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Fixed Point Results for the Fractional Nonlinear Problem with Integral Boundary Condition

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Abstract. In this article, we study an integral boundary condition problem. In fact, we consider a new nonlinear fractional integro-differential equation with arbitrary order and integral boundary condition. We show that there is a unique solution for this type of equation according to Babenko's strategy and the multivariate Mittag–Leffler function. All results, including the existence problem, are proved using Banach's contractive principle and Leray–Schauder's fixed point theorem. Several examples are also presented to demonstrate applications of the proven theorems.

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1. Introduction

Due to the many applications of fractional calculations in various sciences such as engineering, the attention of many researchers has been drawn to this field. For example, boundary value problems including integro-differential equations (IDEs) are suitable tools for modeling multiple events. Cabada and his colleague Hamdi [1], Young, and his coauthors [2] were among those who have investigated two models of IBCPs. These two FDEs with boundary conditions are defined as follows:

$$\begin{cases} {}_{RL}D^{\beta}_{0+}u(x) + f(x, u(x)) = 0, \quad x \in (0,1) \\ u(0) = u'(0) = 0, \quad u(1) = \rho \int_0^1 u(x)\mathrm{d}x, \end{cases} \tag{1.1}$$

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for the continuous function $f(x, u)$ and $2 < \beta \leq 3,\ 0 < \rho < \beta$ and a more general problem in a Banach space E, given by

$$\begin{cases} -({}_{RL}D_{0+}^{\beta})u(x) = f(x, u(x)) + \gamma g(x), & x \in (0, 1) \\ u(0) = u'(0) = \theta, \quad u(1) = \rho \int_0^1 u(x)\mathrm{d}x, \end{cases}$$

for the continuous functions $f(x, u)\colon [0, 1] \times K \to K,\ g\colon [0, 1] \to K$ and $2 < \beta \leq 3,\ 0 < \rho < \beta,\ \gamma \in R$. K and θ are the normal cone and a zero element in the Banach space E, respectively. $\gamma \in R$ is also a variable sign parameter. Cabada and Hamdi proved their results by considering Guo–Krasnoselskii's fixed point theorem (G-KFPT) for Eq. (1.1). The following nonlinear fractional differential equation (N-FDE) is another integral boundary condition problem (IBCP) that has been investigated by Wang et al., using the monotonic iterative method [3]:

$$\begin{cases} ({}_{RL}D_{0+}^{q})u(x) = f(x, u(x)), & x \in [0, T], \quad T > 0, \\ u(0) = \lambda \int_0^T u(x)\mathrm{d}x + d, \quad d \in R, \end{cases}$$

for the continuous function $f\colon [0, T] \times R \to R$ and $\lambda \geq 0$, $0 < q < 1$.

In this article, for $l,\ m \in N$ and $0 \leq \alpha_n < \cdots < \alpha_1 \leq l - 1 < \alpha \leq l$, $0 < \beta_1 < \cdots < \beta_m$, $0 \leq a < b < +\infty$, we consider the following nonlinear fractional integro-differential equation (N-FIDE), which is an IBCP:

$$\begin{cases} {}_{C}D_a^{\alpha}u(x) - \lambda_1\ {}_{C}D_a^{\alpha_1}u(x) - \cdots - \lambda_n\ {}_{C}D_a^{\alpha_n}u(x) - \lambda_{n+1}I_a^{\beta_1}u(x) \\ \qquad - \cdots - \lambda_{n+m}I_a^{\beta_m}u(x) \\ \quad = f(x, u(x)), \quad x \in [a, b] \\ u(a) = u'(a) = \cdots = u^{(l-2)}(a) = 0, \\ u(b) = \lambda_1 I_a^{\alpha-\alpha_1}u(b) + \cdots + \lambda_n I_a^{\alpha-\alpha_n}u(b) + \lambda_{n+1}I_a^{\alpha+\beta_1}u(b) \\ \qquad\qquad + \cdots + \lambda_{n+m}I_a^{\alpha+\beta_m}u(b), \end{cases} \tag{1.2}$$

where $f\colon [a, b] \times \mathbb{R} \to \mathbb{R}$ is a mapping satisfying certain conditions and λ_j is a constant for all $j = 1, 2, \ldots, n + m$. In particular for $l = 1$, Eq. (1.2) turns out to be

$$\begin{cases} {}_{C}D_a^{\alpha}u(x) - \lambda_n\ u(x) - \lambda_{n+1}I_a^{\beta_1}u(x) - \cdots - \lambda_{n+m}I_a^{\beta_m}u(x) \\ \quad = f(x, u(x)), \quad x \in [a, b] \\ u(b) = \lambda_n I_a^{\alpha}u(b) + \lambda_{n+1}I_a^{\alpha+\beta_1}u(b) + \cdots + \lambda_{n+m}I_a^{\alpha+\beta_m}u(b). \end{cases}$$

The remainder of this paper is structured as follows: In Sect. 2, we state the definitions and preliminary results. These definitions include a Banach space $C^{l-1}[a, b]$ which is a subspace of $C[a, b]$, the multivariate Mittag–Leffler function and Babenko's approach. Moreover, an IBCP using Babenko's strategy is also investigated. Then we derive sufficient conditions for the uniqueness and existence of Eq. (1.2) with the help of BCP and Leray–Schauder's fixed point theorem (LS-FPT) in Sect. 3, and further demonstrate applications of main results by several examples in Sect. 4. At the end, we summarize the entire work in Sect. 5.

2. Preliminaries

We begin with all the basic and required concepts. First, we define a Banach space used in our investigation. For $l \in N$, a subset of $C[a,b]$ is a Banach space $C^{l-1}[a,b]$ defined as follows

$$C^{l-1}[a,b] = \left\{ u(x) : [a,b] \to R \text{ such that } u^{(l-1)}(x) \text{ is continuous on } [a,b] \right\},$$

with the norm

$$\|u\| = \max_{x\in[a,b]} |u(x)| < +\infty.$$

Definition 1 [4]. The Riemann–Liouville fractional integral for function $u(x)$ of order $\alpha \in R^+$ is defined as

$$(I_a^\alpha u)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} u(t) \mathrm{d}t.$$

In particular,

$$(I_a^0 u)(x) = u(x).$$

Definition 2. The Liouville–Caputo fractional derivative for function $u(x)$ of order $\alpha \in R^+$ is defined as

$$({}_C D_a^\alpha u)(x) = I_a^{n-\alpha} \frac{d^n}{\mathrm{d}x^n} u(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} u^{(n)}(t) \mathrm{d}t,$$

for $n \in N = \{1, 2, 3, \ldots\}$ and $n-1 < \alpha \leq n$.

In the following, we will define the 2-parameter Mittag–Leffler function and the multivariate Mittag–Leffler function. In relation to linear fractional differential equations with constant coefficients, the multivariate Mittag–Leffler function was defined by Hadid and Luchko. See [5–9] for more details.

Definition 3 [8]. The two-parameter Mittag–Leffler function is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \; z \in C, \; \alpha, \beta > 0,$$

and the multivariate-Mittag–Leffler function is defined as follows

$$E_{(\alpha_1,\ldots,\alpha_m),\beta}(z_1,\ldots,z_m) = \sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_m=k} \binom{k}{k_1,\ldots,k_m} \frac{z_1^{k_1}\cdots z_m^{k_m}}{\Gamma(\alpha_1 k_1 + \cdots + \alpha_m k_m + \beta)},$$

where $\alpha_i, \beta > 0$ for $i = 1, 2, \ldots, m$ and

$$\binom{k}{k_1,\ldots,k_m} = \frac{k!}{k_1!\ldots k_m!}.$$

One of the important tools for solving differential and integral equations with initial conditions is Babenko's strategy. This method is generally the same as the Laplace transform while dealing with equations with constant coefficients. Also, this method is used for differential and integral equations with continuous and certain variable coefficients [10]. In the following, to show the applications of this approach in detail, we will investigate the solutions of two problems in the space $C^{l-1}[a,b]$. These equations are defined as follows:

$$\begin{cases} {}_C D_0^{\alpha} u(x) + a(x) I_0^{\beta} u(x) = g(x), \quad x \in [0,T], \quad l-1 < \alpha \le l, \ \beta \ge 0, \\ u(0) = u'(0) = \cdots = u^{(l-1)}(0) = 0, \end{cases} \tag{2.1}$$

where $a(x)$, $g(x) \in C[0,T]$, and

$$\begin{cases} {}_C D_a^{\alpha} u(x) - \lambda_1 \, {}_C D_a^{\alpha_1} u(x) - \cdots - \lambda_n \, {}_C D_a^{\alpha_n} u(x) - \lambda_{n+1} I_a^{\beta_1} u(x) \\ \qquad - \cdots - \lambda_{n+m} I_a^{\beta_m} u(x) \\ \quad = f(x), \quad x \in [a,b] \\ u(a) = u'(a) = \cdots = u^{(l-2)}(a) = 0, \\ u(b) = \lambda_1 I_a^{\alpha-\alpha_1} u(b) + \cdots + \lambda_n I_a^{\alpha-\alpha_n} u(b) + \lambda_{n+1} I_a^{\alpha+\beta_1} u(b) \\ \qquad\quad + \cdots + \lambda_{n+m} I_a^{\alpha+\beta_m} u(b). \end{cases} \tag{2.2}$$

where $f \in C[a,b]$.

Lemma 4. *Assume that* $a(x)$, $g(x) \in C[0,T]$. *Equation* (2.1) *has a unique solution.*

Proof. First, we can apply the operator I_0^{α} to both sides of equation

$${}_C D_0^{\alpha} u(x) + a(x) I_0^{\beta} u(x) = g(x),$$

then using the initial condition $u(0) = u'(0) = \cdots = u^{(l-1)}(0) = 0$, we have

$$u(x) + I_0^{\alpha} a(x) I_0^{\beta} u(x) = I_0^{\alpha} g(x).$$

Then,

$$(1 + I_0^{\alpha} a(x) I_0^{\beta}) u(x) = u(x) + I_0^{\alpha} a(x) I_0^{\beta} u(x) = I_0^{\alpha} g(x).$$

Due to the boundedness of the variable $I_0^{\alpha} a(x) I_0^{\beta}$ on $C[a,b]$ and treating $1 + I_0^{\alpha} a(x) I_0^{\beta}$ as a normal variable, we use Babenko's strategy. Considering $\left(I_0^{\alpha} a(x) I_0^{\beta}\right)^k I_0^{\alpha} = I_0^{\alpha} \left(a(x) I_0^{\alpha+\beta}\right)^k$, then we have

$$\begin{aligned} u(x) &= \left(1 + I_0^{\alpha} a(x) I_0^{\beta}\right)^{-1} I_0^{\alpha} g(x) = \sum_{k=0}^{\infty} (-1)^k \left(I_0^{\alpha} a(x) I_0^{\beta}\right)^k I_0^{\alpha} g(x) \\ &= \sum_{k=0}^{\infty} (-1)^k I_0^{\alpha} \left(a(x) I_0^{\alpha+\beta}\right)^k g(x). \end{aligned}$$

In the following, we show that in the space $C^{l-1}[a,b]$, the obtained series on the right-hand side of the above equation is convergent. Clearly

$$\|I_0^{\alpha} g(x)\| = \max_{x \in [0,T]} \frac{1}{\Gamma(\alpha)} \left| \int_0^x (x-t)^{\alpha-1} g(t) \mathrm{d}t \right| \le \frac{T^{\alpha}}{\Gamma(\alpha+1)} \|g\|,$$

and

$$\left\| I_0^{\alpha} a(x) I_0^{\beta} g(x) \right\| \le M \frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \|g\|,$$

where $M = \max_{x\in[0,T]} |a(x)|$. Furthermore,

$$\|u\| \le \sum_{k=0}^{\infty} \left\| \left(I_0^{\alpha} a(x) I_0^{\beta} \right)^k I_0^{\alpha} g(x) \right\| \le \|g\| \sum_{k=0}^{\infty} M^k \frac{T^{(\alpha+\beta)k+\alpha}}{\Gamma((\alpha+\beta)k+\alpha+1)}$$
$$= \|g\| \, T^{\alpha} E_{\alpha+\beta,\, \alpha+1}\left(M T^{\alpha+\beta} \right),$$

which infers that $u \in C[0,T]$. Moreover, from the identity

$$u(x) = \sum_{k=0}^{\infty} (-1)^k I_0^{\alpha} \left(a(x) I_0^{\alpha+\beta} \right)^k g(x) = I_0^{\alpha} \sum_{k=0}^{\infty} (-1)^k \left(a(x) I_0^{\alpha+\beta} \right)^k g(x),$$

we deduce that $u \in C^{l-1}[a,b]$ due to the factor I_0^{α}. □

Lemma 5. *Assume that* $0 \le \alpha_n < \cdots < \alpha_1 \le l-1 < \alpha \le l \in N$, $0 < \beta_1 < \cdots < \beta_m$, $0 \le a < b < +\infty$, λ_j *is a constant for all* $j = 1, 2, \ldots, n+m$ *and* $f \in C[a,b]$. *Then, Eq.* (2.2) *has a unique solution*

$$u(x) = \sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_{n+m}=k} \binom{k}{k_1, \ldots, k_{n+m}}$$
$$\lambda_1^{k_1} \cdots \lambda_{n+m}^{k_{n+m}} \cdot I_a^{\alpha+(\alpha-\alpha_1)k_1+\cdots+(\alpha+\beta_m)k_{n+m}} f(x)$$
$$- \frac{1}{(b-a)^{l-1}\Gamma(\alpha)} \int_a^b (b-x)^{\alpha-1} f(x) \mathrm{d}x \cdot \sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_{n+m}=k} \binom{k}{k_1, \ldots, k_{n+m}} \cdot$$
$$\lambda_1^{k_1} \cdots \lambda_{n+m}^{k_{n+m}} \cdot I_a^{(\alpha-\alpha_1)k_1+\cdots+(\alpha+\beta_m)k_{n+m}} (x-a)^{l-1}$$

in the space $C^{l-1}[a,b]$.

Proof. Like the previous lemma, we begin the proof by applying I_a^{α} to both sides of the following equation:

$$ {}_C D_a^{\alpha} u(x) - \lambda_1 \, {}_C D_a^{\alpha_1} u(x) - \cdots - \lambda_n \, {}_C D_a^{\alpha_n} u(x) - \lambda_{n+1} I_a^{\beta_1} u(x)$$
$$- \cdots - \lambda_{n+m} I_a^{\beta_m} u(x) = f(x),$$

then we have

$$u(x) - \lambda_1 \, I_a^{\alpha-\alpha_1} u(x) - \cdots - \lambda_n I_a^{\alpha-\alpha_n} u(x) - \lambda_{n+1} I_a^{\alpha+\beta_1} u(x)$$
$$- \cdots - \lambda_{n+m} I_a^{\alpha+\beta_m} u(x) = I_a^{\alpha} f(x) + c(x-a)^{l-1}, \tag{2.3}$$

by utilizing the condition

$$u(a) = u'(a) = \cdots = u^{(l-2)}(a) = 0, \tag{2.4}$$

and $0 \le \alpha_n < \cdots < \alpha_1 \le l-1 < \alpha \le l \in N$, where c is a constant to be determined by the boundary condition.

Setting $x = b$ in Eq. (2.3) and using the condition

$$u(b) = \lambda_1 I_a^{\alpha-\alpha_1} u(b) + \cdots + \lambda_n I_a^{\alpha-\alpha_n} u(b)$$
$$+ \lambda_{n+1} I_a^{\alpha+\beta_1} u(b) + \cdots + \lambda_{n+m} I_a^{\alpha+\beta_m} u(b),$$

we get

$$I_a^{\alpha} f(b) + c(b-a)^{l-1} = 0,$$

which implies that

$$c = -\frac{1}{(b-a)^{l-1}} I_a^{\alpha} f(b).$$

Hence, Eq. (2.3) turns out to be

$$\left(1 - \lambda_1\, I_a^{\alpha-\alpha_1} - \cdots - \lambda_n I_a^{\alpha-\alpha_n} - \lambda_{n+1} I_a^{\alpha+\beta_1} - \cdots - \lambda_{n+m} I_a^{\alpha+\beta_m}\right) u(x) \\ = I_a^{\alpha} f(x) - \frac{(x-a)^{l-1}}{(b-a)^{l-1}\Gamma(\alpha)} \int_a^b (b-x)^{\alpha-1} f(x)\mathrm{d}x.$$

By Babenko's approach, we come to

$$\begin{aligned} u(x) &= \left(1 - \lambda_1\, I_a^{\alpha-\alpha_1} - \cdots - \lambda_n I_a^{\alpha-\alpha_n} - \lambda_{n+1} I_a^{\alpha+\beta_1} - \cdots - \lambda_{n+m} I_a^{\alpha+\beta_m}\right)^{-1} \cdot \\ &\quad \left[I_a^{\alpha} f(x) - \frac{(x-a)^{l-1}}{(b-a)^{l-1}\Gamma(\alpha)} \int_a^b (b-x)^{\alpha-1} f(x)\mathrm{d}x \right] \\ &= \sum_{k=0}^{\infty} \left(\lambda_1\, I_a^{\alpha-\alpha_1} + \cdots + \lambda_n I_a^{\alpha-\alpha_n} + \lambda_{n+1} I_a^{\alpha+\beta_1} + \cdots + \lambda_{n+m} I_a^{\alpha+\beta_m}\right)^{k} \cdot \\ &\quad \left[I_a^{\alpha} f(x) - \frac{(x-a)^{l-1}}{(b-a)^{l-1}\Gamma(\alpha)} \int_a^b (b-x)^{\alpha-1} f(x)\mathrm{d}x \right]. \end{aligned}$$

It follows from the multinomial theorem that

$$\begin{aligned} &\left(\lambda_1\, I_a^{\alpha-\alpha_1} + \cdots + \lambda_n I_a^{\alpha-\alpha_n} + \lambda_{n+1} I_a^{\alpha+\beta_1} + \cdots + \lambda_{n+m} I_a^{\alpha+\beta_m}\right)^{k} \\ &\quad = \sum_{k_1+\cdots+k_{n+m}=k} \binom{k}{k_1, \ldots, k_{n+m}} \left(\lambda_1\, I_a^{\alpha-\alpha_1}\right)^{k_1} \cdots \left(\lambda_{n+m} I_a^{\alpha+\beta_m}\right)^{k_{n+m}} \\ &\quad = \sum_{k_1+\cdots+k_{n+m}=k} \binom{k}{k_1, \ldots, k_{n+m}} \lambda_1^{k_1} \cdots \lambda_{n+m}^{k_{n+m}}\; I_a^{(\alpha-\alpha_1)k_1+\cdots+(\alpha+\beta_m)k_{n+m}}. \end{aligned}$$

Thus,

$$\begin{aligned} u(x) &= \sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_{n+m}=k} \binom{k}{k_1, \ldots, k_{n+m}} \\ &\quad \lambda_1^{k_1} \cdots \lambda_{n+m}^{k_{n+m}} \cdot I_a^{\alpha+(\alpha-\alpha_1)k_1+\cdots+(\alpha+\beta_m)k_{n+m}} f(x) \\ &\qquad - \frac{1}{(b-a)^{l-1}\Gamma(\alpha)} \int_a^b (b-x)^{\alpha-1} f(x)\mathrm{d}x \cdot \sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_{n+m}=k} \binom{k}{k_1, \ldots, k_{n+m}} \cdot \\ &\qquad \lambda_1^{k_1} \cdots \lambda_{n+m}^{k_{n+m}} \cdot I_a^{(\alpha-\alpha_1)k_1+\cdots+(\alpha+\beta_m)k_{n+m}} (x-a)^{l-1}. \end{aligned}$$

Evidently, $u^{(l-1)}(x)$ is continuous due to the two factors I_a^{α} and $(x-a)^{l-1}$. Further, we consider

$$\begin{aligned}
\|u\| \leq \|f\|\,(b-a)^{\alpha} \sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_{n+m}=k} \binom{k}{k_1,\ldots,k_{n+m}} |\lambda_1|^{k_1} \cdots |\lambda_{n+m}|^{k_{n+m}} \cdot \\
\frac{(b-a)^{(\alpha-\alpha_1)k_1+\cdots+(\alpha+\beta_m)k_{n+m}}}{\Gamma((\alpha-\alpha_1)k_1+\cdots+(\alpha+\beta_m)k_{n+m}+\alpha+1)} + \\
+ \frac{\|f\|\,(b-a)^{\alpha}}{\Gamma(\alpha+1)} \sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_{n+m}=k} \binom{k}{k_1,\ldots,k_{n+m}} |\lambda_1|^{k_1} \cdots |\lambda_{n+m}|^{k_{n+m}} \cdot \\
\frac{(b-a)^{(\alpha-\alpha_1)k_1+\cdots+(\alpha+\beta_m)k_{n+m}}}{\Gamma((\alpha-\alpha_1)k_1+\cdots+(\alpha+\beta_m)k_{n+m}+1)} \\
= \|f\|\,(b-a)^{\alpha} \cdot \Big(E_{(\alpha-\alpha_1,\ldots,\alpha+\beta_m,\ \alpha+1)}\left(|\lambda_1|(b-a)^{\alpha-\alpha_1},\ldots,|\lambda_{n+m}|(b-a)^{\alpha+\beta_m}\right) \\
+ \frac{1}{\Gamma(\alpha+1)} E_{(\alpha-\alpha_1,\ldots,\alpha+\beta_m,\ 1)}\left(|\lambda_1|(b-a)^{\alpha-\alpha_1},\ldots,|\lambda_{n+m}|(b-a)^{\alpha+\beta_m}\right)\Big) < +\infty,
\end{aligned}$$

which claims that $u \in C^{l-1}[a,b]$. Given that the following system

$$\begin{cases}
{}_C D_a^{\alpha} u(x) - \lambda_1\, {}_C D_a^{\alpha_1} u(x) - \cdots - \lambda_n\, {}_C D_a^{\alpha_n} u(x) - \lambda_{n+1} I_a^{\beta_1} u(x) \\
\qquad - \cdots - \lambda_{n+m} I_a^{\beta_m} u(x) \\
\quad = 0, \quad x \in [a,b] \\
u(a) = u'(a) = \cdots = u^{(l-2)}(a) = 0, \\
u(b) = \lambda_1 I_a^{\alpha-\alpha_1} u(b) + \cdots + \lambda_n I_a^{\alpha-\alpha_n} u(b) + \lambda_{n+1} I_a^{\alpha+\beta_1} u(b) \\
\qquad + \cdots + \lambda_{n+m} I_a^{\alpha+\beta_m} u(b),
\end{cases}$$

has only zero solution in $C^{l-1}[a,b]$, therefore the uniqueness is proved. □

3. Existence and Uniqueness

Theorem 6. *We consider the following conditions for the continuous function $f\colon [a,b] \times R \to R$ and θ:*

- *f satisfies in the Lipschitz condition with constant $L \geq 0$, that is*

$$|f(x,y_1) - f(x,y_2)| \leq L|y_1 - y_2|,$$

  for $y_1, y_2 \in R$.
- *We suppose*

$$\begin{aligned}
\theta = L(b-a)^{\alpha} \Big[E_{(\alpha-\alpha_1,\ldots,\alpha+\beta_m,\ \alpha+1)}\left(|\lambda_1|(b-a)^{\alpha-\alpha_1},\ldots,|\lambda_{n+m}|(b-a)^{\alpha+\beta_m}\right) \\
+ \frac{1}{\Gamma(\alpha+1)} E_{(\alpha-\alpha_1,\cdots,\alpha+\beta_m,\ 1)}\left(|\lambda_1|(b-a)^{\alpha-\alpha_1},\ldots,|\lambda_{n+m}|(b-a)^{\alpha+\beta_m}\right)\Big] < 1.
\end{aligned}$$

Then, the boundary value problem (1.2) has a unique solution in the space $C^{l-1}[a,b]$.

Proof. Define a nonlinear mapping T over $C^{l-1}[a,b]$ by

$$\begin{aligned}(Tu)(x) &= \sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_{n+m}=k}\binom{k}{k_1,\ldots,k_{n+m}}\lambda_1^{k_1}\cdots\lambda_{n+m}^{k_{n+m}}\cdot I_a^{\alpha+(\alpha-\alpha_1)k_1+\cdots+(\alpha+\beta_m)k_{n+m}}f(x,u(x)) \\ &\quad - \frac{1}{(b-a)^{l-1}\Gamma(\alpha)}\int_a^b (b-x)^{\alpha-1}f(x,u(x))\mathrm{d}x\cdot\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_{n+m}=k}\binom{k}{k_1,\ldots,k_{n+m}}\cdot \\ &\quad \lambda_1^{k_1}\cdots\lambda_{n+m}^{k_{n+m}}\cdot I_a^{(\alpha-\alpha_1)k_1+\cdots+(\alpha+\beta_m)k_{n+m}}(x-a)^{l-1}.\end{aligned}$$

Clearly,

$$\begin{aligned}\max_{x\in[a,b]}|f(x,u(x))| &= \max_{x\in[a,b]}|f(x,u(x))-f(x,0)+f(x,0)| \\ &\le \max_{x\in[a,b]}L|u(x)| + \max_{x\in[a,b]}|f(x,0)| \\ &= L\,\|u\| + \max_{x\in[a,b]}|f(x,0)| < +\infty,\end{aligned}$$

using the Lipschitz condition and noting that $f(x,0)\in C[a,b]$.

It follows from the proof of Lemma 5 that $(Tu)(x)\in C^{l-1}[a,b]$. We need to prove that T is contractive. Indeed for $u,\ v\in C^{l-1}[a,b]$,

$$\begin{aligned}&|(Tu)(x)-(Tv)(x)| \\ &\le \sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_{n+m}=k}\binom{k}{k_1,\ldots,k_{n+m}}|\lambda_1|^{k_1}\cdots|\lambda_{n+m}|^{k_{n+m}}\cdot \\ &\quad \frac{1}{\Gamma((\alpha-\alpha_1)k_1+\cdots+(\alpha+\beta_m)k_{n+m}+\alpha)}\cdot \\ &\quad \left|\int_a^x (x-t)^{(\alpha-\alpha_1)k_1+\cdots+(\alpha+\beta_m)k_{n+m}+\alpha-1}(f(t,u(t))-f(t,v(t))\mathrm{d}t\right| \\ &\quad + \frac{1}{(b-a)^{l-1}\Gamma(\alpha)}\left|\int_a^b (b-x)^{\alpha-1}(f(x,u(x))-f(x,v(x)))\mathrm{d}x\right|\cdot \\ &\quad \sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_{n+m}=k}\binom{k}{k_1,\ldots,k_{n+m}}\cdot \\ &\quad |\lambda_1|^{k_1}\cdots|\lambda_{n+m}|^{k_{n+m}}\cdot I_a^{(\alpha-\alpha_1)k_1+\cdots+(\alpha+\beta_m)k_{n+m}}(x-a)^{l-1} \\ &\le L(b-a)^{\alpha}\,\|u-v\|\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_{n+m}=k}\binom{k}{k_1,\ldots,k_{n+m}}|\lambda_1|^{k_1}\cdots|\lambda_{n+m}|^{k_{n+m}}\cdot \\ &\quad \frac{(b-a)^{(\alpha-\alpha_1)k_1+\cdots+(\alpha+\beta_m)k_{n+m}}}{\Gamma((\alpha-\alpha_1)k_1+\cdots+(\alpha+\beta_m)k_{n+m}+\alpha+1)} \\ &\quad + L(b-a)^{\alpha}\,\|u-v\|\frac{1}{\Gamma(\alpha+1)}\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_{n+m}=k}\binom{k}{k_1,\ldots,k_{n+m}}|\lambda_1|^{k_1}\cdots|\lambda_{n+m}|^{k_{n+m}}\cdot \\ &\quad \frac{(b-a)^{\Gamma((\alpha-\alpha_1)k_1+\cdots+(\alpha+\beta_m)k_{n+m}}}{\Gamma((\alpha-\alpha_1)k_1+\cdots+(\alpha+\beta_m)k_{n+m}+1)} \\ &= L(b-a)^{\alpha}\,\|u-v\|\,E_{(\alpha-\alpha_1,\ldots,\alpha+\beta_m,\ \alpha+1)}\left(|\lambda_1|(b-a)^{\alpha-\alpha_1},\ldots,|\lambda_{n+m}(b-a)^{\alpha+\beta_m}\right) \\ &\quad + L(b-a)^{\alpha}\,\|u-v\|\frac{1}{\Gamma(\alpha+1)}\cdot \\ &\quad E_{(\alpha-\alpha_1,\ldots,\alpha+\beta_m,\ 1)}\left(|\lambda_1|(b-a)^{\alpha-\alpha_1},\ldots,|\lambda_{n+m}|(b-a)^{\alpha+\beta_m}\right) \\ &= \theta\,\|u-v\|\,.\end{aligned}$$

Since $\theta < 1$, Eq. (1.2) has a unique solution in the space $C^{l-1}[a,b]$ by BCP. □

Regarding the existence of solutions to Eq. (1.2), we have the following theorem.

Theorem 7. *We consider the following conditions*

- *For $M_1 > 0$, $M_2 > 0$ and $y \in R$, the continuous function $f\colon [a,b] \times R \to R$ satisfying*

$$|f(x,y)| \leq M_1 + M_2|y|,$$

- *suppose that*

$$\begin{aligned}\Theta = {} & M_2(b-a)^{\alpha}\Big(E_{(\alpha-\alpha_1,\ldots,\alpha+\beta_m,\ \alpha+1)}\left(|\lambda_1|(b-a)^{\alpha-\alpha_1},\ldots,|\lambda_{n+m}|(b-a)^{\alpha+\beta_m}\right)\\ & +\frac{1}{\Gamma(\alpha+1)}E_{(\alpha-\alpha_1,\ldots,\alpha+\beta_m,\ 1)}\left(|\lambda_1|(b-a)^{\alpha-\alpha_1},\ldots,|\lambda_{n+m}|(b-a)^{\alpha+\beta_m}\right)\Big) < 1.\end{aligned}$$

Then, Eq. (1.2) has at least one solution in the space $C^{l-1}[a,b]$.

Proof. We again consider the nonlinear mapping T over $C^{l-1}[a,b]$ given by

$$\begin{aligned}&(Tu)(x)\\ &= \sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_{n+m}=k}\binom{k}{k_1,\ldots,k_{n+m}}\lambda_1^{k_1}\cdots\lambda_{n+m}^{k_{n+m}}\cdot I_a^{\alpha+(\alpha-\alpha_1)k_1+\cdots+(\alpha+\beta_m)k_{n+m}}f(x,u(x))\\ &\quad -\frac{1}{(b-a)^{l-1}\Gamma(\alpha)}\int_a^b (b-x)^{\alpha-1}f(x,u(x))\mathrm{d}x\cdot\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_{n+m}=k}\binom{k}{k_1,\ldots,k_{n+m}}\cdot\\ &\quad \lambda_1^{k_1}\cdots\lambda_{n+m}^{k_{n+m}}\cdot I_a^{(\alpha-\alpha_1)k_1+\cdots+(\alpha+\beta_m)k_{n+m}}(x-a)^{l-1}.\end{aligned}$$

Obviously,

$$\begin{aligned}&\|Tu\| \leq (M_1 + M_2\|u\|)(b-a)^{\alpha}\cdot\\ &\quad (E_{(\alpha-\alpha_1,\ldots,\alpha+\beta_m,\ \alpha+1)}\left(|\lambda_1|(b-a)^{\alpha-\alpha_1},\ldots,|\lambda_{n+m}|(b-a)^{\alpha+\beta_m}\right)\\ &\quad +\frac{1}{\Gamma(\alpha+1)}E_{(\alpha-\alpha_1,\ldots,\alpha+\beta_m,\ 1)}\left(|\lambda_1|(b-a)^{\alpha-\alpha_1},\ldots,|\lambda_{n+m}|(b-a)^{\alpha+\beta_m}\right)) < +\infty,\end{aligned}$$

which claims that the mapping T is from the space $C^{l-1}[a,b]$ to itself. We are going to prove (i) T is continuous. Indeed,

$$\begin{aligned}&|(Tu)(x)-(Tv)(x)| \leq \max_{x\in[a,b]}|f(x,u(x))-f(x,v(x))|(b-a)^{\alpha}\cdot\\ &\quad (E_{(\alpha-\alpha_1,\ldots,\alpha+\beta_m,\ \alpha+1)}\left(|\lambda_1|(b-a)^{\alpha-\alpha_1},\ldots,|\lambda_{n+m}|(b-a)^{\alpha+\beta_m}\right)\\ &\quad +\frac{1}{\Gamma(\alpha+1)}E_{(\alpha-\alpha_1,\ldots,\alpha+\beta_m,\ 1)}\left(|\lambda_1|(b-a)^{\alpha-\alpha_1},\ldots,|\lambda_{n+m}|(b-a)^{\alpha+\beta_m}\right)) < +\infty,\end{aligned}$$

which claims T is continuous, according to the continuity of f.

(ii) In addition, we need to prove that T defined over $C^{l-1}[a,b]$ is a mapping from bounded sets to bounded sets. In fact, let W be a bounded set in $C^{l-1}[a,b]$, then there exists a constant $Z \in R^+$ such that

$$|f(x,u(x))| \leq M_1 + M_2|u(x)| \leq Z,$$

for all $u \in W$. This implies that

$$\|Tu\| \le Z(b-a)^{\alpha} \cdot \\ \left(E_{(\alpha-\alpha_1,\ldots,\alpha+\beta_m,\ \alpha+1)}\left(|\lambda_1|(b-a)^{\alpha-\alpha_1},\ldots,|\lambda_{n+m}|(b-a)^{\alpha+\beta_m}\right)\right. \\ \left. + \frac{1}{\Gamma(\alpha+1)} E_{(\alpha-\alpha_1,\ldots,\alpha+\beta_m,\ 1)}\left(|\lambda_1|(b-a)^{\alpha-\alpha_1},\ldots,|\lambda_{n+m}|(b-a)^{\alpha+\beta_m}\right)\right) < +\infty.$$

Therefore, the set

$$\{Tu\colon\ u \in W\} = TW$$

is uniformly bounded in $C^{l-1}[a,b]$.

(iii) T is completely continuous from $C^{l-1}[a,b]$ to itself, which is a subspace of $C[a,b]$. Now, we show that T is equicontinuous on every bounded set such as W in $C^{l-1}[a,b]$. For this purpose, we consider the Arzela–Ascoli theorem and assume that $a \le \tau_1 < \tau_2 \le b$ and $u \in W$. Then, we have

$$|(Tu)(\tau_2) - (Tu)(\tau_1)| \\ \le \sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_{n+m}=k} \binom{k}{k_1,\ldots,k_{n+m}} |\lambda_1|^{k_1} \cdots |\lambda_{n+m}|^{k_{n+m}} \cdot \\ \frac{1}{\Gamma(\gamma)} \left| \int_a^{\tau_2} (\tau_2-t)^{\gamma-1} f(t,u(t))\mathrm{d}t - \int_a^{\tau_1} (\tau_1-t)^{\gamma-1} f(t,u(t))\mathrm{d}t \right| \\ +\colon \frac{Z(b-a)^{\alpha}}{(b-a)^{l-1}\Gamma(\alpha+1)} \sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_{n+m}=k} \binom{k}{k_1,\ldots,k_{n+m}} |\lambda_1|^{k_1} \cdots |\lambda_{n+m}|^{k_{n+m}} \cdot \\ \frac{1}{\Gamma(\gamma-\alpha)} \left| \int_a^{\tau_2} (\tau_2-t)^{\gamma-\alpha-1}(t-a)^{l-1}\mathrm{d}t - \int_a^{\tau_1} (\tau_1-t)^{\gamma-\alpha-1}(t-a)^{l-1}\mathrm{d}t \right| \\ = I_1 + I_2,$$

where

$$\gamma = \alpha + (\alpha-\alpha_1)k_1 + \cdots + (\alpha+\beta_m)k_{n+m} \ge \alpha.$$

Regarding I_1, we have

$$\int_a^{\tau_2} (\tau_2-t)^{\gamma-1} f(t,u(t))\mathrm{d}t \\ = \int_a^{\tau_1} (\tau_2-t)^{\gamma-1} f(t,u(t))\mathrm{d}t + \int_{\tau_1}^{\tau_2} (\tau_2-t)^{\gamma-1} f(t,u(t))\mathrm{d}t.$$

Therefore,

$$\int_a^{\tau_2} (\tau_2-t)^{\gamma-1} f(t,u(t))\mathrm{d}t - \int_a^{\tau_1} (\tau_1-t)^{\gamma-1} f(t,u(t))\mathrm{d}t \\ = \int_a^{\tau_1} [(\tau_2-t)^{\gamma-1} - (\tau_1-t)^{\gamma-1}] f(t,u(t))\mathrm{d}t + \int_{\tau_1}^{\tau_2} (\tau_2-t)^{\gamma-1} f(t,u(t))\mathrm{d}t,$$

and by the mean value theorem,

$$\left| \int_a^{\tau_1} [(\tau_2-t)^{\gamma-1} - (\tau_1-t)^{\gamma-1}] f(t,u(t))\mathrm{d}t \right| \\ \le Z \left[\frac{(\tau_2-a)^{\gamma}}{\gamma} - \frac{(\tau_1-a)^{\gamma}}{\gamma} \right] \le Z(\tau_2-\tau_1)(b-a)^{\gamma-1}.$$

Furthermore,

$$\left|\int_{\tau_1}^{\tau_2}(\tau_2-t)^{\gamma-1}f(t,u(t))\mathrm{d}t\right|\leq Z\frac{(\tau_2-\tau_1)^{\gamma}}{\gamma}.$$

In summary,

$$\begin{aligned}
I_1 &\leq Z(\tau_2-\tau_1)(b-a)^{\alpha-1}\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_{n+m}=k}\binom{k}{k_1,\ldots,k_{n+m}}|\lambda_1|^{k_1}\cdots|\lambda_{n+m}|^{k_{n+m}}\cdot\\
&\quad\frac{(b-a)^{(\alpha-\alpha_1)k_1+\cdots+(\alpha+\beta_m)k_{n+m}}}{\Gamma(\alpha+(\alpha-\alpha_1)k_1+\cdots+(\alpha+\beta_m)k_{n+m})}\\
&\quad+Z(\tau_2-\tau_1)^{\alpha}\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_{n+m}=k}\binom{k}{k_1,\ldots,k_{n+m}}|\lambda_1|^{k_1}\cdots|\lambda_{n+m}|^{k_{n+m}}\cdot\\
&\quad\frac{(b-a)^{(\alpha-\alpha_1)k_1+\cdots+(\alpha+\beta_m)k_{n+m}}}{\Gamma(\alpha+1+(\alpha-\alpha_1)k_1+\cdots+(\alpha+\beta_m)k_{n+m})}\\
&=Z(\tau_2-\tau_1)(b-a)^{\alpha-1}E_{(\alpha-\alpha_1,\ldots,\alpha+\beta_m,\ \alpha)}\\
&\quad\left(|\lambda_1|(b-a)^{\alpha-\alpha_1},\ldots,|\lambda_{n+m}|(b-a)^{\alpha+\beta_m}\right)\\
&\quad+Z(\tau_2-\tau_1)^{\alpha}E_{(\alpha-\alpha_1,\ldots,\alpha+\beta_m,\ \alpha+1)}\\
&\quad\left(|\lambda_1|(b-a)^{\alpha-\alpha_1},\cdots,|\lambda_{n+m}|(b-a)^{\alpha+\beta_m}\right).
\end{aligned}$$

As for I_2,

$$\begin{aligned}
&\frac{1}{\Gamma(\gamma-\alpha)}\left|\int_a^{\tau_2}(\tau_2-t)^{\gamma-\alpha-1}(t-a)^{l-1}\mathrm{d}t-\int_a^{\tau_1}(\tau_1-t)^{\gamma-\alpha-1}(t-a)^{l-1}\mathrm{d}t\right|\\
&\leq\frac{(b-a)^{l-1}}{\Gamma(\gamma-\alpha)}\left[\left|\int_a^{\tau_1}((\tau_2-t)^{\gamma-\alpha-1}-(\tau_1-t)^{\gamma-\alpha-1})\mathrm{d}t\right|+\int_{\tau_1}^{\tau_2}(\tau_2-t)^{\gamma-\alpha-1}\mathrm{d}t\right]\\
&\leq\frac{(b-a)^{l-1}}{\Gamma(\gamma+1-\alpha)}\left[(\tau_2-a)^{\gamma-\alpha}-(\tau_1-a)^{\gamma-\alpha}\right]\\
&\quad+\frac{(b-a)^{l-1}}{\Gamma(\gamma+1-\alpha)}(\tau_2-\tau_1)^{\gamma-\alpha},
\end{aligned}$$

for $\gamma>\alpha$.

Clearly,

$$\begin{aligned}
I_2 &\leq\frac{Z(b-a)^{\alpha}}{(b-a)^{l-1}\Gamma(\alpha+1)}\left[(\tau_2-a)^{l-1}-(\tau_1-a)^{l-1}\right]\\
&\quad+\frac{Z(b-a)^{\alpha}}{\Gamma(\alpha+1)}\sum_{k=1}^{\infty}\sum_{k_1+\cdots+k_{n+m}=k}\binom{k}{k_1,\ldots,k_{n+m}}|\lambda_1|^{k_1}\cdots|\lambda_{n+m}|^{k_{n+m}}\cdot\\
&\quad\frac{1}{\Gamma(\gamma+1-\alpha)}\left[(\tau_2-a)^{\gamma-\alpha}-(\tau_1-a)^{\gamma-\alpha}\right]\\
&\quad+\frac{Z(b-a)^{\alpha}}{\Gamma(\alpha+1)}\sum_{k=1}^{\infty}\sum_{k_1+\cdots+k_{n+m}=k}\binom{k}{k_1,\ldots,k_{n+m}}|\lambda_1|^{k_1}\cdots|\lambda_{n+m}|^{k_{n+m}}\cdot\\
&\quad\frac{1}{\Gamma(\gamma+1-\alpha)}(\tau_2-\tau_1)^{\gamma-\alpha}.
\end{aligned}$$

Obviously, the second term above contains the factor $\tau_2 - \tau_1$ from the mean value theorem and third includes the factor $(\tau_2 - \tau_1)^{\alpha-\alpha_1}$, by noting that the index k starts from 1 rather than zero. Thus, T is completely continuous.

(iv) As a final step, we shall show that for some $0 < \lambda < 1$ the set

$$X = \{u \in C^{l-1}[a,b]\colon\ u = \lambda T u\}$$

is bounded.

Indeed,

$$\begin{aligned}\|u\| \le \|Tu\| &\le (M_1 + M_2\,\|u\|)(b-a)^{\alpha}\cdot \\ &\quad (E_{(\alpha-\alpha_1,\ldots,\alpha+\beta_m,\ \alpha+1)}\left(|\lambda_1|(b-a)^{\alpha-\alpha_1},\ldots,|\lambda_{n+m}|(b-a)^{\alpha+\beta_m}\right) \\ &\quad +\frac{1}{\Gamma(\alpha+1)}E_{(\alpha-\alpha_1,\ldots,\alpha+\beta_m,\ 1)} \\ &\quad \left(|\lambda_1|(b-a)^{\alpha-\alpha_1},\ldots,|\lambda_{n+m}|(b-a)^{\alpha+\beta_m}\right)) < +\infty,\end{aligned}$$

which deduces that

$$\begin{aligned}(1-\Theta)\,\|u\| &\le M_1(E_{(\alpha-\alpha_1,\ldots,\alpha+\beta_m,\ \alpha+1)} \\ &\quad \left(|\lambda_1|(b-a)^{\alpha-\alpha_1},\ldots,|\lambda_{n+m}|(b-a)^{\alpha+\beta_m}\right) \\ &\quad + \frac{1}{\Gamma(\alpha+1)}E_{(\alpha-\alpha_1,\ldots,\alpha+\beta_m,\ 1)}\left(|\lambda_1|(b-a)^{\alpha-\alpha_1},\ldots,|\lambda_{n+m}|(b-a)^{\alpha+\beta_m}\right)).\end{aligned}$$

Hence,

$$\begin{aligned}\|u\| &\le \frac{M_1}{1-\Theta}(E_{(\alpha-\alpha_1,\cdots,\alpha+\beta_m,\ \alpha+1)}\left(|\lambda_1|(b-a)^{\alpha-\alpha_1},\ldots,|\lambda_{n+m}|(b-a)^{\alpha+\beta_m}\right) \\ &\quad + \frac{1}{\Gamma(\alpha+1)}E_{(\alpha-\alpha_1,\ldots,\alpha+\beta_m,\ 1)}\left(|\lambda_1|(b-a)^{\alpha-\alpha_1},\ldots,|\lambda_{n+m}|(b-a)^{\alpha+\beta_m}\right)).\end{aligned}$$

This completes the proof of Theorem 7. □

Theorem 8. *Assume $f\colon [a,b]\times R \to R$ is a continuous and bounded function. Then, Eq. (1.2) has at least one solution in the space $C^{l-1}[a,b]$.*

Proof. It follows immediately from the proof of Theorem 7 by setting $M_2 = 0$. □

Remark 9. It would be interesting and challenging to consider the following integral boundary value problem with variable coefficients:

$$\begin{cases} {}_C D_a^{\alpha} u(x) - \lambda_1(x)\ {}_C D_a^{\alpha_1} u(x) - \cdots - \lambda_n(x)\ {}_C D_a^{\alpha_n} u(x) \\ -\lambda_{n+1}(x) I_a^{\beta_1} u(x) - \cdots - \lambda_{n+m}(x) I_a^{\beta_m} u(x) = f(x,u(x)), \quad x \in [a,b] \\ u(a) = u'(a) = \cdots = u^{(l-2)}(a) = 0, \\ u(b) = I_a^{\alpha}\lambda_1(b)\ {}_C D_a^{\alpha_1} u(b) + \cdots + I_a^{\alpha}\lambda_n(b)\ {}_C D_a^{\alpha_n} u(b) \\ +I_a^{\alpha}\lambda_{n+1}(b) I_a^{\beta_1} u(b) + \cdots + I_a^{\alpha}\lambda_{n+m}(b) I_a^{\beta_m} u(b), \end{cases}$$

where $\lambda_j(x) \in C[a,b]$ for all $j = 1, 2, \ldots, n+m$ and $l-1 < \alpha \le l$.

4. Examples

Example 10. We consider the following IBCP

$$\begin{cases} {}_CD_0^{1.5}u(x) - 2\,{}_CD_0^{0.5}u(x) + I_0^{0.1}u(x) = \dfrac{1}{50}\sin(xu(x)) + x^3, \quad x \in [0,1] \\ u(0) = 0, \\ u(1) = 2I_0u(1) - I_0^{1.6}u(1) = 2\int_0^1 u(x)\mathrm{d}x - \frac{1}{\Gamma(1.6)}\int_0^1 (1-x)^{0.6}u(x)\mathrm{d}x. \end{cases} \tag{4.1}$$

According to Theorem 6, Eq. (4.1) has a unique solution in the space $C^1[0,1]$.

Proof. Clearly,

$$f(x,u(x)) = \frac{1}{50}\sin(xu(x)) + x^3,$$

and

$$|f(x,u(x)) - f(x,v(x))| \le \frac{1}{50}|xu(x) - xv(x)| \le \frac{1}{50}|u(x) - v(x)|,$$

which implies that $L = \dfrac{1}{50}$ using the fact $0 \le x \le 1$. In addition, from Theorem 6

$$\theta = \frac{1}{50}\left[E_{(1,1.6,\,2.5)}(2,1) + \frac{1}{\Gamma(2.5)}E_{(1,1.6,\,1)}(2,1)\right].$$

By the definition,

$$E_{(1,1.6,\,2.5)}(2,1) = \sum_{k=0}^{\infty}\sum_{k_1+k_2=k}\binom{k}{k_1,k_2}\frac{2^{k_1}}{\Gamma(k_1+1.6k_2+2.5)}.$$

Evidently,

$$\sum_{k_1+k_2=k}\binom{k}{k_1,k_2} = 2^k,$$

$$\frac{2^{k_1}}{\Gamma(k_1+1.6k_2+2.5)} \le \frac{2^k}{\Gamma(k+2.5)},$$

then, according to the calculations, we have

$$E_{(1,1.6,\,2.5)}(2,1) \le \sum_{k=0}^{\infty}\frac{4^k}{\Gamma(k+2.5)} \approx 6.51075.$$

On the other hand,

$$\frac{1}{\Gamma(2.5)}E_{(1,1.6,\,1)}(2,1) \le 0.752253 * \sum_{k=0}^{\infty}\frac{4^k}{\Gamma(k+1)} \approx 41.0716.$$

In summary,

$$\theta = \frac{1}{50}[6.51075 + 41.0716] < 1.$$

By Theorem 6, the integral boundary value problem has a unique solution in the space $C^1[0,1]$. □

Example 11. Consider the following FDE which is an IBCP.

$$\begin{cases} {}_C D_0^{2.3} u(x) - 4\, {}_C D_0^{1.3} u(x) = \cos^2(x + u(x)) + x^2 + \arctan u^2(x), \\ u(0) = u'(0) = 0, \\ u(1) = 4 I_0 u(1) = 4 \int_0^1 u(x) \mathrm{d}x. \end{cases} \tag{4.2}$$

According to Theorem 8, Eq. (4.2) has at least one solution in the space $C^2[0, 1]$.

Proof. The function

$$f(x, u(x)) = \cos^2(x + u(x)) + x^2 + \arctan u^2(x),$$

is clearly bounded over $[0, 1]$. From Theorem 8, the above equation with the IBCP has at least one solution in the space $C^2[0, 1]$. □

5. Conclusion

In this work, we have first provided several definitions and basic concepts needed to prove the main results of this article. These include the Banach space, Riemann–Liouville fractional integral and Liouville–Caputo fractional derivative, two-parameter Mittag–Leffler function and multivariate Mittag–Leffler function, Babenko's strategy. Then, applying BCP and LS-FPT, we investigated IBCP (1.2) and proved our main results. In the end, by adding some numerical examples, we have shown the applications of the obtained results.

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