International Journal of Modern Physics B
Vol. 38, No. 3 (2024) 2450043 (15 pages)

DOI: 10.1142/S0217979224500437

World Scientific
www.worldscientific.com

The multiple exp-function method to obtain soliton solutions of the conformable Date–Jimbo–Kashiwara–Miwa equations

Zahra Eidinejad* and Reza Saadati†
School of Mathematics, Iran University of Science and Technology, Narmak, Tehran 13114-16846, Iran
**zahra_eidinejad@mathdep.iust.ac.ir*
†rsaadati@iust.ac.ir

Chenkuan Li
Department of Mathematics and Computer Science, Brandon University, Brandon, Manitoba Canada R7A 6A9
lic@brandonu.ca

Mustafa Inc‡
Department of Mathematics, Science Faculty, Firat University, 23119 Elazig, Turkiye
Department of Medical Research, China Medical University, 40402 Taichung, Taiwan
minc@firat.edu.tr

Javad Vahidi
School of Mathematics, Iran University of Science and Technology, Narmak, Tehran 13114-16846, Iran
jvahidi@iust.ac.ir

Received 27 September 2022
Revised 30 November 2022
Accepted 16 December 2022
Published 16 March 2023

Considering the importance of using nonlinear evolution equations in the investigation of many natural phenomena, in this paper, we consider the $(2+1)$-dimensional Date–Jimbo–Kashiwara–Miwa ($(2+1)$-dimensional DJKM) equation, we will investigate the solutions for this equation. Using the multiple exp functions method, we obtain analytical solutions for this equation, which are one-soliton, two-soliton and three-soliton solutions and these solutions include three categories of soliton wave solutions, i.e., one-wave solutions, two-wave solutions and three wave solutions. We have performed all calculations with a computer algebra system such as Maple and have also provided a graphical representation of the obtained solutions.

‡Corresponding author.

Keywords: ME-function method; soliton wave solutions; conformable Date–Jimbo–Kashiwara–Miwa equations; $(2+1)$-dimensional D-J-K-M.

PACS numbers: 05.45.Yv, 02.30.Jr.

1. Introduction

In nonlinear sciences, nonlinear partial equations are widely used in fields such as nonlinear optics, quantum mechanics, molecular biology, surface water waves and other cases. On the other hand, many physical phenomena that exist in nature are described by exact solutions of nonlinear partial differential equations (PDEs). Nonlinear wave solutions play a fundamental role in various scientific fields such as geochemistry, optical fibers, chemical physics, fluid mechanics, plasma physics, solid-state physics, chemical kinematics, etc. Therefore, for nonlinear models in nature and industry, analytical and semi-analytical solutions have been investigated with many methods including the tanh–coth method, the sine-cosine method, the tanh-function method, the homogeneous balance method, inverse scattering transformation, Darboux transformation, Hirota bilinear method and so on.[1–8] For example, the generalized bilinear equation of Kadomtsev–Petviashvili–Benjamin–Bona–Mahony was investigated by Hirota's efficient bilinear method and exact solutions were obtained from this equation. By converting nonlinear equations to ordinary differential equations (ODEs) with variable coefficients, we can use these techniques to obtain logical solutions. Among the methods for accurate solutions of nonlinear equations, there is now a single method that determines the relationship between ordinary solvable differential equations and nonlinear PDEs. Nowadays, various solutions such as periodic waves, single waves and so on, have been proposed for scattered and nonlinear scattered equations. The authors also found in their studies that by using the substitution of high-order polynomial functions in bilinear equations, the exact solutions of some nonlinear equations that are of the type of rogue-type multiple lumps waves can be constructed.[9–13] There are various solutions to nonlinear equations, and these solutions are not limited to traveling wave solutions. Then, there are direct methods to obtain these solutions for nonlinear equations. For example, for important equations such as the Korteweg–de Vries (KdV) equations, the Toda lattice equations and the Hirota bilinear equations, there are multi-soliton multiple wave solutions and multiple periodic wave solutions, respectively.[14–17] Lump-type solutions and interaction phenomenon as well as lump wave solutions have been obtained for the (2+1)-dimensional breaking soliton equation, variable-coefficient Kadomtsev–Petviashvili equation, and Kadomtsev–Petviashvili–Benjamin–Bona–Mahony equation, respectively. Describing the propagation of nonlinear waves with nonuniform velocities is another task provided by the nonlinear evolution equations and their solutions. Many characteristics of natural phenomena with continuous and variable fields are shown by converting a nonlinear evolution equation into a nonlinear evolution equation of a integrated. One of the important long water wave equations describing the propagation of nonlinear and weakly scattered waves in heterogeneous media is Date–Jimbo–Kashiwara–Miwa equation.[18] In this paper, we propose a method to obtain the exact soliton wave solutions of the desired nonlinear equation. We investigate the conformable Date–Jimbo–Kashiwara– Miwa (CDJKM) equation as follows[19]:

$$\Phi_{uuuuv} + 4\Phi_{uuv}\Phi_u + 2\Phi_{uuu}\Phi_v + 6\Phi_{uv}\Phi_{uu} - \lambda\Phi_{vvv} - 2\mu\frac{\partial}{\partial u}\left(\frac{\partial}{\partial u}\left(\frac{\partial^\zeta\Phi}{\partial s^\zeta}\right)\right) = 0, \quad (1)$$

where $\Phi = \Phi(u, v, s)$ and μ is a constant and $0 < \zeta \leq 1$. For $0 < \zeta < 1$, the desired derivative is the conformable fractional derivative which is defined as follows for the function $\mathfrak{u} : [a, \infty) \to \mathbb{R}$ and starting from a of order ζ:

$$\mathfrak{D}_\zeta^a\mathfrak{u}(\mathfrak{d}) = \lim_{\imath\to 0}\frac{\mathfrak{u}(\mathfrak{d} + \imath(\mathfrak{d} - a)^{1-\zeta}) - \mathfrak{u}(\mathfrak{d})}{\imath}, \quad \mathfrak{d} > a, \quad 0 < \zeta < 1.$$

If on (a, b), $\mathfrak{D}_\zeta^a\mathfrak{u}(\mathfrak{d})$ exists, then $\mathfrak{D}_\zeta^a\mathfrak{u}(a) = \lim_{\mathfrak{d}\to a^+}\mathfrak{D}_\zeta^a\mathfrak{u}(\mathfrak{d})$. The CDJKM equation turns out to be $(2 + 1)$-dimensional Date–Jimbo–Kashiwara–Miwa ($(2 + 1)$-dimensional DJKM) equation when the fractional derivative $\zeta = 1$. For the first time, Guo and Lin introduced the CDJKM equation and Kumar *et al.* successfully investigated the solutions for this equation via the sine-Gordon expansion and the improved Bernoulli sub equation methods.[20,21] Many researchers have investigated the $(2 + 1)$-dimensional Date–Jimbo–Kashiwara–Miwa ($(2 + 1)$-dimensional DJKM) equation,[22–24] but no one used our desired method to solve the CDJKM equation. The multiple exp function (MEF) method is a direct and systematic solution method for nonlinear PDEs in mathematical physics that generalizes Hirota's perturbation scheme and obtains exact multiple wavelet solutions. This method is a suitable method for obtaining construct solitary and soliton solutions of nonlinear evolution equations because it has a wider application for use in nonlinear wave equations. This method was first presented by Hi and Wu and it can be used in all kinds of nonlinear evolution equations. Considering the optimality of the MEF method in dealing with nonlinear equations, we choose this method to obtain the exact traveling wave solutions of the desired equation. Therefore, one of our reasons for choosing this method is to emphasize its power because it can be used for various nonlinear equations. This method is a new tool in mathematics to obtain solitary and periodic solutions and also is a coherent framework in which other methods can be placed.

We propose a MEF method and investigate precise wave one-soliton, wave two-soliton and wave three-soliton solutions to the $(2 + 1)$-dimensional Date–Jimbo–Kashiwara–Miwa ($(2 + 1)$-dimensional DJKM) equation.

This paper is arranged as follows. In Sec. 2, we describe the MEF method, which is the method of obtaining multiple wave solutions for nonlinear equations. In Sec. 3, using the method presented in the second section, we obtain multiple soliton wave solutions for the $(2+1)$-dimensional Date–Jimbo–Kashiwara–Miwa ($(2+1)$-dimensional DJKM) equation. Section 4 concludes.

2. A Multiple Exp-Function Method

We consider the $(1+1)$-dimensional PDE

$$\mathcal{H}(u, s, \Phi_u, \Phi_s, \ldots) = 0.$$

We present the proposed method for this equation, which is a type of differential polynomials. This method is also valid for nonlinear equations and higher dimensions of equations.

- In the first step, we introduce a sequence of new variables that includes PDEs with a solution. $\Theta_\ell = \Theta_\ell(u, s), 1 \leq \ell \leq m$, by solvable PDEs, where, $\Theta_{\ell,u} = \gamma_\ell \Theta_\ell, \Theta_{\ell,s} = -\tau_\ell \Theta_\ell, 1 \leq \ell \leq \mathrm{m}$ and $\gamma_\ell, 1 \leq \ell \leq m$, are the angular wave numbers and $\tau_\ell, 1 \leq \ell \leq m$, are the wave frequencies. Since nonlinear differential equations are not directly solvable, by going through the first step, we can make accurate solutions of nonlinear equations. By solving linear equations as follows:

  $$\Theta_\ell = \kappa_\ell \exp(\varkappa_\ell), \quad \varkappa_\ell = \gamma_\ell u - \tau_\ell s, \quad 1 \leq \ell \leq m,$$

  where $\kappa_\ell, 1 \leq \ell \leq m$, are any constants, positive or negative, we can obtain exponential function solutions for these equations.
  By selecting the desired $\kappa_\ell, 1 \leq \ell \leq m$ constants, there will be more choices for the solutions obtained, see Ref. 25 for more details. The function $\Theta_\ell, 1 \leq \ell \leq m$ is a combination of all single waves. Each function Θ_ℓ also describes a single wave and a multiple wave solution.
  According to the obtained solutions, the reason for naming the introduced method is determined solutions obtained above offer reasons as to why the approach is called the MEF method.
  The introduction of linear differential equations and their application to other systems is also used, in which case the solutions are varied.[26] The use of linear differential equations is used in various methods to accurately solve PDEs. For example, in T-F-type method, the mapping method, the F-E-type methods, the JEF method, and G'/G-E method uses this idea.[19,27–30]
- In this step, we consider solutions using the new variables $\Theta_\ell, 1 \leq \ell \leq m$

  $$\begin{aligned} \Phi(u, s) &= \frac{\Psi(\Theta_1, \Theta_2, \ldots, \Theta_m)}{\Upsilon(\Theta_1, \Theta_2, \ldots, \Theta_m)}, \\ \Psi &= \sum_{p,q=1}^{m} \sum_{\ell,j=0}^{I} \Psi_{pq,\ell j} \Theta_p^\ell \Theta_q^j, \\ \Upsilon &= \sum_{p,q=1}^{m} \sum_{\ell,j=0}^{J} \Upsilon_{pq,\ell j} \Theta_p^\ell \Theta_q^j, \end{aligned} \tag{2}$$

where $\Psi_{pq,\ell j}$ and $\Upsilon_{pq,\ell j}$ are constants. These constants are determined by the main $(1+1)$-dimensional PDE ($(1+1)$-dimensional PDE). Consider the differential equations $\Theta_{\ell,u}$ and $\Theta_{\ell,s}$. Using these equations, we can express the partial derivatives of Φ in terms of $\Theta_\ell, 1 \leq \ell \leq m$ with u and s. If Ψ_{Θ_ℓ} is a partial derivative of Ψ in terms of Θ_ℓ and Υ_{Θ_ℓ} is a partial derivative of Υ with respect to Θ_ℓ, then we can have

$$\begin{aligned}\Phi_s &= \frac{\Upsilon \sum_{\ell=1}^{m} \Psi_{\Theta_\ell}\Theta_{\ell,s} - \Psi \sum_{\ell=1}^{m} \Upsilon_{\Theta_\ell}\Theta_{\ell,s}}{\Upsilon^2}\\ &= \frac{-\Upsilon \sum_{\ell=1}^{m} \tau_\ell \Psi_{\Theta_\ell}\Theta_\ell + \Psi \sum_{\ell=1}^{m} \tau_\ell \Upsilon_{\Theta_\ell}\Theta_\ell}{\Upsilon^2}\end{aligned}$$

and

$$\begin{aligned}\Phi_u &= \frac{\Upsilon \sum_{\ell=1}^{m} \Psi_{\Theta_\ell}\Theta_{\ell,u} - \Psi \sum_{\ell=1}^{m} \Upsilon_{\Theta_\ell}\Theta_{\ell,u}}{\Upsilon^2}\\ &= \frac{\Upsilon \sum_{\ell=1}^{m} \gamma_\ell \Psi_{\Theta_\ell}\Theta_\ell - \Psi \sum_{\ell=1}^{m} \gamma_\ell \Upsilon_{\Theta_\ell}\Theta_\ell}{\Upsilon^2}.\end{aligned}$$

On this account, all the partial derivatives, including Φ_s and Φ_u, represented by the new variables $\Theta_\ell, 1 \leq \ell \leq m$ are rational functions. Next, we place the newly obtained partial derivatives in the principal $(1+1)$-dimensional PDE ($(1+1)$-dimensional PDE). Therefore, with the new variables $\Theta_\ell, 1 \leq \ell \leq m$, a logical function equation is created as follows:

$$\mathcal{M}(u, s, \Theta_1, \Theta_2, \ldots, \Theta_m) = 0.$$

This is called the transformed equation of the original equation $(1+1)$-dimensional PDE.

- In this step, we can obtain solutions of differential equations using computer algebraic systems. Here, we come to a system of algebraic equations of all variables $\gamma_\ell, \tau_\ell, \Psi_{pq,\ell j}, \Upsilon_{pq,\ell j}$, which we arrive at by equating the numerator of the logical function $\mathcal{M}(u, s, \Theta_1, \Theta_2, \ldots, \Theta_m)$ with zero. We solve the system obtained from algebraic equations to determine the polynomials of Ψ and Υ as well as the power of the wave $\varkappa_\ell, 1 \leq \ell \leq m$. Since both the algorithms presented in this work and the other algorithms obtained from these algebraic systems may be complex, we can use computer programs to obtain solutions for these systems. Here, we use the Maple algebraic system to obtain the solutions of the desired system. Thus, the multiple wave soliton solution Φ is obtained, which is shown as follows:

$$\Phi(u,s) = \frac{\Psi(\kappa_1 \exp(\gamma_1 u - \tau_1 s), \ldots, \kappa_m \exp(\gamma_m u - \tau_m s))}{\Upsilon(\kappa_1 \exp(\gamma_1 u - \tau_1 s), \ldots, \kappa_m \exp(\gamma_m u - \tau_m s))}. \tag{3}$$

According to the solutions of linear equations obtained at the beginning of the algorithm, which are of exponential function-type, we call this algorithm the MEF method. By considering different linear equations, we can obtain different solutions for nonlinear equations. For example, by considering linear sine-cosine equations, we can obtain multiple periodic wave solutions for nonlinear equations by multiple sine-cosine method. If $m = 1$ is considered in the proposed MEF method, this method becomes the so-called exp-function method provided by He and Wu. See Ref. 31 for more information. The algorithm presented in this paper is a direct and systematic method that is used to obtain multi-soliton wave solutions. To perform complex calculations of this algorithm, we can use computer algebraic systems such as Maple, Mathematica, MuPAD and Matlab. This method also offers a generalization of the Hirotas perturbation scheme to create multi-soliton solution.[15] In this work, we define different Ψ and Υ polynomials in each step to obtain the multiple soliton wave solutions of the $(2+1)$-dimensional Date–Jimbo–Kashiwara–Miwa equation ($(2+1)$-dimensional DJKM).

3. Application of Multiple Exponential Function Method for the (2+1)-Dimensional Date–Jimbo–Kashiwara–Miwa Equation

We consider the $(2+1)$-dimensional Date–Jimbo–Kashiwara–Miwa equation ($(2+1)$-dimensional DJKM), which is defined as follows:

$$\Phi_{uuuuv} + 4\Phi_{uuv}\Phi_u + 2\Phi_{uuu}\Phi_v + 6\Phi_{uv}\Phi_{uu} - \Phi_{vvv} - 2\frac{\partial}{\partial u}\left(\frac{\partial}{\partial u}\left(\frac{\partial \Phi}{\partial s}\right)\right) = 0, \quad (4)$$

where $\lambda = 1, \mu = 1$ and $\zeta = 1$. To obtain the one-soliton wave, two-soliton wave and three-soliton wave solutions of the above equation, we use the proposed MEF method. To obtain these solutions, we introduce different polynomials of Ψ and Υ at each steps. These steps are as follows.

3.1. *One-wave solutions*

We consider the function $\Phi = \Phi(u, v, s)$ as follows:

$$\Phi(u, v, s) = \frac{\Psi(u, v, s)}{\Upsilon(u, v, s)}, \quad (5)$$

where Ψ and Υ are one degree polynomials defined as follows:

$$\Psi(u, v, s) = f_0 + f_1\Theta_1, \quad \Upsilon(u, v, s) = g_0 + g_1\Theta_1, \quad \Theta_1 = \exp(\gamma_1 u + \varsigma_1 v - \tau_1 s), \quad (6)$$

where f_0, f_1, g_0 and g_1 are constants to be determined. Then, due to the (5), we obtain

$$\Phi(u, v, s) = \frac{f_0 + f_1 \exp(\gamma_1 u + \varsigma_1 v - \tau_1 s)}{g_0 + g_1 \exp(\gamma_1 u + \varsigma_1 v - \tau_1 s)}. \quad (7)$$

Now substituting Eq. (7) into Eq. (4), we have

$$\varsigma_1^4\gamma_1^4\Phi^{(5)} + 6\varsigma_1\gamma_1^3\Phi'''\Phi' + 6\varsigma_1\gamma_1^3(\Phi'')^2 - \varsigma_1^3\Phi''' + 2\tau_1\gamma_1^2\Phi''' = 0. \quad (8)$$

By the MEF method and simplifying, we have

$$\frac{1}{\mathcal{Y}}(\mathcal{G}_5\exp(5(\gamma_1 u+\varsigma_1 v-\tau_1 s))+\mathcal{G}_4\exp(4(\gamma_1 u+\varsigma_1 v-\tau_1 s))+\mathcal{G}_3\exp(3(\gamma_1 u+\varsigma_1 v - \tau_1 s))+\mathcal{G}_2\exp(2(\gamma_1 u+\varsigma_1 v-\tau_1 s))+\mathcal{G}_1\exp(1(\gamma_1 u+\varsigma_1 v-\tau_1 s)))=0, \tag{9}$$

where $\mathcal{Y}=(g_0+g_1(\exp(\gamma_1 u+\varsigma_1 v-\tau_1 s)))^6$.

Setting each coefficient of $\exp(m(\gamma_1 u+\varsigma_1 v-\tau_1 s))$, $m=1,2,3,4,5$, we obtain a set of algebraic equations for f_0, f_1, g_0, g_1. Then,

$$\begin{aligned}
\mathcal{G}_5 &= -f_0 g_1^5\gamma_1^4\varsigma_1+f_1 g_0 g_1^4\gamma_1^4\varsigma_1-2f_0 g_1^5\gamma_1^2\tau_1+f_0 g_1^5\varsigma_1^3+2f_1 g_0 g_1^4\gamma_1^2\tau_1-f_1 g_0 g_1^4\varsigma_1^3=0,\\
\mathcal{G}_4 &= 26 f_0 g_0 g_1^4\gamma_1^4\varsigma_1-26 f_1 g_0^2 g_1^3\gamma_1^4\varsigma_1+12\varsigma_1\gamma_1^3 g_1^4 f_0^2-24 f_0 f_1 g_0 g_1^3\gamma_1^3\varsigma_1+12 f_1^2 g_0^2 g_1^2\gamma_1^3\varsigma_1\\
&\quad +4 f_0 g_0 g_1^4\gamma_1^2\tau_1-2 f_0 g_0 g_1^4\varsigma_1^3-4 f_1 g_0^2 g_1^3\gamma_1^2\tau_1+2 f_1 g_0^2 g_1^3\varsigma_1^3=0,\\
\mathcal{G}_3 &= -66 f_0 g_0^2 g_1^3\gamma_1^4\varsigma_1+66 f_1 g_0^3 g_1^2\gamma_1^4\varsigma_1-36 f_0^2 g_0 g_1^3\gamma_1^3\varsigma_1+72 f_0 f_1 g_0^2 g_1^2\gamma_1^3\varsigma_1\\
&\quad -36 f_1^2 g_0^3 g_1\gamma_1^3\varsigma_1+12 f_0 g_0^2 g_1^3\gamma_1^2\tau_1-6 f_0 g_0^2 g_1^3\varsigma_1^3-12 f_1 g_0^3 g_1^2\gamma_1^2\tau_1+6 f_1 g_0^3 g_1^2\varsigma_1^3\\
&= 0,\\
\mathcal{G}_2 &= 26 f_0 g_0^3 g_1^2\gamma_1^4\varsigma_1-26 f_1 g_0^4 g_1\gamma_1^4\varsigma_1+12 f_0^2 g_0^2 g_1^2\gamma_1^3\varsigma_1-24 f_0 f_1 g_0^3 g_1\gamma_1^3\varsigma_1+12 f_1^2 g_0^4\gamma_1^3\varsigma_1\\
&\quad +4 f_0 g_0^3 g_1^2\gamma_1^2\tau_1-2 f_0 g_0^3 g_1^2\varsigma_1^3-4 f_1 g_0^4 g_1\gamma_1^2\tau_1+2 f_1 g_0^4 g_1\varsigma_1^3=0,\\
\mathcal{G}_1 &= -f_0 g_0^4 g_1\gamma_1^4\varsigma_1+f_1 g_0^5\gamma_1^4\varsigma_1-2 f_0 g_0^4 g_1\gamma_1^2\tau_1+f_0 g_0^4 g_1\varsigma_1^3+2 f_1 g_0^5\gamma_1^2\tau_1-f_1 g_0^5\varsigma_1^3=0.
\end{aligned} \tag{10}$$

Solving the system of the aforementioned algebraic equations with the aid of Maple, we obtain

$$\begin{aligned}
&f_0=f_0,\quad f_1=\frac{g_1(2g_0\gamma_1+f_0)}{g_0},\quad g_0=g_0,\quad g_1=g_1,\quad \gamma_1=\gamma_1,\quad \varsigma_1=\varsigma_1,\\
&\tau_1=-\frac{\varsigma_1(\gamma_1^4-\varsigma_1^2)}{2\gamma_1^2}.
\end{aligned} \tag{11}$$

By placing these items in Eq. (7), we have

$$\Phi(u,v,s) = -\frac{\begin{array}{c}2\exp(\gamma_1 u+\varsigma_1 v-\tau_1 s)g_0 g_1^2\gamma_1+\exp(\gamma_1 u+\varsigma_1 v-\tau_1 s)f_0 g_1^2\\ -2\exp(\gamma_1 u+\varsigma_1 v-\tau_1 s)g_0 g_1\gamma_1+2g_0^2 g_1\gamma_1-\exp(\gamma_1 u+\varsigma_1 v-\tau_1 s)f_0 g_1\end{array}}{g_0(g_0+g_1\exp(\gamma_1 u+\varsigma_1 v-\tau_1 s))}. \tag{12}$$

The solution diagram, for some values, is drawn as follows.

3.2. *Two-wave solutions*

Consider the polynomials Ψ and Υ of degree two as follows:

$$\begin{aligned}
\Psi(\Theta_1,\Theta_2)&=2[\gamma_1\Theta_1+\gamma_2\Theta_2+f_{12}(\gamma_1+\gamma_2)\Theta_1\Theta_2],\\
\Upsilon(\Theta_1,\Theta_2)&=1+\Theta_1+\Theta_2+f_{12}\Theta_1\Theta_2,
\end{aligned} \tag{13}$$

where $\Theta_\ell = \kappa_\ell \exp(\gamma_1 u + \varsigma_1 v - \tau_1 s), 1 \leq \ell \leq 2$. Now, using the above polynomials and according to Eq. (5), we define the Φ function, which gives us two-wave solutions, as follows:

$$\Phi(u,v,s) = \frac{2[\gamma_1\Theta_1 + \gamma_2\Theta_2 + f_{12}(\gamma_1+\gamma_2)\Theta_1\Theta_2]}{1+\Theta_1+\Theta_2+f_{12}\Theta_1\Theta_2}. \tag{14}$$

By placing Eq. (14) in Eq. (4) and using the multiple exponential method and solving the algebraic system as in the first case, we have

$$\begin{aligned} f_{12} &= \frac{\gamma_1^4\gamma_2^2 - 2\gamma_1^3\gamma_2^3 + \gamma_1^2\gamma_2^4 + \gamma_1^2\varsigma_2^2 - 2\gamma_1\gamma_2\varsigma_1\varsigma_2 + \gamma_2^2\varsigma_1^2}{\gamma_1^4\gamma_2^2 + 2\gamma_1^3\gamma_2^3 + \gamma_1^2\gamma_2^4 + \gamma_1^2\varsigma_2^2 - 2\gamma_1\gamma_2\varsigma_1\varsigma_2 + \gamma_2^2\varsigma_1^2}, \\ \kappa_1 &= \kappa_1, \quad \kappa_2 = \kappa_2, \quad \gamma_1 = \gamma_1, \quad \gamma_2 = \gamma_2, \quad \varsigma_1 = \varsigma_1, \quad \varsigma_2 = \varsigma_2, \\ \tau_1 &= \frac{\varsigma_1(\gamma_1^4 - \varsigma_1^2)}{2\gamma_1^2}, \quad \tau_2 = \frac{\varsigma_2(\gamma_2^4 - \varsigma_2^2)}{2\gamma_2^2}. \end{aligned} \tag{15}$$

By placing these items in Eq. (14), two-wave solutions are obtained.

3.3. *Three-wave solutions*

Similar to the second case, for $\Theta_\ell = \exp(\gamma_\ell u + \varsigma_\ell v - \tau_\ell s), 1 \leq \ell \leq 3$, we consider the repentance of Ψ and Υ as follows:

$$\begin{aligned} \Psi(\Theta_1,\Theta_1,\Theta_1) &= 2[\gamma_1\Theta_1 + \gamma_2\Theta_2 + \gamma_3\Theta_3 + f_{12}(\gamma_1+\gamma_2)\Theta_1\Theta_2 + f_{13}(\gamma_1+\gamma_3)\Theta_1\Theta_3 \\ &\quad + f_{23}(\gamma_2+\gamma_3)\Theta_2\Theta_3 + f_{12}f_{13}f_{23}(\gamma_1+\gamma_2+\gamma_3)\Theta_1\Theta_2\Theta_3], \\ &\quad \times \Upsilon(\Theta_1,\Theta_2,\Theta_3) = 1 + \Theta_1 + \Theta_2 + \Theta_3 + f_{12}\Theta_1\Theta_2 + f_{13}\Theta_1\Theta_3 \\ &\quad + f_{23}\Theta_2\Theta_3 + f_{12}f_{13}f_{23}\Theta_1\Theta_2\Theta_3, \end{aligned} \tag{16}$$

and using the above polynomials and according to Eq. (5), we define the u function, which gives us three-wave solutions, as follows:

$$\Phi(u,v,s) = \frac{\begin{array}{c} 2[\gamma_1\Theta_1 + \gamma_2\Theta_2 + \gamma_3\Theta_3 + f_{12}(\gamma_1+\gamma_2)\Theta_1\Theta_2 + f_{13}(\gamma_1+\gamma_3)\Theta_1\Theta_3 \\ + f_{23}(\gamma_2+\gamma_3)\Theta_2\Theta_3 + f_{12}f_{13}f_{23}(\gamma_1+\gamma_2+\gamma_3)\Theta_1\Theta_2\Theta_3] \end{array}}{\begin{array}{c} 1 + \Theta_1 + \Theta_2 + \Theta_3 + f_{12}\Theta_1\Theta_2 + f_{13}\Theta_1\Theta_3 + f_{23}\Theta_2\Theta_3 \\ + f_{12}f_{13}f_{23}\Theta_1\Theta_2\Theta_3 \end{array}}. \tag{17}$$

By placing Eq. (17) in Eq. (4) and using the multiple exponential method and solving the algebraic system as in the first case, we have

$$\begin{aligned} f_{12} &= \frac{\gamma_1^4\gamma_2^2 - 2\gamma_1^3\gamma_2^3 + \gamma_1^2\gamma_2^4 + \gamma_1^2\varsigma_2^2 - 2\gamma_1\gamma_2\varsigma_1\varsigma_2 + \gamma_2^2\varsigma_1^2}{\gamma_1^4\gamma_2^2 + 2\gamma_1^3\gamma_2^3 + \gamma_1^2\gamma_2^4 + \gamma_1^2\varsigma_2^2 - 2\gamma_1\gamma_2\varsigma_1\varsigma_2 + \gamma_2^2\varsigma_1^2}, \\ f_{13} &= 0, \quad f_{23} = \frac{\gamma_2^4\gamma_3^2 - 2\gamma_2^3\gamma_3^3 + \gamma_2^2\gamma_3^4 + \gamma_2^2\varsigma_3^2 - 2\gamma_2\gamma_3\varsigma_2\varsigma_3 + \gamma_3^2\varsigma_2^2}{\gamma_2^4\gamma_3^2 + 2\gamma_2^3\gamma_3^3 + \gamma_2^2\gamma_3^4 + \gamma_2^2\varsigma_3^2 - 2\gamma_2\gamma_3\varsigma_2\varsigma_3 + \gamma_3^2\varsigma_2^2}, \\ \kappa_1 &= \kappa_1, \quad \kappa_2 = \kappa_2, \quad \kappa_3 = \kappa_3, \quad \gamma_1 = \gamma_1, \quad \gamma_2 = \gamma_2, \quad \gamma_3 = \gamma_3, \quad \varsigma_1 = \varsigma_1, \\ \varsigma_2 &= \varsigma_2, \quad \varsigma_3 = \varsigma_3, \quad \tau_1 = \frac{\varsigma_1(\gamma_1^4 - \varsigma_1^2)}{2\gamma_1^2}, \quad \tau_2 = \frac{\varsigma_2(\gamma_2^4 - \varsigma_2^2)}{2\gamma_2^2}, \end{aligned}$$

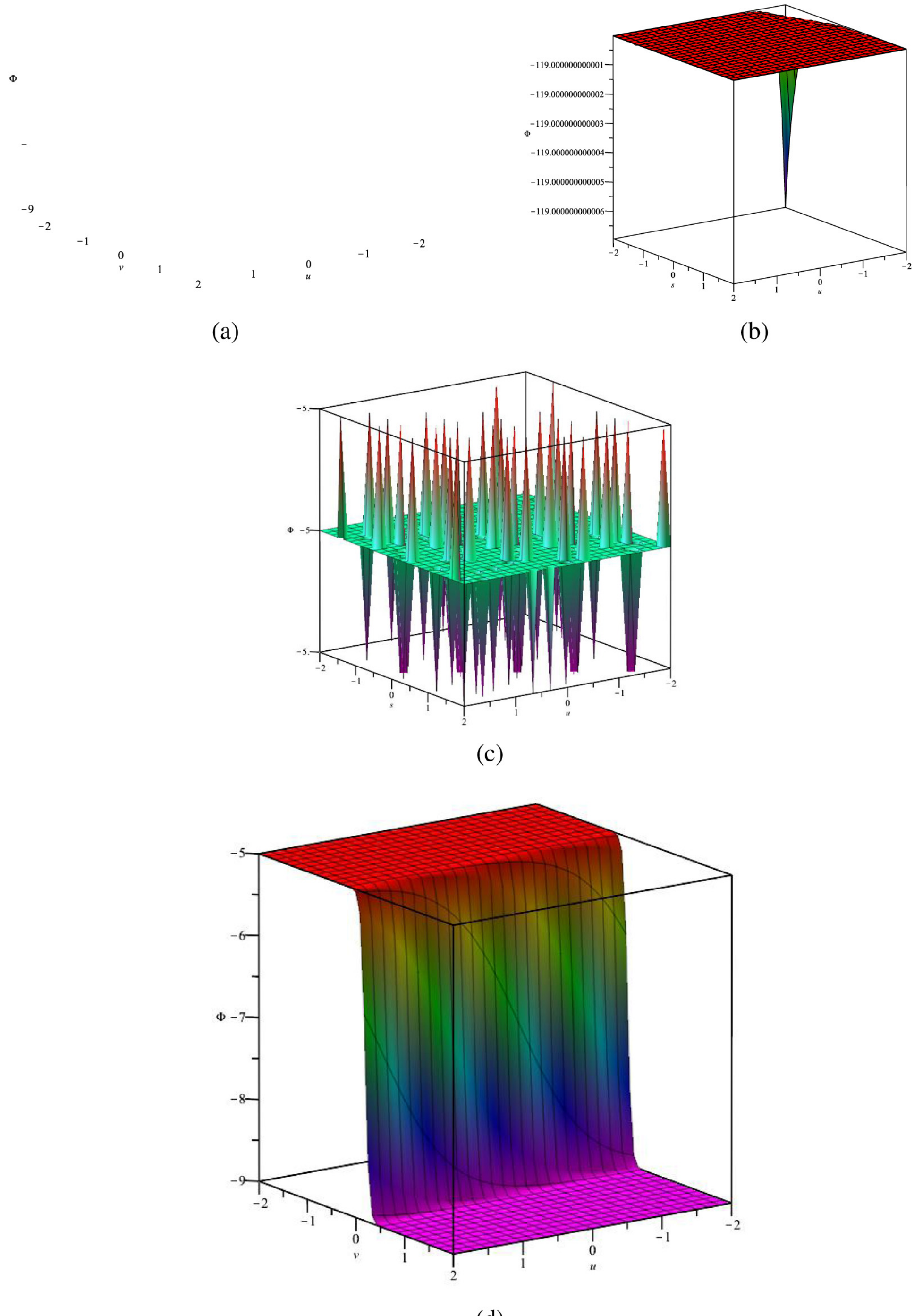


Fig. 1. (Color online) The figure of the one-wave solutions.

$$\tau_3 = \frac{\begin{array}{c}\gamma_1^6\varsigma_3 + 2\gamma_1^5\gamma_3\varsigma_1 - 4\gamma_1^5\gamma_3\varsigma_3 - 5\gamma_1^4\gamma_3^2\varsigma_1 + 6\gamma_1^4\gamma_3^2\varsigma_3 + 4\gamma_1^3\gamma_3^3\varsigma_1 - 4\gamma_1^3\gamma_3^3\varsigma_3 \\ -\ \gamma_1^2\gamma_3^4\varsigma_1 + \gamma_1^2\gamma_3^4\varsigma_3 - 3\gamma_1^2\varsigma_1^2\varsigma_3 + 3\gamma_1^2\varsigma_1\varsigma_3^2 - \gamma_1^2\varsigma_3^3 + 2\gamma_1\gamma_3\varsigma_1^3 - \gamma_3^2\varsigma_1^3\end{array}}{2(\gamma_1 - \gamma_3)^2\gamma_1^2}. \tag{18}$$

By placing these items in Eq. (17), three-wave solutions are obtained. The following diagrams show the solution of three waves for different values.

Remark 1. Here, by considering a CDJKM equation and using the MEFs method, we have obtained solutions for the equation, which are of the type of one-wave soliton, two-waves soliton and three-waves soliton. We have also presented

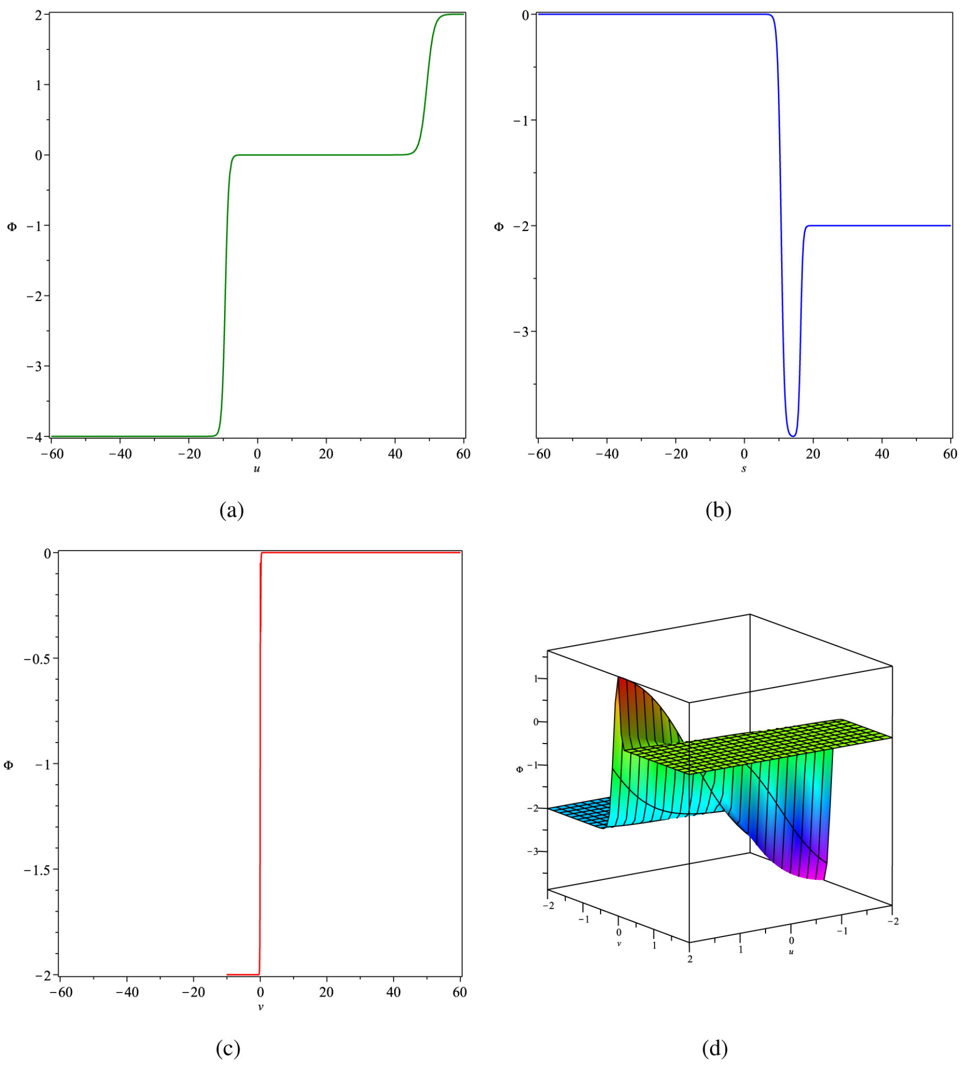


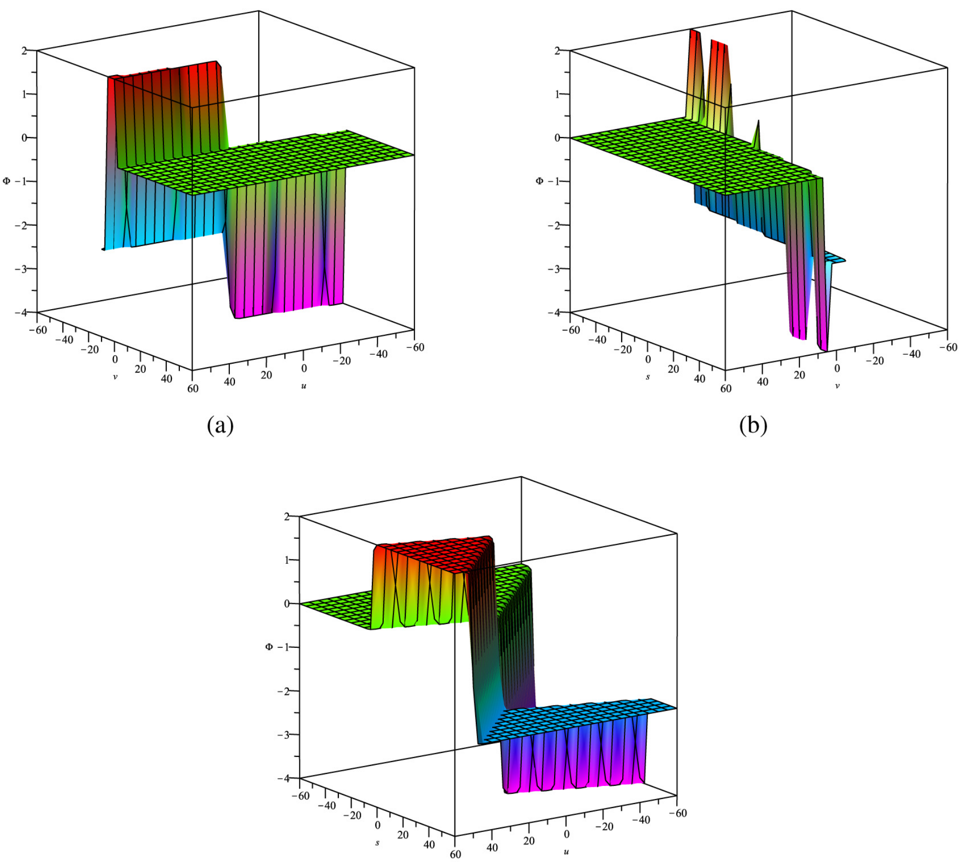
Fig. 2. (Color online) The figure of the two-wave solutions.

two-dimensional and three-dimensional graphical representations of the obtained solutions, and in the following, we state all the values used to draw these graphs. Figure 1 illustrates the one-wave solution for the values of

(a) $g_0 = -1, g_1 = -2, f_0 = -1, \gamma_1 = -2, s = 0, \varsigma_1 = -50, u = [-2, 2], v = [-2, 2]$,
(b) $g_0 = 5, g_1 = 8, f_0 = 5, \gamma_1 = 8, \varsigma_1 = 50, v = 1, \tau_1 = -3, u = [-2, 2], s = [-2, 2]$,
(c) $g_0 = 1, g_1 = 2, f_0 = 1, \gamma_1 = 2, \varsigma_1 = 50, v = 1, \tau[1] = -3, u = [-2, 2], s = [-2, 2]$,
(d) $g_0 = 1, g_1 = 2, f_0 = 1, \gamma_1 = 2, s = 0, \varsigma_1 = -25, u = [-2, 2], v = [-2, 2]$.

Figure 2 illustrates the two-wave solution for the values of

(a) $\gamma_1 = 1, \gamma_2 = -2, \kappa_1 = 1, \kappa_2 = 2, s = \frac{1}{4}, \varsigma_1 = -50, \varsigma_2 = -20, v = 1, \tau_1 = -3, \tau_2 = -2, u = [-60, 60]$,
(b) $\gamma_1 = 1, \gamma_2 = -2, \kappa_1 = 1, \kappa_2 = 2, u = 1, \varsigma_1 = -50, \varsigma_2 = -20, v = 1, \tau_1 = -3, \tau_2 = -2, s = [-60, 60]$,



(a) (b) (c)

Fig. 3. (Color online) The figure of the two-wave solutions.

(c) $\gamma_1 = 1, \gamma_2 = -2, \kappa_1 = 1, \kappa_2 = 2, u = 1, s = 1, \varsigma_1 = -50, \varsigma_2 = -20, \tau_1 = -3,$ $\tau_2 = -2, v = [-60, 60],$

(d) $\gamma_1 = 1, \gamma_2 = -2, \kappa_1 = 1, \kappa_2 = 2, s = \frac{1}{4}, \varsigma_1 = -50, \varsigma_2 = -20, v = 1, \tau_1 = -3,$ $\tau_2 = -2, u = [-2, 2], v = [-2, 2].$

Figure 3 illustrates the two-wave solution for the values of

(a) $\gamma_1 = 1, \gamma_2 = -2, \kappa_1 = 1, \kappa_2 = 2, s = \frac{1}{4}, \varsigma_1 = -50, \varsigma_2 = -20, v = 1, \tau_1 = -3,$ $\tau_2 = -2, u = [-60, 60], v = [-60, 60],$

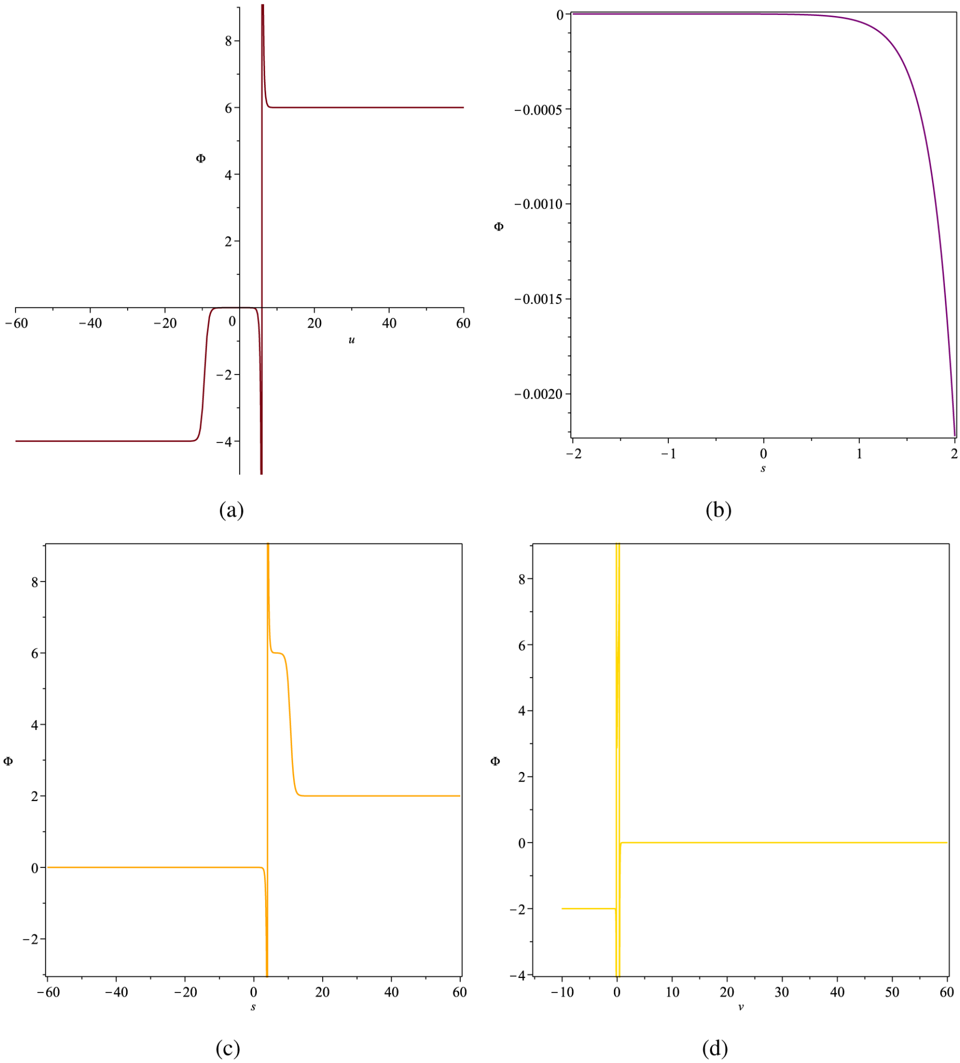


Fig. 4. (Color online) The figure of the three-wave solutions.

(b) $\gamma_1 = 1, \gamma_2 = -2, \kappa_1 = 1, \kappa_2 = 2, u = 1, \varsigma_1 = -50, \varsigma_2 = -20, v = 1, \tau_1 = -3, \tau_2 = -2, v = [-60, 60], s = [-60, 60]$,

(c) $\gamma_1 = 1, \gamma_2 = -2, \kappa_1 = 1, \kappa_2 = 2, \varsigma_1 = -50, \varsigma_2 = -20, v = 1, \tau_1 = -3, \tau_2 = -2, u = [-60, 60], s = [-60, 60]$.

Figure 4 illustrates the three-wave solution for the values of

(a) $\gamma_1 = 1, \gamma_2 = -2, \gamma_3 = 3, \kappa_1 = 1, \kappa_2 = 2, \kappa_3 = -3, s = \frac{1}{4}, \varsigma_1 = -50, \varsigma_2 = -20, \varsigma_3 = -20, v = 1, \tau_1 = -3, \tau_2 = -2, \tau_3 = -4, u = [-60, 60]$,

(b) $\gamma_1 = 1, \gamma_2 = -2, \gamma_3 = 3, \kappa_1 = 1, \kappa_2 = 2, \kappa_3 = -3, u = 1, \varsigma_1 = -50, \varsigma_2 = -20, \varsigma_3 = -20, v = 1, \tau_1 = -3, \tau_2 = -2, \tau_3 = -4, s = [-2, 2]$,

(c) $\gamma_1 = 1, \gamma_2 = -2, \gamma_3 = 3, \kappa_1 = 1, \kappa_2 = 2, \kappa_3 = -3, u = 1, \varsigma_1 = -50, \varsigma_2 = -20, \varsigma_3 = -20, v = 1, \tau_1 = -3, \tau_2 = -2, \tau_3 = -4, s = [-60, 60]$,

(d) $\gamma_1 = 1, \gamma_2 = -2, \gamma_3 = 3, \kappa_1 = 1, \kappa_2 = 2, \kappa_3 = -3, u = 1, s = 1, \varsigma_1 = -50, \varsigma_2 = -20, \varsigma_3 = -20, \tau_1 = -3, \tau_2 = -2, \tau_3 = -4, v = [-60, 60]$.

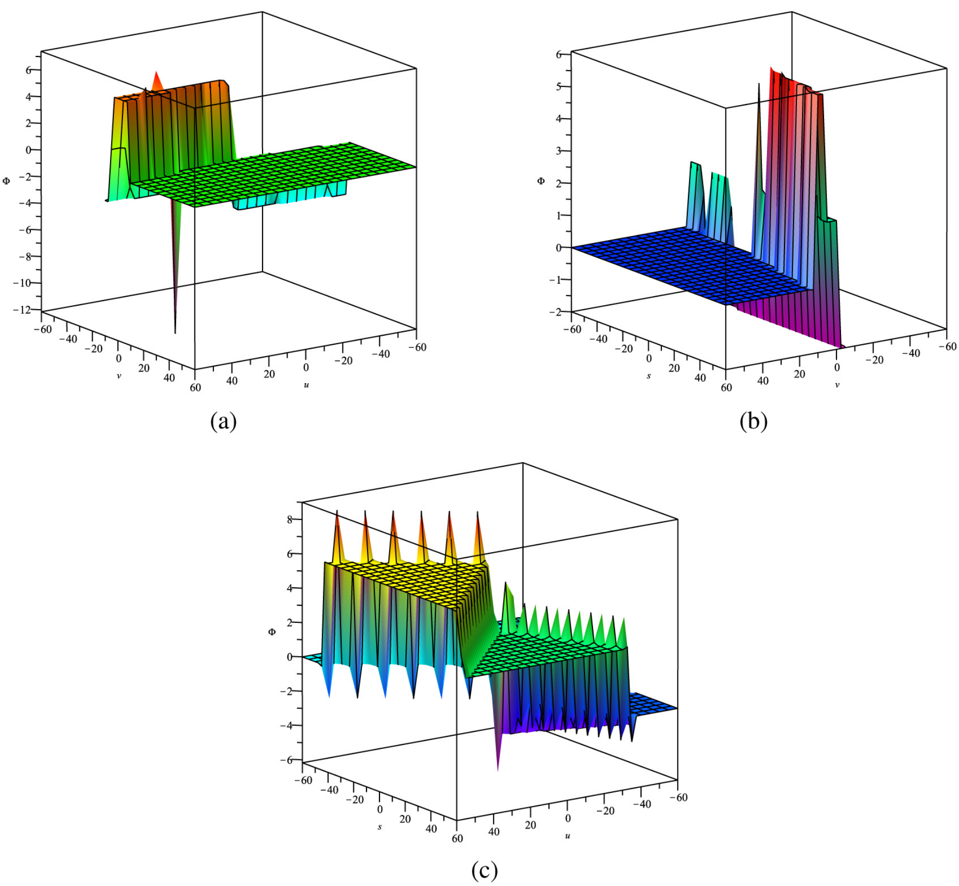


Fig. 5. (Color online) The figure of the three-wave solutions.

Then Fig. 5 is drawn for the following values:

(a) $\gamma_1 = 1, \gamma_2 = -2, \gamma_3 = 3, \kappa_1 = 1, \kappa_2 = 2, \kappa_3 = -3, s = \frac{1}{4}, \varsigma_1 = -50, \varsigma_2 = -20, \varsigma_3 = -20, \tau_1 = -3, \tau_2 = -2, \tau_3 = -4, u = [-60, 60], v = [-60, 60]$,
(b) $\gamma_1 = 1, \gamma_2 = -2, \gamma_3 = 3, \kappa_1 = 1, \kappa_2 = 2, \kappa_3 = -3, u = 1, \varsigma_1 = -50, \varsigma_2 = -20, \varsigma_3 = -20, \tau_1 = -3, \tau_2 = -2, \tau_3 = -4, s = [-60, 60], v = [-60, 60]$,
(c) $\gamma_1 = 1, \gamma_2 = -2, \gamma_3 = 3, \kappa_1 = 1, \kappa_2 = 2, \kappa_3 = -3, v = \frac{1}{4}, \varsigma_1 = -50, \varsigma_2 = -20, \varsigma_3 = -20, v = 1, \tau_1 = -3, \tau_2 = -2, \tau_3 = -4, s = [-60, 60], v = [-60, 60]$,

4. Conclusion

Due to the importance of nonlinear equations in physical and scientific phenomena, this topic has always been the focus of many researchers. Obtaining solutions for these equations with different methods, some of which we mentioned in the introduction, has been one of the important scientific issues. In this paper, considering the better efficiency of the MEFs method and its comprehensiveness compared to other methods, we have obtained analytical solutions for a nonlinear equation. We applied MEF method to obtain the exact soliton wave solutions of the following CDJKM nonlinear equation:

$$\Phi_{uuuuv} + 4\Phi_{uuv}\Phi_u + 2\Phi_{uuu}\Phi_v + 6\Phi_{uv}\Phi_{uu} - \lambda\Phi_{vvv} - 2\mu\frac{\partial}{\partial u}\left(\frac{\partial}{\partial u}\left(\frac{\partial^\zeta\Phi}{\partial s^\theta}\right)\right) = 0, \quad (19)$$

where $\Phi = \Phi(u, v, s)$ and μ is a constant and $0 < \zeta < 1$. The CDJKM equation turns out to be $(2 + 1)$-dimensional DJKM equation when the fractional derivative of the order $\zeta = 1$. First, in the introduction section, we have provided explanations regarding the overall work done, referring to the main equation and the proposed method. Then we have stated the required steps of the proposed method to solve the equation. We consider the $(2 + 1)$-dimensional DJKM equation to perform numerical calculations and we have done all our calculations with computer algebra systems such as Maple. Also, we have provided 2D and 3D graphic representations of the obtained solutions.

References

1. H. B. Lan and K. L. Wang, *J. Phys. A* **23**, 3923 (1990).
2. W. Malfliet, *Am. J. Phys.* **60**, 650 (1992).
3. B. R. Duffy and E. J. Parkes, *Comput. Phys. Commun.* **98**, 288 (1996).
4. W.-X. Ma, *Phys. Lett. A* **180**, 221 (1993).
5. B. R. Duffy and E. J. Parkes, *Phys. Lett. A* **214**, 271 (1996).
6. M. Wenxiu and D. Zhou, *Acta Math. Sci.* **17**, 168 (1997).
7. M. L. Wang, *Phys. Lett. A* **199**, 169 (1995).
8. E. Fan and Z. Hongqing, *Phys. Lett. A* **246**, 403 (1998).
9. M. S. Hashemi, F. Bahrami and R. Najafi, *Optik* **138**, 240 (2017).
10. M. S. Hashemi, *Chaos Solitons Fractals* **152**, 111367 (2021).
11. H. Rezazadeh *et al.*, *Int. J. Mod. Phys. B* **34**, 2050161 (2020).

12. F. L. Xia *et al.*, *Results Phys.* **38**, 105512 (2022).
13. Y. M. Chu *et al.*, *Comput. Appl. Math.* **41**, 1 (2022).
14. W.-X. Ma and J.-H. Lee, *Chaos Solitons Fractals* **42**, 1356 (2009).
15. R. Hirota, *The Direct Method in Soliton Theory*, Vol. 155 (Cambridge University Press, 2004).
16. Y. Zhang *et al.*, *J. Phys. A* **40**, 5539 (2007).
17. W.-X. Ma, R. Zhou and L. Gao, *Mod. Phys. Lett. A* **24**, 1677 (2009).
18. D. V. Tanwar, *Chaos Solitons Fractals* **162**, 112414 (2022).
19. Md Ashik Iqbal *et al.*, *Fractal Fract.* **6**, 4 (2021).
20. F. Guo and J. Lin, *Nonlinear Dyn.* **96**, 1233 (2019).
21. A. Kumar *et al.*, *AIMS Math.* **6**, 4238 (2021).
22. Y.-Q. Yuan *et al.*, *Comput. Math. Appl.* **74**, 873 (2017).
23. M. Singh and R. K. Gupta, *Int. J. Appl. Comput. Math.* **4**, 88 (2018).
24. S.-N. Wang and J. Hu, *Mod. Phys. Lett. B* **33**, 1950119 (2019).
25. N. A. Kudryashov, *Commun. Nonlinear Sci. Numer. Simul.* **14**, 3507 (2009).
26. W.-X. Ma, H. Wu and J. He, *Phys. Lett. A* **364**, 29 (2007).
27. E. J. Parkes and B. R. Duffy, *Comput. Phys. Commun.* **98**, 288 (1996).
28. Mustafa Inc *et al.*, *AIMS Math.* **5**, 6726 (2020).
29. L.-X. Li, E.-Q. Li and M.-L. Wang, *Appl. Math. J. Chin. Univ. Ser. B* **25**, 454 (2010).
30. Y.-Q. Yuan *et al.*, *Comput. Math. Appl.* **74**, 873 (2017).
31. J.-H. He and X.-H. Wu, *Chaos Solitons Fractals* **30**, 700 (2006).
32. A. Akbulut *et al.*, *Int. J. Mod. Phys. B* **37**, 2350063 (2023).
33. A. Akgül, M. S. Hashemi and F. Jarad, *Fractal Fract.* **6**, 202 (2022).
34. K. Hosseini *et al.*, *Int. J. Geomathematics* **13**, 1 (2022).
35. S. Abbagari *et al.*, *Phys. Lett. A* **442**, 128191 (2022).
36. A. Houwe *et al.*, *Phys. Scr.* **97**, 085206 (2022).
37. D. Ntiamoah, W. Ofori-Atta and L. Akinyemi, *J. Ocean Eng. Sci.* (2022).
38. M. M. Khater *et al.*, *Mod. Phys. Lett. B* **35**, 2150381 (2021).
39. K. S. Nisar *et al.*, *Results Phys.* **33**, 105200 (2022).
40. A. C. Cevikel *et al.*, *Front. Phys.* 1064 (2022).
41. M. Raheel *et al.*, *Opt. Quantum Electron.* **54**, 1 (2022).
42. A. C. Cevikel and A. Bekir, *Int. J. Mod. Phys. B* 2350084 (2022).
43. M. Raheel *et al.*, *Int. J. Mod. Phys. B* 2350003 (2022).
44. N. Raza *et al.*, *Int. J. Mod. Phys. B* **33**, 1950360 (2019).
45. A. Akbulut and F. Taşcan, *Chaos Solitons Fractals* **104**, 33 (2017).
46. A. Akbulut and M. Kaplan, *Comput. Math. Appl.* **75**, 876 (2018).
47. M. Kaplan and A. Akbulut, *Opt. Quantum Electron.* **53**, 1 (2021).