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On a New Approach for Stability and Controllability Analysis of Functional Equations

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Abstract: We consider a new approach to approximate stability analysis for a tri-additive functional inequality and to obtain the optimal approximation for permuting tri-derivations and tri-homomorphisms in unital matrix algebras via the vector-valued alternative fixed-point theorem, which is a popular technique of proving the stability of functional equations. We also present a small list of aggregation functions on the classical, well-known special functions to investigate the best approximation error estimates using a different concept of perturbation stability.

Keywords: multi stability; approximation

MSC: 17A40; 39B52; 47B47; 39B62; 46L57

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1. Introduction and Preliminaries

In recent years, one of the attractive fields of research in the area of functional equations has been devoted to stability analysis. Stability analysis is a basic character of mathematical analysis and has paramount importance in different areas of science and engineering. In the nineteenth century, Ulam [1] proposed the popular Ulam stability of functional equations that was partially solved by Hyers in the Banach space setting [2]. The problem presented by Ulam inspired well-known mathematician such as Brzdek, Cieplinski, Brillouët–Belluot, Gajda, Ger, Šmerl, Sikorska, Fechner, Forti and others; for details and further references see [3–14], in particular Bourgin, who presented in [15] some remarks concerning approximately additive mappings.

In 1978, a new concept of Ulam stability was presented by Rassias which led to the improvement of what is known as Hyers–Ulam–Rassias stability of linear mappings [16]. The results were then improved by Aoki who weakened the condition for the bound of the norm of Cauchy difference [17]. As far as we know, works by Obloza [18,19] were among the first contributions dealing with the Ulam-type stability of ODEs. Since then, the stability results of different classes of ODEs and PDEs of fractional order were explored by using a wide spectrum of methods, see [20–26]. There are now many research papers in the literature which consider generalizations of Hyers–Ulam–Rassias stability for different types of functional equations, functional inequalities and fractional equations [27–31]. For example, in [32,33] Mittag–Leffler–Hyers–Ulam–Rassias stability, hypergeometric-Hyers–Ulam–Rassias stability, Wright–Hyers–Ulam–Rassias stability, and Fox–Hyers–Ulam–Rassias stability are presented.

Hyers's method, which was applied in [2], is usually named the direct technique, and has been used for investigating the stability of functional equations. However, this technique sometimes does not work (see [34]). Nevertheless, there are other techniques proving the stability results; for instance: the technique applying the notion of shadowing [35], the technique of invariant means [36] and the technique according to sandwich theorems [37]. In this paper, we propose the fixed-point technique that is the most important technique of proposing the stability of diverse mathematical equations. Although it was applyed for the first time by J. A. Baker [38], who used this technique to gain the stability of a functional equation, most authors follow Radu's technique [39] and apply the Diaz and Margolis theorem.

The major issue we are studying in this paper is that of aggregation maps which play an important role in several technical tasks scholars are faced with nowadays. The mentioned maps refer to the procedure of combining some inputs into one output. The oldest example is the notion of arithmetic mean which has been used throughout the history of empirical sciences. These maps are applied in both applied and pure mathematics (like: probability,theory of means), social sciences (like: psychology), engineering sciences (like: artificial intelligence, image analysis), as well as many other natural sciences [40,41]. In this paper, we apply n-ary aggregation functions on special functions to define a class of matrix-valued controllers which help us to present a concept of Ulam-type stability. The aggregation functions allow us to obtain the best approximation errors [42]. Recently, special functions like Mittag–Leffler function, hypergeometric function, Wright function, Fox H-function, Fox–Wright function, Meijer G-function, G-function and others have received a lot of attention because of their important roles in finding optimal solutions for different types of mathematical equations and their close relations to problems which come from applications [43].

In the present paper, we propose some novel notions concerning a new type of stability of a tri-additive $\boldsymbol{\lambda}$-fuzzy operator inequality in the Mittag–Leffler–Hyers–Ulam–Rassias sense using some special function which include the one-parameter Mittag–Leffler function, the one-parameter pre-supersine function generated by the Mittag–Leffler function, the one-parameter pre-superhyperbolic supersine function generated by the Mittag–Leffler function, the one-parameter pre-supercosine function generated by the Mittag–Leffler function, and the one-parameter pre-superhyperbolic supercosine function generated by the Mittag–Leffler function. In particular, in this paper, we consider the tri-additive $\boldsymbol{\lambda}$-fuzzy operator inequality

$$\begin{aligned}
&\mathcal{N}\Big(\chi(\lambda_1+\lambda_2,\lambda_3-\lambda_4,\lambda_5-\lambda_6)+\chi(\lambda_1-\lambda_2,\lambda_3+\lambda_4,\lambda_5-\lambda_6) \\
&\qquad -2\chi(\lambda_1,\lambda_3,\lambda_5)+2\chi(\lambda_1,\lambda_4,\lambda_6)-2\chi(\lambda_2,\lambda_3,\lambda_6)+2\chi(\lambda_2,\lambda_4,\lambda_5),\mathcal{T}\Big) \\
&\succeq \mathcal{N}\Big(\boldsymbol{\lambda}[2\chi(\frac{\lambda_1+\lambda_2}{2},\lambda_3-\lambda_4,\lambda_5+\lambda_6)+2\chi(\frac{\lambda_1-\lambda_2}{2},\lambda_3+\lambda_4,\lambda_5-\lambda_6) \\
&\qquad -2\chi(\lambda_1,\lambda_3,\lambda_5)+2\chi(\lambda_1,\lambda_4,\lambda_6)-2\chi(\lambda_2,\lambda_3,\lambda_6)+2\chi(\lambda_2,\lambda_4,\lambda_5)],\mathcal{T}\Big)
\end{aligned} \tag{1}$$

where $0 \neq \boldsymbol{\lambda} \in \mathbb{C}$ is fixed and $|\boldsymbol{\lambda}| < 1$. Further, we get an estimation for permuting tri-homomorphisms and tri-derivations in unital matrix FC-$\diamond$-algebras, associated with the above inequality. As an application, we present a small list of aggregation functions to get diverse estimates depending on the input values and to study optimum stability results and minimal errors that provide a unique optimal solution.

Let $n \in \mathbb{N}$, $\Phi := [0,1]$ and the following diagonal matrix defined by

$$\text{Diagonal}\Upsilon_n(\Phi) = \left\{ \begin{bmatrix} \psi_{11} & 0 & \cdots & 0 \\ 0 & \psi_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \psi_{nn} \end{bmatrix} = \text{Diagonal}[\psi_{11},\cdots,\psi_{nn}],\ \psi_{ii} \in \Phi,\ i = 1,\cdots,n \right\},$$

where $\text{DiagonalY}_n(\Phi)$ is equipped with the partial order relation:

$$\psi := \text{Diagonal}[\psi_{11}, \cdots, \psi_{nn}], \alpha := \text{Diagonal}[\alpha_{11}, \cdots, \alpha_{nn}] \in \text{DiagonalY}_n(\Phi),$$
$$\psi \preceq \alpha \Longleftrightarrow \psi_{ii} \leq \alpha_{ii}, \quad \forall i \in \mathbb{N}.$$

Also, $\psi \prec \alpha$ denotes that $\psi \preceq \alpha$ and $\psi \neq \alpha$; $\psi \ll \alpha$ and $\psi_{ii} < \alpha_{ii}$, for all $i \in \mathbb{N}$. We define $\boldsymbol{\varrho} := \text{Diagonal}[\varrho, \cdots, \varrho]$ in $\text{DiagonalY}_n(\Phi)$ in which $\varrho \in \Phi$. For example,

$$\mathbf{0} := \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix}_{n \times n} = \text{Diagonal}[0, \cdots, 0]_{n \times n},$$

and

$$\mathbf{1} := \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}_{n \times n} = \text{Diagonal}[1, \cdots, 1]_{n \times n}.$$

Here, we generalize the t-norm $\bigotimes_{\texttt{TN}}$ [44] on $\text{DiagonalY}_n(\Phi)$.

Definition 1 ([45])**.** *A generalized triangular norm (GTN) on* $\text{DiagonalY}_n(\Phi)$ *is an operation* $\bigotimes_{\texttt{GTN}} : \text{DiagonalY}_n(\Phi) \times \text{DiagonalY}_n(\Phi) \to \text{DiagonalY}_n(\Phi)$ *s.t.,*

(1) $(\forall \psi \in \text{DiagonalY}_n(\Phi))(\psi \bigotimes_{\texttt{GTN}} \mathbf{1}) = \psi)$ *(boundary condition);*

(2) $(\forall (\psi, \alpha) \in (\text{DiagonalY}_n(\Phi))^2)(\psi \bigotimes_{\texttt{GTN}} \alpha = \alpha \bigotimes_{\texttt{GTN}} \psi)$ *(commutativity);*

(3) $(\forall (\psi, \alpha, \phi) \in (\text{DiagonalY}_n(\Phi)^3)(\psi \bigotimes_{\texttt{GTN}} (\alpha \bigotimes_{\texttt{GTN}} \phi) = (\psi \bigotimes_{\texttt{GTN}} \alpha) \bigotimes_{\texttt{GTN}} \phi)$ *(associativity);*

(4) $(\forall (\psi, \psi', \alpha, \alpha') \in (\text{DiagonalY}_n(\Phi)^4(\psi \preceq \psi'$ *and* $\alpha \preceq \alpha' \Longrightarrow \psi \bigotimes_{\texttt{GTN}} \alpha \preceq \psi' \bigotimes_{\texttt{GTN}} \alpha'$ *(monotonicity).*

For all $\psi := \text{Diagonal}[\psi_{11}, \cdots, \psi_{nn}], \alpha := \text{Diagonal}[\alpha_{11}, \cdots, \alpha_{nn}] \in \text{DiagonalY}_n(\Phi)$ and all sequences $\{(\psi_{ii})_k\}$ and $\{(\alpha_{ii})_k\}$, with $1 \leq i \leq n$ and $k > 0$, converging to ψ_{ii} and α_{ii}, suppose we have

$$\lim_{k \longrightarrow \infty} \left(\text{Diagonal}[(\psi_{11})_k, \cdots, (\psi_{nn})_k] \bigotimes_{\texttt{GTN}} \text{Diagonal}[(\alpha_{11})_k, \cdots, (\alpha_{nn})_k] \right)$$
$$= \text{Diagonal}[\psi_{11}, \cdots, \psi_{nn}] \bigotimes_{\texttt{GTN}} \text{Diagonal}[\alpha_{11}, \cdots, \alpha_{nn}],$$

then, $\bigotimes_{\texttt{GTN}}$ on $\text{DiagonalY}_n(\Phi)$ is a continuous generalized triangular norm (CGTN). Now, we present some examples of continuous generalized triangular norms.

(1) Let $\bigotimes^{P}_{\texttt{GTN}} : \text{DiagonalY}_n(\Phi) \times \text{DiagonalY}_n(\Phi) \to \text{DiagonalY}_n(\Phi)$, such that,

$$\begin{aligned} \psi \bigotimes^{P}_{\texttt{GTN}} \alpha &= \text{Diagonal}[\psi_{11}, \cdots, \psi_{nn}] \bigotimes^{P}_{\texttt{GTN}} \text{Diagonal}[\alpha_{11}, \cdots, \alpha_{nn}] \\ &= \text{Diagonal}[\psi_{11}.\alpha_{11}, \cdots, \psi_{nn}.\alpha_{nn}], \end{aligned}$$

then, $\bigotimes^{P}_{\texttt{GTN}}$ is a CGTN.

(2) Let $\bigotimes^{M}_{\mathtt{GTN}}$: $\text{DiagonalY}_n(\Phi) \times \text{DiagonalY}_n(\Phi) \to \text{DiagonalY}_n(\Phi)$, such that,

$$\begin{aligned}\psi \overset{M}{\underset{\mathtt{GTN}}{\bigotimes}} \alpha &= \text{Diagonal}[\psi_{11}, \cdots, \psi_{nn}] \overset{M}{\underset{\mathtt{GTN}}{\bigotimes}} \text{Diagonal}[\alpha_{11}, \cdots, \alpha_{nn}] \\ &= \text{Diagonal}[\min\{\psi_{11}, \alpha_{11}\}, \cdots, \min\{\psi_{nn}, \alpha_{nn}\}],\end{aligned}$$

then, $\bigotimes^{M}_{\mathtt{GTN}}$ is a CGTN.

(3) Let $\bigotimes^{L}_{\mathtt{GTN}}$: $\text{DiagonalY}_n(\Phi) \times \text{DiagonalY}_n(\Phi) \to \text{DiagonalY}_n(\Phi)$, such that,

$$\begin{aligned}\psi \overset{L}{\underset{\mathtt{GTN}}{\bigotimes}} \alpha &= \text{Diagonal}[\psi_{11}, \cdots, \psi_{nn}] \overset{L}{\underset{\mathtt{GTN}}{\bigotimes}} \text{Diagonal}[\alpha_{11}, \cdots, \alpha_{nn}] \\ &= \text{Diagonal}[\max\{\psi_{11} + \alpha_{11} - 1, 0\}, \cdots, \max\{\psi_{nn} + \alpha_{nn} - 1, 0\}],\end{aligned}$$

then, $\bigotimes^{L}_{\mathtt{GTN}}$ is a CGTN.

Here, we present some numeric examples:

$$\text{Diagonal}\left[\frac{1}{2}, 0.2, 1\right] \overset{M}{\underset{\mathtt{GTN}}{\bigotimes}} \text{Diagonal}\left[\frac{3}{10}, 0.7, 0\right] = \begin{bmatrix}\frac{1}{2} & & \\ & 0.2 & \\ & & 1\end{bmatrix} \overset{M}{\underset{\mathtt{GTN}}{\bigotimes}} \begin{bmatrix}\frac{3}{10} & & \\ & 0.7 & \\ & & 0\end{bmatrix} = \begin{bmatrix}\frac{3}{10} & & \\ & 0.2 & \\ & & 0\end{bmatrix}$$

$$\text{Diagonal}\left[\frac{1}{2}, 0.2, 1\right] \overset{P}{\underset{\mathtt{GTN}}{\bigotimes}} \text{Diagonal}\left[\frac{3}{10}, 0.7, 0\right] = \begin{bmatrix}\frac{1}{2} & & \\ & 0.2 & \\ & & 1\end{bmatrix} \overset{P}{\underset{\mathtt{GTN}}{\bigotimes}} \begin{bmatrix}\frac{3}{10} & & \\ & 0.7 & \\ & & 0\end{bmatrix} = \begin{bmatrix}\frac{3}{20} & & \\ & \frac{7}{50} & \\ & & 0\end{bmatrix}$$

$$\text{Diagonal}\left[\frac{1}{2}, 0.2, 1\right] \overset{L}{\underset{\mathtt{GTN}}{\bigotimes}} \text{Diagonal}\left[\frac{3}{10}, 0.7, 0\right] = \begin{bmatrix}\frac{1}{2} & & \\ & 0.2 & \\ & & 1\end{bmatrix} \overset{L}{\underset{\mathtt{GTN}}{\bigotimes}} \begin{bmatrix}\frac{3}{10} & & \\ & 0.7 & \\ & & 0\end{bmatrix} = \begin{bmatrix}0 & & \\ & 0 & \\ & & 0\end{bmatrix}$$

Then, we get

$$\begin{aligned}& \text{Diagonal}\left[\frac{1}{2}, 0.2, 1\right] \overset{M}{\underset{\mathtt{GTN}}{\bigotimes}} \text{Diagonal}\left[\frac{3}{10}, 0.7, 0\right] \\ \succeq\ & \text{Diagonal}\left[\frac{1}{2}, 0.2, 1\right] \overset{P}{\underset{\mathtt{GTN}}{\bigotimes}} \text{Diagonal}\left[\frac{3}{10}, 0.7, 0\right] \\ \succeq\ & \text{Diagonal}\left[\frac{1}{2}, 0.2, 1\right] \overset{L}{\underset{\mathtt{GTN}}{\bigotimes}} \text{Diagonal}\left[\frac{3}{10}, 0.7, 0\right].\end{aligned}$$

We are interested in defining a multi-control function using some known special functions, and to achieve it, we apply diagonal matrices as the values of control functions instead of finite sequences, because the value of a control function must be a member of a topological monoid with a unit (i.e., **1**) with the monotonicity property (Definition 1, (4)).

Let ξ be a vector space and $\mathcal{T} > 0$. We denote the set of a matrix valued fuzzy set by Λ^*. Now, $\mathscr{N} \in \Lambda^*$ denotes $\mathscr{N} : \xi \times (0, +\infty) \longrightarrow \text{DiagonalY}_n(\Phi)$ s.t.,

- $\mathscr{N}$ is continuous;
- $\mathscr{N}(\zeta, .)$ is non-decreasing, for every $\zeta \in \xi$;
- $\lim_{\mathcal{T} \to +\infty}(\zeta, \mathcal{T}) = \mathbf{1}$, for every $\zeta \in \xi$.

In Λ^*, we denote " $\preceq$ " as follows:

$$\mathscr{N} \preceq \mathscr{N}_\circ \iff \mathscr{N}(\zeta, \mathcal{T}) \preceq \mathscr{N}_\circ(\zeta, \mathcal{T}'), \quad \forall \mathcal{T}', \mathcal{T} > 0, \text{ and } \zeta \in \xi.$$

Definition 2 ([45]). *Consider the CGTN $\bigotimes_{\texttt{GTN}}$, a vector space ξ and a matrix valued fuzzy set $\mathscr{N} : \xi \times (0, +\infty) \longrightarrow \text{Diagonal}\Upsilon_n(\Phi)$. In this case, we define a matrix valued fuzzy norm $\mathscr{N}$ as follows:*

(D-1) $\mathscr{N}(\zeta, \mathcal{T}) = \mathbf{1}$ *for any* $\mathcal{T} > 0$ *if and only if* $\zeta = 0$;
(D-2) $\mathscr{N}(\jmath\zeta, \mathcal{T}) = \mathscr{N}(\zeta, \frac{\mathcal{T}}{|\jmath|})$ *for any* $\zeta \in \xi, \mathcal{T} > 0$, *and* $0 \neq \jmath \in \mathbb{C}$;
(D-3) $\mathscr{N}(\zeta + \zeta', \mathcal{T} + \mathcal{T}') \succeq \mathscr{N}(\zeta, \mathcal{T}) \bigotimes_{\texttt{GTN}} \mathscr{N}(\zeta', \mathcal{T}')$ *for all* $\zeta, \zeta' \in \xi$ *and* $\mathcal{T}, \mathcal{T}' \geq 0$;
(D-4) $\lim_{\mathcal{T} \to +\infty} \mathscr{N}(\zeta, \mathcal{T}) = \mathbf{1}$, *for all* $\zeta \in \xi$.
Now, $(\xi, \mathscr{N}, \bigotimes_{\texttt{GTN}})$, *is called a matrix valued fuzzy normed space (MFN-space).*

For example, the matrix valued fuzzy set $\mathscr{N}$

$$\mathscr{N}(\zeta, \mathcal{T}) = \text{Diagonal}\left[\exp(-\frac{\|\zeta\|}{\mathcal{T}}), \frac{\mathcal{T}}{\mathcal{T} + \|\zeta\|}\right],$$

is a matrix valued fuzzy norm, where $\mathcal{T} > 0$ and $(\xi, \mathscr{N}, \bigotimes^{M}_{\texttt{GTN}})$ is an MFN-space and $(\xi, \|\cdot\|)$ is a linear normed space.

Definition 3 ([45]). *Consider the MFN-space $(\xi, \mathscr{N}, \bigotimes_{\texttt{GTN}})$ and the CGTNs $\bigotimes_{\texttt{GTN}}$ and $\bigoplus_{\texttt{GTN}}$. If*
(D-5) $\mathscr{N}(\zeta\zeta', \mathcal{T}\mathcal{T}') \succeq \mathscr{N}(\zeta, \mathcal{T}) \bigoplus_{\texttt{GTN}} \mathscr{N}(\zeta', \mathcal{T}')$ *for any* $\zeta, \zeta' \in \xi$ *and any* $\mathcal{T}', \mathcal{T} > 0$,
then $(\xi, \mathscr{N}, \bigotimes_{\texttt{GTN}}, \bigoplus_{\texttt{GTN}})$ is called a matrix fuzzy normed algebra (MFN-algebra).

If
$$\|\zeta\zeta'\| \leq \|\zeta\|\|\zeta'\| + \mathcal{T}'\|\zeta'\| + \mathcal{T}\|\zeta\| \quad (\zeta, \zeta' \in (\xi, \|\cdot\|); \quad \mathcal{T}, \mathcal{T}' > 0),$$
then,
$$\mathscr{N}(\zeta, \mathcal{T}) = \text{Diagonal}\left[\frac{1}{e^{\frac{\|\zeta\|}{\mathcal{T}}}}, \frac{1}{1 + \frac{\|\zeta\|}{\mathcal{T}}}\right],$$
for any $\mathcal{T} > 0$, is a matrix fuzzy normed algebra $(\xi, \mathscr{N}, \bigotimes^{M}_{\texttt{GTN}}, \bigotimes^{P}_{\texttt{GTN}})$ and vice versa. A complete matrix fuzzy normed algebra is called a matrix fuzzy Banach algebra (or MFB-algebra).

Let $(\xi, \mathscr{N}, \bigotimes_{\texttt{GTN}}, \bigoplus_{\texttt{GTN}})$ be an MFB-algebra. An involution on ξ is a mapping $\zeta \to \zeta^{\diamond}$ from ξ into ξ, s.t.,

(1) $\zeta^{\diamond\diamond} = \zeta$ for any $\zeta \in \xi$;
(2) $(\varphi\zeta + \phi\zeta)^{\diamond} = \bar{\varphi}\zeta^{\diamond} + \bar{\phi}\zeta^{\diamond}$, for any $\zeta \in \xi$;
(3) $(\zeta\zeta')^{\diamond} = \zeta'^{\diamond} m^{\diamond}$ for any $\zeta, \zeta' \in \xi$.

Then, ξ is called an MFB-$\diamond$-algebra. In addition, if $\mathscr{N}(\zeta^{\diamond}\zeta, \mathcal{T}) = \mathscr{N}(\zeta, \mathcal{T})$ for all $\zeta \in \xi$ and $\mathcal{T} > 0$, then ξ is called an MFC-$\diamond$-algebra.

Here, we denote the unital MFC-$\diamond$-algebra $(\xi, \mathscr{N}, \bigotimes_{\texttt{GTN}}, \bigoplus_{\texttt{GTN}})$ with unit e and the unitary group $U(\xi) = \{\theta \in \xi : \theta^{\diamond}\theta = \theta\theta^{\diamond} = e\}$.

Definition 4 ([45]). *A mapping $\varpi : \nu^3 \to \nu$ is called tri-additive, if*

$$\varpi(\alpha + \epsilon, \beta, \gamma) = \varpi(\alpha, \beta, \gamma) + \varpi(\epsilon, \beta, \gamma),$$
$$\varpi(\alpha, \beta + \epsilon, \gamma) = \varpi(\alpha, \beta, \gamma) + \varpi(\alpha, \epsilon, \gamma),$$
$$\varpi(\alpha, \beta, \gamma + \epsilon) = \varpi(\alpha, \beta, \gamma) + \varpi(\alpha, \beta, \epsilon),$$

for every $\alpha, \gamma, \beta, \epsilon \in \nu$.

Definition 5 ([45]). *Consider the ring ν. A tri-additive mapping $\varpi : \nu^3 \to \nu$ is a permuting tri-derivation on ν if we have*

$$\begin{aligned} \varpi(\lambda_1\lambda_2, \lambda_3, \lambda_4) &= \varpi(\lambda_1, \lambda_3, \lambda_4)\lambda_2 + \lambda_1\varpi(\lambda_2, \lambda_3, \lambda_4), \\ \varpi(\ell_{\beta(1)}, \ell_{\beta(2)}, \ell_{\beta(3)}) &= \varpi(\ell_1, \ell_2, \ell_3) \end{aligned}$$

for all permutations $(\beta(1), \beta(2), \beta(3))$ *of* $(1,2,3)$, *and for all* $\lambda_1, \cdots, \lambda_4, \ell_1, \ell_2, \ell_3 \in \nu$.

Definition 6 ([45]). *Consider two complex Banach algebras* ν *and* Θ. *A* $\mathbb{C}-$*trilinear mapping* $\rho : \nu^3 \to \Theta$ *is a permuting tri-homomorphism if we have*

$$\begin{aligned} \rho(\lambda_1\lambda_2, \lambda_3\lambda_4, \lambda_5\lambda_6) &= \rho(\lambda_1, \lambda_3, \lambda_5)\rho(\lambda_2, \lambda_4, \lambda_6), \\ \rho(\ell_{\beta(1)}, \ell_{\beta(2)}, \ell_{\beta(3)}) &= \rho(\ell_1, \ell_2, \ell_3) \end{aligned}$$

for all permutations $(\beta(1), \beta(2), \beta(3))$ *of* $(1,2,3)$, *and for all* $\lambda_1, \cdots, \lambda_4, \ell_1, \ell_2, \ell_3 \in \nu$.

Next, we propose vector valued generalized metric spaces.

Definition 7. *Let* $\hbar = (\hbar_1, \ldots, \hbar_m)$ *and* $\gamma = (\gamma_1, \ldots, \gamma_m), m \in \mathbb{N}$. *Thus, we have*

$$\hbar \preceq \gamma \Longleftrightarrow \hbar_j \leq \gamma_j, \quad j = 1, \cdots, m;$$

and also

$$\hbar \to 0 \Longleftrightarrow \hbar_j \to 0, \quad j = 1, \cdots, m.$$

Definition 8 ([46]). *Consider the nonempty set* $\mathfrak{J}$ *and a given mapping* $\hbar : \mathfrak{J}^2 \to [0, +\infty]^m$, $m \in \mathbb{N}$. *A generalized metric* $\hbar$ *on* $\mathfrak{J}$ *is a function s.t.,*

(1) *for all* $(\epsilon_1, \epsilon_2) \in \mathfrak{J}^2$, *we get*

$$\hbar(\epsilon_1, \epsilon_2) = \mathbf{0} = \underbrace{(0, \cdots, 0)}_{m} \Longleftrightarrow \epsilon_1 = \epsilon_2;$$

(2) *for all* $(\epsilon_1, \epsilon_2) \in \mathfrak{J}^2$, *we get*

$$\hbar(\epsilon_1, \epsilon_2) = \hbar(\epsilon_1, \epsilon_2) \Longleftrightarrow \epsilon_1 = \epsilon_2;$$

(3) *for all* $\epsilon_1, \epsilon_2, \iota \in \mathfrak{J}$, *we get*

$$\hbar(\epsilon_1, \iota) + \hbar(\iota, \epsilon_2) \succeq \hbar(\epsilon_1, \epsilon_2).$$

Theorem 1 ([46]). *Let* $m \in \mathbb{N}$ *and consider a function* $\hbar : \mathfrak{J}^2 \to [0, +\infty]^m$, *and a complete generalized metric space* $(\mathfrak{J}, \hbar)$. *Consider a strictly contractive function* $\Gamma : \mathfrak{J} \to \mathfrak{J}$ *with Lipschitz constant* $T < 1$. *Then, for any* $\vartheta \in \mathfrak{J}$, *either*

$$\hbar(\Gamma^n\vartheta, \Gamma^{n+1}\vartheta) = \underbrace{(+\infty, \cdots, +\infty)}_{m}$$

for all $n \in \mathbb{N} \cup \{0\}$ *or there exists an* $n_0 \in \mathbb{N}$ *s.t.*

(1) $\hbar(\Gamma^n\vartheta, \Gamma^{n+1}\vartheta) \leq \underbrace{(+\infty, \cdots, +\infty)}_{m}, \qquad \forall n \geq n_0;$

(2) *The fixed point* κ^* *of* Γ *is a convergence point of the sequence* $\{\Gamma^n\vartheta\}$ *and is unique in the set* $\mathfrak{J}' = \{\kappa \in \mathfrak{J} \mid \hbar(\Gamma^{n_0}\vartheta, \kappa) \leq \underbrace{(+\infty, \cdots, +\infty)}_{m}\}$;

(3) $\hbar(\kappa, \kappa^*) \leq \frac{1}{1-T}\hbar(\kappa, \Gamma\kappa)$ *for every* $\kappa \in \mathfrak{J}'$.

Consider the MFN-space $\mathfrak{G}$, the MFB-space $\mathfrak{P}$ and the MFB-algebras ν and Θ and also let $\boldsymbol{\lambda} \in \mathbb{C}$ s.t. $|\boldsymbol{\lambda}| < 1$.

2. Tri-Additive λ-Functional Inequality (1)

Using Theorem 1, we study the multi stability of the functional Equation (1) in MFB-algebras.

Lemma 1. *Let W be a linear space and $\chi : W^3 \to W$ be a function satisfying (1), for any $\lambda_i \in W, i = 1, \cdots, 6$. Let $\chi(0, \lambda_3, \lambda_5) = \chi(\lambda_1, 0, \lambda_5) = \chi(\lambda_1, \lambda_3, 0) = \chi(\lambda_1, 0, 0) = \chi(\lambda_2, 0, \lambda_5) = \chi(\lambda_2, \lambda_3, 0) = 0$, for any $\lambda_i \in W, i = 1, 2, 3, 5$. Then, $\chi : W \to W$ is tri-additive.*

Proof. Putting $\lambda_1 = \lambda_2$ and $\lambda_4 = \lambda_6 = 0$ in (1), we have

$$\chi(2\lambda_1, \lambda_3, \lambda_5) = 2\chi(\lambda_1, \lambda_3, \lambda_5), \quad \forall \lambda_1, \lambda_3, \lambda_5 \in W.$$

Thus,

$$\begin{aligned} &\mathscr{N}\Big(\chi(\lambda_1 + \lambda_2, \lambda_3 - \lambda_4, \lambda_5 - \lambda_6) + \chi(\lambda_1 - \lambda_2, \lambda_3 + \lambda_4, \lambda_5 - \lambda_6) \\ &-2\chi(\lambda_1, \lambda_3, \lambda_5) + 2\chi(\lambda_1, \lambda_4, \lambda_6) - 2\chi(\lambda_2, \lambda_3, \lambda_6) + 2\chi(\lambda_2, \lambda_4, \lambda_5), \mathcal{T}\Big) \\ &\succeq \mathscr{N}\Big(\boldsymbol{\lambda}[\chi(\lambda_1 + \lambda_2, \lambda_3 - \lambda_4, \lambda_5 + \lambda_6) + \chi(\lambda_1 - \lambda_2, \lambda_3 + \lambda_4, \lambda_5 - \lambda_6) \\ &-2\chi(\lambda_1, \lambda_3, \lambda_5) + 2\chi(\lambda_1, \lambda_4, \lambda_6) - 2\chi(\varpi, \lambda_2, \lambda_3, \lambda_6) + 2\chi(\lambda_2, \lambda_4, \lambda_5)], \mathcal{T}\Big), \end{aligned} \tag{2}$$

and so

$$\begin{aligned} &\chi(\lambda_1 + \lambda_2, \lambda_3 - \lambda_4, \lambda_5 - \lambda_6) + \chi(\lambda_1 - \lambda_2, \lambda_3 + \lambda_4, \lambda_5 - \lambda_6) \\ &-2\chi(\lambda_1, \lambda_3, \lambda_5) + 2\chi(\lambda_1, \lambda_4, \lambda_6) - 2\chi(\lambda_2, \lambda_3, \lambda_6) + 2\chi(\lambda_2, \lambda_4, \lambda_5) = 0, \end{aligned} \tag{3}$$

for all $\lambda_i \in W, i = 1, \cdots, 6$. Now, putting $\lambda_4 = \lambda_6 = 0$ in (3), we have

$$\begin{aligned} &\chi(\lambda_1 + \lambda_2, \lambda_3, \lambda_5) + \chi(\lambda_1 - \lambda_2, \lambda_3, \lambda_5) \\ &= 2\chi(\lambda_1, \lambda_3, \lambda_5), \end{aligned} \tag{4}$$

and so

$$\begin{aligned} &\chi(\lambda, \lambda_3, \lambda_5) + \chi(\ell, \lambda_3, \lambda_5) \\ &= 2\chi(\frac{\lambda + \ell}{2}, \lambda_3, \lambda_5) = \chi(\lambda + \ell, \lambda_3, \lambda_5), \end{aligned} \tag{5}$$

for all $\lambda = \lambda_1 + \lambda_2, \ell = \lambda_1 - \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \in W$. Since $|\boldsymbol{\lambda}| < 1$, $\chi : W^3 \to W$ is additive in the second variable. In a similar way, we have that $\chi : W^3 \to W$ is additive both in the second and in the third variable. Thus, $\chi : W^3 \to W$ is tri-additive. $\square$

Theorem 2. *Suppose $i = 1, \cdots, n$ and $n \in \mathbb{N}$. Let $(\mathfrak{G}, \mathscr{N}, \bigotimes^M_{\texttt{GTN}}, \bigotimes^M_{\texttt{GTN}})$ be an MFB-algebra, and $\mathcal{E}_i : \mathfrak{G}^6 \times (0, +\infty) \to \Phi$ be a fuzzy control function s.t. there exists a $\vartheta_i < 1$ with*

$$\mathcal{E}_i\left(\frac{\lambda_1}{2}, \frac{\lambda_2}{2}, \frac{\lambda_3}{2}, \frac{\lambda_4}{2}, \frac{\lambda_5}{2}, \frac{\lambda_6}{2}, \mathcal{T}\right) \succeq \mathcal{E}_i\left(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \frac{2}{\vartheta_i}\mathcal{T}\right) \tag{6}$$

and

$$\lim_{j\to\infty} \mathcal{E}_i\left(\frac{\lambda_1}{2^j}, \frac{\lambda_2}{2^j}, \frac{\lambda_3}{2^j}, \frac{\lambda_4}{2^j}, \frac{\lambda_5}{2^j}, \frac{\lambda_6}{2^j}, \frac{\mathcal{T}}{2^j}\right) = \mathbf{1}, \tag{7}$$

for all $\lambda_i \in \mathfrak{G}, i = 1, \ldots, 6$ and $\mathcal{T} > 0$. Let the fuzzy operator $\chi : \mathfrak{G}^3 \to \mathfrak{P}$ satisfying $\chi(0, \lambda_3, \lambda_5) = \chi(\lambda_1, 0, \lambda_5) = \chi(\lambda_1, \lambda_3, 0) = \chi(\lambda_1, 0, 0) = \chi(\lambda_2, 0, \lambda_5) = \chi(\lambda_2, \lambda_3, 0) = 0$ and

$$\begin{aligned} &\mathscr{N}\Big(\chi(\lambda_1+\lambda_2, \lambda_3-\lambda_4, \lambda_5-\lambda_6) + \chi(\lambda_1-\lambda_2, \lambda_3+\lambda_4, \lambda_5-\lambda_6) \\ &\quad -2\chi(\lambda_1, \lambda_3, \lambda_5) + 2\chi(\lambda_1, \lambda_4, \lambda_6) - 2\chi(\lambda_2, \lambda_3, \lambda_6) + 2\chi(\lambda_2, \lambda_4, \lambda_5), \mathcal{T}\Big) \\ &\succeq \mathscr{N}\Big(\mathfrak{I}[2\chi(\frac{\lambda_1+\lambda_2}{2}, \lambda_3-\lambda_4, \lambda_5+\lambda_6) + 2\chi(\frac{\lambda_1-\lambda_2}{2}, \lambda_3+\lambda_4, \lambda_5-\lambda_6) \\ &\quad -2\chi(\lambda_1, \lambda_3, \lambda_5) + 2\chi(\lambda_1, \lambda_4, \lambda_6) - 2\chi(\lambda_2, \lambda_3, \lambda_6) + 2\chi(\lambda_2, \lambda_4, \lambda_5)], \mathcal{T}\Big) \\ &\bigotimes_{\mathtt{GTN}} Diagonal\Big[\mathcal{E}_1\Big(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \mathcal{T}\Big), \cdots, \mathcal{E}_n\Big(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \mathcal{T}\Big)\Big]. \end{aligned} \tag{8}$$

Then, there exists a unique tri-additive mapping $\chi' : \mathfrak{G}^3 \to \mathfrak{P}$ satisfying

$$\begin{aligned} &\mathscr{N}\Big(\chi(\lambda_1, \lambda_3, \lambda_5) - \chi'(\lambda_1, \lambda_3, \lambda_5), \mathcal{T}\Big) \\ &\succeq Diagonal\Big[\mathcal{E}_1\Big(\lambda_1, 0, \frac{\lambda_3}{2}, \frac{\lambda_3}{2}, \lambda_5, 0, (1-4\vartheta_1)\mathcal{T}\Big) \\ &\quad \bigotimes_{\mathtt{TN}} \mathcal{E}_1\Big(\frac{\lambda_1}{2}, \frac{\lambda_1}{2}, \frac{\lambda_3}{2}, 0, \lambda_5, 0, \frac{(1-4\vartheta_1)\mathcal{T}}{2}\Big) \\ &\quad \bigotimes_{\mathtt{TN}} \mathcal{E}_1\Big(\frac{\lambda_1}{2}, 0, \frac{\lambda_3}{2}, 0, \frac{\lambda_5}{2}, \frac{\lambda_5}{2}, \frac{(1-4\vartheta_1)\mathcal{T}}{4}\Big), \cdots, \\ &\quad \mathcal{E}_n\Big(\lambda_1, 0, \frac{\lambda_3}{2}, \frac{\lambda_3}{2}, \lambda_5, 0, (1-4\vartheta_n)\mathcal{T}\Big) \\ &\quad \bigotimes_{\mathtt{TN}} \mathcal{E}_n\Big(\frac{\lambda_1}{2}, \frac{\lambda_1}{2}, \frac{\lambda_3}{2}, 0, \lambda_5, 0, \frac{(1-4\vartheta_n)\mathcal{T}}{2}\Big) \\ &\quad \bigotimes_{\mathtt{TN}} \mathcal{E}_n\Big(\frac{\lambda_1}{2}, 0, \frac{\lambda_3}{2}, 0, \frac{\lambda_5}{2}, \frac{\lambda_5}{2}, \frac{(1-4\vartheta_n)\mathcal{T}}{4}\Big)\Big], \end{aligned} \tag{9}$$

for all $\lambda_1, \lambda_3, \lambda_5 \in \mathfrak{G}$.

Proof. Letting $\lambda_4 = \lambda_6 = 0$ and $\lambda_2 = \lambda_1$ in (8), we get

$$\begin{aligned} &\mathscr{N}\Big(\chi(2\lambda_1, \lambda_3, \lambda_5) - 2\chi(\lambda_1, \lambda_3, \lambda_5), \mathcal{T}\Big) \\ &\succeq \text{Diagonal}\Big[\mathcal{E}_1\Big(\lambda_1, \lambda_1, \lambda_3, 0, \lambda_5, 0, \mathcal{T}\Big), \cdots, \mathcal{E}_n\Big(\lambda_1, \lambda_1, \lambda_3, 0, \lambda_5, 0, \mathcal{T}\Big)\Big], \end{aligned}$$

and so

$$\begin{aligned} &\mathscr{N}\Big(\chi(2\lambda_1, \lambda_3, 2\lambda_5) - 2\chi(\lambda_1, \lambda_3, 2\lambda_5), \mathcal{T}\Big) \\ &\succeq \text{Diagonal}\Big[\mathcal{E}_1\Big(\lambda_1, \lambda_1, \lambda_3, 0, 2\lambda_5, 0, \mathcal{T}\Big), \cdots, \mathcal{E}_n\Big(\lambda_1, \lambda_1, \lambda_3, 0, 2\lambda_5, 0, \mathcal{T}\Big)\Big], \end{aligned}$$

for all $\lambda_1, \lambda_3, \lambda_5 \in \mathfrak{G}$.

Letting $\lambda_2 = \lambda_4 = 0$ and $\lambda_6 = \lambda_5$ in (8), we get

$$\begin{aligned}&\mathscr{N}\Big(\chi(\lambda_1,\lambda_3,2\lambda_5) - 2\chi(\lambda_1,\lambda_3,\lambda_5), \mathcal{T}\Big)\\ &\succeq \text{Diagonal}\Big[\mathcal{E}_1\Big(\lambda_1,0,\lambda_3,0,\lambda_5,\lambda_5,\mathcal{T}\Big),\cdots,\mathcal{E}_n\Big(\lambda_1,0,\lambda_3,0,\lambda_5,\lambda_5,\mathcal{T}\Big)\Big],\end{aligned}$$

and so

$$\begin{aligned}&\mathscr{N}\Big(\chi(2\lambda_1,\lambda_3,2\lambda_5) - 4\chi(\lambda_1,\lambda_3,\lambda_5), \mathcal{T}\Big)\\ &\succeq \text{Diagonal}\Big[\mathcal{E}_1\Big(\lambda_1,\lambda_1,\lambda_3,0,2\lambda_5,0,\mathcal{T}\Big)\bigotimes_{\mathtt{TN}}\mathcal{E}_1\Big(\lambda_1,0,\lambda_3,0,\lambda_5,\lambda_5,\frac{\mathcal{T}}{2}\Big),\cdots,\\ &\qquad\mathcal{E}_n\Big(\lambda_1,\lambda_1,\lambda_3,0,2\lambda_5,0,\mathcal{T}\Big)\bigotimes_{\mathtt{TN}}\mathcal{E}_n\Big(\lambda_1,0,\lambda_3,0,\lambda_5,\lambda_5,\frac{\mathcal{T}}{2}\Big)\Big]\end{aligned}\tag{10}$$

for all $\lambda_1, \lambda_3, \lambda_5 \in \mathfrak{G}$.

Letting $\lambda_2 = \lambda_6 = 0$ and $\lambda_4 = \lambda_3$ in (8), we get

$$\begin{aligned}&\mathscr{N}\Big(\chi(\lambda_1,2\lambda_3,\lambda_5) - 2\chi(\lambda_1,\lambda_3,\lambda_5), \mathcal{T}\Big)\\ &\succeq \text{Diagonal}\Big[\mathcal{E}_1\Big(\lambda_1,0,\lambda_3,\lambda_3,\lambda_5,0,\mathcal{T}\Big),\cdots,\mathcal{E}_n\Big(\lambda_1,0,\lambda_3,\lambda_3,\lambda_5,0,\mathcal{T}\Big)\Big],\end{aligned}$$

and so

$$\begin{aligned}&\mathscr{N}\Big(\chi(2\lambda_1,2\lambda_3,2\lambda_5) - 2\chi(2\lambda_1,\lambda_3,2\lambda_5), \mathcal{T}\Big)\\ &\succeq \text{Diagonal}\Big[\mathcal{E}_1\Big(2\lambda_1,0,\lambda_3,\lambda_3,2\lambda_5,0,\mathcal{T}\Big),\cdots,\mathcal{E}_n\Big(2\lambda_1,0,\lambda_3,\lambda_3,2\lambda_5,0,\mathcal{T}\Big)\Big],\end{aligned}\tag{11}$$

for all $\lambda_1, \lambda_3, \lambda_5 \in \mathfrak{G}$.

According to (10) and (11)

$$\begin{aligned}&\mathscr{N}\Big(\chi(2\lambda_1,2\lambda_3,2\lambda_5) - 8\chi(\lambda_1,\lambda_3,\lambda_5), \mathcal{T}\Big)\\ &\succeq \text{Diagonal}\Big[\mathcal{E}_1\Big(2\lambda_1,0,\lambda_3,\lambda_3,2\lambda_5,0,\mathcal{T}\Big)\\ &\qquad\bigotimes_{\mathtt{TN}}\mathcal{E}_1\Big(\lambda_1,\lambda_1,\lambda_3,0,2\lambda_5,0,\frac{\mathcal{T}}{2}\Big)\\ &\qquad\qquad\bigotimes_{\mathtt{TN}}\mathcal{E}_1\Big(\lambda_1,0,\lambda_3,0,\lambda_5,\lambda_5,\frac{\mathcal{T}}{4}\Big),\ldots,\\ &\qquad\mathcal{E}_n\Big(2\lambda_1,0,\lambda_3,\lambda_3,2\lambda_5,0,\mathcal{T}\Big)\\ &\qquad\bigotimes_{\mathtt{TN}}\mathcal{E}_n\Big(\lambda_1,\lambda_1,\lambda_3,0,2\lambda_5,0,\frac{\mathcal{T}}{2}\Big)\\ &\qquad\qquad\bigotimes_{\mathtt{TN}}\mathcal{E}_n\Big(\lambda_1,0,\lambda_3,0,\lambda_5,\lambda_5,\frac{\mathcal{T}}{4}\Big)\Big],\end{aligned}\tag{12}$$

and so

$$\begin{aligned}
&\mathscr{N}\left(\chi(\lambda_1,\lambda_3,\lambda_5)-8\chi\left(\frac{\lambda_1}{2},\frac{\lambda_3}{2},\frac{\lambda_5}{2}\right),\mathcal{T}\right) \\
&\succeq \text{Diagonal}\Bigg[\mathcal{E}_1\left(\lambda_1,0,\frac{\lambda_3}{2},\frac{\lambda_3}{2},\lambda_5,0,\mathcal{T}\right) \\
&\qquad \bigotimes_{\mathtt{TN}}\mathcal{E}_1\left(\frac{\lambda_1}{2},\frac{\lambda_1}{2},\frac{\lambda_3}{2},0,\lambda_5,0,\frac{\mathcal{T}}{2}\right) \\
&\qquad\quad \bigotimes_{\mathtt{TN}}\mathcal{E}_1\left(\frac{\lambda_1}{2},0,\frac{\lambda_3}{2},0,\frac{\lambda_5}{2},\frac{\lambda_5}{2},\frac{\mathcal{T}}{4}\right),\dots, \\
&\qquad \mathcal{E}_n\left(\lambda_1,0,\frac{\lambda_3}{2},\frac{\lambda_3}{2},\lambda_5,0,\mathcal{T}\right) \\
&\qquad \bigotimes_{\mathtt{TN}}\mathcal{E}_n\left(\frac{\lambda_1}{2},\frac{\lambda_1}{2},\frac{\lambda_3}{2},0,\lambda_5,0,\frac{\mathcal{T}}{2}\right) \\
&\qquad\quad \bigotimes_{\mathtt{TN}}\mathcal{E}_n\left(\frac{\lambda_1}{2},0,\frac{\lambda_3}{2},0,\frac{\lambda_5}{2},\frac{\lambda_5}{2},\frac{\mathcal{T}}{4}\right)\Bigg],
\end{aligned} \tag{13}$$

for all $\lambda_1,\lambda_3,\lambda_5\in\mathfrak{G}$.

Consider the set

$$\eta := \{\Xi:\mathfrak{G}^3\to\mathfrak{P},\ \ \Xi(\lambda_1,0,\lambda_5)=\Xi(0,\lambda_3,\lambda_5)=\Xi(\lambda_1,\lambda_3,0)=0\}$$

and the following vector valued generalized metric $\hbar:\eta\times\eta\longrightarrow[0,+\infty]^{3n}$ given by

$$\begin{aligned}
\hbar(\Xi,\varsigma) &= \Big(\hbar_1(\Xi,\varsigma),\hbar_2(\Xi,\varsigma),\hbar_3(\Xi,\varsigma)\Big) \\
&= \inf\Bigg\{(\mu_1,\mu_2,\mu_3)^n\in\mathbb{R}_+^{3n}:\mathscr{N}\Big(\Xi(\lambda_1,\lambda_3,\lambda_5)-\varsigma(\lambda_1,\lambda_3,\lambda_5),\mathcal{T}\Big) \\
&\qquad \preceq \mathcal{E}_1\left(\lambda_1,0,\frac{\lambda_3}{2},\frac{\lambda_3}{2},\lambda_5,0,\frac{\mathcal{T}}{\mu_1}\right) \\
&\qquad \bigotimes_{\mathtt{TN}}\mathcal{E}_1\left(\frac{\lambda_1}{2},\frac{\lambda_1}{2},\frac{\lambda_3}{2},0,\lambda_5,0,\frac{\mathcal{T}}{\mu_2}\right) \\
&\qquad\quad \bigotimes_{\mathtt{TN}}\mathcal{E}_1\left(\frac{\lambda_1}{2},0,\frac{\lambda_3}{2},0,\frac{\lambda_5}{2},\frac{\lambda_5}{2},\frac{\mathcal{T}}{\mu_3}\right),\dots, \\
&\qquad \mathcal{E}_n\left(\lambda_1,0,\frac{\lambda_3}{2},\frac{\lambda_3}{2},\lambda_5,0,\frac{\mathcal{T}}{\mu_1}\right) \\
&\qquad \bigotimes_{\mathtt{TN}}\mathcal{E}_n\left(\frac{\lambda_1}{2},\frac{\lambda_1}{2},\frac{\lambda_3}{2},0,\lambda_5,0,\frac{\mathcal{T}}{\mu_2}\right) \\
&\qquad\quad \bigotimes_{\mathtt{TN}}\mathcal{E}_n\left(\frac{\lambda_1}{2},0,\frac{\lambda_3}{2},0,\frac{\lambda_5}{2},\frac{\lambda_5}{2},\frac{\mathcal{T}}{\mu_3}\right), \\
&\qquad\qquad \forall\lambda_1,\lambda_3,\lambda_5\in\mathfrak{G},\Bigg\},
\end{aligned} \tag{14}$$

where, as usual, $\inf\emptyset=(\inf\emptyset,\inf\emptyset,\inf\emptyset)=(+\infty,+\infty,+\infty)$. We prove that $(\eta,\hbar)$ is a complete generalized metric space. We first prove the inequality $\hbar(\Xi,\varsigma)\preceq\hbar(\Xi,\veebar)+\hbar(\veebar,\varsigma)$, as follows:

$$
\begin{aligned}
\hbar(\Xi,\varsigma) &= \Big(\hbar_1(\Xi,\varsigma),\hbar_2(\Xi,\varsigma),\hbar_3(\Xi,\varsigma)\Big) \\
&= \inf\Bigg\{(\mu_1,\mu_2,\mu_3)^n \in \mathbb{R}_+^{3n} : \mathscr{N}\Big(\Xi(\lambda_1,\lambda_3,\lambda_5)-\varsigma(\lambda_1,\lambda_3,\lambda_5),\mathcal{T}\Big) \\
&\succeq \text{Diagonal}\Bigg[\mathcal{E}_1\Big(\lambda_1,0,\frac{\lambda_3}{2},\frac{\lambda_3}{2},\lambda_5,0,\frac{\mathcal{T}}{\mu_1}\Big) \\
&\quad \bigotimes_{\texttt{TN}} \mathcal{E}_1\Big(\frac{\lambda_1}{2},\frac{\lambda_1}{2},\frac{\lambda_3}{2},0,\lambda_5,0,\frac{\mathcal{T}}{\mu_2}\Big) \\
&\quad \bigotimes_{\texttt{TN}} \mathcal{E}_1\Big(\frac{\lambda_1}{2},0,\frac{\lambda_3}{2},0,\frac{\lambda_5}{2},\frac{\lambda_5}{2},\frac{\mathcal{T}}{\mu_3}\Big),\cdots, \\
&\quad \mathcal{E}_n\Big(\lambda_1,0,\frac{\lambda_3}{2},\frac{\lambda_3}{2},\lambda_5,0,\frac{\mathcal{T}}{\mu_1}\Big) \\
&\quad \bigotimes_{\texttt{TN}} \mathcal{E}_n\Big(\frac{\lambda_1}{2},\frac{\lambda_1}{2},\frac{\lambda_3}{2},0,\lambda_5,0,\frac{\mathcal{T}}{\mu_2}\Big) \\
&\quad \bigotimes_{\texttt{TN}} \mathcal{E}_n\Big(\frac{\lambda_1}{2},0,\frac{\lambda_3}{2},0,\frac{\lambda_5}{2},\frac{\lambda_5}{2},\frac{\mathcal{T}}{\mu_3}\Big)\Bigg],\ \forall\lambda_1,\lambda_3,\lambda_5\in\mathfrak{G}\Bigg\} \\
&\preceq \inf\Bigg\{(\mu_1,\mu_2,\mu_3)^n \in \mathbb{R}_+^{3n} : \mathscr{N}\Big(\Xi(\lambda_1,\lambda_3,\lambda_5)-\veebar(\lambda_1,\lambda_3,\lambda_5),\mathcal{T}\Big) \\
&\quad \bigotimes_{\texttt{TN}} \mathscr{N}\Big(\veebar(\lambda_1,\lambda_3,\lambda_5)-\varsigma(\lambda_1,\lambda_3,\lambda_5),\mathcal{T}\Big) \\
&\succeq \text{Diagonal}\Bigg[\mathcal{E}_1\Big(\lambda_1,0,\frac{\lambda_3}{2},\frac{\lambda_3}{2},\lambda_5,0,\frac{\mathcal{T}}{\mu_1}\Big) \\
&\quad \bigotimes_{\texttt{TN}} \mathcal{E}_1\Big(\frac{\lambda_1}{2},\frac{\lambda_1}{2},\frac{\lambda_3}{2},0,\lambda_5,0,\frac{\mathcal{T}}{\mu_2}\Big) \\
&\quad \bigotimes_{\texttt{TN}} \mathcal{E}_1\Big(\frac{\lambda_1}{2},0,\frac{\lambda_3}{2},0,\frac{\lambda_5}{2},\frac{\lambda_5}{2},\frac{\mathcal{T}}{\mu_3}\Big),\cdots, \\
&\quad \mathcal{E}_n\Big(\lambda_1,0,\frac{\lambda_3}{2},\frac{\lambda_3}{2},\lambda_5,0,\frac{\mathcal{T}}{\mu_1}\Big) \\
&\quad \bigotimes_{\texttt{TN}} \mathcal{E}_n\Big(\frac{\lambda_1}{2},\frac{\lambda_1}{2},\frac{\lambda_3}{2},0,\lambda_5,0,\frac{\mathcal{T}}{\mu_2}\Big) \\
&\quad \bigotimes_{\texttt{TN}} \mathcal{E}_n\Big(\frac{\lambda_1}{2},0,\frac{\lambda_3}{2},0,\frac{\lambda_5}{2},\frac{\lambda_5}{2},\frac{\mathcal{T}}{\mu_3}\Big)\Bigg],\ \forall\lambda_1,\lambda_3,\lambda_5\in\mathfrak{G}\Bigg\} \\
&\preceq \inf\Bigg\{(\mu_1,\mu_2,\mu_3)^n \in \mathbb{R}_+^{3n} : \mathscr{N}\Big(\Xi(\lambda_1,\lambda_3,\lambda_5)-\veebar(\lambda_1,\lambda_3,\lambda_5),\mathcal{T}\Big) \\
&\succeq \text{Diagonal}\Bigg[\mathcal{E}_1\Big(\lambda_1,0,\frac{\lambda_3}{2},\frac{\lambda_3}{2},\lambda_5,0,\frac{\mathcal{T}}{\mu_1}\Big) \\
&\quad \bigotimes_{\texttt{TN}} \mathcal{E}_1\Big(\frac{\lambda_1}{2},\frac{\lambda_1}{2},\frac{\lambda_3}{2},0,\lambda_5,0,\frac{\mathcal{T}}{\mu_2}\Big) \\
&\quad \bigotimes_{\texttt{TN}} \mathcal{E}_1\Big(\frac{\lambda_1}{2},0,\frac{\lambda_3}{2},0,\frac{\lambda_5}{2},\frac{\lambda_5}{2},\frac{\mathcal{T}}{\mu_3}\Big),\cdots, \\
&\quad \mathcal{E}_n\Big(\lambda_1,0,\frac{\lambda_3}{2},\frac{\lambda_3}{2},\lambda_5,0,\frac{\mathcal{T}}{\mu_1}\Big) \\
&\quad \bigotimes_{\texttt{TN}} \mathcal{E}_n\Big(\frac{\lambda_1}{2},\frac{\lambda_1}{2},\frac{\lambda_3}{2},0,\lambda_5,0,\frac{\mathcal{T}}{\mu_2}\Big) \qquad (15)
\end{aligned}
$$

$$
\begin{aligned}
& \bigotimes_{\texttt{TN}} \mathcal{E}_n\left(\frac{\lambda_1}{2}, 0, \frac{\lambda_3}{2}, 0, \frac{\lambda_5}{2}, \frac{\lambda_5}{2}, \frac{\mathcal{T}}{\mu_3}\right), \ \forall \lambda_1, \lambda_3, \lambda_5 \in \mathfrak{G}\Bigg\}, \\
& + \inf\Bigg\{(\mu_1, \mu_2, \mu_3)^n \in \mathbb{R}_+^{3n} : \mathcal{N}\left(\veebar(\lambda_1, \lambda_3, \lambda_5) - \varsigma(\lambda_1, \lambda_3, \lambda_5), \mathcal{T}\right) \\
& \succeq \text{Diagonal}\Bigg[\mathcal{E}_1\left(\lambda_1, 0, \frac{\lambda_3}{2}, \frac{\lambda_3}{2}, \lambda_5, 0, \frac{\mathcal{T}}{\mu_1}\right) \\
& \bigotimes_{\texttt{TN}} \mathcal{E}_1\left(\frac{\lambda_1}{2}, \frac{\lambda_1}{2}, \frac{\lambda_3}{2}, 0, \lambda_5, 0, \frac{\mathcal{T}}{\mu_2}\right) \\
& \bigotimes_{\texttt{TN}} \mathcal{E}_1\left(\frac{\lambda_1}{2}, 0, \frac{\lambda_3}{2}, 0, \frac{\lambda_5}{2}, \frac{\lambda_5}{2}, \frac{\mathcal{T}}{\mu_3}\right), \cdots, \\
& \mathcal{E}_n\left(\lambda_1, 0, \frac{\lambda_3}{2}, \frac{\lambda_3}{2}, \lambda_5, 0, \frac{\mathcal{T}}{\mu_1}\right) \\
& \bigotimes_{\texttt{TN}} \mathcal{E}_n\left(\frac{\lambda_1}{2}, \frac{\lambda_1}{2}, \frac{\lambda_3}{2}, 0, \lambda_5, 0, \frac{\mathcal{T}}{\mu_2}\right) \\
& \bigotimes_{\texttt{TN}} \mathcal{E}_n\left(\frac{\lambda_1}{2}, 0, \frac{\lambda_3}{2}, 0, \frac{\lambda_5}{2}, \frac{\lambda_5}{2}, \frac{\mathcal{T}}{\mu_3}\right), \ \forall \lambda_1, \lambda_3, \lambda_5 \in \mathfrak{G}\Bigg\} \\
= \ & \hbar(\Xi, \veebar) + \hbar(\veebar, \varsigma).
\end{aligned}
$$

We now prove $(\eta, \hbar)$ is complete. Let ω_w be a Cauchy sequence in $(\eta, \hbar)$. Thus, for every $\epsilon_1, \epsilon_2, \epsilon_3 > 0$, there exists an $\aleph_{\epsilon_1,\epsilon_2,\epsilon_3} \in \mathbb{N}$ such that $\hbar(\omega_m, \omega_w) \preceq (\epsilon_1, \epsilon_2, \epsilon_3)^n$ for every $m, w \geq \aleph_{\epsilon_1,\epsilon_2,\epsilon_3}$. According to (14), we have

$$
\begin{aligned}
& \mathcal{N}\left(\omega_m(\lambda_1, \lambda_3, \lambda_5) - \omega_w(\lambda_1, \lambda_3, \lambda_5), \mathcal{T}\right) \\
& \succeq \text{Diagonal}\Bigg[\mathcal{E}_1\left(\lambda_1, 0, \frac{\lambda_3}{2}, \frac{\lambda_3}{2}, \lambda_5, 0, \frac{\mathcal{T}}{\epsilon_1}\right) \\
& \qquad \bigotimes_{\texttt{TN}} \mathcal{E}_1\left(\frac{\lambda_1}{2}, \frac{\lambda_1}{2}, \frac{\lambda_3}{2}, 0, \lambda_5, 0, \frac{\mathcal{T}}{\epsilon_2}\right) \\
& \qquad \quad \bigotimes_{\texttt{TN}} \mathcal{E}_1\left(\frac{\lambda_1}{2}, 0, \frac{\lambda_3}{2}, 0, \frac{\lambda_5}{2}, \frac{\lambda_5}{2}, \frac{\mathcal{T}}{\epsilon_3}\right), \cdots, \\
& \qquad \mathcal{E}_n\left(\lambda_1, 0, \frac{\lambda_3}{2}, \frac{\lambda_3}{2}, \lambda_5, 0, \frac{\mathcal{T}}{\epsilon_1}\right) \\
& \qquad \bigotimes_{\texttt{TN}} \mathcal{E}_n\left(\frac{\lambda_1}{2}, \frac{\lambda_1}{2}, \frac{\lambda_3}{2}, 0, \lambda_5, 0, \frac{\mathcal{T}}{\epsilon_2}\right) \\
& \qquad \quad \bigotimes_{\texttt{TN}} \mathcal{E}_n\left(\frac{\lambda_1}{2}, 0, \frac{\lambda_3}{2}, 0, \frac{\lambda_5}{2}, \frac{\lambda_5}{2}, \frac{\mathcal{T}}{\epsilon_3}\right)\Bigg],
\end{aligned} \tag{16}
$$

for all $\lambda_1, \lambda_3, \lambda_5 \in \mathfrak{G}$. If $\lambda_1, \lambda_3, \lambda_5$ are fixed, (16) implies that $\{\omega_w(\lambda_1, \lambda_3, \lambda_5)\}$ is a Cauchy sequence in $\mathfrak{G}$. Since $\mathfrak{G}$ is complete, $\{\omega_w(\lambda_1, \lambda_3, \lambda_5)\}$ converges for any $\lambda_1, \lambda_3, \lambda_5 \in \mathfrak{G}$. Thus, we can obtain a function ω by

$$
\omega(\lambda_1, \lambda_2, \lambda_3) := \lim_{w \to \infty} \omega_w(\lambda_1, \lambda_3, \lambda_5), \quad (\lambda_1, \lambda_3, \lambda_5 \in \mathfrak{G}). \tag{17}
$$

It is straightforward to show $\omega \in \eta$. If we let $m \to \infty$ we conclude from (16) that

$$\begin{aligned}
&\mathscr{N}\left(\omega(\lambda_1,\lambda_3,\lambda_5)-\omega_w(\lambda_1,\lambda_3,\lambda_5),\mathcal{T}\right) \\
&\succeq \mathrm{Diagonal}\Bigg[\mathcal{E}_1\left(\lambda_1,0,\frac{\lambda_3}{2},\frac{\lambda_3}{2},\lambda_5,0,\frac{\mathcal{T}}{\epsilon_1}\right) \\
&\qquad \bigotimes_{\mathtt{TN}}\mathcal{E}_1\left(\frac{\lambda_1}{2},\frac{\lambda_1}{2},\frac{\lambda_3}{2},0,\lambda_5,0,\frac{\mathcal{T}}{\epsilon_2}\right) \\
&\qquad\qquad \bigotimes_{\mathtt{TN}}\mathcal{E}_1\left(\frac{\lambda_1}{2},0,\frac{\lambda_3}{2},0,\frac{\lambda_5}{2},\frac{\lambda_5}{2},\frac{\mathcal{T}}{\epsilon_3}\right),\cdots, \\
&\qquad \mathcal{E}_n\left(\lambda_1,0,\frac{\lambda_3}{2},\frac{\lambda_3}{2},\lambda_5,0,\frac{\mathcal{T}}{\epsilon_1}\right) \\
&\qquad \bigotimes_{\mathtt{TN}}\mathcal{E}_n\left(\frac{\lambda_1}{2},\frac{\lambda_1}{2},\frac{\lambda_3}{2},0,\lambda_5,0,\frac{\mathcal{T}}{\epsilon_2}\right) \\
&\qquad\qquad \bigotimes_{\mathtt{TN}}\mathcal{E}_n\left(\frac{\lambda_1}{2},0,\frac{\lambda_3}{2},0,\frac{\lambda_5}{2},\frac{\lambda_5}{2},\frac{\mathcal{T}}{\epsilon_3}\right)\Bigg].
\end{aligned} \tag{18}$$

Considering (14), we get

$$\hbar(\omega,\omega_w) \preceq (\epsilon_1,\epsilon_2,\epsilon_3)^n.$$

Therefore, the Cauchy sequence $\{\omega_w\}$ is convergent to ω in $(\eta,\hbar)$. Hence, $(\eta,\hbar)$ is complete.

We now consider the linear mapping $\Gamma:\eta\to\eta$ as follows:

$$\Gamma(\Xi(\lambda_1,\lambda_3,\lambda_5)) := 8\Xi\left(\frac{\lambda_1}{2},\frac{\lambda_3}{2},\frac{\lambda_5}{2}\right)$$

for all $\lambda_1,\lambda_3,\lambda_5 \in \mathfrak{G}$.

Let $\Xi,\varsigma \in \eta$ be given such that $\hbar(\Xi,\varsigma) = (\varepsilon_1,\varepsilon_2,\varepsilon_3)^n$. Then, we get

$$\begin{aligned}
&\mathscr{N}\left(\Xi(\lambda_1,\lambda_3,\lambda_5)-\varsigma(\lambda_1,\lambda_3,\lambda_5),\mathcal{T}\right) \\
&\succeq \mathrm{Diagonal}\Bigg[\mathcal{E}_1\left(\lambda_1,0,\frac{\lambda_3}{2},\frac{\lambda_3}{2},\lambda_5,0,\frac{\mathcal{T}}{\epsilon_1}\right) \\
&\qquad \bigotimes_{\mathtt{TN}}\mathcal{E}_1\left(\frac{\lambda_1}{2},\frac{\lambda_1}{2},\frac{\lambda_3}{2},0,\lambda_5,0,\frac{\mathcal{T}}{\epsilon_2}\right) \\
&\qquad\qquad \bigotimes_{\mathtt{TN}}\mathcal{E}_1\left(\frac{\lambda_1}{2},0,\frac{\lambda_3}{2},0,\frac{\lambda_5}{2},\frac{\lambda_5}{2},\frac{\mathcal{T}}{\epsilon_3}\right),\cdots, \\
&\qquad \mathcal{E}_n\left(\lambda_1,0,\frac{\lambda_3}{2},\frac{\lambda_3}{2},\lambda_5,0,\frac{\mathcal{T}}{\epsilon_1}\right) \\
&\qquad \bigotimes_{\mathtt{TN}}\mathcal{E}_n\left(\frac{\lambda_1}{2},\frac{\lambda_1}{2},\frac{\lambda_3}{2},0,\lambda_5,0,\frac{\mathcal{T}}{\epsilon_2}\right) \\
&\qquad\qquad \bigotimes_{\mathtt{TN}}\mathcal{E}_n\left(\frac{\lambda_1}{2},0,\frac{\lambda_3}{2},0,\frac{\lambda_5}{2},\frac{\lambda_5}{2},\frac{\mathcal{T}}{\epsilon_3}\right)\Bigg],
\end{aligned}$$

for all $\lambda_1,\lambda_3,\lambda_5 \in \mathfrak{G}$.

Thus, we have that

$$
\begin{aligned}
& \mathscr{N}\left(\Gamma(\Xi(\lambda_1,\lambda_3,\lambda_5))-\Gamma(\varsigma(\lambda_1,\lambda_3,\lambda_5)),\mathcal{T}\right) \qquad (19)\\
= & \mathscr{N}\left(8\Xi(\frac{\lambda_1}{2},\frac{\lambda_3}{2},\frac{\lambda_5}{2})-8\varsigma(\frac{\lambda_1}{2},\frac{\lambda_3}{2},\frac{\lambda_5}{2}),\mathcal{T}\right)\\
\succeq & \text{Diagonal}\Big[\mathcal{E}_1\left(\frac{\lambda_1}{2},0,\frac{\lambda_3}{4},\frac{\lambda_3}{4},\frac{\lambda_5}{2},0,\frac{1}{8\varepsilon_1}\mathcal{T}\right)\\
& \bigotimes_{\mathtt{TN}}\mathcal{E}_1\left(\frac{\lambda_1}{4},\frac{\lambda_1}{4},\frac{\lambda_3}{4},0,\frac{\lambda_5}{2},0,\frac{1}{8\varepsilon_2}\mathcal{T}\right)\\
& \bigotimes_{\mathtt{TN}}\mathcal{E}_1\left(\frac{\lambda_1}{4},0,\frac{\lambda_3}{4},0,\frac{\lambda_5}{4},\frac{\lambda_5}{4},\frac{1}{8\varepsilon_3}\mathcal{T}\right),\cdots,\\
& \mathcal{E}_n\left(\frac{\lambda_1}{2},0,\frac{\lambda_3}{4},\frac{\lambda_3}{4},\frac{\lambda_5}{2},0,\frac{1}{8\varepsilon_1}\mathcal{T}\right)\\
& \bigotimes_{\mathtt{TN}}\mathcal{E}_n\left(\frac{\lambda_1}{4},\frac{\lambda_1}{4},\frac{\lambda_3}{4},0,\frac{\lambda_5}{2},0,\frac{1}{8\varepsilon_2}\mathcal{T}\right)\\
& \bigotimes_{\mathtt{TN}}\mathcal{E}_n\left(\frac{\lambda_1}{4},0,\frac{\lambda_3}{4},0,\frac{\lambda_5}{4},\frac{\lambda_5}{4},\frac{1}{8\varepsilon_3}\mathcal{T}\right)\Big]\\
\succeq & \text{Diagonal}\Big[\mathcal{E}_1\left(\lambda_1,0,\frac{\lambda_3}{2},\frac{\lambda_3}{2},\lambda_5,0,\frac{1}{4\vartheta_1\varepsilon_1}\mathcal{T}\right)\\
& \bigotimes_{\mathtt{TN}}\mathcal{E}_1\left(\frac{\lambda_1}{2},\frac{\lambda_1}{2},\frac{\lambda_3}{2},0,\lambda_5,0,\frac{1}{4\vartheta_1\varepsilon_2}\mathcal{T}\right)\\
& \bigotimes_{\mathtt{TN}}\mathcal{E}_1\left(\frac{\lambda_1}{2},0,\frac{\lambda_3}{2},0,\frac{\lambda_5}{2},\frac{\lambda_5}{2},\frac{1}{4\vartheta_1\varepsilon_3}\mathcal{T}\right),\cdots,\\
& \mathcal{E}_n\left(\lambda_1,0,\frac{\lambda_3}{2},\frac{\lambda_3}{2},\lambda_5,0,\frac{1}{4\vartheta_n\varepsilon_1}\mathcal{T}\right)\\
& \bigotimes_{\mathtt{TN}}\mathcal{E}_n\left(\frac{\lambda_1}{2},\frac{\lambda_1}{2},\frac{\lambda_3}{2},0,\lambda_5,0,\frac{1}{4\vartheta_n\varepsilon_2}\mathcal{T}\right)\\
& \bigotimes_{\mathtt{TN}}\mathcal{E}_n\left(\frac{\lambda_1}{2},0,\frac{\lambda_3}{2},0,\frac{\lambda_5}{2},\frac{\lambda_5}{2},\frac{1}{4\vartheta_n\varepsilon_3}\mathcal{T}\right)\Big],
\end{aligned}
$$

for all $\lambda_1,\lambda_3,\lambda_5\in\mathfrak{G}$. So we have that $\hbar(\Gamma\Xi,\Gamma\varsigma)\preceq\left(4\vartheta_1(\varepsilon_1,\varepsilon_2,\varepsilon_3),\cdots,4\vartheta_n(\varepsilon_1,\varepsilon_2,\varepsilon_3)\right)$. This means that

$$\hbar(\Gamma\Xi,\Gamma\varsigma)\preceq(4\vartheta_1,\cdots,4\vartheta_n)\hbar(\Xi,\varsigma),$$

for all $\Xi,\varsigma\in\eta$.

It follows from (13) that

$$\begin{aligned}
&\mathscr{N}\left(\chi(\lambda_1,\lambda_3,\lambda_5)-8\chi\left(\frac{\lambda_1}{2},\frac{\lambda_3}{2},\frac{\lambda_5}{2}\right),\mathcal{T}\right) \\
&\succeq \text{Diagonal}\Bigg[\mathcal{E}_1\left(\lambda_1,0,\frac{\lambda_3}{2},\frac{\lambda_3}{2},\lambda_5,0,\mathcal{T}\right) \\
&\qquad \bigotimes_{\texttt{TN}} \mathcal{E}_1\left(\frac{\lambda_1}{2},\frac{\lambda_1}{2},\frac{\lambda_3}{2},0,\lambda_5,0,2\mathcal{T}\right) \\
&\qquad\qquad \bigotimes_{\texttt{TN}} \mathcal{E}_1\left(\frac{\lambda_1}{2},0,\frac{\lambda_3}{2},0,\frac{\lambda_5}{2},\frac{\lambda_5}{2},4\mathcal{T}\right),\cdots, \\
&\quad \mathcal{E}_n\left(\lambda_1,0,\frac{\lambda_3}{2},\frac{\lambda_3}{2},\lambda_5,0,\mathcal{T}\right) \\
&\qquad \bigotimes_{\texttt{TN}} \mathcal{E}_n\left(\frac{\lambda_1}{2},\frac{\lambda_1}{2},\frac{\lambda_3}{2},0,\lambda_5,0,2\mathcal{T}\right) \\
&\qquad\qquad \bigotimes_{\texttt{TN}} \mathcal{E}_n\left(\frac{\lambda_1}{2},0,\frac{\lambda_3}{2},0,\frac{\lambda_5}{2},\frac{\lambda_5}{2},4\mathcal{T}\right)\Bigg],
\end{aligned} \tag{20}$$

for all $\lambda_1,\lambda_3,\lambda_5 \in \mathfrak{G}$ and $\chi \in \eta$. So $\hbar(\chi,\Gamma\chi) \preceq (1,2,4)^n$.

Based on Theorem 1, we can obtain a mapping $\chi' : \mathfrak{G}^3 \to \mathfrak{P}$ s.t.

(1) χ' is a fixed point of Γ, i.e.,

$$\Gamma(\chi'(\lambda_1,\lambda_3,\lambda_5)) = \chi'(\lambda_1,\lambda_3,\lambda_5) := 8\chi\left(\frac{\lambda_1}{2},\frac{\lambda_3}{2},\frac{\lambda_5}{2}\right) \tag{21}$$

for all $\lambda_1,\lambda_3,\lambda_5 \in \mathfrak{G}$. The mapping χ' is a unique fixed point of Γ. We conclude that χ' is a unique mapping satisfying (21) s.t. there exists $(\mu_1,\mu_2,\mu_3)^n \in (0,\infty)^{3n}$ satisfying

$$\begin{aligned}
&\mathscr{N}\left(\chi(\lambda_1,\lambda_3,\lambda_5)-\chi'(\lambda_1,\lambda_3,\lambda_5),\mathcal{T}\right) \\
\succeq\ &\text{Diagonal}\Bigg[\mathcal{E}_1\left(\lambda_1,0,\frac{\lambda_3}{2},\frac{\lambda_3}{2},\lambda_5,0,\frac{\mathcal{T}}{\mu_1}\right) \\
&\qquad \bigotimes_{\texttt{TN}} \mathcal{E}_1\left(\frac{\lambda_1}{2},\frac{\lambda_1}{2},\frac{\lambda_3}{2},0,\lambda_5,0,\frac{\mathcal{T}}{\mu_2}\right) \\
&\qquad\qquad \bigotimes_{\texttt{TN}} \mathcal{E}_1\left(\frac{\lambda_1}{2},0,\frac{\lambda_3}{2},0,\frac{\lambda_5}{2},\frac{\lambda_5}{2},\frac{\mathcal{T}}{\mu_3}\right),\cdots, \\
&\quad \mathcal{E}_n\left(\lambda_1,0,\frac{\lambda_3}{2},\frac{\lambda_3}{2},\lambda_5,0,\frac{\mathcal{T}}{\mu_1}\right) \\
&\qquad \bigotimes_{\texttt{TN}} \mathcal{E}_n\left(\frac{\lambda_1}{2},\frac{\lambda_1}{2},\frac{\lambda_3}{2},0,\lambda_5,0,\frac{\mathcal{T}}{\mu_2}\right) \\
&\qquad\qquad \bigotimes_{\texttt{TN}} \mathcal{E}_n\left(\frac{\lambda_1}{2},0,\frac{\lambda_3}{2},0,\frac{\lambda_5}{2},\frac{\lambda_5}{2},\frac{\mathcal{T}}{\mu_3}\right)\Bigg],
\end{aligned}$$

for all $\lambda_1,\lambda_3,\lambda_5 \in \mathfrak{G}$;

(2) $\hbar(\Gamma^k\mathcal{L},\mathcal{L}') \to (0,0,0)^n$ as $k \to \infty$. This implies the following equality

$$\chi'(\lambda_1,\lambda_3,\lambda_5) = \lim_{k\to\infty} 8^k\chi\left(\frac{\lambda_1}{2^k},\frac{\lambda_3}{2^k},\frac{\lambda_5}{2^k}\right)$$

for all $\lambda_1,\lambda_3,\lambda_5 \in \mathfrak{G}$;

(3) $\hbar(\chi,\chi') \preceq (\frac{1}{1-4\vartheta_1},\cdots,\frac{1}{1-4\vartheta_n})\hbar(\chi,\Gamma\chi)$, which implies that

$$\begin{aligned}&\mathscr{N}\left(\chi(\lambda_1,\lambda_3,\lambda_5)-\chi'(\lambda_1,\lambda_3,\lambda_5),\mathcal{T}\right)\\&\succeq \text{Diagonal}\Bigg[\mathcal{E}_1\left(\lambda_1,0,\frac{\lambda_3}{2},\frac{\lambda_3}{2},\lambda_5,0,(1-4\vartheta_1)\mathcal{T}\right)\\&\qquad\bigotimes_{\mathtt{TN}}\mathcal{E}_1\left(\frac{\lambda_1}{2},\frac{\lambda_1}{2},\frac{\lambda_3}{2},0,\lambda_5,0,\frac{1-4\vartheta_1}{2}\mathcal{T}\right)\\&\qquad\qquad\bigotimes_{\mathtt{TN}}\mathcal{E}_1\left(\frac{\lambda_1}{2},0,\frac{\lambda_3}{2},0,\frac{\lambda_5}{2},\frac{\lambda_5}{2},\frac{1-4\vartheta_1}{4}\mathcal{T}\right),\cdots,\\&\quad\mathcal{E}_n\left(\lambda_1,0,\frac{\lambda_3}{2},\frac{\lambda_3}{2},\lambda_5,0,(1-4\vartheta_n)\mathcal{T}\right)\\&\qquad\bigotimes_{\mathtt{TN}}\mathcal{E}_n\left(\frac{\lambda_1}{2},\frac{\lambda_1}{2},\frac{\lambda_3}{2},0,\lambda_5,0,\frac{1-4\vartheta_n}{2}\mathcal{T}\right)\\&\qquad\qquad\bigotimes_{\mathtt{TN}}\mathcal{E}_n\left(\frac{\lambda_1}{2},0,\frac{\lambda_3}{2},0,\frac{\lambda_5}{2},\frac{\lambda_5}{2},\frac{1-4\vartheta_n}{4}\mathcal{T}\right)\Bigg],\end{aligned}$$

for all $\lambda_1,\lambda_3,\lambda_5\in\mathfrak{G}$.

Now, let $\chi'':\mathfrak{G}^3\to\mathfrak{P}$ be another additive mapping satisfying (9). Thus, we get

$$\begin{aligned}&\mathscr{N}\left(\chi''(\lambda_1,\lambda_3,\lambda_5)-\chi'(\lambda_1,\lambda_3,\lambda_5,\mathcal{T}\right)\\&=\mathscr{N}\left(8^q\chi''\left(\frac{\lambda_1}{2^q},\frac{\lambda_3}{2^q},\frac{\lambda_5}{2^q}\right)-8^q\chi'\left(\frac{\lambda_1}{2^q},\frac{\lambda_3}{2^q},\frac{\lambda_5}{2^q}\right),\mathcal{T}\right)\\&\succeq\mathscr{N}\left(8^q\chi''\left(\frac{\lambda_1}{2^q},\frac{\lambda_3}{2^q},\frac{\lambda_5}{2^q}\right)-8^q\chi\left(\frac{\lambda_1}{2^q},\frac{\lambda_3}{2^q},\frac{\lambda_5}{2^q}\right),\mathcal{T}\right)\\&\qquad\bigotimes_{\mathtt{GTN}}\mathscr{N}\left(8^q\chi'\left(\frac{\lambda_1}{2^q},\frac{\lambda_3}{2^q},\frac{\lambda_5}{2^q}\right)-8^q\chi\left(\frac{\lambda_1}{2^q},\frac{\lambda_3}{2^q},\frac{\lambda_5}{2^q}\right),\mathcal{T}\right)\\&\succeq\text{Diagonal}\Bigg[\mathcal{E}_1\left(\frac{\lambda_1}{2^q},0,\frac{\lambda_3}{2^{q+1}},\frac{\lambda_3}{2^{q+1}},\frac{\lambda_5}{2^q},0,\frac{1-4\vartheta}{2.8^q}\mathcal{T}\right)\\&\qquad\bigotimes_{\mathtt{TN}}\mathcal{E}_1\left(\frac{\lambda_1}{2^{q+1}},\frac{\lambda_1}{2^{q+1}},\frac{\lambda_3}{2^{q+1}},0,\frac{\lambda_5}{2^q},0,\frac{1-4\vartheta}{4.8^q}\mathcal{T}\right)\\&\qquad\qquad\bigotimes_{\mathtt{TN}}\mathcal{E}_1\left(\frac{\lambda_1}{2^{q+1}},0,\frac{\lambda_3}{2^{q+1}},0,\frac{\lambda_5}{2^{q+1}},\frac{\lambda_5}{2^{q+1}},\frac{1-4\vartheta}{8.8^q}\mathcal{T}\right),\cdots,\\&\quad\mathcal{E}_n\left(\frac{\lambda_1}{2^q},0,\frac{\lambda_3}{2^{q+1}},\frac{\lambda_3}{2^{q+1}},\frac{\lambda_5}{2^q},0,\frac{1-4\vartheta}{2.8^q}\mathcal{T}\right)\\&\qquad\bigotimes_{\mathtt{TN}}\mathcal{E}_n\left(\frac{\lambda_1}{2^{q+1}},\frac{\lambda_1}{2^{q+1}},\frac{\lambda_3}{2^{q+1}},0,\frac{\lambda_5}{2^q},0,\frac{1-4\vartheta}{4.8^q}\mathcal{T}\right)\\&\qquad\qquad\bigotimes_{\mathtt{TN}}\mathcal{E}_n\left(\frac{\lambda_1}{2^{q+1}},0,\frac{\lambda_3}{2^{q+1}},0,\frac{\lambda_5}{2^{q+1}},\frac{\lambda_5}{2^{q+1}},\frac{1-4\vartheta}{8.8^q}\mathcal{T}\right)\Bigg],\end{aligned}$$

which tends to **1** as $q\to\infty$, for all $\lambda_1,\lambda_3,\lambda_5\in\mathfrak{G}$. Hence, we can infer $\chi''(\lambda_1,\lambda_3,\lambda_5)=\chi'(\lambda_1,\lambda_3,\lambda_5)$ for all $\lambda_1,\lambda_3,\lambda_5\in\mathfrak{G}$. This shows the uniqueness of χ'.

Making use of (8), we get

$$\begin{aligned}
&\mathscr{N}\Big(\chi'(\lambda_1+\lambda_2,\lambda_3-\lambda_4,\lambda_5+\lambda_6)+\chi'(\lambda_1-\lambda_2,\lambda_3+\lambda_4,\lambda_5-\lambda_6),\\
&\quad -2\chi'(\lambda_1,\lambda_3,\lambda_5)+2\chi'(\lambda_1,\lambda_4,\lambda_6)-2\chi'(\lambda_2,\lambda_3,\lambda_6)+2\chi'(\lambda_2,\lambda_4,\lambda_5),\mathcal{T}\Big)\\
&= \lim_{n\to\infty}\mathscr{N}\Big(8^n[\chi(\frac{\lambda_1+\lambda_2}{2^n},\frac{\lambda_3-\lambda_4}{2^n},\frac{\lambda_5+\lambda_6}{2^n})+\chi(\frac{\lambda_1-\lambda_2}{2^n},\frac{\lambda_3+\lambda_4}{2^n},\frac{\lambda_5-\lambda_6}{2^n})\\
&\quad -2\chi(\frac{\lambda_1}{2^n},\frac{\lambda_3}{2^n},\frac{\lambda_5}{2^n})+2\chi(\frac{\lambda_1}{2^n},\frac{\lambda_4}{2^n},\frac{\lambda_6}{2^n})\\
&\quad -2\chi(\frac{\lambda_2}{2^n},\frac{\lambda_3}{2^n},\frac{\lambda_6}{2^n})+2\chi(\frac{\lambda_2}{2^n},\frac{\lambda_4}{2^n},\frac{\lambda_5}{2^n})],\mathcal{T}\Big)\\
&\succeq \lim_{n\to\infty}\mathscr{N}\Big(8^n\boldsymbol{\lambda}[2\chi(\frac{\lambda_1+\lambda_2}{2^{n+1}},\frac{\lambda_3-\lambda_4}{2^n},\frac{\lambda_5+\lambda_6}{2^n})+2\chi(\frac{\lambda_1-\lambda_2}{2^{n+1}},\frac{\lambda_3+\lambda_4}{2^n},\frac{\lambda_5-\lambda_6}{2^n})\\
&\quad -2\chi(\frac{\lambda_1}{2^n},\frac{\lambda_3}{2^n},\frac{\lambda_5}{2^n})+2\chi(\frac{\lambda_1}{2^n},\frac{\lambda_4}{2^n},\frac{\lambda_6}{2^n})\\
&\quad -2\chi(\frac{\lambda_2}{2^n},\frac{\lambda_3}{2^n},\frac{\lambda_6}{2^n})+2\chi(\frac{\lambda_2}{2^n},\frac{\lambda_4}{2^n},\frac{\lambda_5}{2^n})],\mathcal{T}\Big)\\
&\quad \bigotimes_{\mathtt{GTN}}\text{Diagonal}\Big[\lim_{n\to\infty}\mathcal{E}_1\Big(\frac{\lambda_1}{2^n},\frac{\lambda_2}{2^n},\frac{\lambda_3}{2^n},\frac{\lambda_4}{2^n},\frac{\lambda_5}{2^n},\frac{\lambda_6}{2^n},\frac{\mathcal{T}}{8^n}\Big),\cdots,\\
&\quad \lim_{n\to\infty}\mathcal{E}_n\Big(\frac{\lambda_1}{2^n},\frac{\lambda_2}{2^n},\frac{\lambda_3}{2^n},\frac{\lambda_4}{2^n},\frac{\lambda_5}{2^n},\frac{\lambda_6}{2^n},\frac{\mathcal{T}}{8^n}\Big)\Big]\\
&\succeq \mathscr{N}\Big(\boldsymbol{\lambda}[2\chi'(\frac{\lambda_1+\lambda_2}{2},\lambda_3-\lambda_4,\lambda_5+\lambda_6)+2\chi'(\frac{\lambda_1-\lambda_2}{2},\lambda_3+\lambda_4,\lambda_5-\lambda_6)\\
&\quad -2\chi'(\lambda_1,\lambda_3,\lambda_5)+2\chi'(\lambda_1,\lambda_4,\lambda_6)\\
&\quad -2\chi'(\lambda_2,\lambda_3,\lambda_6)+2\chi'(\lambda_2,\lambda_4,\lambda_5)],\mathcal{T}\Big)
\end{aligned}$$

for all $\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6\in\mathfrak{G}$. So, we have

$$\begin{aligned}
&\mathscr{N}\Big(\chi'(\lambda_1+\lambda_2,\lambda_3-\lambda_4,\lambda_5+\lambda_6)+\chi'(\lambda_1-\lambda_2,\lambda_3+\lambda_4,\lambda_5-\lambda_6)\\
&\quad -2\chi'(\lambda_1,\lambda_3,\lambda_5)+2\chi'(\lambda_1,\lambda_4,\lambda_6)\\
&\quad -2\chi'(\lambda_2,\lambda_3,\lambda_6)+2\chi'(\lambda_2,\lambda_4,\lambda_5),\mathcal{T}\Big)\\
&\succeq \mathscr{N}\Big(\boldsymbol{\lambda}[2\chi'(\frac{\lambda_1+\lambda_2}{2},\lambda_3-\lambda_4,\lambda_5+\lambda_6)+2\chi'(\frac{\lambda_1-\lambda_2}{2},\lambda_3+\lambda_4,\lambda_5-\lambda_6)\\
&\quad -2\chi'(\lambda_1,\lambda_3,\lambda_5)+2\chi'(\lambda_1,\lambda_4,\lambda_6)\\
&\quad -2\chi'(\lambda_2,\lambda_3,\lambda_6)+2\chi'(\lambda_2,\lambda_4,\lambda_5)],\mathcal{T}\Big)
\end{aligned}$$

for all $\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6\in\mathfrak{G}$. Using Lemma 1, we infer the mapping $\chi':\mathfrak{G}^3\to\mathfrak{P}$ is tri-additive. □

Theorem 3. *Suppose $i=1,\cdots,n$ and $n\in\mathbb{N}$. Let $\mathcal{E}_i:\mathfrak{G}^6\times(0,+\infty)\to\Phi$ be a function such that*

$$\mathcal{E}_i\Big(2\lambda_1,2\lambda_2,2\lambda_3,2\lambda_4,2\lambda_5,2\lambda_6,\mathcal{T}\Big)\succeq\mathcal{E}_i\Big(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\frac{\mathcal{T}}{2\vartheta_i}\Big), \tag{22}$$

for all $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \in \mathfrak{G}$ and $0 < \vartheta_i < 1$. Let $\chi : \mathfrak{G}^3 \to \mathfrak{P}$ be a mapping satisfying (8) and $\chi(\lambda_1, 0, \lambda_5) = \chi(0, \lambda_3, \lambda_5) = \chi(\lambda_1, \lambda_3, 0) = 0$ for all $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \in \mathfrak{G}$. Then, we can find a unique tri-additive mapping $\chi' : \mathfrak{G}^3 \to \mathfrak{P}$ satisfying

$$\begin{aligned}
&\mathscr{N}\Big(\chi(\lambda_1, \lambda_3, \lambda_5) - \chi'(\lambda_1, \lambda_3, \lambda_5), \mathcal{T}\Big) \\
&\succeq \mathit{Diagonal}\Big[\mathcal{E}_1\Big(\lambda_1, 0, \lambda_3, \lambda_3, 2\lambda_5, 0, (8 - 2\vartheta_1)\mathcal{T}\Big) \\
&\qquad \bigotimes_{\mathtt{TN}} \mathcal{E}_1\Big(\lambda_1, \lambda_1, \lambda_3, 0, 2\lambda_5, 0, (4 - \vartheta_1)\mathcal{T}\Big) \\
&\qquad\qquad \bigotimes_{\mathtt{TN}} \mathcal{E}_1\Big(\lambda_1, 0, \lambda_3, 0, \lambda_5, \lambda_5, \frac{4 - \vartheta_1}{2}\mathcal{T}\Big), \cdots, \\
&\quad \mathcal{E}_n\Big(\lambda_1, 0, \lambda_3, \lambda_3, 2\lambda_5, 0, (8 - 2\vartheta_n)\mathcal{T}\Big) \\
&\qquad \bigotimes_{\mathtt{TN}} \mathcal{E}_n\Big(\lambda_1, \lambda_1, \lambda_3, 0, 2\lambda_5, 0, (4 - \vartheta_n)\mathcal{T}\Big) \\
&\qquad\qquad \bigotimes_{\mathtt{TN}} \mathcal{E}_n\Big(\lambda_1, 0, \lambda_3, 0, \lambda_5, \lambda_5, \frac{4 - \vartheta_n}{2}\mathcal{T}\Big)\Big],
\end{aligned}$$

for all $\lambda_1, \lambda_3, \lambda_5 \in \mathfrak{G}$.

Proof. According to (12) we have

$$\begin{aligned}
&\mathscr{N}\Big(\chi(\lambda_1, \lambda_3, \lambda_5) - \frac{1}{8}\chi(2\lambda_1, 2\lambda_3, 2\lambda_5), \mathcal{T}\Big) \\
&\succeq \text{Diagonal}\Big[\mathcal{E}_1\Big(2\lambda_1, 0, \lambda_3, \lambda_3, 2\lambda_5, 0, 8\mathcal{T}\Big) \\
&\qquad \bigotimes_{\mathtt{TN}} \mathcal{E}_1\Big(\lambda_1, \lambda_1, \lambda_3, 0, 2\lambda_5, 0, 4\mathcal{T}\Big) \\
&\qquad\qquad \bigotimes_{\mathtt{TN}} \mathcal{E}_1\Big(\lambda_1, 0, \lambda_3, 0, \lambda_5, \lambda_5, 2\mathcal{T}\Big), \cdots, \\
&\quad \mathcal{E}_n\Big(2\lambda_1, 0, \lambda_3, \lambda_3, 2\lambda_5, 0, 8\mathcal{T}\Big) \\
&\qquad \bigotimes_{\mathtt{TN}} \mathcal{E}_n\Big(\lambda_1, \lambda_1, \lambda_3, 0, 2\lambda_5, 0, 4\mathcal{T}\Big) \\
&\qquad\qquad \bigotimes_{\mathtt{TN}} \mathcal{E}_n\Big(\lambda_1, 0, \lambda_3, 0, \lambda_5, \lambda_5, 2\mathcal{T}\Big)\Big],
\end{aligned}$$

for all $\lambda_1, \lambda_3, \lambda_5 \in \mathfrak{G}$.

By using a similar method as in the proof of Theorem 2, the proof will be completed. □

3. Permuting Tri-Derivations on MFB-Algebras

Here, we study the multi-stability of permuting triderivations on unital MFC-⋄-algebras and complex MFB-algebras related to the functional Equation (1).

Lemma 2 ([47])**.** *Let $\chi : \mathfrak{G}^2 \to \mathfrak{P}$ be a bi-additive mapping s.t. $\chi(\Lambda_1\lambda_1, \Lambda_2\lambda_3) = \Lambda_1\Lambda_2\chi(\lambda_1, \lambda_3)$ for all $\lambda_1, \lambda_3 \in \mathfrak{I}$ and $\Lambda_1, \Lambda_2 \in \Delta^1 := \{\mathtt{C} \in \mathbb{C} \ : \ |\mathtt{C}| = 1\}$. Then, χ is $\mathbb{C}-$bilinear.*

Lemma 3. *Consider the tri-additive mapping $\chi : \mathfrak{G}^3 \to \mathfrak{P}$ s.t. $\chi(\Lambda_1\lambda_1, \Lambda_2\lambda_3, \Lambda_3\lambda_5) = \Lambda_1\Lambda_2\Lambda_3\chi(\lambda_1, \lambda_3, \lambda_5)$ for all $\lambda_1, \lambda_3, \lambda_5 \in \mho_1$ and $\Lambda_1, \Lambda_2, \Lambda_3 \in \Delta^1$. Then, χ is $\mathbb{C}-$trilinear.*

Proof. It follows from a similar method as in the proof of Theorem [47] (Lemma 2.1). □

Theorem 4. *Suppose $i = 1, \cdots, n$ and $n \in \mathbb{N}$. Let $\boldsymbol{\mathcal{E}}_i : \nu^6 \times (0, +\infty) \to \Phi$ be a function such that there exists a $0 < \vartheta_i < 1$ with*

$$\mathcal{E}_i\left(\frac{\lambda_1}{2}, \frac{\lambda_2}{2}, \frac{\lambda_3}{2}, \frac{\lambda_4}{2}, \frac{\lambda_5}{2}, \frac{\lambda_6}{2}, \mathcal{T}\right) \succeq \mathcal{E}_i\left(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \frac{2}{\vartheta}\mathcal{T}\right) \tag{23}$$

and let $\chi : \nu^3 \to \nu$ be a mapping satisfying $\chi(\lambda_1, 0, \lambda_5) = \chi(0, \lambda_3, \lambda_5) = \chi(\lambda_1, \lambda_3, 0) = 0$ and

$$\begin{aligned}
&\mathscr{N}\Big(\chi(\Lambda_1(\lambda_1+\lambda_2), \Lambda_2(\lambda_3-\lambda_4), \Lambda_3(\lambda_5+\lambda_6)) \\
&\qquad +\chi(\Lambda_1(\lambda_1-\lambda_2), \Lambda_2(\lambda_3+\lambda_4), \Lambda_3(\lambda_5-\lambda_6)) \\
&\qquad\qquad -\Lambda_1\Lambda_2\Lambda_3(2\chi(\lambda_1, \lambda_3, \lambda_5) - 2\chi(\lambda_1, \lambda_4, \lambda_6) \\
&\qquad\qquad\qquad +2\chi(\lambda_2, \lambda_3, \lambda_6) - 2\chi(\lambda_2, \lambda_4, \lambda_5)), \mathcal{T}\Big) \\
&\succeq \mathscr{N}\Big(\boldsymbol{\lambda}[2\chi(\frac{\lambda_1+\lambda_2}{2}, \lambda_3-\lambda_4, \lambda_5+\lambda_6) + 2\chi(\frac{\lambda_1-\lambda_2}{2}, \lambda_3+\lambda_4, \lambda_5-\lambda_6) \\
&\qquad -2\chi(\lambda_1, \lambda_3, \lambda_5) + 2\chi(\lambda_1, \lambda_4, \lambda_6) \\
&\qquad\qquad -2\chi(\lambda_2, \lambda_3, \lambda_6) + 2\chi(\lambda_2, \lambda_4, \lambda_5))], \mathcal{T}\Big) \\
&\qquad \bigotimes_{\mathtt{GTN}} Diagonal\Big[\mathcal{E}_1\Big(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \mathcal{T}\Big), \cdots, \mathcal{E}_n\Big(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \mathcal{T}\Big)\Big],
\end{aligned} \tag{24}$$

for all $\Lambda_1, \Lambda_2, \Lambda_3 \in \Delta^1$ and all $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \in \nu$. Then, we obtain a unique $\mathbb{C}-$trilinear mapping $\varpi : \nu^3 \to \nu$ satisfying

$$\begin{aligned}
&\mathscr{N}\Big(\chi(\lambda_1, \lambda_3, \lambda_5) - \varpi(\lambda_1, \lambda_3, \lambda_5), \mathcal{T}\Big) \\
&\succeq Diagonal\Big[\mathcal{E}_1\Big(\lambda_1, 0, \frac{\lambda_3}{2}, \frac{\lambda_3}{2}, \lambda_5, 0, (1-4\vartheta_1)\mathcal{T}\Big) \\
&\qquad \bigotimes_{\mathtt{TN}} \mathcal{E}_1\Big(\frac{\lambda_1}{2}, \frac{\lambda_1}{2}, \frac{\lambda_3}{2}, 0, \lambda_5, 0, \frac{(1-4\vartheta_1)}{2}\mathcal{T}\Big) \\
&\qquad\qquad \bigotimes_{\mathtt{TN}} \mathcal{E}_1\Big(\frac{\lambda_1}{2}, 0, \frac{\lambda_3}{2}, 0, \frac{\lambda_5}{2}, \frac{\lambda_5}{2}, \frac{(1-4\vartheta_1)}{4}\mathcal{T}\Big), \cdots, \\
&\quad \mathcal{E}_n\Big(\lambda_1, 0, \frac{\lambda_3}{2}, \frac{\lambda_3}{2}, \lambda_5, 0, (1-4\vartheta_n)\mathcal{T}\Big) \\
&\qquad \bigotimes_{\mathtt{TN}} \mathcal{E}_n\Big(\frac{\lambda_1}{2}, \frac{\lambda_1}{2}, \frac{\lambda_3}{2}, 0, \lambda_5, 0, \frac{(1-4\vartheta_n)}{2}\mathcal{T}\Big) \\
&\qquad\qquad \bigotimes_{\mathtt{TN}} \mathcal{E}_n\Big(\frac{\lambda_1}{2}, 0, \frac{\lambda_3}{2}, 0, \frac{\lambda_5}{2}, \frac{\lambda_5}{2}, \frac{(1-4\vartheta_n)}{4}\mathcal{T}\Big)\Big],
\end{aligned} \tag{25}$$

for any $\lambda_1, \lambda_3, \lambda_5 \in \nu$.

Besides, if the mapping $\chi : \nu^3 \to \nu$ satisfies $\chi(\lambda_1, \lambda_3, \lambda_5) = 2\chi(\lambda_1, \lambda_3, \lambda_5)$ and

$$\mathcal{N}\Big(\chi(\lambda_1\lambda_2, \lambda_3, \lambda_5) - \chi(\lambda_1, \lambda_3, \lambda_5)\lambda_2 - \lambda_1\chi(\lambda_2, \lambda_3, \lambda_5), \mathcal{T}\Big) \tag{26}$$
$$\succeq Diagonal\Big[\mathcal{E}_1\Big(\lambda_1, \lambda_2, \lambda_3, \lambda_3, \lambda_5, \lambda_5, \mathcal{T}\Big), \cdots, \mathcal{E}_n\Big(\lambda_1, \lambda_2, \lambda_3, \lambda_3, \lambda_5, \lambda_5, \mathcal{T}\Big)\Big],$$

$$\mathcal{N}\Big(\chi(\ell_{\beta(1)}, \ell_{\beta(2)}, \ell_{\beta(3)}) - \chi(\ell_1, \ell_2, \ell_3), \mathcal{T}\Big) \tag{27}$$
$$\succeq \mathcal{E}\Big(\ell_1, \ell_1, \ell_2, \ell_2, \ell_3, \ell_3, \mathcal{T}\Big)$$

for all permutations $(\beta(1), \beta(2), \beta(3))$ of $(1, 2, 3)$, and for all $\lambda_1, \lambda_2, \lambda_3, \ell_1, \ell_2, \ell_3, \lambda_5 \in \nu$, then, the $\mathbb{C}-$trilinear mapping $\varpi : \nu^3 \to \nu$ is a permuting tri-derivation.

Proof. Suppose $\Lambda_1 = \Lambda_2 = \Lambda_3 = 1$ in (24). Theorem 2 and [48] (Theorem 3.3) establish the theorem. □

Theorem 5. *Suppose $i = 1, \cdots, n$ and $n \in \mathbb{N}$. Let $\mathcal{E}_i : \nu^6 \times (0, +\infty) \to \Phi$ be a function such that there exists an $0 < \vartheta_i < 1$ with*

$$\mathcal{E}_i\Big(2\lambda_1, 2\lambda_2, 2\lambda_3, 2\lambda_4, 2\lambda_5, 2\lambda_6, \mathcal{T}\Big) \succ \mathcal{E}\Big(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \frac{\mathcal{T}}{2\vartheta_i}\Big), \tag{28}$$

for all $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \in \nu$. Let $\chi : \nu^3 \to \nu$ be a mapping satisfying (24) and $\chi(\lambda_1, 0, \lambda_5) = \chi(0, \lambda_3, \lambda_5) = \chi(\lambda_1, \lambda_3, 0) = 0$ for all $\lambda_1, \lambda_3, \lambda_5 \in \nu$. Then, we obtain a unique $\mathbb{C}-$trilinear mapping $\varpi : \nu^3 \to \nu$ satisfying

$$\mathcal{N}\Big(\chi(\lambda_1, \lambda_3, \lambda_5) - \varpi(\lambda_1, \lambda_3, \lambda_5), \mathcal{T}\Big) \tag{29}$$
$$\succeq Diagonal\Big[\mathcal{E}_1\Big(2\lambda_1, 0, \lambda_3, \lambda_3, 2\lambda_5, 0, 2(4-\vartheta_1)\mathcal{T}\Big)$$
$$\bigotimes \mathcal{E}_1\Big(\lambda_1, \lambda_1, \lambda_3, 0, 2\lambda_5, 0, (4-\vartheta_1)\mathcal{T}\Big)$$
$$\bigotimes \mathcal{E}_1\Big(\lambda_1, 0, \lambda_3, 0, \lambda_5, \lambda_5, \frac{(4-\vartheta_1)}{2}\mathcal{T}\Big), \cdots,$$
$$\mathcal{E}_n\Big(2\lambda_1, 0, \lambda_3, \lambda_3, 2\lambda_5, 0, 2(4-\vartheta_n)\mathcal{T}\Big)$$
$$\bigotimes \mathcal{E}_n\Big(\lambda_1, \lambda_1, \lambda_3, 0, 2\lambda_5, 0, (4-\vartheta_n)\mathcal{T}\Big)$$
$$\bigotimes \mathcal{E}_n\Big(\lambda_1, 0, \lambda_3, 0, \lambda_5, \lambda_5, \frac{(4-\vartheta_n)}{2}\mathcal{T}\Big)\Big],$$

for all $\lambda_1, \lambda_3, \lambda_5 \in \nu$.

Also, if the mapping $\chi : \nu^3 \to \nu$, satisfies (26), (27) and $\chi(2\lambda_1, \lambda_3, \lambda_5) = 2\chi(\lambda_1, \lambda_3, \lambda_5)$ for all $\lambda_1, \lambda_3, \lambda_5 \in \nu$, then the $\mathbb{C}-$trilinear mapping $\varpi : \nu^3 \to \nu$ is a permuting tri-derivation.

Proof. This follows from an analogous technique as in the proof of Theorem 4. □

Now, let ν and $U(\nu)$ be a unital MFC-$\diamond-$algebra with unit e and unitary group, respectively.

Theorem 6. *Suppose $i = 1, \cdots, n$ and $n \in \mathbb{N}$. Consider a function $\mathcal{E}_i : \nu^6 \times (0, +\infty) \to \Phi$ which satisfies (23) and a mapping $\chi : \nu^3 \to \nu$ which satisfies (24) and $\chi(\lambda_1, 0, \lambda_5) = \chi(0, \lambda_3, \lambda_5) =$*

$\chi(\lambda_1, \lambda_3, 0) = 0$, *for all* $\lambda_1, \lambda_3, \lambda_5 \in \nu$. *Then, we obtain a unique* $\mathbb{C}-$*trilinear mapping* $\varpi : \nu^3 \to \nu$ *which satisfies* (25).

Also, if the mapping $\chi : \nu^3 \to \nu$ *satisfies* (27), $\chi(2\phi, \lambda_3, \lambda_5) = 2\chi(\phi, \lambda_3, \lambda_5)$ *and*

$$\begin{aligned} &\mathscr{N}\Big(\chi(\phi\lambda_2, \lambda_3, \lambda_5) - \chi(\phi, \lambda_3, \lambda_5)\lambda_2 - \phi\chi(\lambda_2, \lambda_3, \lambda_5), \mathcal{T}\Big) \\ &\succeq Diagonal\Big[\mathcal{E}_1\Big(\phi, \lambda_2, \lambda_3, \lambda_3, \lambda_5, \lambda_5\Big), \cdots, \mathcal{E}_n\Big(\phi, \lambda_2, \lambda_3, \lambda_3, \lambda_5, \lambda_5\Big)\Big], \end{aligned} \tag{30}$$

for all $\lambda_2, \lambda_3, \lambda_5 \in \nu$ *and every* $\phi \in U(\nu)$, *then, the* $\mathbb{C}-$*trilinear mapping* $\varpi : \nu^3 \to \nu$, *is a permuting tri-derivation.*

Proof. Theorem 4 and [48] (Theorem 3.7) establish the theorem. □

Remark 1. *By a similar method as in the proof of the last theorem, we can conclude that if* (30) *in Theorem 6 is replaced by*

$$\begin{aligned} &\mathscr{N}\Big(\chi(\phi\varphi, \varphi_1, \varphi_2) - \chi(\phi, \varphi_1, \varphi_2)\varphi - \phi\chi(\varphi, \varphi_1, \varphi_2), \mathcal{T}\Big) \\ &\succeq Diagonal\Big[\mathcal{E}_1\Big(\phi, \varphi, \varphi_1, \varphi_1, \varphi_2, \varphi_2, \mathcal{T}\Big), \cdots, \mathcal{E}_n\Big(\phi, \varphi, \varphi_1, \varphi_1, \varphi_2, \varphi_2, \mathcal{T}\Big)\Big], \end{aligned}$$

for all $\phi, \varphi, \varphi_1, \varphi_2 \in U(\nu)$, *then, the* $\mathbb{C}-$*trilinear mapping* $\varpi : \nu^3 \to \nu$ *is a permuting triderivation.*

Theorem 7. *Suppose* $i = 1, \cdots, n$ *and* $n \in \mathbb{N}$. *Let* $\mathcal{E}_i : \nu^6 \times (0, +\infty) \to \Phi$ *be a function satisfying* (28) *and* $\chi : \nu^3 \to \nu$ *be a mapping satisfying* (24) *and* $\chi(\lambda_1, 0, \lambda_5) = \chi(0, \lambda_3, \lambda_5) = \chi(\lambda_1, \lambda_3, 0) = 0$, *for all* $\lambda_1, \lambda_3, \lambda_5 \in \nu$. *Then, we can find a unique* $\mathbb{C}-$*trilinear mapping* $\varpi : \nu^3 \to \nu$ *satisfying* (29).

Also, if the mapping $\chi : \nu^3 \to \nu$ *satisfies* (27), (30) *and* $\chi(2\phi, \lambda_3, \lambda_5) = 2\chi(\phi, \lambda_3, \lambda_5)$ *for all* $\lambda_3, \lambda_5 \in \nu$ *and every* $\phi \in U(\nu)$, *then, the* $\mathbb{C}-$*trilinear mapping* $\varpi : \nu^3 \to \nu$ *is a permuting tri-derivation.*

Proof. An analogous technique as in the proof of Theorem 6 proves the result. □

4. Permuting Tri-Homomorphisms in MFC-⋄-Aalgebras

Here, we prove the multi-stability results of permuting tri-homomorphisms in unital MFC-⋄-algebras related to the functional inequality (1).

Theorem 8. *Suppose* $i = 1, \cdots, n$ *and* $n \in \mathbb{N}$. *Let* $\mathcal{E}_i : \nu^6 \times (0, +\infty) \to \Phi$ *be a function such that there exists a* $0 < \vartheta_i < 1$ *with*

$$\mathcal{E}_i\left(\frac{\lambda_1}{2}, \frac{\lambda_2}{2}, \frac{\lambda_3}{2}, \frac{\lambda_4}{2}, \frac{\lambda_5}{2}, \frac{\lambda_6}{2}, \mathcal{T}\right) \succ \mathcal{E}_i\left(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \frac{2}{\vartheta_i}\mathcal{T}\right) \tag{31}$$

and let $\chi : \nu^3 \to \Theta$ *be a mapping satisfying* (24) *and* $\chi(\lambda_1, 0, \lambda_5) = \chi(0, \lambda_3, \lambda_5) = \chi(\lambda_1, \lambda_3, 0) = 0$, *for all* $\lambda_1, \lambda_3, \lambda_4 \in \nu$. *Then, we obtain a unique* $\mathbb{C}-$*trilinear mapping* $\rho : \nu^3 \to \Theta$ *satisfying*

$$\begin{aligned}
&\mathcal{N}\left(\chi(\lambda_1,\lambda_3,\lambda_5)-\rho(\lambda_1,\lambda_3,\lambda_5),\mathcal{T}\right) \\
&\succeq Diagonal\Bigg[\mathcal{E}_1\left(\lambda_1,0,\frac{\lambda_3}{2},\frac{\lambda_3}{2},\lambda_5,0,(1-4\vartheta_1)\mathcal{T}\right) \\
&\qquad \bigotimes_{\mathtt{TN}}\mathcal{E}_1\left(\frac{\lambda_1}{2},\frac{\lambda_1}{2},\frac{\lambda_3}{2},0,\lambda_5,0,\frac{(1-4\vartheta_1)}{2}\mathcal{T}\right) \\
&\qquad\qquad \bigotimes_{\mathtt{TN}}\mathcal{E}_1\left(\frac{\lambda_1}{2},0,\frac{\lambda_3}{2},0,\frac{\lambda_5}{2},\frac{\lambda_5}{2},\frac{(1-4\vartheta_1)}{4}\mathcal{T}\right),\cdots, \\
&\quad \mathcal{E}_n\left(\lambda_1,0,\frac{\lambda_3}{2},\frac{\lambda_3}{2},\lambda_5,0,(1-4\vartheta_n)\mathcal{T}\right) \\
&\qquad \bigotimes_{\mathtt{TN}}\mathcal{E}_n\left(\frac{\lambda_1}{2},\frac{\lambda_1}{2},\frac{\lambda_3}{2},0,\lambda_5,0,\frac{(1-4\vartheta_n)}{2}\mathcal{T}\right) \\
&\qquad\qquad \bigotimes_{\mathtt{TN}}\mathcal{E}_n\left(\frac{\lambda_1}{2},0,\frac{\lambda_3}{2},0,\frac{\lambda_5}{2},\frac{\lambda_5}{2},\frac{(1-4\vartheta_n)}{4}\mathcal{T}\right)\Bigg],
\end{aligned} \tag{32}$$

for all $\lambda_1,\lambda_3,\lambda_5\in\nu$, where $\mathcal{E}_i$ is given in Theorem 2.

Also, if the mapping $\chi:\nu^3\to\Theta$ satisfies (27) and

$$\begin{aligned}
&\mathcal{N}\left(\chi(\lambda_1\lambda_2,\lambda_3\lambda 4,\lambda_5\lambda_6)-\chi(\lambda_1,\lambda_3,\lambda_5)\chi(\lambda_2,\lambda_4,\lambda_6),\mathcal{T}\right) \\
&\succeq Diagonal\left[\mathcal{E}_1\left(\lambda_1,\lambda_2,\lambda_3,\lambda_3,\lambda_5,\lambda_5,\mathcal{T}\right),\cdots,\mathcal{E}_n\left(\lambda_1,\lambda_2,\lambda_3,\lambda_3,\lambda_5,\lambda_5,\mathcal{T}\right)\right],
\end{aligned} \tag{33}$$

for all $\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6\in\nu$, then, the $\mathbb{C}-$trilinear mapping $\rho:\nu^3\to\Theta$ is a permuting tri-homomorphism.

Proof. Theorem 4 and [48] (Theorem 4.1) establish the theorem. □

Theorem 9. *Suppose $i=1,\cdots,n$ and $n\in\mathbb{N}$. Let $\mathcal{E}_i:\nu^6\times(0,+\infty)\to\Phi$ be a function which satisfies (28) and $\chi:\nu^3\to\Theta$ be a mapping which satisfies (24) and $\chi(\lambda_1,0,\lambda_5)=\chi(0,\lambda_3,\lambda_5)=\chi(\lambda_1,\lambda_3,0)=0$ for all $\lambda_1,\lambda_3,\lambda_4\in\nu$. Then, we can obtain a unique $\mathbb{C}-$trilinear mapping $\rho:\nu^3\to\Theta$ satisfying*

$$\mathcal{N}\left(\chi(\lambda_1,\lambda_3,\lambda_5)-\rho(\lambda_1,\lambda_3,\lambda_5),\mathcal{T}\right) \tag{34}$$

$$\begin{aligned}
&\succeq Diagonal\Bigg[\mathcal{E}_1\left(2\lambda_1,0,\lambda_3,\lambda_3,2\lambda_5,0,2(4-\vartheta_1)\mathcal{T}\right) \\
&\qquad \bigotimes_{\mathtt{TN}}\mathcal{E}_1\left(\lambda_1,\lambda_1,\lambda_3,0,2\lambda_5,0,(4-\vartheta_1)\mathcal{T}\right) \\
&\qquad\qquad \bigotimes_{\mathtt{TN}}\mathcal{E}_1\left(\lambda_1,0,\lambda_3,0,\lambda_5,\lambda_5,\frac{(4-\vartheta_1)}{2}\mathcal{T}\right),\cdots, \\
&\quad \mathcal{E}_n\left(2\lambda_1,0,\lambda_3,\lambda_3,2\lambda_5,0,2(4-\vartheta_n)\mathcal{T}\right) \\
&\qquad \bigotimes_{\mathtt{TN}}\mathcal{E}_n\left(\lambda_1,\lambda_1,\lambda_3,0,2\lambda_5,0,(4-\vartheta_n)\mathcal{T}\right) \\
&\qquad\qquad \bigotimes_{\mathtt{TN}}\mathcal{E}_n\left(\lambda_1,0,\lambda_3,0,\lambda_5,\lambda_5,\frac{(4-\vartheta_n)}{2}\mathcal{T}\right)\Bigg],
\end{aligned} \tag{35}$$

for all $\lambda_1,\lambda_3,\lambda_5\in\nu$.

Also, if the mapping $\chi : \nu^3 \to \Theta$ satisfies (27) and (33), then, the $\mathbb{C}-$trilinear mapping $\rho : \nu^3 \to \Theta$ is a permuting tri-homomorphism.

Proof. By using a similar method as in the proof of Theorem 8, we obtain the result. □

Now, let ν and $U(\nu)$ be a unital C^* MFB-algebra with unit e and unitary group, respectively.

Theorem 10. *Suppose $i = 1, \cdots, n$ and $n \in \mathbb{N}$. Let $\mathcal{E}_i : \nu^6 \times (0, +\infty) \to \Phi$ be a function satisfying (31) and $\chi : \nu^3 \to \Theta$ be a mapping satisfying (24) and $\chi(\lambda_1, 0, \lambda_5) = \chi(0, \lambda_3, \lambda_5) = \chi(\lambda_1, \lambda_3, 0) = 0$ for all $\lambda_1, \lambda_3, \lambda_5 \in \nu$. Then, we can find a unique $\mathbb{C}-$trilinear mapping $\rho : \nu^3 \to \Theta$ satisfying (32).*

Also, if $\chi : \nu^3 \to \Theta$ satisfies (27) and

$$\mathcal{N}\Big(\chi(\phi_1\varphi_1, \phi_2\varphi_2, \phi_3\varphi_3) - \chi(\phi_1, \phi_2, \phi_3)\chi(\varphi_1, \varphi_2, \varphi_3), \mathcal{T}\Big) \succeq Diagonal\Big[\mathcal{E}_1\Big(\phi_1, \varphi_1, \phi_2, \varphi_2, \phi_3, \varphi_3, \mathcal{T}\Big), \cdots, \mathcal{E}_n\Big(\phi_1, \varphi_1, \phi_2, \varphi_2, \phi_3, \varphi_3, \mathcal{T}\Big)\Big], \tag{36}$$

for all $\phi_i, \varphi_i \in U(\nu), i = 1, 2, 3$, then, the $\mathbb{C}-$trilinear mapping $\rho : \nu^3 \to \Theta$ is a permuting tri-homomorphism.

Proof. Theorem 4 and [48] (Theorem 4.5) establish the theorem. □

Theorem 11. *Suppose $i = 1, \cdots, n$ and $n \in \mathbb{N}$. Let $\mathcal{E}_i : \nu^6 \times (0, +\infty) \to \Phi$ be a function which satisfies (28) and $\chi : \nu^3 \to \Theta$ be a mapping which satisfies (24) and $\chi(\lambda_1, 0, \lambda_5) = \chi(0, \lambda_3, \lambda_5) = \chi(\lambda_1, \lambda_3, 0) = 0$, for all $\lambda_1, \lambda_3, \lambda_5 \in \nu$. Then, we can find a unique $\mathbb{C}-$trilinear mapping $\rho : \nu^3 \to \Theta$ satisfying (34).*

Also, if the mapping $\chi : \nu^3 \to \Theta$ satisfies (27) and (36), then, the $\mathbb{C}-$trilinear mapping $\rho : \nu^3 \to \Theta$ is a permuting tri-homomorphism.

Proof. By using a similar method as in the proof of Theorem 10, we obtain the result. □

5. Application

First, we present the concept of aggregation functions. Next, we propose a small list of aggregation functions on some special functions to obtain optimal stability and minimal error which enable us to present a unique optimum solution. We refer to [49–57] for more applications.

Let $n \in \mathbb{N}$, and $[n] := \{1, \cdots, n\}$. We will use bold symbols to denote $n-$tuples. For example Diagonal$[y_1, \cdots, y_n]_{n\times n}$ will often be written $\mathbf{Y}$.

Definition 9 ([42]). *A function $A^{(n)} : Diagonal[\Phi, \cdots, \Phi]_{n\times n} \longrightarrow \Phi$ is called an aggregation function if it is nondecreasing in each variable and also fulfills the boundary conditions*

$$\inf_{\mathbf{Y}\in\Phi^n} A^{(n)}(\mathbf{Y}) = \inf \Phi, \quad and \quad \sup_{\mathbf{Y}\in\Phi^n} A^{(n)}(\mathbf{Y}) = \sup \Phi. \tag{37}$$

The $n \in \mathbb{N}$ displays the arity of the aggregation function or the number of its variables. Note that we will denote the aggregation functions as A instead of $A^{(n)}$.

We now give a common list of aggregation functions.

- The arithmetic mean function $\mathbb{AM}$: Diagonal$[\Phi, \cdots, \Phi]_{n\times n} \longrightarrow \Phi$ and the geometric mean function $\mathbb{GM}$: Diagonal$[\Phi, \cdots, \Phi]_{n\times n} \longrightarrow \Phi$ are respectively given by

$$\mathtt{AG}_1(\mathbf{Y}) := \mathbb{AM}(\mathbf{Y}) := \frac{1}{n}\sum_{i=1}^{n} y_i, \tag{38}$$

$$\mathtt{AG}_2(\mathbf{Y}) := \mathbb{GM}(\mathbf{Y}) := (\prod_{i=1}^{n} y_i)^{\frac{1}{n}}. \tag{39}$$

- For every $\Bbbk \in [n]$, the projection function $\mathbb{P}_{\Bbbk}$: Diagonal$[\Phi, \cdots, \Phi]_{n\times n} \longrightarrow \Phi$ and the order statistic function $\mathbb{OS}_{\Bbbk}$: Diagonal$[\Phi, \cdots, \Phi]_{n\times n} \longrightarrow \Phi$ associated with the $\Bbbk^{th}$ argument, are respectively given by

$$\mathtt{AG}_3(\mathbf{Y}) := \mathbb{P}_{\Bbbk}(\mathbf{Y}) := y_{\Bbbk}, \tag{40}$$

$$\mathtt{AG}_4(\mathbf{Y}) := \mathbb{OS}_{\Bbbk}(\mathbf{Y}) := (y)_{\Bbbk}, \tag{41}$$

where $(y)_{\Bbbk}$ is the $\Bbbk^{th}$ lowest coordinate of y, that is,

$$y_{(1)} \leqslant \cdots \leqslant y_{(k)} \leqslant \cdots y_{(n)}.$$

The projections onto the first and the last coordinates are given by

$$\mathtt{AG}_5(\mathbf{Y}) := \mathbb{P}_F(\mathbf{Y}) := \mathbb{P}_1(\mathbf{Y}) = y_1, \tag{42}$$

$$\mathtt{AG}_6(\mathbf{Y}) := \mathbb{P}_L(\mathbf{Y}) := \mathbb{P}_n(\mathbf{Y}) = y_n. \tag{43}$$

Also, the extreme order statistics y_1 and y_n are the minimum and maximum functions, respectively,

$$\mathtt{AG}_7(\mathbf{Y}) := \mathbb{MIN}(\mathbf{Y}) := \mathbb{OS}_1(\mathbf{Y}) = \min\{y_1, \cdots, y_n\}, \tag{44}$$

$$\mathtt{AG}_8(\mathbf{Y}) := \mathbb{MAX}(\mathbf{Y}) := \mathbb{OS}_n(\mathbf{Y}) = \max\{y_1, \cdots, y_n\}, \tag{45}$$

which will sometimes be written by means of the lattice operations $\vee$ and $\wedge$, respectively, that is,

$$\mathbb{MIN}(\mathbf{Y}) = \bigwedge_{i=1}^{n} y_i, \quad \text{and} \quad \mathbb{MAX}(\mathbf{Y}) = \bigvee_{i=1}^{n} y_i.$$

Note that $\mathbb{OS}_{\Bbbk}$ can be shown in terms of only minima and maxima as follows

$$\mathbb{OS}_{\Bbbk}(\mathbf{Y}) = \bigwedge_{\substack{K\subseteq[n] \\ |K|=\Bbbk}} \bigvee_{i\in K} y_i = \bigvee_{\substack{K\subseteq[n] \\ |K|=n-\Bbbk+1}} \bigwedge_{i\in K} y_i.$$

Similarly, the median of an odd number of values Diagonal$[y_1, \cdots, y_{2\Bbbk-1}]_{(2\Bbbk-1)\times(2\Bbbk-1)}$ is given by

$$\mathbb{MED}\Big(\text{Diagonal}[y_1, \cdots, y_{2\Bbbk-1}]_{(2\Bbbk-1)\times(2\Bbbk-1)}\Big) = y_{(\Bbbk)},$$

that can be shown by

$$\mathbb{MED}\Big(\text{Diagonal}[y_1, \cdots, y_{2\Bbbk-1}]_{(2\Bbbk-1)\times(2\Bbbk-1)}\Big) = \bigwedge_{\substack{K\subseteq[2\Bbbk-1] \\ |K|=\Bbbk}} \bigvee_{i\in K} y_i = \bigvee_{\substack{K\subseteq[2\Bbbk-1] \\ |K|=\Bbbk}} \bigwedge_{i\in K} y_i.$$

For instance, we get

$$\begin{aligned}\mathbb{MED}(\text{Diagonal}[y_1, y_2, y_3]_{3\times 3}) &= (y_1 \wedge y_2) \vee (y_1 \wedge y_3) \vee (y_2 \wedge y_3) \\ &= (y_1 \vee y_2) \wedge (y_1 \vee y_3) \wedge (y_2 \vee y_3).\end{aligned}$$

For an even number of values Diagonal$[y_1, \cdots, y_{2\Bbbk}]$, the median is given by

$$\mathbb{MED}\Big(\text{Diagonal}[y_1, \cdots, y_{2\Bbbk}]_{2\Bbbk\times 2\Bbbk}\Big) := \mathbb{AM}\Big(\text{Diagonal}[y_{(\Bbbk)}, y_{(\Bbbk+1)}]_{2\times 2}\Big) = \frac{y_{(\Bbbk)} + y_{(\Bbbk+1)}}{2}.$$

For every $\varphi \in \Phi$, we also define the $\varphi-$median, $\mathbb{MED}_\varphi : \text{Diagonal}[\Phi, \cdots, \Phi]_{n\times n} \longrightarrow \Phi$, by

$$\begin{aligned}\mathtt{AG}_9(\mathbf{Y}) := \mathbb{MED}_\varphi(\mathbf{Y}) &= \mathbb{MED}\Big(\text{Diagonal}[y_1, \cdots, y_n, \underbrace{\varphi, \cdots, \varphi}_{n-1}]_{(2n-1)\times(2n-1)}\Big) \\ &= \mathbb{MED}(\mathbb{MIN}(\mathbf{Y}), \varphi, \mathbb{MAX}(\mathbf{Y})).\end{aligned}$$

- For every $\emptyset \neq K \subseteq [n]$, the partial minimum $\mathbb{MIN}_\Bbbk : \text{Diagonal}[\Phi, \cdots, \Phi]_{n\times n} \longrightarrow \Phi$ and the partial maximum $\mathbb{MAX}_\Bbbk : \text{Diagonal}[\Phi, \cdots, \Phi]_{n\times n} \longrightarrow \Phi$, associated with K, are respectively given by

$$\mathtt{AG}_{10}(\mathbf{Y}) := \mathbb{MIN}_\Bbbk(\mathbf{Y}) := \bigwedge_{i\in K} y_i, \tag{46}$$

$$\mathtt{AG}_{11}(\mathbf{Y}) := \mathbb{MAX}_\Bbbk(\mathbf{Y}) := \bigvee_{i\in K} y_i. \tag{47}$$

- For every weight vector $\mathbf{V} = \text{Diagonal}[v_1, \cdots, v_n]_{n\times n} \in \text{Diagonal}[\Phi, \cdots, \Phi]_{n\times n}$ s.t. $\sum_{i=1}^n v_i = 1$, the weighted arithmetic mean function $\mathbb{WAM}_\mathbf{V} : \text{Diagonal}[\Phi, \cdots, \Phi]_{n\times n} \longrightarrow \Phi$ and the ordered weighted averaging function $\mathbb{OWA}_\mathbf{V} : \text{Diagonal}[\Phi, \cdots, \Phi]_{n\times n} \longrightarrow \Phi$, associated with $\mathbf{V}$, are respectively given by

$$\mathtt{AG}_{12}(\mathbf{Y}) := \mathbb{WAM}_\mathbf{V}(\mathbf{Y}) := \sum_{i=1}^n v_i y_i, \tag{48}$$

$$\mathtt{AG}_{13}(\mathbf{Y}) := \mathbb{OWA}_\mathbf{V}(\mathbf{Y}) := \sum_{i=1}^n v_i y_{(i)}. \tag{49}$$

- The sum $\sum : \text{Diagonal}[\Phi, \cdots, \Phi]_{n\times n} \longrightarrow \Phi$ and product $\Pi : \text{Diagonal}[\Phi, \cdots, \Phi]_{n\times n} \longrightarrow \Phi$ functions are respectively given by

$$\mathtt{AG}_{14}(\mathbf{Y}) := \sum(\mathbf{Y}) := \sum_{i=1}^n y_i, \tag{50}$$

$$\mathtt{AG}_{15}(\mathbf{Y}) := \Pi(\mathbf{Y}) := \prod_{i=1}^n y_i. \tag{51}$$

The main issue we are investigating in this section is that of aggregation which refers to the process of merging and combining various values into a single one. Now, we apply the above aggregation functions on Mittag-Leffler-type functions to present a class of controller to study the multi stability for the governing model.

Assume the following Mittag-Leffler-type functions:

- The one parameter Mittag-Leffler function [43]:

$$\mathcal{E}_1(Y) := \nabla_\flat(Y) = \sum_{i=0}^\infty \frac{Y^i}{\Gamma(i\flat + 1)}, \tag{52}$$

where $\flat, Y \in \mathbb{C}, i \in \mathbb{N}$, and $\Re(\wp) > 0$.

- The pre-superhyperbolic supercosine through (52) [43]:

$$\begin{aligned}\mathcal{E}_2(Y) &:= precosh_\flat(Y)\\ &= 0.5\Big(\nabla_\flat(Y) + \nabla_\flat(-Y)\Big)\\ &= \sum_{i=0}^{\infty} \frac{Y^{2i}}{\Gamma((2i)\flat + 1)},\end{aligned}$$

where $Y, \flat \in \mathbb{C}$, and $\Re(\flat) > 0$.

- The pre-supercosine function through (52) [43]:

$$\begin{aligned}\mathcal{E}_3(Y) &:= precos_\flat(Y)\\ &= \frac{1}{2}\Big(\nabla_\flat(iY) + \nabla_\flat(-iY)\Big)\\ &= \sum_{i=0}^{\infty} \frac{(-1)^i Y^{2i}}{\Gamma((2i)\flat + 1)},\end{aligned}$$

where $Y, \flat \in \mathbb{C}$, and $\Re(\flat) > 0$.

- The pre-superhyperbolic supersine through (52) [43]:

$$\begin{aligned}\mathcal{E}_4(Y) &:= presinh_\flat(Y)\\ &= \frac{1}{2}\Big(\nabla_\flat(Y) - \nabla_\flat(-Y)\Big)\\ &= \sum_{i=0}^{\infty} \frac{Y^{2i+1}}{\Gamma((2i+1)\flat + 1)},\end{aligned}$$

where $Y, \flat \in \mathbb{C}$, and $\Re(\flat) > 0$.

- The pre-supersine function through (52) [43]:

$$\begin{aligned}\mathcal{E}_5(Y) &:= presin_\flat(Y)\\ &= \frac{1}{2i}\Big(\nabla_\flat(iY) - \nabla_\flat(-iY)\Big)\\ &= \sum_{i=0}^{\infty} \frac{(-1)^i Y^{2i+1}}{\Gamma((2i+1)\flat + 1)},\end{aligned}$$

where $Y, \flat \in \mathbb{C}$, and $\Re(\flat) > 0$.

Now, we have the following results, for every $i = 1, \cdots, 5$.

Corollary 1. *Let $\mathcal{T} \geq 0$, $r_i > 3$ and $\Psi_i > 0$ be in $\mathbb{R}$ and $\chi : \mathfrak{G}^3 \to \mathfrak{P}$ be a mapping satisfying $\chi(\lambda_1, 0, \lambda_5) = \chi(0, \lambda_3, \lambda_5) = \chi(\lambda_1, \lambda_3, 0) = 0$ and*

$$\begin{aligned}
&\mathscr{N}\Bigg(\chi(\lambda_1+\lambda_2,\lambda_3-\lambda_4,\lambda_5+\lambda_6)+\chi(\lambda_1-\lambda_2,\lambda_3+\lambda_4,\lambda_5-\lambda_6) \\
&\qquad -2\chi(\lambda_1,\lambda_3,\lambda_5)+2\chi(\lambda_1,\lambda_4,\lambda_6) \\
&\qquad -2\chi(\lambda_2,\lambda_3,\lambda_6)+2\chi(\lambda_2,\lambda_4,\lambda_5),\mathcal{T}\Bigg) \\
&\succeq \mathscr{N}\Bigg(\boldsymbol{\lambda}[2\chi(\frac{\lambda_1+\lambda_2}{2},\lambda_3-\lambda_4,\lambda_5+\lambda_6)+2\chi(\frac{\lambda_1-\lambda_2}{2},\lambda_3+\lambda_4,\lambda_5-\lambda_6) \\
&\qquad -2\chi(\lambda_1,\lambda_3,\lambda_5)+2\chi(\lambda_1,\lambda_4,\lambda_6) \\
&\qquad -2\chi(\lambda_2,\lambda_3,\lambda_6)+2\chi(\lambda_2,\lambda_4,\lambda_5)],\mathcal{T}\Bigg) \\
&\bigotimes_{\mathtt{GTN}} Diagonal\Bigg[\mathtt{AG}_1\Bigg(Diagonal\Bigg[\\
&\mathcal{E}_1\Bigg(-\frac{\Psi_1[\|\lambda_1\|^{r_1}+\|\lambda_2\|^{r_1}+\|\lambda_3\|^{r_1}+\|\lambda_4\|^{r_1}+\|\lambda_5\|^{r_1}+\|\lambda_6\|^{r_1}]}{\mathcal{T}}\Bigg),\cdots, \\
&\mathcal{E}_5\Bigg(-\frac{\Psi_5[\|\lambda_1\|^{r_5}+\|\lambda_2\|^{r_5}+\|\lambda_3\|^{r_5}+\|\lambda_4\|^{r_5}+\|\lambda_5\|^{r_5}+\|\lambda_6\|^{r_5}]}{\mathcal{T}}\Bigg)\Bigg]\Bigg),\ldots, \\
&\mathtt{AG}_{15}\Bigg(Diagonal\Bigg[\mathcal{E}_1\Bigg(-\frac{\Psi_1[\|\lambda_1\|^{r_1}+\|\lambda_2\|^{r_1}+\|\lambda_3\|^{r_1}+\|\lambda_4\|^{r_1}+\|\lambda_5\|^{r_1}+\|\lambda_6\|^{r_1}]}{\mathcal{T}}\Bigg),\cdots, \\
&\mathcal{E}_5\Bigg(-\frac{\Psi_5[\|\lambda_1\|^{r_5}+\|\lambda_2\|^{r_5}+\|\lambda_3\|^{r_5}+\|\lambda_4\|^{r_5}+\|\lambda_5\|^{r_5}+\|\lambda_6\|^{r_5}]}{\mathcal{T}}\Bigg)\Bigg]\Bigg)\ \Bigg],
\end{aligned} \tag{53}$$

for all $\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6\in\mathfrak{G}$. Then, we can obtain a unique tri-additive mapping $\chi':\mathfrak{G}^3\to\mathfrak{P}$ satisfying

$$\begin{aligned}
&\mathscr{N}\Big(\chi(\lambda_1,\lambda_3,\lambda_5)-\chi'(\lambda_1,\lambda_3,\lambda_5),\mathcal{T}\Big) \\
&\succeq Diagonal[\sigma_1,\cdots,\sigma_{15}],
\end{aligned} \tag{54}$$

in which

$$
\begin{aligned}
&Diagonal[\sigma_1, \cdots, \sigma_{15}] \qquad (55)\\
&:= Diagonal\Bigg[\mathtt{AG}_1\Bigg(Diagonal\Bigg[\mathcal{E}_1\Bigg(-\frac{\Psi_1[\|\lambda_1\|^{r_1} + 2\frac{\|\lambda_3\|^{r_1}}{2^{r_1}} + \|\lambda_5\|^{r_1}]}{(1-4\vartheta_1)\mathcal{T}}\Bigg)\\
&\bigotimes_{\mathtt{TN}} \mathcal{E}_1\Bigg(-\frac{\Psi_1[\frac{2\|\lambda_1\|^{r_1}}{2^{r_1}} + 2\frac{\|\lambda_3\|^{r_1}}{2^{r_1}} + \|\lambda_5\|^{r_1}]}{\frac{(1-4\vartheta_1)}{2}\mathcal{T}}\Bigg)\\
&\bigotimes_{\mathtt{TN}} \mathcal{E}_1\Bigg(-\frac{\Psi_1[\frac{\|\lambda_1\|^{r_1}}{2^{r_1}} + \frac{\|\lambda_3\|^{r_1}}{2^{r_1}} + \frac{\|\lambda_5\|^{r_1}}{2^{r_1}}]}{\frac{(1-4\vartheta_1)}{4}\mathcal{T}}\Bigg), \cdots,\\
&\mathcal{E}_5\Bigg(-\frac{\Psi_5[\|\lambda_1\|^{r_5} + 2\frac{\|\lambda_3\|^{r_5}}{2^{r_5}} + \|\lambda_5\|^{r_5}]}{(1-4\vartheta_5)\mathcal{T}}\Bigg)\\
&\bigotimes_{\mathtt{TN}} \mathcal{E}_5\Bigg(-\frac{\Psi_5[\frac{2\|\lambda_1\|^{r_5}}{2^{r_5}} + 2\frac{\|\lambda_3\|^{r_5}}{2^{r_5}} + \|\lambda_5\|^{r_5}]}{\frac{(1-4\vartheta_5)}{2}\mathcal{T}}\Bigg)\\
&\bigotimes_{\mathtt{TN}} \mathcal{E}_5\Bigg(-\frac{\Psi_5[\frac{\|\lambda_1\|^{r_5}}{2^{r_5}} + \frac{\|\lambda_3\|^{r_5}}{2^{r_5}} + \frac{\|\lambda_5\|^{r_5}}{2^{r_5}}]}{\frac{(1-4\vartheta_5)}{4}\mathcal{T}}\Bigg)\Bigg]\Bigg), \cdots,\\
&\mathtt{AG}_{15}\Bigg(Diagonal\Bigg[\mathcal{E}_1\Bigg(-\frac{\Psi_1[\|\lambda_1\|^{r_1} + 2\frac{\|\lambda_3\|^{r_1}}{2^{r_1}} + \|\lambda_5\|^{r_1}]}{(1-4\vartheta_1)\mathcal{T}}\Bigg)\\
&\bigotimes_{\mathtt{TN}} \mathcal{E}_1\Bigg(-\frac{\Psi_1[\frac{2\|\lambda_1\|^{r_1}}{2^{r_1}} + 2\frac{\|\lambda_3\|^{r_1}}{2^{r_1}} + \|\lambda_5\|^{r_1}]}{\frac{(1-4\vartheta_1)}{2}\mathcal{T}}\Bigg)\\
&\bigotimes_{\mathtt{TN}} \mathcal{E}_1\Bigg(-\frac{\Psi_1[\frac{\|\lambda_1\|^{r_1}}{2^{r_1}} + \frac{\|\lambda_3\|^{r_1}}{2^{r_1}} + \frac{\|\lambda_5\|^{r_1}}{2^{r_1}}]}{\frac{(1-4\vartheta_1)}{4}\mathcal{T}}\Bigg), \cdots,\\
&\mathcal{E}_5\Bigg(-\frac{\Psi_5[\|\lambda_1\|^{r_5} + 2\frac{\|\lambda_3\|^{r_5}}{2^{r_5}} + \|\lambda_5\|^{r_5}]}{(1-4\vartheta_5)\mathcal{T}}\Bigg)\\
&\bigotimes_{\mathtt{TN}} \mathcal{E}_5\Bigg(-\frac{\Psi_5[\frac{2\|\lambda_1\|^{r_5}}{2^{r_5}} + 2\frac{\|\lambda_3\|^{r_5}}{2^{r_5}} + \|\lambda_5\|^{r_5}]}{\frac{(1-4\vartheta_5)}{2}\mathcal{T}}\Bigg)\\
&\bigotimes_{\mathtt{TN}} \mathcal{E}_5\Bigg(-\frac{\Psi_5[\frac{\|\lambda_1\|^{r_5}}{2^{r_5}} + \frac{\|\lambda_3\|^{r_5}}{2^{r_5}} + \frac{\|\lambda_5\|^{r_5}}{2^{r_5}}]}{\frac{(1-4\vartheta_5)}{4}\mathcal{T}}\Bigg)\Bigg]\Bigg)\Bigg],
\end{aligned}
$$

where $\vartheta_i = 2^{r_i-1}$, *for all* $\lambda_1, \lambda_3, \lambda_5 \in \mathfrak{G}$.

Corollary 2. *Let $\mathcal{T} \geq 0$ and $r_i < 3$, $\Psi_i > 0$ be in $\mathbb{R}$, and $\chi : \mathfrak{G}^3 \to \mathfrak{P}$ be a mapping satisfying $\chi(\lambda_1, 0, \lambda_5) = \chi(0, \lambda_3, \lambda_5) = \chi(\lambda_1, \lambda_3, 0) = 0$ and (53). Then, we can obtain a unique tri-additive mapping $\chi' : \mathfrak{G}^3 \to \mathfrak{P}$ satisfying*

$$\mathscr{N}\left(\chi(\lambda_1, \lambda_3, \lambda_5) - \chi'(\lambda_1, \lambda_3, \lambda_5), \mathcal{T}\right) \succeq Diagonal[\sigma', \ldots, \sigma']_{15\times 15} \tag{56}$$

in which

$$\begin{aligned} &Diagonal[\sigma', \ldots, \sigma']_{15\times 15} \\ &:= Diagonal\Bigg[\mathtt{AG}_1\Bigg(Diagonal\Bigg[\mathcal{E}_1\left(-\frac{\Psi_1[\|\lambda_1\|^{r_1} + 2\|\lambda_3\|^{r_1} + \|\lambda_5\|^{r_1}]}{(8-2\vartheta_1)\mathcal{T}}\right) \\ &\qquad \bigotimes_{\mathtt{TN}} \mathcal{E}_1\left(-\frac{\Psi_1[\|\lambda_1\|^{r_1} + \|\lambda_3\|^{r_1} + 2^{r_1}\|\lambda_5\|^{r_1}]}{(4-\vartheta_1)\mathcal{T}}\right) \\ &\qquad\qquad \bigotimes_{\mathtt{TN}} \mathcal{E}_1\left(-\frac{\Psi_1[\|\lambda_1\|^{r_1} + \|\lambda_3\|^{r_1} + 2\|\lambda_5\|^{r_1}]}{\frac{(4-\vartheta_1)}{2}\mathcal{T}}\right), \cdots, \\ &\quad \mathcal{E}_5\left(-\frac{\Psi_5[\|\lambda_1\|^{r_5} + 2\|\lambda_3\|^{r_5} + \|\lambda_5\|^{r_5}]}{(8-2\vartheta_5)\mathcal{T}}\right) \\ &\qquad \bigotimes_{\mathtt{TN}} \mathcal{E}_5\left(-\frac{\Psi_5[\|\lambda_1\|^{r_5} + \|\lambda_3\|^{r_5} + 2^{r_5}\|\lambda_5\|^{r_5}]}{(4-\vartheta_5)\mathcal{T}}\right) \\ &\qquad\qquad \bigotimes_{\mathtt{TN}} \mathcal{E}_5\left(-\frac{\Psi_5[\|\lambda_1\|^{r_5} + \|\lambda_3\|^{r_5} + 2\|\lambda_5\|^{r_5}]}{\frac{(4-\vartheta_5)}{2}\mathcal{T}}\right)\Bigg]_{5\times 5}\Bigg), \cdots, \\ &\mathtt{AG}_{15}\Bigg(Diagonal\Bigg[\mathcal{E}_1\left(-\frac{\Psi_1[\|\lambda_1\|^{r_1} + 2\|\lambda_3\|^{r_1} + \|\lambda_5\|^{r_1}]}{(8-2\vartheta_1)\mathcal{T}}\right) \\ &\qquad \bigotimes_{\mathtt{TN}} \mathcal{E}_1\left(-\frac{\Psi_1[\|\lambda_1\|^{r_1} + \|\lambda_3\|^{r_1} + 2^{r_1}\|\lambda_5\|^{r_1}]}{(4-\vartheta_1)\mathcal{T}}\right) \\ &\qquad\qquad \bigotimes_{\mathtt{TN}} \mathcal{E}_1\left(-\frac{\Psi_1[\|\lambda_1\|^{r_1} + \|\lambda_3\|^{r_1} + 2\|\lambda_5\|^{r_1}]}{\frac{(4-\vartheta_1)}{2}\mathcal{T}}\right), \cdots, \\ &\quad \mathcal{E}_5\left(-\frac{\Psi_5[\|\lambda_1\|^{r_5} + 2\|\lambda_3\|^{r_5} + \|\lambda_5\|^{r_5}]}{(8-2\vartheta_5)\mathcal{T}}\right) \\ &\qquad \bigotimes_{\mathtt{TN}} \mathcal{E}_5\left(-\frac{\Psi_5[\|\lambda_1\|^{r_5} + \|\lambda_3\|^{r_5} + 2^{r_5}\|\lambda_5\|^{r_5}]}{(4-\vartheta_5)\mathcal{T}}\right) \\ &\qquad\qquad \bigotimes_{\mathtt{TN}} \mathcal{E}_5\left(-\frac{\Psi_5[\|\lambda_1\|^{r_5} + \|\lambda_3\|^{r_5} + 2\|\lambda_5\|^{r_5}]}{\frac{(4-\vartheta_5)}{2}\mathcal{T}}\right)\Bigg]_{5\times 5}\Bigg)\Bigg]_{15\times 15}, \end{aligned} \tag{57}$$

where $\vartheta_i = 2^{r_i+1}$, for all $\lambda_1, \lambda_3, \lambda_5 \in \mathfrak{G}$.

Corollary 3. *Let $\mathcal{T} \geq 0$, $r_i > 4$ and $\Psi_i > 0$ be in $\mathbb{R}$, and $\chi : \nu^3 \to \nu$ be a mapping satisfying $\chi(\lambda_1, 0, \lambda_5) = \chi(0, \lambda_3, \lambda_5) = \chi(\lambda_1, \lambda_3, 0) = 0$ and*

$$\begin{aligned}
&\mathcal{N}\Big(\chi(\Lambda_1(\lambda_1+\lambda_2), \Lambda_2(\lambda_3-\lambda_4), \Lambda_3(\lambda_5+\lambda_6)) \\
&+\chi(\Lambda_1(\lambda_1-\lambda_2), \Lambda_2(\lambda_3+\lambda_4), \Lambda_3(\lambda_5-\lambda_6)) \\
&\qquad -\Lambda_1\Lambda_2\Lambda_3(2\chi(\lambda_1,\lambda_3,\lambda_5) - 2\chi(\lambda_1,\lambda_4,\lambda_6) \\
&\qquad +2\chi(\lambda_2,\lambda_3,\lambda_6) - 2\chi(\lambda_2,\lambda_4,\lambda_5)), \mathcal{T}\Big) \\
&\succeq \mathcal{N}\Big(\boldsymbol{\lambda}[2\chi(\frac{\lambda_1+\lambda_2}{2}, \lambda_3-\lambda_4, \lambda_5+\lambda_6) + 2\chi(\frac{\lambda_1-\lambda_2}{2}, \lambda_3+\lambda_4, \lambda_5-\lambda_6] \\
&-2\chi(\lambda_1,\lambda_3,\lambda_5) + 2\chi(\lambda_1,\lambda_4,\lambda_6) \\
&-2\chi(\lambda_2,\lambda_3,\lambda_6) + 2\chi(\lambda_2,\lambda_4,\lambda_5)], \mathcal{T}\Big) \\
&\bigotimes_{\mathtt{GTN}} Diagonal\Big[\mathtt{AG}_1\Big(Diagonal\Big[\\
&\mathcal{E}_1\Big(-\frac{\Psi_1[\|\lambda_1\|^{r_1}+\|\lambda_2\|^{r_1}+\|\lambda_3\|^{r_1}+\|\lambda_4\|^{r_1}+\|\lambda_5\|^{r_1}+\|\lambda_6\|^{r_1}]}{\mathcal{T}}\Big), \cdots, \\
&\mathcal{E}_5\Big(-\frac{\Psi_5[\|\lambda_1\|^{r_5}+\|\lambda_2\|^{r_5}+\|\lambda_3\|^{r_5}+\|\lambda_4\|^{r_5}+\|\lambda_5\|^{r_5}+\|\lambda_6\|^{r_5}]}{\mathcal{T}}\Big)\Big]_{5\times 5}\Big), \cdots, \\
&\mathtt{AG}_{15}\Big(Diagonal\Big[\mathcal{E}_1\Big(-\frac{\Psi_1[\|\lambda_1\|^{r_1}+\|\lambda_2\|^{r_1}+\|\lambda_3\|^{r_1}+\|\lambda_4\|^{r_1}+\|\lambda_5\|^{r_1}+\|\lambda_6\|^{r_1}]}{\mathcal{T}}\Big), \cdots, \\
&\mathcal{E}_5\Big(-\frac{\Psi_5[\|\lambda_1\|^{r_5}+\|\lambda_2\|^{r_5}+\|\lambda_3\|^{r_5}+\|\lambda_4\|^{r_5}+\|\lambda_5\|^{r_5}+\|\lambda_6\|^{r_5}]}{\mathcal{T}}\Big)\Big]_{5\times 5}\Big)\Big]_{15\times 15}
\end{aligned} \tag{58}$$

for all $\Lambda_1, \Lambda_2, \Lambda_3 \in \Delta^1$ and all $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \in \nu$. Then, we can obtain a unique $\mathbb{C}-$trilinear mapping $\varpi : \nu^3 \to \nu$ satisfying

$$\mathcal{N}\Big(\chi(\lambda_1,\lambda_3,\lambda_5) - \varpi(\lambda_1,\lambda_3,\lambda_5), \mathcal{T}\Big) \succeq Diagonal[\sigma_1, \ldots, \sigma_{15}]$$

where $Diagonal[\sigma_1, \ldots, \sigma_{15}]$ satisfies (55) for all $\lambda_1, \lambda_3, \lambda_5 \in \nu$.

In addition, if the mapping $\chi : \nu^3 \to \nu$ satisfies $\chi(2\lambda_1, \lambda_3, \lambda_5) = 2\chi(\lambda_1, \lambda_3, \lambda_5)$ and

$$\begin{aligned}
&\mathcal{N}\Big(\chi(\lambda_1\lambda_2,\lambda_3,\lambda_5) - \chi(\lambda_1,\lambda_3,\lambda_5)\lambda_2 - \lambda_1\chi(\lambda_2,\lambda_3,\lambda_5), \mathcal{T}\Big) \\
&\succeq Diagonal\Big[\mathtt{AG}_1\Big(Diagonal\Big[\\
&\mathcal{E}_1\Big(-\frac{\Psi_1[\|\lambda_1\|^{r_1}+\|\lambda_2\|^{r_1}+\|\lambda_3\|^{r_1}+\|\lambda_4\|^{r_1}+\|\lambda_5\|^{r_1}+\|\lambda_6\|^{r_1}]}{\mathcal{T}}\Big), \cdots, \\
&\mathcal{E}_5\Big(-\frac{\Psi_5[\|\lambda_1\|^{r_5}+\|\lambda_2\|^{r_5}+\|\lambda_3\|^{r_5}+\|\lambda_4\|^{r_5}+\|\lambda_5\|^{r_5}+\|\lambda_6\|^{r_5}]}{\mathcal{T}}\Big)\Big]_{5\times 5}\Big), \cdots, \\
&\mathtt{AG}_{15}\Big(Diagonal\Big[\mathcal{E}_1\Big(-\frac{\Psi_1[\|\lambda_1\|^{r_1}+\|\lambda_2\|^{r_1}+\|\lambda_3\|^{r_1}+\|\lambda_4\|^{r_1}+\|\lambda_5\|^{r_1}+\|\lambda_6\|^{r_1}]}{\mathcal{T}}\Big), \cdots, \\
&\mathcal{E}_5\Big(-\frac{\Psi_5[\|\lambda_1\|^{r_5}+\|\lambda_2\|^{r_5}+\|\lambda_3\|^{r_5}+\|\lambda_4\|^{r_5}+\|\lambda_5\|^{r_5}+\|\lambda_6\|^{r_5}]}{\mathcal{T}}\Big)\Big]_{5\times 5}\Big)\Big]_{15\times 15},
\end{aligned} \tag{59}$$

and

$$\mathscr{N}\left(\chi(\ell_{\beta(1)},\ell_{\beta(2)},\ell_{\beta(3)})-\chi(\ell_1,\ell_2,\ell_3),\mathcal{T}\right) \tag{60}$$
$$\succeq Diagonal\Bigg[\mathtt{AG}_1\Bigg(Diagonal\Bigg[$$
$$\mathcal{E}_1\left(-\frac{\Psi_1[\|\lambda_1\|^{r_1}+\|\lambda_2\|^{r_1}+\|\lambda_3\|^{r_1}+\|\lambda_4\|^{r_1}+\|\lambda_5\|^{r_1}+\|\lambda_6\|^{r_1}]}{\mathcal{T}}\right),\cdots,$$
$$\mathcal{E}_5\left(-\frac{\Psi_5[\|\lambda_1\|^{r_5}+\|\lambda_2\|^{r_5}+\|\lambda_3\|^{r_5}+\|\lambda_4\|^{r_5}+\|\lambda_5\|^{r_5}+\|\lambda_6\|^{r_5}]}{\mathcal{T}}\right)\Bigg]_{5\times5}\Bigg),\cdots,$$
$$\mathtt{AG}_{15}\Bigg(Diagonal\Bigg[\mathcal{E}_1\left(-\frac{\Psi_1[\|\lambda_1\|^{r_1}+\|\lambda_2\|^{r_1}+\|\lambda_3\|^{r_1}+\|\lambda_4\|^{r_1}+\|\lambda_5\|^{r_1}+\|\lambda_6\|^{r_1}]}{\mathcal{T}}\right),\cdots,$$
$$\mathcal{E}_5\left(-\frac{\Psi_5[\|\lambda_1\|^{r_5}+\|\lambda_2\|^{r_5}+\|\lambda_3\|^{r_5}+\|\lambda_4\|^{r_5}+\|\lambda_5\|^{r_5}+\|\lambda_6\|^{r_5}]}{\mathcal{T}}\right)\Bigg]_{5\times5}\Bigg)\Bigg]_{15\times15}$$

for all permutations $(\beta(1),\beta(2),\beta(3))$ of $(1,2,3)$, and for all $\lambda_1,\lambda_2,\lambda_3,\ell_1,\ell_2,\ell_3,\lambda_5\in\nu$, then, the $\mathbb{C}-$trilinear mapping $\varpi:\nu^3\to\nu$ is a permuting tri-derivation.

Corollary 4. *Let $\mathcal{T}\geq 0$, $r_i<3$ and $\Psi_i>0$ be in $\mathbb{R}$, and $\chi:\nu^3\to\nu$ be a mapping satisfying (58) and $\chi(\lambda_1,0,\lambda_5)=\chi(0,\lambda_3,\lambda_5)=\chi(\lambda_1,\lambda_3,0)=0$, for all $\lambda_1,\lambda_3,\lambda_5\in\nu$. Then, we can find a unique $\mathbb{C}-$trilinear mapping $\varpi:\nu^3\to\nu$ satisfying*

$$\mathscr{N}\left(\chi(\lambda_1,\lambda_3,\lambda_5)-\varpi(\lambda_1,\lambda_3,\lambda_5),\mathcal{T}\right)\succeq Diagonal[\sigma'_1,\ldots,\sigma'_{15}], \tag{61}$$

where $Diagonal[\sigma'_1,\ldots,\sigma'_n]$ satisfies (57), for all $\lambda_1,\lambda_3,\lambda_5\in\nu$.

Also, if the mapping $\chi:\nu^3\to\nu$ satisfies (59), (60) and $\chi(2\lambda_1,\lambda_3,\lambda_5)=2\chi(\lambda_1,\lambda_3,\lambda_5)$ for all $\lambda_1,\lambda_3,\lambda_5\in\nu$, then the $\mathbb{C}-$trilinear mapping $\varpi:\nu^3\to\nu$ is a permuting tri-derivation.

Corollary 5. *Let $\mathcal{T}\geq 0$, $r_i>4$ and $\Psi_i>0$ be in $\mathbb{R}$, and $\chi:\nu^3\to\nu$ be a mapping satisfying (58) and $\chi(\lambda_1,0,\lambda_5)=\chi(0,\lambda_3,\lambda_5)=\chi(\lambda_1,\lambda_3,0)=0$ for all $\lambda_1,\lambda_3,\lambda_5\in\nu$. Then, we can find a unique $\mathbb{C}-$trilinear mapping $\varpi:\nu^3\to\nu$ satisfying (59).*

Also, if the mapping $\chi:\nu^3\to\nu$ satisfies (60), $\chi(2\phi,\lambda_3,\lambda_5)=2\chi(\phi,\lambda_3,\lambda_5)$ and

$$\mathscr{N}\left(\chi(\phi\lambda_2,\lambda_3,\lambda_5)-\chi(\phi,\lambda_3,\lambda_5)\lambda_2-\phi\chi(\lambda_2,\lambda_3,\lambda_5),\mathcal{T}\right) \tag{62}$$
$$\succeq Diagonal\Bigg[\mathtt{AG}_1\Bigg(Diagonal\Bigg[\mathcal{E}_1\left(-\frac{\Psi_1[1+\|\lambda_2\|^{r_1}+2\|\lambda_3\|^{r_1}+2\|\lambda_5\|^{r_1}]}{\mathcal{T}}\right),\cdots,$$
$$\mathcal{E}_5\left(-\frac{\Psi_5[1+\|\lambda_2\|^{r_5}+2\|\lambda_3\|^{r_5}+2\|\lambda_5\|^{r_5}]}{\mathcal{T}}\right)\Bigg]_{5\times5}\Bigg),\cdots,$$
$$\mathtt{AG}_{15}\Bigg(Diagonal\Bigg[\mathcal{E}_1\left(-\frac{\Psi_1[1+\|\lambda_2\|^{r_1}+2\|\lambda_3\|^{r_1}+2\|\lambda_5\|^{r_1}]}{\mathcal{T}}\right),\cdots,$$
$$\mathcal{E}_5\left(-\frac{\Psi_5[1+\|\lambda_2\|^{r_5}+2\|\lambda_3\|^{r_5}+2\|\lambda_5\|^{r_5}]}{\mathcal{T}}\right)\Bigg]_{5\times5}\Bigg)\Bigg]_{15\times15},$$

for any $\phi\in U(\nu)$ and all $\lambda_2,\lambda_3,\lambda_5\in\nu$, then the $\mathbb{C}-$trilinear mapping $\varpi:\nu^3\to\nu$ is a permuting tri-derivation.

Corollary 6. *Let $\mathcal{T}\geq 0$, $r_i<3$ and $\Psi_i>0$ be in $\mathbb{R}$, and $\chi:\nu^3\to\nu$ be a mapping satisfying (58) and $\chi(\lambda_1,0,\lambda_5)=\chi(0,\lambda_3,\lambda_5)=\chi(\lambda_1,\lambda_3,0)=0$ for all $\lambda_1,\lambda_3,\lambda_5\in\nu$. Then, we can obtain a unique $\mathbb{C}-$trilinear mapping $\varpi:\nu^3\to\nu$ satisfying (61).*

In addition, if the mapping $\chi : \nu^3 \to \nu$ satisfies (60), (62) and $\chi(2\phi, \lambda_3, \lambda_5) = 2\chi(\phi, \lambda_3, \lambda_5)$ for all $\phi \in U(\nu)$ and all $\lambda_3, \lambda_5 \in \nu$, then the $\mathbb{C}-$trilinear mapping $\varpi : \nu^3 \to \nu$ is a permuting tri-derivation.

Corollary 7. *Let $\mathcal{T} \geq 0$, $r_i > 6$ and $\Psi_i > 0$ be in $\mathbb{R}$, and $\chi : \nu^3 \to \Theta$ be a mapping satisfying (58) and $\chi(\lambda_1, 0, \lambda_5) = \chi(0, \lambda_3, \lambda_5) = \chi(\lambda_1, \lambda_3, 0) = 0$ for all $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \in \nu$. Then, we can find a unique $\mathbb{C}-$trilinear mapping $\rho : \nu^3 \to \Theta$ satisfying*

$$\mathcal{N}\Big(\chi(\lambda_1, \lambda_3, \lambda_5) - \rho(\lambda_1, \lambda_3, \lambda_4), \mathcal{T}\Big) \succeq Diagonal[\sigma_1, \ldots, \sigma_{15}] \tag{63}$$

in which $Diagonal[\sigma_1, \ldots, \sigma_{15}]$ satisfies (55), for all $\lambda_1, \lambda_3, \lambda_5 \in \nu$.

Also, if the mapping $\chi : \nu^3 \to \Theta$ satisfies (60) and

$$\begin{aligned} &\mathcal{N}\Big(\chi(\lambda_1\lambda_2, \lambda_3\lambda_4, \lambda_5\lambda_6) - \chi(\lambda_1, \lambda_3, \lambda_5)\chi(\lambda_2, \lambda_4, \lambda_6), \mathcal{T}\Big) \\ &\succeq Diagonal\Bigg[\mathtt{AG}_1\Bigg(Diagonal\Bigg[\\ &\mathcal{E}_1\Bigg(-\frac{\Psi_1[\|\lambda_1\|^{r_1} + \|\lambda_2\|^{r_1} + \|\lambda_3\|^{r_1} + \|\lambda_4\|^{r_1} + \|\lambda_5\|^{r_1} + \|\lambda_6\|^{r_1}]}{\mathcal{T}}\Bigg), \cdots, \\ &\mathcal{E}_5\Bigg(-\frac{\Psi_5[\|\lambda_1\|^{r_5} + \|\lambda_2\|^{r_5} + \|\lambda_3\|^{r_5} + \|\lambda_4\|^{r_5} + \|\lambda_5\|^{r_5} + \|\lambda_6\|^{r_5}]}{\mathcal{T}}\Bigg)\Bigg]_{5\times 5}\Bigg), \cdots, \\ &\mathtt{AG}_{15}\Bigg(Diagonal\Bigg[\mathcal{E}_1\Bigg(-\frac{\Psi_1[\|\lambda_1\|^{r_1} + \|\lambda_2\|^{r_1} + \|\lambda_3\|^{r_1} + \|\lambda_4\|^{r_1} + \|\lambda_5\|^{r_1} + \|\lambda_6\|^{r_1}]}{\mathcal{T}}\Bigg), \cdots, \\ &\mathcal{E}_5\Bigg(-\frac{\Psi_5[\|\lambda_1\|^{r_5} + \|\lambda_2\|^{r_5} + \|\lambda_3\|^{r_5} + \|\lambda_4\|^{r_5} + \|\lambda_5\|^{r_5} + \|\lambda_6\|^{r_5}]}{\mathcal{T}}\Bigg)\Bigg]_{5\times 5}\Bigg)\Bigg]_{15\times 15}, \end{aligned} \tag{64}$$

for all $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \in \nu$, then, the $\mathbb{C}-$trilinear mapping $\rho : \nu^3 \to \Theta$ is a permuting tri-homomorphism.

Corollary 8. *Let $\mathcal{T} \geq 0$, $r_i < 3$ and $\Psi_i > 0$ be in $\mathbb{R}$, and $\chi : \nu^3 \to \Theta$ be a mapping satisfying (58) and $\chi(\lambda_1, 0, \lambda_5) = \chi(0, \lambda_3, \lambda_5) = \chi(\lambda_1, \lambda_3, 0) = 0$ for all $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \in \nu$. Then, we can find a unique $\mathbb{C}-$trilinear mapping $\rho : \nu^3 \to \Theta$ satisfying*

$$\mathcal{N}\Big(\chi(\lambda_1, \lambda_3, \lambda_4) - \rho(\lambda_1, \lambda_3, \lambda_4), \mathcal{T}\Big) \succeq Diagonal[\sigma_1', \ldots, \sigma_1']_{15\times 15} \tag{65}$$

where $Diagonal[\sigma_1', \ldots, \sigma_1']_{15\times 15}$ satisfies (57), for all $\lambda_1, \lambda_3, \lambda_5 \in \nu$.

Besides, if the mapping $\chi : \nu^3 \to \Theta$ satisfies (60) and (64), then the $\mathbb{C}-$trilinear mapping $\rho : \nu^3 \to \Theta$ is a permuting tri-homomorphism.

Corollary 9. *Let $\mathcal{T} \geq 0$, $r_i > 6$ and $\Psi_i > 0$ be in $\mathbb{R}$, and $\chi : \nu^3 \to \Theta$ be a mapping satisfying (58) and $\chi(\lambda_1, 0, \lambda_5) = \chi(0, \lambda_3, \lambda_5) = \chi(\lambda_1, \lambda_3, 0) = 0$ for all $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \in \nu$. Then, we can find a unique $\mathbb{C}-$trilinear mapping $\rho : \nu^3 \to \Theta$ satisfying (63).*

Also, if the mapping $\chi : \nu^3 \to \Theta$ satisfies (60) and

$$\mathcal{N}\Big(\chi(\phi_1\varphi_1, \phi_2\varphi_2, \phi_3\varphi_3) - \chi(\phi_1, \phi_2, \phi_3)\chi(\varphi_1, \varphi_2, \varphi_3), \mathcal{T}\Big) \succeq Diagonal[6\mathcal{T}, \ldots, 6\mathcal{T}]_{15\times 15}, \tag{66}$$

for all $\phi_i, \varphi_i \in U(\nu)$, $i = 1, 2, 3$, then, the $\mathbb{C}-$trilinear mapping $\rho : \nu^3 \to \Theta$ is a permuting tri-homomorphism.

Corollary 10. *Let $\mathcal{T} \geq 0$, $r_i < 3$ and $\Psi_i > 0$ be in $\mathbb{R}$, and $\chi : \nu^3 \to \Theta$ be a mapping satisfying* (58) *and $\chi(\lambda_1, 0, \lambda_5) = \chi(0, \lambda_3, \lambda_5) = \chi(\lambda_1, \lambda_3, 0) = 0$ for all $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \in \nu$. Then, we can find a unique $\mathbb{C}-$trilinear mapping $\rho : \nu^3 \to \Theta$ satisfying* (65)*.*

Besides, if $\chi : \nu^3 \to \Theta$ satisfies (60) *and* (66)*, then, the $\mathbb{C}-$trilinear mapping $\rho : \nu^3 \to \Theta$ is a permuting tri-homomorphism.*

6. Conclusions

The main goal of the paper is to propose a new concept of Ulam-type stability, i.e., multi-stability, through the classical, well-known special functions and aggregation maps, and to gain the best approximation error estimates by a diverse concept of perturbation stability in fuzzy spaces. This stability allows us to get various approximations depending on the different special functions and aggregation maps that are initially chosen and to evaluate optimal stability and minimal error which enable us to obtain a unique optimum solution of functional equations. Stability analysis, in the sense of Ulam and others, has received considerable attention from researchers. However, the effective generalization of Ulam stability problems and evaluating optimized controllability and stability are new issues. The multi-stability covers not only the previous concepts, but also considers the optimization of the problem. Multi-stability provides a comprehensive discussion of optimizing the different types of Ulam stabilities of mathematical models used in the natural sciences (like: physics, earth science, biology, chemistry), social sciences (like: psychology, economics, political science, sociology) and engineering sciences (like: electrical engineering, computer science). This stability allows us to obtain the best approximation results of optimal control problems through classes of special functions.

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