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RESEARCH ARTICLE

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Some results on a nonlinear fractional equation with nonlocal boundary condition

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The aim of this paper is to derive sufficient conditions for the existence, uniqueness, and Hyers–Ulam stability of solutions to a new nonlinear fractional integro-differential equation with functional boundary conditions, using several fixed-point theorems, the multivariate Mittag-Leffler function and Babenko's approach. A few examples are also presented to illustrate the applications of our results based on approximate values of a couple of Mittag-Leffler functions calculated by Python codes. Furthermore, the approaches used have a wide range of applications to various fractional differential equations with initial or boundary conditions or integral equations in complete spaces.

KEYWORDS

Babenko's approach, Banach's contractive principle, Hyers–Ulam's stability, Leray–Schauder's fixed-point theorem, multivariate Mittag-Leffler function, nonlinear fractional integro-differential equation

MSC CLASSIFICATION

34B15, 34A12, 34K20, 26A33

1 | INTRODUCTION

The Riemann–Liouville fractional integral I^α of order $\alpha \in R^+$ is defined for the function $\zeta(x)$ as

$$(I^\alpha \zeta)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1} \zeta(\tau) d\tau, \ x \in [0,1].$$

In particular,

$$I^0 \phi = \phi,$$

from [1].

The Liouville–Caputo fractional derivative ${}_C D^\beta$ of order $\beta \in (1,2]$ of the function $\phi(x, \zeta(x))$ is defined as [2, 3]

$$({}_C D^\beta \phi)(x) = \left(I^{2-\beta} \frac{d^2}{dx^2} \phi \right)(x) = \frac{1}{\Gamma(2-\beta)} \int_0^x (x-\tau)^{1-\beta} \phi^{(2)}(\tau, \zeta(\tau)) d\tau.$$

Assume $m, l \in \mathbb{N}, \phi_i, \psi : [0,1] \times \mathbb{R} \to \mathbb{R}$ are mappings for $i = 1, 2, \cdots, l$, and $\gamma : C[0,1] \to \mathbb{R}$ is a functional. The aim of this paper is to study the uniqueness, existence, and Hyers–Ulam stability for the following nonlinear fractional integro-differential equation with a functional boundary condition for $1 < \alpha \le 2$ and constants λ_i:

$$\begin{cases} {}_C D^{\alpha}\left[\zeta(x) - \sum_{i=1}^{m} \lambda_i \, I^{\beta_i}\zeta(x) - \sum_{i=1}^{l} I^{\alpha_i}\phi_i(x,\zeta(x))\right] = \psi(x,\zeta(x)), \quad x \in [0,1], \\ \zeta(0) = \zeta_1, \quad \zeta(1) = \gamma(\zeta), \end{cases} \tag{1.1}$$

where ζ_1 is a constant in $\mathbb{R}$, $\beta_i > 0$ and $\alpha_i > 0$.

Equation (1.1) is new and, to the best of our knowledge, has not been previously investigated. Another motivation of considering this equation is to demonstrate how the use of an inverse operator of a bounded integral in a complete space can be used to study the nonlinear fractional differential equation with functional boundary condition.

Fractional differential equations including PDEs with boundary conditions have played an important role in diverse disciplines of applied sciences and engineering. For example, computational fluid dynamics methods are directly related to boundary data [4], and the assumptions of circular cross-sections in fluid flow problems cannot be justified in many cases. The concept of nonlocal boundary conditions acts as a natural tool to handle this issue since such conditions can be applied to arbitrarily shaped structures. There have been many interesting studies in the area dealing with various nonlinear boundary value problems [5–15]. Ahmad and Ntouyas [16] considered the following nonlocal boundary value problem of hybrid fractional integro-differential equation by means of several fixed-point theorems:

$$\begin{cases} {}_C D^{\alpha}\left[\frac{\zeta(x) - \sum_{i=1}^{m} I^{\beta_i} h_i(x,\zeta(x))}{f(x,\zeta(x))}\right] = g(x,\zeta(x)), \quad x \in [0,1], \\ \zeta(0) = \mu(\zeta), \quad \zeta(1) = A, \end{cases}$$

where $1 < \alpha \leq 2$, $f \in C([0,1]\times\mathbb{R}, \mathbb{R}\backslash\{0\})$, $g \in C([0,1]\times\mathbb{R})$, $h_i \in C([0,1]\times\mathbb{R})$, $1 < \beta_i \leq 2$, $i = 1, 2, \cdots, m$, $\mu : C[0,1] \to \mathbb{R}$, and $A \in \mathbb{R}$.

As usual, we define the Banach space $C[0,1]$ with the norm

$$\|\zeta\| = \max_{x\in[0,1]} |\zeta(x)| < +\infty.$$

The multivarite Mittag-Leffler function [3] is defined by

$$E_{(\alpha_1,\cdots,\alpha_m),\ \beta}(z_1,\cdots,z_m) = \sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_m=k} \binom{k}{k_1,\cdots,k_m} \frac{z_1^{k_1}\cdots z_m^{k_m}}{\Gamma(\alpha_1 k_1 + \cdots + \alpha_m k_m + \beta)},$$

where $z = (z_1,\cdots,z_m) \in \mathbb{C}^m$, $\alpha_i, \beta > 0$ for $i = 1, 2, \cdots, m$, and

$$\binom{k}{k_1,\cdots,k_m} = \frac{k!}{k_1!\cdots k_m!}.$$

Babenko's approach [17] is a useful tool for studying the uniqueness and existence of integro-differential equations with initial or boundary conditions. To illustrate this in detail, we consider the following nonlinear fractional integro-differential equation with an integral boundary condition:

$$\begin{cases} {}_C D^{\alpha} y(x) + a I^{\beta} y(x) = f(x, y(x)), \quad x \in [0,1], \quad 0 < \alpha \leq 1, \beta > 0, \\ y(0) = b \int_0^{1/2} y(x)dx, \end{cases} \tag{1.2}$$

where a, b are constants.

Applying the operator I^{α} to Equation (1.2), we get

$$y(x) - y(0) + aI^{\alpha+\beta} y(x) = I^{\alpha} f(x, y(x)),$$

which derives

$$\left(1 + aI^{\alpha+\beta}\right) y(x) = I^{\alpha} f(x, y(x)) + b \int_0^{1/2} y(x)dx.$$

To use Babenko's approach, we first define an operator A by

$$A = \sum_{k=0}^{\infty} (-1)^k a^k I^{(\alpha+\beta)k}$$

over the space $C[0, 1]$ and shall prove this operator is uniformly convergent. In fact, we have the following for any $\phi \in C[0, 1]$:

$$\begin{aligned} \|A\phi\| &= \left\| \sum_{k=0}^{\infty} (-1)^k a^k I^{(\alpha+\beta)k} \phi \right\| \le \sum_{k=0}^{\infty} |a|^k \left\| I^{(\alpha+\beta)k} \right\| \|\phi\| \\ &\le \|\phi\| \sum_{k=0}^{\infty} |a|^k \frac{1^{(\alpha+\beta)k}}{\Gamma((\alpha+\beta)k+1)} = \|\phi\| \, E_{\alpha+\beta,1}(|a|) < +\infty. \end{aligned}$$

Furthermore,

$$\begin{aligned} \left(1 + aI^{\alpha+\beta}\right) A &= A \left(1 + aI^{\alpha+\beta}\right) \\ &= \sum_{k=0}^{\infty} (-1)^k a^k I^{(\alpha+\beta)k} + aI^{\alpha+\beta} \sum_{k=0}^{\infty} (-1)^k a^k I^{(\alpha+\beta)k} \\ &= 1 + \sum_{k=1}^{\infty} (-1)^k a^k I^{(\alpha+\beta)k} + \sum_{k=0}^{\infty} (-1)^k a^{k+1} I^{(\alpha+\beta)(k+1)} \\ &= 1, \end{aligned}$$

by using the identity

$$aI^{\alpha+\beta} \sum_{k=0}^{\infty} (-1)^k a^k I^{(\alpha+\beta)k} = \sum_{k=0}^{\infty} (-1)^k a^{k+1} I^{(\alpha+\beta)(k+1)},$$

since A is uniformly convergent over $C[0, 1]$. Assume that B is another operator satisfying

$$B\left(1 + aI^{\alpha+\beta}\right) = \left(1 + aI^{\alpha+\beta}\right) B = 1.$$

Then, clearly, $A = B$ by applying the operator A to both sides. Hence, A is a unique inverse operator of $\left(1 + aI^{\alpha+\beta}\right)$. This implies that

$$\begin{aligned} y(x) &= \left(1 + aI^{\alpha+\beta}\right)^{-1} \left(I^{\alpha} f(x, y(x)) + b \int_0^{1/2} y(x)dx \right) \\ &= \sum_{k=0}^{\infty} (-1)^k a^k I^{(\alpha+\beta)k+\alpha} f(x, y(x)) + b \int_0^{1/2} y(x)dx \sum_{k=0}^{\infty} (-1)^k a^k I^{(\alpha+\beta)k} \\ &= \sum_{k=0}^{\infty} (-1)^k a^k I^{(\alpha+\beta)k+\alpha} f(x, y(x)) + b \int_0^{1/2} y(x)dx \sum_{k=0}^{\infty} (-1)^k a^k \frac{x^{(\alpha+\beta)k}}{\Gamma((\alpha+\beta)k+1)}. \end{aligned}$$

The above integral equation is clearly equivalent to Equation (1.2) with the initial condition, since

$$y(0) = 0 + b \int_0^{1/2} y(x)dx(1 + 0) = b \int_0^{1/2} y(x)dx,$$

by noting that $\alpha > 0$ and

$$ {}_{C}D^{\alpha} y(x) + aI^{\beta} y(x) = \sum_{k=0}^{\infty} (-1)^k a^k I^{(\alpha+\beta)k} f(x, y(x)) + b \int_0^{1/2} y(x)dx \sum_{k=1}^{\infty} (-1)^k a^k \frac{x^{(\alpha+\beta)k-\alpha}}{\Gamma((\alpha+\beta)k+1-\alpha)} + \sum_{k=0}^{\infty} (-1)^k a^{k+1} I^{(\alpha+\beta)(k+1)} f(x, y(x)) + b \int_0^{1/2} y(x)dx \sum_{k=0}^{\infty} (-1)^k a^{k+1} \frac{x^{(\alpha+\beta)k+\beta}}{\Gamma((\alpha+\beta)k+1+\beta)}. $$

Clearly,

$$ \begin{aligned} \sum_{k=0}^{\infty} (-1)^k a^k I^{(\alpha+\beta)k} f(x, y(x)) &= f(x, y(x)) + \sum_{k=1}^{\infty} (-1)^k a^k I^{(\alpha+\beta)k} f(x, y(x)) \\ &= f(x, y(x)) + \sum_{k=0}^{\infty} (-1)^{k+1} a^{k+1} I^{(\alpha+\beta)(k+1)} f(x, y(x)) \\ &= f(x, y(x)) - \sum_{k=0}^{\infty} (-1)^k a^{k+1} I^{(\alpha+\beta)(k+1)} f(x, y(x)). \end{aligned} $$

Similarly,

$$ b \int_0^{1/2} y(x)dx \sum_{k=1}^{\infty} (-1)^k a^k \frac{x^{(\alpha+\beta)k-\alpha}}{\Gamma((\alpha+\beta)k+1-\alpha)} = -b \int_0^{1/2} y(x)dx \sum_{k=0}^{\infty} (-1)^k a^{k+1} \frac{x^{(\alpha+\beta)k+\beta}}{\Gamma((\alpha+\beta)k+1+\beta)}. $$

Therefore, $y(x)$ satisfies Equation (1.2).

Furthermore, we assume f is a continuous and bounded function over $[0, 1] \times \mathbb{R}$, and

$$ D = 1 - \frac{|b|}{2} E_{(\alpha+\beta,\ 1)}(|a|) > 0. $$

Then, y is uniformly bounded on $[0, 1]$. Indeed,

$$ \|y\| \leq \sum_{k=0}^{\infty} \frac{|a|^k}{\Gamma((\alpha+\beta)k+\alpha+1)} \sup_{(x,y)\in[0,1]\times\mathbb{R}} |f(x, y)| + \frac{|b|}{2} \|y\| E_{(\alpha+\beta,\ 1)}(|a|). $$

Thus,

$$ \|y\| \leq \frac{1}{D} E_{\alpha+\beta,\alpha+1}(|a|) \sup_{(x,y)\in[0,1]\times\mathbb{R}} |f(x, y)| < +\infty, $$

which claims y is uniformly bounded. If the function f further satisfies the following Lipschitz condition:

$$ |f(x, z_1) - f(x, z_2)| \leq \mathbb{L}|z_1 - z_2|, \quad z_1, z_2 \in \mathbb{R}, $$

and

$$ q = \mathbb{L}E_{(\alpha+\beta,\alpha+1)}(|a|) + \frac{|b|}{2} E_{(\alpha+\beta,\ 1)}(|a|) < 1, $$

then Equation (1.2) has a unique solution in $C[0, 1]$ by Banach's contractive principle. To prove this, we define a mapping $\mathbb{M}$ over $C[0, 1]$ as

$$ (\mathbb{M}y)(x) = \sum_{k=0}^{\infty} (-1)^k a^k I^{(\alpha+\beta)k+\alpha} f(x, y(x)) + b \int_0^{1/2} y(x)dx \sum_{k=0}^{\infty} (-1)^k a^k \frac{x^{(\alpha+\beta)k}}{\Gamma((\alpha+\beta)k+1)}. $$

Then, $\mathbb{M}y \in C[0, 1]$. We need to show $\mathbb{M}$ is contractive. Clearly,

$$ \|\mathbb{M}y_1 - \mathbb{M}y_2\| \leq \mathbb{L}E_{(\alpha+\beta,\alpha+1)}(|a|) \|y_1 - y_2\| + \frac{|b|}{2} E_{(\alpha+\beta,\ 1)}(|a|) \|y_1 - y_2\| = q \|y_1 - y_2\|. $$

Since $q < 1$, Equation (1.2) has a unique solution in $C[0, 1]$ using Banach's contractive principle.

2 | UNIQUENESS

In this section, we are going to provide sufficient conditions for the uniqueness of Equation (1.1) based on Banach's contractive principle, an implicit integral equation, and Babenko's approach.

Theorem 1. *Assume $m, l \in \mathbb{N}$, $\phi_i, \psi : [0,1]\times\mathbb{R} \to \mathbb{R}$ are continuous and bounded functions for $i = 1, 2, \cdots, l$, and $\gamma : C[0,1] \to \mathbb{R}$ is a functional. In addition, we suppose*

$$v = 1 - E_{(\beta_1,\cdots,\beta_m),\ 2}(|\lambda_1|,\cdots,|\lambda_m|)\sum_{i=1}^{m}\frac{|\lambda_i|}{\Gamma(\beta_i+1)} > 0.$$

Then, Equation (1.1) with the boundary conditions is equivalent to the following implicit integral equation in $C[0,1]$:

$$\begin{aligned}
\zeta(x) = & \sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,\cdots,k_m}\lambda_1^{k_1}\cdots\lambda_m^{k_m} I^{\beta_1 k_1+\cdots+\beta_m k_m+\alpha}\psi(x,\zeta(x)) \\
& + \sum_{i=1}^{l}\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,\cdots,k_m}\lambda_1^{k_1}\cdots\lambda_m^{k_m} I^{\beta_1 k_1+\cdots+\beta_m k_m+\alpha_i}\phi_i(x,\zeta(x)) \\
& + \zeta_1\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,\cdots,k_m}\lambda_1^{k_1}\cdots\lambda_m^{k_m} \\
& \cdot\left[\frac{x^{\beta_1 k_1+\cdots+\beta_m k_m}}{\Gamma(\beta_1 k_1+\cdots+\beta_m k_m+1)} - \frac{x^{\beta_1 k_1+\cdots+\beta_m k_m+1}}{\Gamma(\beta_1 k_1+\cdots+\beta_m k_m+2)}\right] \\
& + \gamma(\zeta)\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,\cdots,k_m}\lambda_1^{k_1}\cdots\lambda_m^{k_m}\frac{x^{\beta_1 k_1+\cdots+\beta_m k_m+1}}{\Gamma(\beta_1 k_1+\cdots+\beta_m k_m+2)} \\
& - I^{\alpha}_{x=1}\psi(x,\zeta(x))\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,\cdots,k_m}\lambda_1^{k_1}\cdots\lambda_m^{k_m}\frac{x^{\beta_1 k_1+\cdots+\beta_m k_m+1}}{\Gamma(\beta_1 k_1+\cdots+\beta_m k_m+2)} \\
& - \sum_{i=1}^{m}\lambda_i\ I^{\beta_i}_{x=1}\zeta(x)\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,\cdots,k_m}\lambda_1^{k_1}\cdots\lambda_m^{k_m}\frac{x^{\beta_1 k_1+\cdots+\beta_m k_m+1}}{\Gamma(\beta_1 k_1+\cdots+\beta_m k_m+2)} \\
& - \sum_{i=1}^{l} I^{\alpha_i}_{x=1}\phi_i(x,\zeta(x))\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,\cdots,k_m}\lambda_1^{k_1}\cdots\lambda_m^{k_m}\frac{x^{\beta_1 k_1+\cdots+\beta_m k_m+1}}{\Gamma(\beta_1 k_1+\cdots+\beta_m k_m+2)}.
\end{aligned} \tag{2.1}$$

Furthermore,

$$\begin{aligned}
\|\zeta\| \leq & \frac{1}{v}E_{(\beta_1,\cdots,\beta_m),\ \alpha+1}(|\lambda_1|,\cdots,|\lambda_m|)\sup_{(x,y)\in[0,1]\times\mathbb{R}}|\psi(x,y)| \\
& + \frac{1}{v}\sum_{i=1}^{l}\sup_{(x,y)\in[0,1]\times\mathbb{R}}|\phi_i(x,y)|E_{(\beta_1,\cdots,\beta_m),\ \alpha_i+1}(|\lambda_1|,\cdots,|\lambda_m|) \\
& + \frac{|\zeta_1|}{v}E_{(\beta_1,\cdots,\beta_m),\ 1}(|\lambda_1|,\cdots,|\lambda_m|) + \frac{|\gamma(\zeta)|}{v}E_{(\beta_1,\cdots,\beta_m),\ 2}(|\lambda_1|,\cdots,|\lambda_m|) \\
& + \frac{\sup_{(x,y)\in[0,1]\times\mathbb{R}}|\psi(x,y)|}{v\,\Gamma(\alpha+1)}E_{(\beta_1,\cdots,\beta_m),\ 2}(|\lambda_1|,\cdots,|\lambda_m|) \\
& + \frac{1}{v}E_{(\beta_1,\cdots,\beta_m),\ 2}(|\lambda_1|,\cdots,|\lambda_m|)\sum_{i=1}^{l}\frac{\sup_{(x,y)\in[0,1]\times\mathbb{R}}|\phi_i(x,y)|}{\Gamma(\alpha_i+1)} < +\infty.
\end{aligned} \tag{2.2}$$

Proof. We start by applying the integral operator I^{α} to equation

$${}_{C}D^{\alpha}\left[\zeta(x) - \sum_{i=1}^{m}\lambda_i\ I^{\beta_i}\zeta(x) - \sum_{i=1}^{l}I^{\alpha_i}\phi_i(x,\zeta(x))\right] = \psi(x,\zeta(x)),$$

to get

$$\zeta(x) - \sum_{i=1}^{m} \lambda_i\ I^{\beta_i}\zeta(x) - \sum_{i=1}^{l} I^{\alpha_i}\phi_i(x,\zeta(x)) + c_0 + c_1 x = I^{\alpha}\psi(x,\zeta(x)),$$

where c_0, c_1 are to be determined using the boundary conditions.

It follows from setting $x = 0$ that

$$\zeta(0) + c_0 = 0,$$

which deduces $c_0 = -\zeta_1$. From $x = 1$, we have

$$c_1 = I^{\alpha}_{x=1}\psi(x,\zeta(x)) + \sum_{i=1}^{m} \lambda_i\ I^{\beta_i}_{x=1}\zeta(x) + \sum_{i=1}^{l} I^{\alpha_i}_{x=1}\phi_i(x,\zeta(x)) + \zeta_1 - \gamma(\zeta).$$

Thus,

$$\begin{aligned}\left(1 - \sum_{i=1}^{m} \lambda_i\ I^{\beta_i}\right)\zeta(x) &= I^{\alpha}\psi(x,\zeta(x)) + \sum_{i=1}^{l} I^{\alpha_i}\phi_i(x,\zeta(x)) + \zeta_1(1-x) + \gamma(\zeta)x \\ &\quad - I^{\alpha}_{x=1}\psi(x,\zeta(x))x - \sum_{i=1}^{m} \lambda_i\ I^{\beta_i}_{x=1}\zeta(x)x - \sum_{i=1}^{l} I^{\alpha_i}_{x=1}\phi_i(x,\zeta(x))x.\end{aligned}$$

Following the proof in Section 1, we can show that the inverse operator of $\left(1 - \sum_{i=1}^{m} \lambda_i\ I^{\beta_i}\right)$ uniquely exists in $C[0,1]$.

By Babenko's approach described above, we come to

$$\begin{aligned}\zeta(x) &= \left(1 - \sum_{i=1}^{m} \lambda_i\ I^{\beta_i}\right)^{-1} \left[I^{\alpha}\psi(x,\zeta(x)) + \sum_{i=1}^{l} I^{\alpha_i}\phi_i(x,\zeta(x)) + \zeta_1(1-x) + \gamma(\zeta)x\right. \\ &\quad \left. - I^{\alpha}_{x=1}\psi(x,\zeta(x))x - \sum_{i=1}^{m} \lambda_i\ I^{\beta_i}_{x=1}\zeta(x)x - \sum_{i=1}^{l} I^{\alpha_i}_{x=1}\phi_i(x,\zeta(x))x\right] \\ &= \sum_{k=0}^{\infty}\left(\sum_{i=1}^{m} \lambda_i\ I^{\beta_i}\right)^{k} \left[I^{\alpha}\psi(x,\zeta(x)) + \sum_{i=1}^{l} I^{\alpha_i}\phi_i(x,\zeta(x)) + \zeta_1(1-x) + \gamma(\zeta)x\right. \\ &\quad \left. - I^{\alpha}_{x=1}\psi(x,\zeta(x))x - \sum_{i=1}^{m} \lambda_i\ I^{\beta_i}_{x=1}\zeta(x)x - \sum_{i=1}^{l} I^{\alpha_i}_{x=1}\phi_i(x,\zeta(x))x\right] \\ &= \sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,\cdots,k_m}\lambda_1^{k_1}\cdots\lambda_m^{k_m} I^{\beta_1 k_1+\cdots+\beta_m k_m+\alpha}\psi(x,\zeta(x)) \\ &\quad + \sum_{i=1}^{l}\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,\cdots,k_m}\lambda_1^{k_1}\cdots\lambda_m^{k_m} I^{\beta_1 k_1+\cdots+\beta_m k_m+\alpha_i}\phi_i(x,\zeta(x)) \\ &\quad + \zeta_1\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,\cdots,k_m}\lambda_1^{k_1}\cdots\lambda_m^{k_m} I^{\beta_1 k_1+\cdots+\beta_m k_m}(1-x) \\ &\quad + \gamma(\zeta)\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,\cdots,k_m}\lambda_1^{k_1}\cdots\lambda_m^{k_m} I^{\beta_1 k_1+\cdots+\beta_m k_m}x \\ &\quad - I^{\alpha}_{x=1}\psi(x,\zeta(x))\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,\cdots,k_m}\lambda_1^{k_1}\cdots\lambda_m^{k_m} I^{\beta_1 k_1+\cdots+\beta_m k_m}x \\ &\quad - \sum_{i=1}^{m}\lambda_i\ I^{\beta_i}_{x=1}\zeta(x)\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,\cdots,k_m}\lambda_1^{k_1}\cdots\lambda_m^{k_m} I^{\beta_1 k_1+\cdots+\beta_m k_m}x \\ &\quad - \sum_{i=1}^{l} I^{\alpha_i}_{x=1}\phi_i(x,\zeta(x))\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,\cdots,k_m}\lambda_1^{k_1}\cdots\lambda_m^{k_m} I^{\beta_1 k_1+\cdots+\beta_m k_m}x \\ &= I_1 + \cdots + I_7.\end{aligned}$$

Clearly,

$$\begin{aligned}
I_3 &= \zeta_1 \sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_m=k} \binom{k}{k_1,\cdots,k_m} \lambda_1^{k_1}\cdots\lambda_m^{k_m} I^{\beta_1 k_1+\cdots+\beta_m k_m}(1-x) \\
&= \zeta_1 \sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_m=k} \binom{k}{k_1,\cdots,k_m} \lambda_1^{k_1}\cdots\lambda_m^{k_m} \\
&\quad \cdot \left[\frac{x^{\beta_1 k_1+\cdots+\beta_m k_m}}{\Gamma(\beta_1 k_1+\cdots+\beta_m k_m+1)} - \frac{x^{\beta_1 k_1+\cdots+\beta_m k_m+1}}{\Gamma(\beta_1 k_1+\cdots+\beta_m k_m+2)} \right], \\
I_4 &= \gamma(\zeta) \sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_m=k} \binom{k}{k_1,\cdots,k_m} \lambda_1^{k_1}\cdots\lambda_m^{k_m} \frac{x^{\beta_1 k_1+\cdots+\beta_m k_m+1}}{\Gamma(\beta_1 k_1+\cdots+\beta_m k_m+2)}, \\
I_5 &= -I_{x=1}^{\alpha}\psi(x,\zeta(x)) \sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_m=k} \binom{k}{k_1,\cdots,k_m} \lambda_1^{k_1}\cdots\lambda_m^{k_m} \frac{x^{\beta_1 k_1+\cdots+\beta_m k_m+1}}{\Gamma(\beta_1 k_1+\cdots+\beta_m k_m+2)}, \\
I_6 &= -\sum_{i=1}^{m} \lambda_i\, I_{x=1}^{\beta_i}\zeta(x) \sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_m=k} \binom{k}{k_1,\cdots,k_m} \lambda_1^{k_1}\cdots\lambda_m^{k_m} \frac{x^{\beta_1 k_1+\cdots+\beta_m k_m+1}}{\Gamma(\beta_1 k_1+\cdots+\beta_m k_m+2)},
\end{aligned}$$

and

$$I_7 = -\sum_{i=1}^{l} I_{x=1}^{\alpha_i}\phi_i(x,\zeta(x)) \sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_m=k} \binom{k}{k_1,\cdots,k_m} \lambda_1^{k_1}\cdots\lambda_m^{k_m} \frac{x^{\beta_1 k_1+\cdots+\beta_m k_m+1}}{\Gamma(\beta_1 k_1+\cdots+\beta_m k_m+2)}.$$

In summary,

$$\begin{aligned}
\zeta(x) &= \sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_m=k} \binom{k}{k_1,\cdots,k_m} \lambda_1^{k_1}\cdots\lambda_m^{k_m} I^{\beta_1 k_1+\cdots+\beta_m k_m+\alpha}\psi(x,\zeta(x)) \\
&\quad + \sum_{i=1}^{l}\sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_m=k} \binom{k}{k_1,\cdots,k_m} \lambda_1^{k_1}\cdots\lambda_m^{k_m} I^{\beta_1 k_1+\cdots+\beta_m k_m+\alpha_i}\phi_i(x,\zeta(x)) \\
&\quad + \zeta_1 \sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_m=k} \binom{k}{k_1,\cdots,k_m} \lambda_1^{k_1}\cdots\lambda_m^{k_m} \\
&\quad \cdot \left[\frac{x^{\beta_1 k_1+\cdots+\beta_m k_m}}{\Gamma(\beta_1 k_1+\cdots+\beta_m k_m+1)} - \frac{x^{\beta_1 k_1+\cdots+\beta_m k_m+1}}{\Gamma(\beta_1 k_1+\cdots+\beta_m k_m+2)} \right] \\
&\quad + \gamma(\zeta) \sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_m=k} \binom{k}{k_1,\cdots,k_m} \lambda_1^{k_1}\cdots\lambda_m^{k_m} \frac{x^{\beta_1 k_1+\cdots+\beta_m k_m+1}}{\Gamma(\beta_1 k_1+\cdots+\beta_m k_m+2)} \\
&\quad - I_{x=1}^{\alpha}\psi(x,\zeta(x)) \sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_m=k} \binom{k}{k_1,\cdots,k_m} \lambda_1^{k_1}\cdots\lambda_m^{k_m} \frac{x^{\beta_1 k_1+\cdots+\beta_m k_m+1}}{\Gamma(\beta_1 k_1+\cdots+\beta_m k_m+2)} \\
&\quad - \sum_{i=1}^{m} \lambda_i\, I_{x=1}^{\beta_i}\zeta(x) \sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_m=k} \binom{k}{k_1,\cdots,k_m} \lambda_1^{k_1}\cdots\lambda_m^{k_m} \frac{x^{\beta_1 k_1+\cdots+\beta_m k_m+1}}{\Gamma(\beta_1 k_1+\cdots+\beta_m k_m+2)} \\
&\quad - \sum_{i=1}^{l} I_{x=1}^{\alpha_i}\phi_i(x,\zeta(x)) \sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_m=k} \binom{k}{k_1,\cdots,k_m} \lambda_1^{k_1}\cdots\lambda_m^{k_m} \frac{x^{\beta_1 k_1+\cdots+\beta_m k_m+1}}{\Gamma(\beta_1 k_1+\cdots+\beta_m k_m+2)},
\end{aligned}$$

which is equivalent to Equation (1.1) with the boundary conditions since all the above steps are reversible.

Furthermore,

$$\begin{aligned}
\|\zeta\| \leq & \sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_m=k} \binom{k}{k_1,\cdots,k_m} |\lambda_1|^{k_1}\cdots|\lambda_m|^{k_m} \frac{\sup\limits_{(x,y)\in[0,1]\times\mathbb{R}} |\psi(x,y)|}{\Gamma(\beta_1 k_1+\cdots+\beta_m k_m+\alpha+1)} \\
& + \sum_{i=1}^{l} \sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_m=k} \binom{k}{k_1,\cdots,k_m} |\lambda_1|^{k_1}\cdots|\lambda_m|^{k_m} \frac{\sup\limits_{(x,y)\in[0,1]\times\mathbb{R}} |\phi_i(x,y)|}{\Gamma(\beta_1 k_1+\cdots+\beta_m k_m+\alpha_i+1)} \\
& + |\zeta_1| \sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_m=k} \binom{k}{k_1,\cdots,k_m} |\lambda_1|^{k_1}\cdots|\lambda_m|^{k_m} \frac{1}{\Gamma(\beta_1 k_1+\cdots+\beta_m k_m+1)} \\
& + |\gamma(\zeta)| \sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_m=k} \binom{k}{k_1,\cdots,k_m} |\lambda_1|^{k_1}\cdots|\lambda_m|^{k_m} \frac{1}{\Gamma(\beta_1 k_1+\cdots+\beta_m k_m+2)} \\
& + \frac{1}{\Gamma(\alpha+1)} \sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_m=k} \binom{k}{k_1,\cdots,k_m} |\lambda_1|^{k_1}\cdots|\lambda_m|^{k_m} \frac{\sup\limits_{(x,y)\in[0,1]\times\mathbb{R}} |\psi(x,y)|}{\Gamma(\beta_1 k_1+\cdots+\beta_m k_m+2)} \\
& + \|\zeta\| \sum_{i=1}^{m} \frac{|\lambda_i|}{\Gamma(\beta_i+1)} \sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_m=k} \binom{k}{k_1,\cdots,k_m} |\lambda_1|^{k_1}\cdots|\lambda_m|^{k_m} \\
& \cdot \frac{1}{\Gamma(\beta_1 k_1+\cdots+\beta_m k_m+2)} + \sum_{i=1}^{l} \frac{\sup\limits_{(x,y)\in[0,1]\times\mathbb{R}} |\phi_i(x,y)|}{\Gamma(\alpha_i+1)} \sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_m=k} \\
& \cdot \binom{k}{k_1,\cdots,k_m} |\lambda_1|^{k_1}\cdots|\lambda_m|^{k_m} \frac{1}{\Gamma(\beta_1 k_1+\cdots+\beta_m k_m+2)} \\
= & E_{(\beta_1,\cdots,\beta_m),\ \alpha+1}(|\lambda_1|,\cdots,|\lambda_m|) \sup_{(x,y)\in[0,1]\times\mathbb{R}} |\psi(x,y)| \\
& + \sum_{i=1}^{l} \sup_{(x,y)\in[0,1]\times\mathbb{R}} |\phi_i(x,y)| E_{(\beta_1,\cdots,\beta_m),\ \alpha_i+1}(|\lambda_1|,\cdots,|\lambda_m|) \\
& + |\zeta_1| E_{(\beta_1,\cdots,\beta_m),\ 1}(|\lambda_1|,\cdots,|\lambda_m|) + |\gamma(\zeta)| E_{(\beta_1,\cdots,\beta_m),\ 2}(|\lambda_1|,\cdots,|\lambda_m|) \\
& + \frac{\sup\limits_{(x,y)\in[0,1]\times\mathbb{R}} |\psi(x,y)|}{\Gamma(\alpha+1)} E_{(\beta_1,\cdots,\beta_m),\ 2}(|\lambda_1|,\cdots,|\lambda_m|) \\
& + \|\zeta\| E_{(\beta_1,\cdots,\beta_m),\ 2}(|\lambda_1|,\cdots,|\lambda_m|) \sum_{i=1}^{m} \frac{|\lambda_i|}{\Gamma(\beta_i+1)} \\
& + E_{(\beta_1,\cdots,\beta_m),\ 2}(|\lambda_1|,\cdots,|\lambda_m|) \sum_{i=1}^{l} \frac{\sup\limits_{(x,y)\in[0,1]\times\mathbb{R}} |\phi_i(x,y)|}{\Gamma(\alpha_i+1)}.
\end{aligned}$$

Since

$$v = 1 - E_{(\beta_1,\cdots,\beta_m),\ 2}(|\lambda_1|,\cdots,|\lambda_m|) \sum_{i=1}^{m} \frac{|\lambda_i|}{\Gamma(\beta_i+1)} > 0,$$

which infers that

$$\begin{aligned}
\|\zeta\| \leq & \frac{1}{v} E_{(\beta_1,\cdots,\beta_m),\ \alpha+1}(|\lambda_1|,\cdots,|\lambda_m|) \sup_{(x,y)\in[0,1]\times\mathbb{R}} |\psi(x,y)| \\
& + \frac{1}{v} \sum_{i=1}^{l} \sup_{(x,y)\in[0,1]\times\mathbb{R}} |\phi_i(x,y)| E_{(\beta_1,\cdots,\beta_m),\ \alpha_i+1}(|\lambda_1|,\cdots,|\lambda_m|) \\
& + \frac{|\zeta_1|}{v} E_{(\beta_1,\cdots,\beta_m),\ 1}(|\lambda_1|,\cdots,|\lambda_m|) + \frac{|\gamma(\zeta)|}{v} E_{(\beta_1,\cdots,\beta_m),\ 2}(|\lambda_1|,\cdots,|\lambda_m|)
\end{aligned}$$

$$+\frac{\sup\limits_{(x,y)\in[0,1]\times\mathbb{R}}|\psi(x,y)|}{\nu\,\Gamma(\alpha+1)}E_{(\beta_1,\cdots,\beta_m),\,2}(|\lambda_1|,\cdots,|\lambda_m|)$$
$$+\frac{1}{\nu}E_{(\beta_1,\cdots,\beta_m),\,2}(|\lambda_1|,\cdots,|\lambda_m|)\sum_{i=1}^{l}\frac{\sup\limits_{(x,y)\in[0,1]\times\mathbb{R}}|\phi_i(x,y)|}{\Gamma(\alpha_i+1)}<+\infty,$$

by noting that all ϕ_i and ψ are bounded functions over $[0,1]\times\mathbb{R}$. This completes the proof. □

The following is the theorem regarding the uniqueness of Equation (1.1).

Theorem 2. *Assume $m, l \in \mathbb{N}$, $\phi_i, \psi : [0,1]\times\mathbb{R}\to\mathbb{R}$ are continuous and bounded functions for $i = 1,2,\cdots,l$, satisfying the following Lipschitz conditions:*

$$|\psi(x,y_1)-\psi(x,y_2)|\le\mathcal{L}_0|y_1-y_2|,\quad y_1,y_2\in\mathbb{R},$$

and

$$|\phi_i(x,z_1)-\phi_i(x,z_2)|\le\mathcal{L}_i|z_1-z_2|,\quad z_1,z_2\in\mathbb{R},$$

where $\mathcal{L}_0$ and $\mathcal{L}_i$ are nonnegative constants. Let $\gamma : C[0,1]\to\mathbb{R}$ be a functional satisfying the following condition for a nonnegative constant γ_0:

$$|\gamma(\zeta_1)-\gamma(\zeta_2)|\le\gamma_0\,\|\zeta_1-\zeta_2\|,\quad \zeta_1,\ \zeta_2\in C[0,1].$$

Furthermore, we suppose

$$\begin{aligned}u=&\mathcal{L}_0E_{(\beta_1,\cdots,\beta_m),\,\alpha+1}(|\lambda_1|,\cdots,|\lambda_m|)+\sum_{i=1}^{l}\mathcal{L}_iE_{(\beta_1,\cdots,\beta_m),\,\alpha_i+1}(|\lambda_1|,\cdots,|\lambda_m|)\\&+\left[\gamma_0+\frac{\mathcal{L}_0}{\Gamma(\alpha+1)}+\sum_{i=1}^{m}\frac{|\lambda_i|}{\Gamma(\beta_i+1)}+\sum_{i=1}^{l}\frac{\mathcal{L}_i}{\Gamma(\alpha_i+1)}\right]E_{(\beta_1,\cdots,\beta_m),\,2}(|\lambda_1|,\cdots,|\lambda_m|)<1.\end{aligned}\tag{2.3}$$

Then, Equation (1.1) has a unique solution in $C[0,1]$.

Proof. We begin by defining a nonlinear mapping $\mathbb{M}_1$ over the space $C[0,1]$ as

$$\begin{aligned}(\mathbb{M}_1\zeta)(x)=&\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,\cdots,k_m}\lambda_1^{k_1}\cdots\lambda_m^{k_m}I^{\beta_1k_1+\cdots+\beta_mk_m+\alpha}\psi(x,\zeta(x))\\
&+\sum_{i=1}^{l}\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,\cdots,k_m}\lambda_1^{k_1}\cdots\lambda_m^{k_m}I^{\beta_1k_1+\cdots+\beta_mk_m+\alpha_i}\phi_i(x,\zeta(x))\\
&+\zeta_1\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,\cdots,k_m}\lambda_1^{k_1}\cdots\lambda_m^{k_m}\\
&\cdot\left[\frac{x^{\beta_1k_1+\cdots+\beta_mk_m}}{\Gamma(\beta_1k_1+\cdots+\beta_mk_m+1)}-\frac{x^{\beta_1k_1+\cdots+\beta_mk_m+1}}{\Gamma(\beta_1k_1+\cdots+\beta_mk_m+2)}\right]\\
&+\gamma(\zeta)\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,\cdots,k_m}\lambda_1^{k_1}\cdots\lambda_m^{k_m}\frac{x^{\beta_1k_1+\cdots+\beta_mk_m+1}}{\Gamma(\beta_1k_1+\cdots+\beta_mk_m+2)}\\
&-I^{\alpha}_{x=1}\psi(x,\zeta(x))\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,\cdots,k_m}\lambda_1^{k_1}\cdots\lambda_m^{k_m}\frac{x^{\beta_1k_1+\cdots+\beta_mk_m+1}}{\Gamma(\beta_1k_1+\cdots+\beta_mk_m+2)}\\
&-\sum_{i=1}^{m}\lambda_i\ I^{\beta_i}_{x=1}\zeta(x)\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,\cdots,k_m}\lambda_1^{k_1}\cdots\lambda_m^{k_m}\frac{x^{\beta_1k_1+\cdots+\beta_mk_m+1}}{\Gamma(\beta_1k_1+\cdots+\beta_mk_m+2)}\\
&-\sum_{i=1}^{l}I^{\alpha_i}_{x=1}\phi_i(x,\zeta(x))\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,\cdots,k_m}\lambda_1^{k_1}\cdots\lambda_m^{k_m}\frac{x^{\beta_1k_1+\cdots+\beta_mk_m+1}}{\Gamma(\beta_1k_1+\cdots+\beta_mk_m+2)}.\end{aligned}$$

It follows from the proof of Theorem 1 that $\mathbb{M}_1\zeta \in C[0,1]$. It remains to be shown that $\mathbb{M}_1$ is contractive. To do so, we consider

$$\begin{aligned}(\mathbb{M}_1\zeta_1)(x) - (\mathbb{M}_1\zeta_2)(x) &= \sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,\cdots,k_m}\lambda_1^{k_1}\cdots\lambda_m^{k_m} I^{\beta_1k_1+\cdots+\beta_mk_m+\alpha}(\psi(x,\zeta_1(x)) - \psi(x,\zeta_2(x)))\\ &+ \sum_{i=1}^{l}\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,\cdots,k_m}\lambda_1^{k_1}\cdots\lambda_m^{k_m} I^{\beta_1k_1+\cdots+\beta_mk_m+\alpha_i}\\ &\cdot(\phi_i(x,\zeta_1(x)) - \phi_i(x,\zeta_2(x)))\\ &+ \Bigg[\gamma(\zeta_1) - \gamma(\zeta_2) - I_{x=1}^{\alpha}(\psi(x,\zeta_1(x)) - \psi(x,\zeta_2(x))) - \sum_{i=1}^{m}\lambda_i\, I_{x=1}^{\beta_i}(\zeta_1(x) - \zeta_2(x))\\ &- \sum_{i=1}^{l} I_{x=1}^{\alpha_i}(\phi_i(x,\zeta_1(x)) - \phi_i(x,\zeta_2(x)))\Bigg]\\ &\cdot\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,\cdots,k_m}\lambda_1^{k_1}\cdots\lambda_m^{k_m}\frac{x^{\beta_1k_1+\cdots+\beta_mk_m+1}}{\Gamma(\beta_1k_1+\cdots+\beta_mk_m+2)}.\end{aligned}$$

This implies that

$$\begin{aligned}\|\mathbb{M}_1\zeta_1 - \mathbb{M}_1\zeta_2\| &\le \mathcal{L}_0E_{(\beta_1,\cdots,\beta_m),\,\alpha+1}(|\lambda_1|,\cdots,|\lambda_m|) + \sum_{i=1}^{l}\mathcal{L}_iE_{(\beta_1,\cdots,\beta_m),\,\alpha_i+1}(|\lambda_1|,\cdots,|\lambda_m|)\,\|\zeta_1-\zeta_2\|\\ &+ \left[\gamma_0 + \frac{\mathcal{L}_0}{\Gamma(\alpha+1)} + \sum_{i=1}^{m}\frac{|\lambda_i|}{\Gamma(\beta_i+1)} + \sum_{i=1}^{l}\frac{\mathcal{L}_i}{\Gamma(\alpha_i+1)}\right]E_{(\beta_1,\cdots,\beta_m),\,2}(|\lambda_1|,\cdots,|\lambda_m|)\\ &\cdot\|\zeta_1-\zeta_2\| = u\,\|\zeta_1-\zeta_2\|.\end{aligned}$$

Since $u < 1$, Equation (1.1) has a unique solution in $C[0,1]$ by Banach's contractive principle. This completes the proof. □

Example 3. The following fractional integro-differential equation with a functional boundary condition:

$$\begin{cases} {}_CD^{1.5}\left[\zeta(x) - \frac{1}{17}\,I^{0.5}\zeta(x) + \frac{1}{51}\,I^{1.7}\zeta(x) - \frac{1}{15}I^{2.5}\sin\frac{|x\zeta(x)|}{2} - \frac{1}{19}I^{1.1}\frac{\zeta(x)}{1+\zeta^2(x)}\right]\\ = \frac{1}{48}\arctan(x^2+3\zeta(x)) + 1,\quad x\in[0,1],\\ \zeta(0)=1,\quad \zeta(1) = \frac{1}{57}\cos\zeta(1/2), \end{cases} \tag{2.4}$$

has a unique solution in $C[0,1]$.

Proof. Evidently,

$$\psi(x,y) = \frac{1}{48}\arctan(x^2+3y) + 1$$

is a continuous and bounded function over $[0,1]\times\mathbb{R}$, satisfying the condition

$$|\psi(x,y_1) - \psi(x,y_2)| = \frac{1}{48}|\arctan(x^2+3y_1) - \arctan(x^2+3y_2)| \le \frac{1}{16}|y_1-y_2|,\quad y_1,y_2\in\mathbb{R},$$

by using the fact that

$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2} \le 1,$$

and the mean value theorem. Similarly,

$$\phi_1(x,y)=\frac{1}{15}\sin\frac{|xy|}{2} \text{ and } \phi_2(x,y)=\frac{y}{19(1+y^2)}$$

are continuous and bounded functions over $[0,1]\times\mathbb{R}$, satisfying the conditions:

$$|\phi_1(x,y_1)-\phi_1(x,y_2)|\le\frac{1}{30}|y_1-y_2|, \quad y_1,y_2\in\mathbb{R},$$

and

$$|\phi_2(x,y_1)-\phi_2(x,y_2)|\le\frac{1}{19}|y_1-y_2|,$$

by noting that

$$\left|\frac{d}{dy}\left(\frac{y}{1+y^2}\right)\right|=\left|\frac{1-y^2}{(1+y^2)^2}\right|\le 1.$$

Moreover,

$$\gamma(\zeta)=\frac{1}{57}\cos\zeta(1/2),$$

is a functional from $C[0,1]$ to $\mathbb{R}$, satisfying the inequality

$$|\gamma(\zeta_1)-\gamma(\zeta_2)|\le\frac{1}{57}|\cos\zeta_1(1/2)-\cos\zeta_2(1/2)|\le\frac{1}{57}\|\zeta_1-\zeta_2\|,$$

if $\zeta_1,\zeta_2\in C[0,1]$. From the above, we have

$$\alpha=1.5,\ \beta_1=0.5,\ \beta_2=1.7,\ \mathcal{L}_0=1/16,\ \mathcal{L}_1=1/30,\ \mathcal{L}_2=1/19,$$
$$\gamma_0=1/57,\ |\lambda_1|=1/17,\ |\lambda_2|=1/51,\ \alpha_1=2.5,\ \alpha_2=1.1.$$

So

$$u=\frac{1}{16}E_{(0.5,1.7),\ 2.5}(1/17,1/51)+\frac{1}{30}E_{(0.5,1.7),\ 3.5}(1/17,1/51)+\frac{1}{19}E_{(0.5,1.7),\ 2.1}(1/17,1/51)$$
$$+\left[\frac{1}{57}+\frac{1}{16\Gamma(1.5+1)}+\frac{1}{17\Gamma(0.5+1)}+\frac{1}{51\Gamma(1.7+1)}+\frac{1}{30\Gamma(2.5+1)}+\frac{1}{19\Gamma(1.1+1)}\right]\cdot E_{(0.5,1.7),\ 2}(1/17,1/51).$$

By the following Python codes, we get

$$u\approx 0.326624406640162<1.$$

Thus, Equation (2.4) has a unique solution in $C[0,1]$ by Theorem 1. □

```
# beginning codes
import math
from sympy import gamma

def partition(n, m):
    if m == 1:
        yield (n,)
   else:
       for i in range(n+1):
           for j in partition(n-i, m-1):
             yield (i,) + j
```

```
def MML(zeta, epsilon, z): #multivariate mittag-leffler function
    m = len(zeta)
    if m != len(z):
      return "zeta and z have unequal lengths"
      result = 0
      for l in range(0, 40): #approximate upper bound
        for l_partition in partition(l, m):
          if all(map(lambda x: x >= 0, l_partition)):
            combination = 1
            for i in range(m):
                combination *= math.factorial(l_partition[i])
            combination = math.factorial(l) / combination
            gaminput = sum([zeta[i] * l_partition[i]
            for i in range(m)]) + epsilon
            numerator = 1
            for i in range(m):
                numerator *= z[i] ** l_partition[i]
            result += numerator / (gamma(gaminput)) * combination
    return result
zeta = [0.5, 1.7]
epsilon_1 = 2.5
epsilon_2 = 3.5
epsilon_3 = 2.1
epsilon_4 = 2
z = [1/17, 1/51]

M1 = MML(zeta, epsilon_1, z)
M2 = MML(zeta, epsilon_2, z)
M3 = MML(zeta, epsilon_3, z)
M4 = MML(zeta, epsilon_4, z)

u = (1/16) * M1 + (1/30) * M2 + (1/19) * M3 + (1/57 + 1/(16 * gamma(2.5))
+ 1/(17 * gamma(1.5)) + 1/(51 * gamma(2.7)) + 1/(30 * gamma(3.5))
+ 1/(19 * gamma(2.1))) * M4 print(u)
#end codes.
```

3 | EXISTENCE

The following is the theorem regarding the existence of Equation (1.1) based on Leray–Schauder's fixed-point theorem.

Theorem 4. *Assume $m, l \in \mathbb{N}$, $\phi_i, \psi : [0,1] \times \mathbb{R} \to \mathbb{R}$ are continuous and bounded functions for $i = 1, 2, \cdots, l$. Let $\gamma : C[0,1] \to \mathbb{R}$ be a functional satisfying the following condition for a nonnegative constant γ_0:*

$$|\gamma(\zeta_1) - \gamma(\zeta_2)| \leq \gamma_0 \|\zeta_1 - \zeta_2\|, \quad \zeta_1,\ \zeta_2 \in C[0,1].$$

Furthermore, we suppose

$$\nu = 1 - \sum_{i=1}^{m} \frac{|\lambda_i|}{\Gamma(\beta_i + 1)} E_{(\beta_1, \cdots, \beta_m),\ 2}(|\lambda_1|, \cdots, |\lambda_m|) > 0.$$

Then, Equation (1.1) has a solution in $C[0,1]$.

Proof. Again, consider the nonlinear mapping $\mathbb{M}_1$ over the space $C[0,1]$ given by

$$
\begin{aligned}
(\mathbb{M}_1\zeta)(x) =& \sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,\cdots,k_m}\lambda_1^{k_1}\cdots\lambda_m^{k_m} I^{\beta_1k_1+\cdots+\beta_mk_m+\alpha}\psi(x,\zeta(x))\\
&+\sum_{i=1}^{l}\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,\cdots,k_m}\lambda_1^{k_1}\cdots\lambda_m^{k_m} I^{\beta_1k_1+\cdots+\beta_mk_m+\alpha_i}\phi_i(x,\zeta(x))\\
&+\zeta_1\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,\cdots,k_m}\lambda_1^{k_1}\cdots\lambda_m^{k_m}\\
&\cdot\left[\frac{x^{\beta_1k_1+\cdots+\beta_mk_m}}{\Gamma(\beta_1k_1+\cdots+\beta_mk_m+1)}-\frac{x^{\beta_1k_1+\cdots+\beta_mk_m+1}}{\Gamma(\beta_1k_1+\cdots+\beta_mk_m+2)}\right]\\
&+\gamma(\zeta)\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,\cdots,k_m}\lambda_1^{k_1}\cdots\lambda_m^{k_m}\frac{x^{\beta_1k_1+\cdots+\beta_mk_m+1}}{\Gamma(\beta_1k_1+\cdots+\beta_mk_m+2)}\\
&-I_{x=1}^{\alpha}\psi(x,\zeta(x))\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,\cdots,k_m}\lambda_1^{k_1}\cdots\lambda_m^{k_m}\frac{x^{\beta_1k_1+\cdots+\beta_mk_m+1}}{\Gamma(\beta_1k_1+\cdots+\beta_mk_m+2)}\\
&-\sum_{i=1}^{m}\lambda_i\, I_{x=1}^{\beta_i}\zeta(x)\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,\cdots,k_m}\lambda_1^{k_1}\cdots\lambda_m^{k_m}\frac{x^{\beta_1k_1+\cdots+\beta_mk_m+1}}{\Gamma(\beta_1k_1+\cdots+\beta_mk_m+2)}\\
&-\sum_{i=1}^{l}I_{x=1}^{\alpha_i}\phi_i(x,\zeta(x))\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,\cdots,k_m}\lambda_1^{k_1}\cdots\lambda_m^{k_m}\frac{x^{\beta_1k_1+\cdots+\beta_mk_m+1}}{\Gamma(\beta_1k_1+\cdots+\beta_mk_m+2)}.
\end{aligned}
$$

Since ϕ_i and ψ are continuous and bounded functions for all $i=1,2,\cdots,l$, we derive that

$$
\begin{aligned}
\|\mathbb{M}_1\zeta\| \le& E_{(\beta_1,\cdots,\beta_m),\,\alpha+1}(|\lambda_1|,\cdots,|\lambda_m|)\sup_{(x,y)\in[0,1]\times\mathbb{R}}|\psi(x,y)|\\
&+\sum_{i=1}^{l}\sup_{(x,y)\in[0,1]\times\mathbb{R}}|\phi_i(x,y)|E_{(\beta_1,\cdots,\beta_m),\,\alpha_i+1}(|\lambda_1|,\cdots,|\lambda_m|)\\
&+|\zeta_1|E_{(\beta_1,\cdots,\beta_m),\,1}(|\lambda_1|,\cdots,|\lambda_m|)+\left[|\gamma(\zeta)|+\frac{\sup_{(x,y)\in[0,1]\times\mathbb{R}}|\psi(x,y)|}{\Gamma(\alpha+1)}+\|\zeta\|\sum_{i=1}^{m}\frac{|\lambda_i|}{\Gamma(\beta_i+1)}\right.\\
&\left.+\sum_{i=1}^{l}\frac{\sup_{(x,y)\in[0,1]\times\mathbb{R}}|\phi_i(x,y)|}{\Gamma(\alpha_i+1)}\right]E_{(\beta_1,\cdots,\beta_m),\,2}(|\lambda_1|,\cdots,|\lambda_m|)<+\infty.
\end{aligned}
\tag{3.1}
$$

Thus, $\mathbb{M}_1\zeta\in C[0,1]$. We first prove that (i) $\mathbb{M}_1$ is continuous. Indeed, we have

$$
\begin{aligned}
\|\mathbb{M}_1\zeta_1-\mathbb{M}_1\zeta_2\| \le& E_{(\beta_1,\cdots,\beta_m),\,\alpha+1}(|\lambda_1|,\cdots,|\lambda_m|)\sup_{x\in[0,1]}|\psi(x,\zeta_1(x))-\psi(x,\zeta_2(x))|\\
&+\sum_{i=1}^{l}\sup_{x\in[0,1]}|\phi_i(x,\zeta_1(x))-\phi_i(x,\zeta_2(x))|E_{(\beta_1,\cdots,\beta_m),\,\alpha_i+1}(|\lambda_1|,\cdots,|\lambda_m|)\\
&+\left[|\gamma(\zeta_1)-\gamma(\zeta_2)|+\frac{\sup_{x\in[0,1]}|\psi(x,\zeta_1(x))-\psi(x,\zeta_2(x))|}{\Gamma(\alpha+1)}\right.\\
&\left.+\|\zeta_1-\zeta_2\|\sum_{i=1}^{m}\frac{|\lambda_i|}{\Gamma(\beta_i+1)}+\sum_{i=1}^{l}\frac{\sup_{x\in[0,1]}|\phi_i(x,\zeta_1(x))-\phi_i(x,\zeta_2(x))|}{\Gamma(\alpha_i+1)}\right]\\
&\cdot E_{(\beta_1,\cdots,\beta_m),\,2}(|\lambda_1|,\cdots,|\lambda_m|).
\end{aligned}
$$

It follows from continuity of ψ, ϕ_i and

$$|\gamma(\zeta_1)-\gamma(\zeta_2)| \le \gamma_0 \|\zeta_1-\zeta_2\|, \quad \zeta_1,\ \zeta_2 \in C[0,1],$$

that $\mathbb{M}_1$ is continuous.

(ii) Moreover, we show that $\mathbb{M}_1$ is a mapping from bounded sets to bounded sets in $C[0,1]$. Let W be a bounded set in $C[0,1]$. Then,

$$|\gamma(\zeta)| = |\gamma(\zeta)-\gamma(0)+\gamma(0)| \le \gamma_0\|\zeta\| + |\gamma(0)| \le \mathcal{B},$$

where $\mathcal{B}$ is a positive constant and $\zeta \in W$. From inequality (3.1), we imply that $\mathbb{M}_1 W$ is bounded in $C[0,1]$.

(iii) We are going to prove that $\mathbb{M}_1$ is equicontinuous on every bounded set W of $C[0,1]$. Then, we claim that $\mathbb{M}_1$ is completely continuous using the Arzela–Ascoli theorem. For $0 \le \tau_1 < \tau_2 \le 1$ and $\zeta \in W$, we consider

$$\begin{aligned}
&|(\mathbb{M}_1\zeta)(\tau_2)-(\mathbb{M}_1\zeta)(\tau_1)| \\
&\le \sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,\cdots,k_m}|\lambda_1|^{k_1}\cdots|\lambda_m|^{k_m}\frac{1}{\Gamma(\beta_1k_1+\cdots+\beta_mk_m+\alpha)} \\
&\quad\cdot\left|\int_0^{\tau_2}(\tau_2-t)^{\beta_1k_1+\cdots+\beta_mk_m+\alpha-1}\psi(t,\zeta(t))dt-\int_0^{\tau_1}(\tau_1-t)^{\beta_1k_1+\cdots+\beta_mk_m+\alpha-1}\psi(t,\zeta(t))dt\right| \\
&\quad+\sum_{i=1}^{l}\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,\cdots,k_m}|\lambda_1|^{k_1}\cdots|\lambda_m|^{k_m}\frac{1}{\Gamma(\beta_1k_1+\cdots+\beta_mk_m+\alpha_i)} \\
&\quad\cdot\left|\int_0^{\tau_2}(\tau_2-t)^{\beta_1k_1+\cdots+\beta_mk_m+\alpha_i-1}\phi_i(t,\zeta(t))dt-\int_0^{\tau_1}(\tau_1-t)^{\beta_1k_1+\cdots+\beta_mk_m+\alpha_i-1}\phi_i(t,\zeta(t))dt\right| \\
&\quad+|\zeta_1|\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,\cdots,k_m}|\lambda_1|^{k_1}\cdots|\lambda_m|^{k_m} \\
&\quad\cdot\left|\frac{\tau_2^{\beta_1k_1+\cdots+\beta_mk_m}-\tau_1^{\beta_1k_1+\cdots+\beta_mk_m}}{\Gamma(\beta_1k_1+\cdots+\beta_mk_m+1)}+\frac{\tau_2^{\beta_1k_1+\cdots+\beta_mk_m+1}-\tau_1^{\beta_1k_1+\cdots+\beta_mk_m+1}}{\Gamma(\beta_1k_1+\cdots+\beta_mk_m+2)}\right| \\
&\quad+\left[|\gamma(\zeta)|+|I_{x=1}^{\alpha}\psi(x,\zeta(x))|+\sum_{i=1}^{m}|\lambda_i|\ |I_{x=1}^{\beta_i}\zeta(x)|+\sum_{i=1}^{l}|I_{x=1}^{\alpha_i}\phi_i(x,\zeta(x))|\right] \\
&\quad\cdot\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,\cdots,k_m}|\lambda_1|^{k_1}\cdots|\lambda_m|^{k_m}\frac{\tau_2^{\beta_1k_1+\cdots+\beta_mk_m+1}-\tau_1^{\beta_1k_1+\cdots+\beta_mk_m+1}}{\Gamma(\beta_1k_1+\cdots+\beta_mk_m+2)} \\
&= I_1+I_2+I_3+I_4.
\end{aligned}$$

As for I_1, we have

$$\begin{aligned}
&\int_0^{\tau_2}(\tau_2-t)^{\beta_1k_1+\cdots+\beta_mk_m+\alpha-1}\psi(t,\zeta(t))dt \\
&=\int_0^{\tau_1}(\tau_2-t)^{\beta_1k_1+\cdots+\beta_mk_m+\alpha-1}\psi(t,\zeta(t))dt+\int_{\tau_1}^{\tau_2}(\tau_2-t)^{\beta_1k_1+\cdots+\beta_mk_m+\alpha-1}\psi(t,\zeta(t))dt,
\end{aligned}$$

which implies

$$\begin{aligned}
&\left|\int_0^{\tau_2}(\tau_2-t)^{\beta_1k_1+\cdots+\beta_mk_m+\alpha-1}\psi(t,\zeta(t))dt-\int_0^{\tau_1}(\tau_1-t)^{\beta_1k_1+\cdots+\beta_mk_m+\alpha-1}\psi(t,\zeta(t))dt\right|\\
&\le\int_0^{\tau_1}[(\tau_2-t)^{\beta_1k_1+\cdots+\beta_mk_m+\alpha-1}-(\tau_1-t)^{\beta_1k_1+\cdots+\beta_mk_m+\alpha-1}]dt\sup_{t\in[0,1]}|\psi(t,\zeta(t))|\\
&\quad+\int_{\tau_1}^{\tau_2}(\tau_2-t)^{\beta_1k_1+\cdots+\beta_mk_m+\alpha-1}dt\sup_{t\in[0,1]}|\psi(t,\zeta(t))|\\
&\le\left[\frac{\tau_2^{\beta_1k_1+\cdots+\beta_mk_m+\alpha}-\tau_1^{\beta_1k_1+\cdots+\beta_mk_m+\alpha}}{\beta_1k_1+\cdots+\beta_mk_m+\alpha}+(\tau_2-\tau_1)\right]\sup_{t\in[0,1]}|\psi(t,\zeta(t))|,
\end{aligned}$$

which clearly contains the factor $(\tau_2-\tau_1)$ by noting that $\alpha>1$. Hence, I_1 is equicontinuous.

It follows similarly that I_2 is equicontinuous. Consider

$$\begin{aligned}
I_3=&|\zeta_1|\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,\cdots,k_m}|\lambda_1|^{k_1}\cdots|\lambda_m|^{k_m}\\
&\cdot\left|\frac{\tau_2^{\beta_1k_1+\cdots+\beta_mk_m}-\tau_1^{\beta_1k_1+\cdots+\beta_mk_m}}{\Gamma(\beta_1k_1+\cdots+\beta_mk_m+1)}+\frac{\tau_2^{\beta_1k_1+\cdots+\beta_mk_m+1}-\tau_1^{\beta_1k_1+\cdots+\beta_mk_m+1}}{\Gamma(\beta_1k_1+\cdots+\beta_mk_m+2)}\right|,
\end{aligned}$$

which contains the factor $(\tau_2-\tau_1)^{\alpha_0}$, where

$$\alpha_0=\min\{\beta_1,\beta_2,\cdots,\beta_m,1\}>0.$$

Therefore, I_3 is equicontinuous.

Regarding I_4, we claim the term

$$|\gamma(\zeta)|+|I_{x=1}^{\alpha}\psi(x,\zeta(x))|+\sum_{i=1}^{m}|\lambda_i|\ |I_{x=1}^{\beta_i}\zeta(x)|+\sum_{i=1}^{l}|I_{x=1}^{\alpha_i}\phi_i(x,\zeta(x))|$$

is bounded if $\zeta\in W$. Furthermore,

$$\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,\cdots,k_m}|\lambda_1|^{k_1}\cdots|\lambda_m|^{k_m}\frac{\tau_2^{\beta_1k_1+\cdots+\beta_mk_m+1}-\tau_1^{\beta_1k_1+\cdots+\beta_mk_m+1}}{\Gamma(\beta_1k_1+\cdots+\beta_mk_m+2)}$$

contains the factor $\tau_2-\tau_1$. From the above, we deduce $\mathbb{M}_1$ is equicontinuous.

Finally, we will prove that the set for $0<\lambda<1$

$$Y=\{\ \zeta\in C[0,1]\ :\ \zeta=\lambda\mathbb{M}_1\zeta\ \}$$

is bounded. By inequality (3.1), we come to

$$\begin{aligned}
\|\zeta\|\le\|\mathbb{M}_1\zeta\|\le&\ E_{(\beta_1,\cdots,\beta_m),\ \alpha+1}(|\lambda_1|,\cdots,|\lambda_m|)\sup_{(x,y)\in[0,1]\times\mathbb{R}}|\psi(x,y)|\\
&+\sum_{i=1}^{l}\sup_{(x,y)\in[0,1]\times\mathbb{R}}|\phi_i(x,y)|E_{(\beta_1,\cdots,\beta_m),\ \alpha_i+1}(|\lambda_1|,\cdots,|\lambda_m|)
\end{aligned}$$

$$+|\zeta_1|E_{(\beta_1,\cdots,\beta_m),\,1}(|\lambda_1|,\cdots,|\lambda_m|)+\left[|\gamma(\zeta)|+\frac{\sup\limits_{(x,y)\in[0,1]\times\mathbb{R}}|\psi(x,y)|}{\Gamma(\alpha+1)}+\|\zeta\|\sum_{i=1}^{m}\frac{|\lambda_i|}{\Gamma(\beta_i+1)}\right.$$
$$\left.+\sum_{i=1}^{l}\frac{\sup\limits_{(x,y)\in[0,1]\times\mathbb{R}}|\phi_i(x,y)|}{\Gamma(\alpha_i+1)}\right]E_{(\beta_1,\cdots,\beta_m),\,2}(|\lambda_1|,\cdots,|\lambda_m|).$$

Since

$$v=1-\sum_{i=1}^{m}\frac{|\lambda_i|}{\Gamma(\beta_i+1)}E_{(\beta_1,\cdots,\beta_m),\,2}(|\lambda_1|,\cdots,|\lambda_m|)>0,$$

we get

$$\|\zeta\|\leq\frac{1}{v}E_{(\beta_1,\cdots,\beta_m),\,\alpha+1}(|\lambda_1|,\cdots,|\lambda_m|)\sup_{(x,y)\in[0,1]\times\mathbb{R}}|\psi(x,y)|$$
$$+\frac{1}{v}\sum_{i=1}^{l}\sup_{(x,y)\in[0,1]\times\mathbb{R}}|\phi_i(x,y)|E_{(\beta_1,\cdots,\beta_m),\,\alpha_i+1}(|\lambda_1|,\cdots,|\lambda_m|)$$
$$+\frac{|\zeta_1|}{v}E_{(\beta_1,\cdots,\beta_m),\,1}(|\lambda_1|,\cdots,|\lambda_m|)+\frac{1}{v}\left[|\gamma(\zeta)|+\frac{\sup\limits_{(x,y)\in[0,1]\times\mathbb{R}}|\psi(x,y)|}{\Gamma(\alpha+1)}\right.$$
$$\left.+\sum_{i=1}^{l}\frac{\sup\limits_{(x,y)\in[0,1]\times\mathbb{R}}|\phi_i(x,y)|}{\Gamma(\alpha_i+1)}\right]E_{(\beta_1,\cdots,\beta_m),\,2}(|\lambda_1|,\cdots,|\lambda_m|)<+\infty,$$

which infers that Y is bounded. So there exists a solution in $C[0,1]$ to Equation (1.1) by Leray–Schauder's fixed-point theorem. This completes the proof. □

Example 5. The following nonlinear fractional equation with a functional boundary condition:

$$\begin{cases} {}_CD^{1.2}\left[\zeta(x)-\frac{1}{14}I^{1.5}\zeta(x)+\frac{1}{49}I^{3.1}\zeta(x)+\frac{1}{12}I^{2.1}\frac{1}{|\zeta(x)|+1}-\frac{1}{9}I^{1.6}\frac{x^2+21}{1+\zeta^2(x)}\right] \\ =\frac{1}{4}\cos\frac{\zeta^2(x)+2}{2x+1},\ \ x\in[0,1], \\ \zeta(0)=1,\ \ \zeta(1)=\frac{1}{13}\sin\zeta(t_1)+\frac{1}{14}\sin\zeta(t_2),\ \ 0<t_1<t_2<1, \end{cases} \tag{3.2}$$

has a solution in $C[0,1]$.

Proof. Clearly,

$$\psi(x,y)=\frac{1}{4}\cos\frac{y^2+2}{2x+1},\quad \phi_1(x,y)=-\frac{1}{12(|y|+1)},\quad \phi_2(x,y)=\frac{x^2+21}{9(1+y^2)},$$

are continuous and bounded over $[0,1]\times\mathbb{R}$, and

$$\gamma(\zeta)=\frac{1}{13}\sin\zeta(t_1)+\frac{1}{14}\sin\zeta(t_2)$$

is a functional from $C[0,1]$ to $\mathbb{R}$, satisfying

$$|\gamma(\zeta_1)-\gamma(\zeta_2)|\leq\frac{1}{13}|\sin\zeta_1(t_1)-\sin\zeta_2(t_1)|+\frac{1}{14}|\sin\zeta_1(t_2)-\sin\zeta_2(t_2)|\leq\frac{2}{13}\|\zeta_1-\zeta_2\|.$$

We need to find the value

$$
\begin{aligned}
v &= 1-\sum_{i=1}^{m}\frac{|\lambda_i|}{\Gamma(\beta_i+1)}E_{(\beta_1,\cdots,\beta_m),\ 2}(|\lambda_1|,\cdots,|\lambda_m|)\\
&= 1-\left[\frac{1}{14\Gamma(1.5+1)}+\frac{1}{49\Gamma(3.1+1)}\right]E_{(1.5,3.1),\ 2}(1/14,1/49)\\
&\approx 0.941998702743921<1.
\end{aligned}
$$

By Theorem 4, Equation (3.2) has a solution in $C[0,1]$. This completes the proof. □

4 | THE HYERS–ULAM STABILITY

Hyers–Ulam's stability is an important concept that refers to the stability of a differential equation with a perturbation [18]. We will study the Hyers–Ulam stability of Equation (1.1) based on the implicit integral equation (2.1).

Definition 6. Equation (1.1) is Hyers–Ulam stable if there is a constant $\mathcal{W}>0$ such that for all $\epsilon>0$ and a continuously differentiable function ζ satisfying the boundary conditions $\zeta(0)=\zeta_1$ and $\zeta(1)=\gamma(\zeta)$, and the inequality

$$
\left\|{}_CD^{\alpha}\left[\zeta(x)-\sum_{i=1}^{m}\lambda_i\ I^{\beta_i}\zeta(x)-\sum_{i=1}^{l}I^{\alpha_i}\phi_i(x,\zeta(x))\right]-\psi(x,\zeta(x))\right\|<\epsilon,
$$

then there exists a solution ζ_0 of Equation (1.1) such that

$$
\|\zeta-\zeta_0\|<\mathcal{W}\epsilon,
$$

where $\mathcal{W}$ is a Hyers–Ulam stability constant.

Theorem 7. *Assume $m,l\in\mathbb{N}$, $\phi_i,\psi:[0,1]\times\mathbb{R}\to\mathbb{R}$ are continuous and bounded functions for $i=1,2,\cdots,l$, satisfying the following Lipschitz conditions:*

$$
|\psi(x,y_1)-\psi(x,y_2)|\le\mathcal{L}_0|y_1-y_2|,\quad y_1,y_2\in\mathbb{R},
$$

and

$$
|\phi_i(x,z_1)-\phi_i(x,z_2)|\le\mathcal{L}_i|z_1-z_2|,\quad z_1,z_2\in\mathbb{R},
$$

where $\mathcal{L}_0$ and $\mathcal{L}_i$ are nonnegative constants. Let $\gamma:C[0,1]\to\mathbb{R}$ be a functional satisfying the following condition for a nonnegative constant γ_0:

$$
|\gamma(\zeta_1)-\gamma(\zeta_2)|\le\gamma_0\,\|\zeta_1-\zeta_2\|,\quad \zeta_1,\ \zeta_2\in C[0,1].
$$

Furthermore, we suppose

$$
\begin{aligned}
u =&\mathcal{L}_0E_{(\beta_1,\cdots,\beta_m),\ \alpha+1}(|\lambda_1|,\cdots,|\lambda_m|)+\sum_{i=1}^{l}\mathcal{L}_iE_{(\beta_1,\cdots,\beta_m),\ \alpha_i+1}(|\lambda_1|,\cdots,|\lambda_m|)\\
&+\left[\gamma_0+\frac{\mathcal{L}_0}{\Gamma(\alpha+1)}+\sum_{i=1}^{m}\frac{|\lambda_i|}{\Gamma(\beta_i+1)}+\sum_{i=1}^{l}\frac{\mathcal{L}_i}{\Gamma(\alpha_i+1)}\right]E_{(\beta_1,\cdots,\beta_m),\ 2}(|\lambda_1|,\cdots,|\lambda_m|)<1.
\end{aligned}
$$

Then, Equation (1.1) is also Hyers–Ulam stable in $C[0,1]$.

Proof. We begin by letting

$$\psi_0(x,\zeta(x)) = {}_C D^{\alpha}\left[\zeta(x) - \sum_{i=1}^{m}\lambda_i\ I^{\beta_i}\zeta(x) - \sum_{i=1}^{l} I^{\alpha_i}\phi_i(x,\zeta(x))\right] - \psi(x,\zeta(x)).$$

Then, $\|\psi_0\| < \epsilon$. Using the integral equation (2.1), we arrive at

$$\begin{aligned}
\zeta(x) = & \sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,\cdots,k_m}\lambda_1^{k_1}\cdots\lambda_m^{k_m} I^{\beta_1k_1+\cdots+\beta_mk_m+\alpha}(\psi(x,\zeta(x))-\psi_0)\\
& + \sum_{i=1}^{l}\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,\cdots,k_m}\lambda_1^{k_1}\cdots\lambda_m^{k_m} I^{\beta_1k_1+\cdots+\beta_mk_m+\alpha_i}\phi_i(x,\zeta(x))\\
& + \zeta_1\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,\cdots,k_m}\lambda_1^{k_1}\cdots\lambda_m^{k_m}\\
& \cdot\left[\frac{x^{\beta_1k_1+\cdots+\beta_mk_m}}{\Gamma(\beta_1k_1+\cdots+\beta_mk_m+1)} - \frac{x^{\beta_1k_1+\cdots+\beta_mk_m+1}}{\Gamma(\beta_1k_1+\cdots+\beta_mk_m+2)}\right]\\
& + \gamma(\zeta)\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,\cdots,k_m}\lambda_1^{k_1}\cdots\lambda_m^{k_m}\frac{x^{\beta_1k_1+\cdots+\beta_mk_m+1}}{\Gamma(\beta_1k_1+\cdots+\beta_mk_m+2)}\\
& - I^{\alpha}_{x=1}(\psi(x,\zeta(x))-\psi_0)\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,\cdots,k_m}\lambda_1^{k_1}\cdots\lambda_m^{k_m}\frac{x^{\beta_1k_1+\cdots+\beta_mk_m+1}}{\Gamma(\beta_1k_1+\cdots+\beta_mk_m+2)}\\
& - \sum_{i=1}^{m}\lambda_i\ I^{\beta_i}_{x=1}\zeta(x)\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,\cdots,k_m}\lambda_1^{k_1}\cdots\lambda_m^{k_m}\frac{x^{\beta_1k_1+\cdots+\beta_mk_m+1}}{\Gamma(\beta_1k_1+\cdots+\beta_mk_m+2)}\\
& - \sum_{i=1}^{l} I^{\alpha_i}_{x=1}\phi_i(x,\zeta(x))\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,\cdots,k_m}\lambda_1^{k_1}\cdots\lambda_m^{k_m}\frac{x^{\beta_1k_1+\cdots+\beta_mk_m+1}}{\Gamma(\beta_1k_1+\cdots+\beta_mk_m+2)},
\end{aligned}$$

and

$$\begin{aligned}
\zeta_0(x) = & \sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,\cdots,k_m}\lambda_1^{k_1}\cdots\lambda_m^{k_m} I^{\beta_1k_1+\cdots+\beta_mk_m+\alpha}\psi(x,\zeta_0(x))\\
& + \sum_{i=1}^{l}\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,\cdots,k_m}\lambda_1^{k_1}\cdots\lambda_m^{k_m} I^{\beta_1k_1+\cdots+\beta_mk_m+\alpha_i}\phi_i(x,\zeta_0(x))\\
& + \zeta_1\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,\cdots,k_m}\lambda_1^{k_1}\cdots\lambda_m^{k_m}\\
& \cdot\left[\frac{x^{\beta_1k_1+\cdots+\beta_mk_m}}{\Gamma(\beta_1k_1+\cdots+\beta_mk_m+1)} - \frac{x^{\beta_1k_1+\cdots+\beta_mk_m+1}}{\Gamma(\beta_1k_1+\cdots+\beta_mk_m+2)}\right]\\
& + \gamma(\zeta_0)\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,\cdots,k_m}\lambda_1^{k_1}\cdots\lambda_m^{k_m}\frac{x^{\beta_1k_1+\cdots+\beta_mk_m+1}}{\Gamma(\beta_1k_1+\cdots+\beta_mk_m+2)}\\
& - I^{\alpha}_{x=1}\psi(x,\zeta_0(x))\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,\cdots,k_m}\lambda_1^{k_1}\cdots\lambda_m^{k_m}\frac{x^{\beta_1k_1+\cdots+\beta_mk_m+1}}{\Gamma(\beta_1k_1+\cdots+\beta_mk_m+2)}
\end{aligned}$$

$$
- \sum_{i=1}^{m} \lambda_i \, I_{x=1}^{\beta_i} \zeta_0(x) \sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_m=k} \binom{k}{k_1, \cdots, k_m} \lambda_1^{k_1} \cdots \lambda_m^{k_m} \frac{x^{\beta_1 k_1+\cdots+\beta_m k_m+1}}{\Gamma(\beta_1 k_1 + \cdots + \beta_m k_m + 2)}
$$

$$
- \sum_{i=1}^{l} I_{x=1}^{\alpha_i} \phi_i(x, \zeta_0(x)) \sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_m=k} \binom{k}{k_1, \cdots, k_m} \lambda_1^{k_1} \cdots \lambda_m^{k_m} \frac{x^{\beta_1 k_1+\cdots+\beta_m k_m+1}}{\Gamma(\beta_1 k_1 + \cdots + \beta_m k_m + 2)},
$$

as ζ_0 is a solution of Equation (1.1). It follows from the proof of Theorem 2 that

$$
\begin{aligned}
\|\zeta - \zeta_0\| \leq & \mathcal{L}_0 E_{(\beta_1,\cdots,\beta_m),\, \alpha+1}(|\lambda_1|, \cdots, |\lambda_m|) + \sum_{i=1}^{l} \mathcal{L}_i E_{(\beta_1,\cdots,\beta_m),\, \alpha_i+1}(|\lambda_1|, \cdots, |\lambda_m|) \, \|\zeta - \zeta_0\| \\
& + \left[\gamma_0 + \frac{\mathcal{L}_0}{\Gamma(\alpha+1)} + \sum_{i=1}^{m} \frac{|\lambda_i|}{\Gamma(\beta_i+1)} + \sum_{i=1}^{l} \frac{\mathcal{L}_i}{\Gamma(\alpha_i+1)} \right] E_{(\beta_1,\cdots,\beta_m),\, 2}(|\lambda_1|, \cdots, |\lambda_m|) \\
& \cdot \|\zeta - \zeta_0\| + \left[E_{(\beta_1,\cdots,\beta_m),\, \alpha+1}(|\lambda_1|, \cdots, |\lambda_m|) + \frac{1}{\Gamma(\alpha+1)} E_{(\beta_1,\cdots,\beta_m),\, 2} \right] \|\psi_0\| \\
= & \, u \, \|\zeta - \zeta_0\| + \left[E_{(\beta_1,\cdots,\beta_m),\, \alpha+1}(|\lambda_1|, \cdots, |\lambda_m|) + \frac{1}{\Gamma(\alpha+1)} E_{(\beta_1,\cdots,\beta_m),\, 2} \right] \|\psi_0\| .
\end{aligned}
$$

This implies that

$$
\|\zeta - \zeta_0\| \leq \mathcal{W} \epsilon,
$$

where

$$
\mathcal{W} = \frac{E_{(\beta_1,\cdots,\beta_m),\, \alpha+1}(|\lambda_1|, \cdots, |\lambda_m|) + \frac{1}{\Gamma(\alpha+1)} E_{(\beta_1,\cdots,\beta_m),\, 2}}{1 - u},
$$

is a Hyers–Ulam stability constant. This completes the proof. □

5 | CONCLUSION

We have obtained sufficient conditions for the existence, uniqueness, and Hyers–Ulam stability of solutions to Equation (1.1) with nonlocal boundary condition, by several fixed-point theorems, the multivariate Mittag-Leffler function, and Babenko's approach. In addition, two examples were given to demonstrate the applications of our key results based on approximate values of a couple of Mittag-Leffler functions computed by Python codes. The techniques used also work for various fractional differential equations with initial or boundary conditions and integral equations in Banach spaces.

AUTHOR CONTRIBUTIONS

Chenkuan Li: Writing—original draft; methodology; software; supervision; resources; investigation; validation. **Reza Saadati:** Validation; investigation; formal analysis. **Joshua Beaudin:** Validation; software; resources. **Elisha Tariq:** validation; software; resources. **McKayla Brading:** Validation; software; resources.

All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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CONFLICT OF INTEREST STATEMENT

The authors declare that they have no competing interests.

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