

$$|f(x,y,Z_1) - f(x,y,Z_2)| \leq \mathcal{B}|Z_1 - Z_2|. \tag{34}$$

Since $D < 1$, the problem (5) has a unique solution in the space $\mathcal{S}([0,1] \times \Omega)$ using Banach's contractive principle. This completes the proof of Theorem 2. □

3. Example

Example 1. Consider the following equation:

$$\begin{cases} \frac{{}_{c}\partial^{1.5}}{\partial x^{1.5}} Z(x,y) + \sum_{k=1}^{3} \frac{1}{2^k} I_1^{\alpha_{1k}} \cdots I_4^{\alpha_{4k}} Z(x,y) = \frac{19}{50}\cos(x^2 y_2 Z(z,y)) + x^2 + |y| \sin x, \\ Z(0,y) = Z(1,y) = 0, \quad (x,y) \in [0,1] \times [0,1]^4, \end{cases} \tag{35}$$

where

$$(\alpha_{ij})_{1 \le i \le 3, 1 \le j \le 4} = \begin{bmatrix} 1.1 & 1.2 & 1.3 & 1.4 \\ 1.2 & 1.3 & 1.4 & 1.5 \\ 1.3 & 1.4 & 1.5 & 1.6 \end{bmatrix}. \tag{36}$$

Then, the problem (35) has a unique solution in the space $\mathcal{S}([0,1] \times [0,1]^4)$.

Proof. Clearly,

$$f(x,y,Z) = \frac{19}{50}\cos(x^2 y_2 Z(x,y)) + x^2 + |y| \sin x, \tag{37}$$

and

$$|f(x,y,z_1) - f(x,y,z_2)| \le \frac{19}{50} |\cos(x^2 y_2 z_1) - \cos(x^2 y_2 z_2)| \le \frac{19}{50} |z_1 - z_2|, \tag{38}$$

for all $z_1, z_2 \in \mathbb{R}$, noting that $(x,y) \in [0,1] \times [0,1]^4$. Therefore, $\mathcal{B} = 19/50$. We evaluate the following D given in Theorem 2 using our Python code to get

$$D = \frac{19}{50}\left(E_{Q_{1.5}}(1/2, 1/4, 1/8) + \frac{1}{\Gamma(2.5)} E_{Q_1}(1/2, 1/4, 1/8) \right) + \frac{1}{\Gamma(2.5)} \sum_{k=1}^{3} \frac{1}{2^k} E_{M_k}(1/2, 1/4, 1/8) \approx 0.9896 < 1. \tag{39}$$

Using Theorem 2, the problem (35) has a unique solution in the space $\mathcal{C}([0,1] \times [0,1]^4)$. □

The following is our Python code used to evaluate D given in Theorem 2:

```
import math
from sympy import gamma
def partition(n, m):
    if m == 1:
        yield (n,)
    else:
        for i in range(n+1):
            for j in partition(n-i, m-1):
               yield (i,) + j
def ME(M, z): #Matrix Mittag-Leffler function
    m = len(M)
    n = len(M[0])
    zl = len(z)
    result = 0
    for l in range(0, 40): #approximate upper bound
        for l_partition in partition(l, zl):
            if all(map(lambda x: x >= 0, l_partition)):
                combination = 1
                for i in range(zl):
                    combination *= math.factorial(l_partition[i])
                combination = math.factorial(l) / combination
                gamproduct = 1
```

```
                for i in range(m):
                  gaminput = sum([M[i][j] * l_partition[j]
                  for j in range(zl)]) + M[i][zl]
                  gamproduct *= gamma(gaminput)
                numerator = 1
                for i in range(zl):
                    numerator *= z[i] ** l_partition[i]
                result += (numerator / gamproduct) * combination
    return~result

#Matrices
MQ15 = [[1.5, 1.5, 1.5, 2.5], [1.1, 1.2, 1.3, 1],
[1.2, 1.3, 1.4, 1], [1.3, 1.4, 1.5, 1], [1.4, 1.5, 1.6, 1]]
MQ1 = [[1.5, 1.5, 1.5, 2], [1.1, 1.2, 1.3, 1], [1.2, 1.3, 1.4, 1],
[1.3, 1.4, 1.5, 1], [1.4, 1.5, 1.6, 1]]
M1 = [[1.5, 1.5, 1.5, 2], [1.1, 1.2, 1.3, 2.1], [1.2, 1.3, 1.4, 2.2],
[1.3, 1.4, 1.5, 2.3], [1.4, 1.5, 1.6, 2.4]]
M2 = [[1.5, 1.5, 1.5, 2], [1.1, 1.2, 1.3, 2.2], [1.2, 1.3, 1.4, 2.3],
[1.3, 1.4, 1.5, 2.4], [1.4, 1.5, 1.6, 2.5]]
M3 = [[1.5, 1.5, 1.5, 2], [1.1, 1.2, 1.3, 2.3], [1.2, 1.3, 1.4, 2.4],
[1.3, 1.4, 1.5, 2.5], [1.4, 1.5, 1.6, 2.6]]
z = [0.5, 0.25, 0.125]
#Calculation in example
result = 0.38*(ME(MQ15, z) + (1/gamma(2.5))*ME(MQ1, z)) + (1/gamma(2.5))
*(0.5*ME(M1, z) + 0.25*ME(M2, z) + 0.125*ME(M3, z))
print(result)
```

Remark 2. *The Python language is quite powerful in computing values of the multivariate Mittag–Leffler function or the newly introduced matrix multivariate Mittag–Leffler function. Indeed, these functions often appear in various fields and play an important role in studying integral or differential equations with initial or boundary conditions, as well as in finding approximate solutions.*

4. Conclusions

We have derived the sufficient condition for the uniqueness of the solutions to the new boundary value problem of the fractional nonlinear partial integro-differential Equation (5) in $\mathbb{R}^n$ using the matrix Mittag–Leffler function, Babenko's approach as well as Banach's contractive principle. Finally, we presented one example showing the applications of the key results derived using the Python code, computing the approximate value of the matrix Mittag–Leffler function.

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