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IFAC PapersOnLine 58-12 (2024) 296–301

Uniqueness and existence for a fractional differential equation with functional boundary condition ⋆

Chenkuan Li

Department of Mathematics and Computer Science, Brandon University, Brandon, Manitoba Canada R7A 6A9 (e-mail: lic@brandonu.ca).

Abstract: Using Banach's contractive principle and Leray–Schauder's fixed point theorem, we study the uniqueness and existence for a nonlinear fractional differential equation with functional boundary condition based on the two-parameter Mittag-Leffler function and an implicit integral equation. Several examples are also presented to demonstrate applications of our key theorems. The methods used can also deal with other types of differential equations with various initial or boundary conditions.

Keywords: Fractional differential equation; Banach's contractive principle; Mittag-Leffler function; Babenko's approach; Leray–Schauder's fixed point theorem.

1. INTRODUCTION

The Riemann-Liouville fractional integral I^β of order $\beta \in R^+$ is defined for the function $\zeta(x)$ as

$$(I^\beta \zeta)(x) = \frac{1}{\Gamma(\beta)} \int_0^x (x-\tau)^{\beta-1} \zeta(\tau) d\tau, \quad x \in [0,1].$$

In particular,

$$(I^0 \zeta)(x) = \zeta(x),$$

from Li (2015).

Let $n \in \mathcal{N} = \{1, 2, 3, \cdots\}$. The Caputo fractional derivative of order $\beta \in R^+$ of the function $\zeta(x)$ is defined as (see Li (2023))

$$({}_C D^\beta \zeta)(x) = \frac{1}{\Gamma(n-\beta)} \int_0^x (x-\tau)^{n-\beta-1} \zeta^{(n)}(\tau) d\tau,$$

where $n - 1 < \beta \leq n$.

Assume $\eta : [0,1] \times \mathcal{R} \to \mathcal{R}$ is a mapping and $\phi : C[0,1] \to \mathcal{R}$ is a functional. We shall study the uniqueness and existence for the following nonlinear fractional differential equation with a nonlocal boundary condition for $1 < \beta \leq 2$ and a constant λ:

$$ {}_C D^\beta \zeta(x) + \lambda \ {}_C D^\gamma \zeta(x) = \eta(x, \zeta(x)), \quad x \in [0,1], \quad (1)$$
$$\zeta(0) = 0, \quad \zeta(1) = \phi(\zeta),$$

where $0 < \gamma \leq 1$ is a constant.

Equation (1) is a particular instance of equation (1.2) discussed in C. Li (2023), but with a new functional boundary condition which extends many integral boundary conditions.

Nonlinear boundary value problems, including those with nonlocal conditions, often appear in the mathematical models of real world phenomena. The study of boundary value problems is important due to their extensive applications in diverse disciplines of applied sciences and engineering. There have been many interesting investigations in the area dealing with different boundary conditions (J. Tariboon (2014), S.K. Ntouyas (2020), Li (2023)).

We define the Banach space $C[0,1]$ of all continuous functions from $[0,1]$ to $\mathcal{R}$ with the norm

$$||\zeta|| = \max_{x\in[0,1]} |\zeta(x)| < +\infty.$$

The two-parameter Mittag-Leffler function is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + \beta)}, \ z \in C, \ \alpha, \beta > 0.$$

Babenko's approach (see Babenko (1986)) is a powerful tool for studying uniqueness and existence of differential equations with initial or boundary conditions. To demonstrate this in detail, we consider the following nonlinear fractional differential equation with a nonlocal boundary condition for $0 < \alpha \leq 1$:

$$ {}_C D^\alpha \zeta(x) + \lambda \ \zeta(x) = g(x, \zeta(x)), \quad x \in [0,1], \quad (2)$$
$$\zeta(0) = s \int_0^1 \zeta(x) dx,$$

where s is a constant.

Applying the operator I^α to equation (2), we get

$$\zeta(x) - \zeta(0) + \lambda I^\alpha \zeta(x) = I^\alpha g(x, \zeta(x)),$$

⋆ This research is supported by the Natural Sciences and Engineering Research Council of Canada (Grant No. 2019-03907).

10.1016/j.ifacol.2024.08.206

which deduces

$$(1+\lambda I^{\alpha})\,\zeta(x) = I^{\alpha} g(x,\zeta(x)) + s\int_0^1 \zeta(x)dx.$$

Treating the factor $(1+\lambda I^{\alpha})$ as a normal variable (Babenko's approach), we informally obtain

$$\begin{aligned}\zeta(x) &= (1+\lambda I^{\alpha})^{-1}\left(I^{\alpha} g(x,\zeta(x)) + s\int_0^1 \zeta(x)dx\right)\\ &= \sum_{k=0}^{\infty}(-1)^k\lambda^k I^{\alpha k+\alpha} g(x,\zeta(x))\\ &\quad + s\int_0^1 \zeta(x)dx \sum_{k=0}^{\infty}(-1)^k\lambda^k I^{\alpha k}1\\ &= \sum_{k=0}^{\infty}(-1)^k\lambda^k I^{\alpha k+\alpha} g(x,\zeta(x))\\ &\quad + s\int_0^1 \zeta(x)dx \sum_{k=0}^{\infty}(-1)^k\lambda^k \frac{x^{\alpha k}}{\Gamma(\alpha k+1)}.\end{aligned}$$

The above integral equation is clearly equivalent to equation (2) with the initial condition. Furthermore, we assume g is a continuous and bounded function over $[0,1]\times\mathcal{R}$, and

$$d = 1 - |s|E_{(\alpha,\,1)}(|\lambda|) > 0.$$

Then ζ is uniformly bounded on $[0,1]$. Indeed,

$$\begin{aligned}||\zeta|| \le \sum_{k=0}^{\infty}\frac{|\lambda|^k}{\Gamma(\alpha k+\alpha+1)}\sup_{(x,y)\in[0,1]\times\mathcal{R}}|g(x,y)|\\ +|s|||\zeta||E_{(\alpha,\,1)}(|\lambda|).\end{aligned}$$

Thus,

$$||\zeta|| \le \frac{1}{d}E_{\alpha,\,\alpha+1}(|\lambda|)\sup_{(x,y)\in[0,1]\times\mathcal{R}}|g(x,y)| < +\infty,$$

which claims ζ is uniformly bounded. If the function g further satisfies the following Lipschitz condition:

$$|g(x,y_1)-g(x,y_2)| \le L|y_1-y_2|, \quad y_1,y_2\in\mathcal{R},$$

and

$$Q = LE_{(\alpha,\,\alpha+1)}(|\lambda|) + |s|E_{(\alpha,\,1)}(|\lambda|) < 1,$$

then equation (2) has a unique solution in $C[0,1]$ by Banach's contractive principle. To show this, we start by defining a mapping $\mathcal{T}$ over $C[0,1]$ as

$$\begin{aligned}\mathcal{T}\zeta(x) &= \sum_{k=0}^{\infty}(-1)^k\lambda^k I^{\alpha k+\alpha} g(x,\zeta(x))\\ &\quad + s\int_0^1 \zeta(x)dx \sum_{k=0}^{\infty}(-1)^k\lambda^k \frac{x^{\alpha k}}{\Gamma(\alpha k+1)}.\end{aligned}$$

Then $\mathcal{T}\zeta \in C[0,1]$. It remains to be shown that $\mathcal{T}$ is contractive. Evidently,

$$\begin{aligned}||\mathcal{T}\zeta_1-\mathcal{T}\zeta_2|| \le LE_{(\alpha,\,\alpha+1)}(|\lambda|)||\zeta_1-\zeta_2||\\ +|s|E_{(\alpha,\,1)}(|\lambda|)||\zeta_1-\zeta_2|| = Q||\zeta_1-\zeta_2||.\end{aligned}$$

Since $Q<1$, equation (2) has a unique solution in $C[0,1]$ from Banach's contractive principle.

We will first convert equation (1) into an equivalent implicit integral equation in a series by Babenko's approach in Section 2, and then further study the uniqueness of solutions via Banach's contractive principle in the space $C([0,1])$ with an illustrative example. In Section 3, we derive an existence theorem based on the implicit integral equation and Leray–Schauder's fixed point theorem, and present an example demonstrating application of the theorem obtained. Finally, we summarize the entire work in Section 4.

2. UNIQUENESS

Theorem 1. Let η be a continuous and bounded function on $[0,1]\times\mathcal{R}$, $\phi: C[0,1]\to\mathcal{R}$ be a functional and

$$w = 1 - \frac{|\lambda|}{\Gamma(\beta-\gamma+1)}E_{(\beta-\gamma,\,2)}(|\lambda|) > 0.$$

Then ζ is a solution of equation (1) if and only if it satisfies the following integral equation:

$$\begin{aligned}\zeta(x) &= \sum_{k=0}^{\infty}(-1)^k\lambda^k I^{k(\beta-\gamma)+\beta}\eta(x,\zeta(x))\\ &\quad+\phi(\zeta)\sum_{k=0}^{\infty}(-1)^k\lambda^k\frac{x^{k(\beta-\gamma)+1}}{\Gamma(k(\beta-\gamma)+2)}\\ &\quad+I_{x=1}^{\beta-\gamma}\zeta(x)\sum_{k=0}^{\infty}(-1)^k\lambda^{k+1}\frac{x^{k(\beta-\gamma)+1}}{\Gamma(k(\beta-\gamma)+2)}\\ &\quad-I_{x=1}^{\beta}\eta(x,\zeta(x))\sum_{k=0}^{\infty}(-1)^k\lambda^k\frac{x^{k(\beta-\gamma)+1}}{\Gamma(k(\beta-\gamma)+2)}.\end{aligned}\qquad(3)$$

In addition,

$$\begin{aligned}||\zeta|| \le \frac{1}{w}(E_{(\beta-\gamma,\,\beta+1)}(|\lambda|)\\ +\frac{1}{\Gamma(\beta+1)}E_{(\beta-\gamma,\,2)}(|\lambda|))\sup_{(x,y)\in[0,1]\times\mathcal{R}}|\eta(x,y)|\\ +\frac{1}{w}|\phi(\zeta)|E_{(\beta-\gamma,\,2)}(|\lambda|) < +\infty.\end{aligned}$$

Proof. Let $1<\beta\le 2$. It follows from S.G. Samko (1993) that

$$I^{\beta}({}_CD^{\beta})\zeta(x) = \zeta(x)-\zeta(0)-\zeta'(0)x = \zeta(x)-\zeta'(0)x,$$

using $\zeta(0)=0$. Thus, applying the integral operator I^{β} to the equation

$${}_CD^{\beta}\zeta(x) + \lambda\ {}_CD^{\gamma}\zeta(x) = \eta(x,\zeta(x)),$$

we come to

$$\zeta(x)-\zeta'(0)x+\lambda I^{\beta-\gamma}(\zeta(x)-\zeta(0)) = I^{\beta}\eta(x,\zeta(x)),$$

by noting that $0<\gamma\le 1$. It follows from setting $x=1$ that

$$\phi(\zeta)-\zeta'(0)+\lambda I_{x=1}^{\beta-\gamma}\zeta(x) = I_{x=1}^{\beta}\eta(x,\zeta(x)),$$

and hence

$$\zeta'(0) = \phi(\zeta) + \lambda I^{\beta-\gamma}_{x=1}\zeta(x) - I^{\beta}_{x=1}\eta(x, \zeta(x)).$$

So we have

$$\left(1 + \lambda I^{\beta-\gamma}\right)\zeta(x) = I^{\beta}\eta(x, \zeta(x)) + x\phi(\zeta) + \lambda x I^{\beta-\gamma}_{x=1}\zeta(x) - x I^{\beta}_{x=1}\eta(x, \zeta(x)).$$

Using Babenko's approach, we get

$$\begin{aligned}
\zeta(x) &= \left(1 + \lambda I^{\beta-\gamma}\right)^{-1}\left(I^{\beta}\eta(x, \zeta(x)) + x\phi(\zeta) + \lambda x I^{\beta-\gamma}_{x=1}\zeta(x) - x I^{\beta}_{x=1}\eta(x, \zeta(x))\right)\\
&= \sum_{k=0}^{\infty}(-1)^k\lambda^k I^{k(\beta-\gamma)}\left(I^{\beta}\eta(x, \zeta(x)) + x\phi(\zeta) + \lambda x I^{\beta-\gamma}_{x=1}\zeta(x) - x I^{\beta}_{x=1}\eta(x, \zeta(x))\right)\\
&= \sum_{k=0}^{\infty}(-1)^k\lambda^k I^{k(\beta-\gamma)+\beta}\eta(x, \zeta(x)) + \phi(\zeta)\sum_{k=0}^{\infty}(-1)^k\lambda^k I^{k(\beta-\gamma)}x\\
&\quad + I^{\beta-\gamma}_{x=1}\zeta(x)\sum_{k=0}^{\infty}(-1)^k\lambda^{k+1} I^{k(\beta-\gamma)}x - I^{\beta}_{x=1}\eta(x, \zeta(x))\sum_{k=0}^{\infty}(-1)^k\lambda^k I^{k(\beta-\gamma)}x\\
&= \sum_{k=0}^{\infty}(-1)^k\lambda^k I^{k(\beta-\gamma)+\beta}\eta(x, \zeta(x)) + \phi(\zeta)\sum_{k=0}^{\infty}(-1)^k\lambda^k\frac{x^{k(\beta-\gamma)+1}}{\Gamma(k(\beta-\gamma)+2)}\\
&\quad + I^{\beta-\gamma}_{x=1}\zeta(x)\sum_{k=0}^{\infty}(-1)^k\lambda^{k+1}\frac{x^{k(\beta-\gamma)+1}}{\Gamma(k(\beta-\gamma)+2)} - I^{\beta}_{x=1}\eta(x, \zeta(x))\sum_{k=0}^{\infty}(-1)^k\lambda^k\frac{x^{k(\beta-\gamma)+1}}{\Gamma(k(\beta-\gamma)+2)}.
\end{aligned}$$

Hence, ζ is a solution of equation (1) if and only if it satisfies the integral equation (3) since all above steps are reversible.

Furthermore,

$$\begin{aligned}
||\zeta|| &\le \sum_{k=0}^{\infty}\frac{|\lambda|^k}{\Gamma(k(\beta-\gamma)+\beta+1)}\sup_{(x,y)\in[0,1]\times\mathcal{R}}|\eta(x, y)| + |\phi(\zeta)|\sum_{k=0}^{\infty}\frac{|\lambda|^k}{\Gamma(k(\beta-\gamma)+2)}\\
&\quad + \frac{||\zeta||}{\Gamma(\beta-\gamma+1)}|\lambda|\sum_{k=0}^{\infty}\frac{|\lambda|^k}{\Gamma(k(\beta-\gamma)+2)} + \frac{1}{\gamma(\beta+1)}\sum_{k=0}^{\infty}\frac{|\lambda|^k}{\Gamma(k(\beta-\gamma)+2)}\sup_{(x,y)\in[0,1]\times\mathcal{R}}|\eta(x, y)|\\
&= E_{(\beta-\gamma,\, \beta+1)}(|\lambda|)\sup_{(x,y)\in[0,1]\times\mathcal{R}}|\eta(x, y)| + |\phi(\zeta)|E_{(\beta-\gamma,\, 2)}(|\lambda|) + \frac{||\zeta||}{\Gamma(\beta-\gamma+1)}|\lambda|E_{(\beta-\gamma,\, 2)}(|\lambda|)\\
&\quad + \frac{1}{\gamma(\beta+1)}E_{(\beta-\gamma,\, 2)}(|\lambda|)\sup_{(x,y)\in[0,1]\times\mathcal{R}}|\eta(x, y)|.
\end{aligned}$$

Since

$$w = 1 - \frac{|\lambda|}{\Gamma(\beta-\gamma+1)}E_{(\beta-\gamma,\, 2)}(|\lambda|) > 0,$$

we deduce

$$||\zeta|| \le \frac{1}{w}\left(E_{(\beta-\gamma,\, \beta+1)}(|\lambda|) + \frac{1}{\gamma(\beta+1)}E_{(\beta-\gamma,\, 2)}(|\lambda|)\right) \cdot \sup_{(x,y)\in[0,1]\times\mathcal{R}}|\eta(x, y)| + \frac{1}{w}|\phi(\zeta)|E_{(\beta-\gamma,\, 2)}(|\lambda|) < +\infty.$$

This completes the proof.

The following is a theorem regarding the uniqueness to equation (1) based on Banach's contractive principle.

Theorem 2. Let η be a continuous and bounded function on $[0, 1] \times \mathcal{R}$, satisfying the following Lipschitz condition for a nonnegative constant $\mathcal{L}_1$:

$$|\eta(x, y_1) - \eta(x, y_2)| \le \mathcal{L}_1|y_1 - y_2|, \quad y_1, y_2 \in \mathcal{R},$$

$\phi : C[0, 1] \to \mathcal{R}$ be a functional satisfying the condition for a nonnegative constant $\mathcal{L}_2$

$$|\phi(\zeta_1) - \phi(\zeta_2)| \le \mathcal{L}_2||\zeta_1 - \zeta_2||,$$

for $\zeta_1, \zeta_2 \in C[0, 1]$. Furthermore, we assume

$$S = \mathcal{L}_1 E_{(\beta-\gamma,\, \beta+1)}(|\lambda|) + (\mathcal{L}_2 + \frac{\mathcal{L}_1}{\Gamma(\beta+1)} + \frac{|\lambda|}{\Gamma(\beta-\gamma+1)})E_{(\beta-\gamma,\, 2)}(|\lambda|) < 1.$$

Then equation (1) has a unique solution in $C[0, 1]$.

Proof. Define a nonlinear mapping $\mathcal{M}$ over $C[0, 1]$ as

$$\begin{aligned}
\mathcal{M}\zeta &= \sum_{k=0}^{\infty}(-1)^k\lambda^k I^{k(\beta-\gamma)+\beta}\eta(x, \zeta(x)) + \phi(\zeta)\sum_{k=0}^{\infty}(-1)^k\lambda^k\frac{x^{k(\beta-\gamma)+1}}{\Gamma(k(\beta-\gamma)+2)}\\
&\quad + I^{\beta-\gamma}_{x=1}\zeta(x)\sum_{k=0}^{\infty}(-1)^k\lambda^{k+1}\frac{x^{k(\beta-\gamma)+1}}{\Gamma(k(\beta-\gamma)+2)} - I^{\beta}_{x=1}\eta(x, \zeta(x))\sum_{k=0}^{\infty}(-1)^k\lambda^k\frac{x^{k(\beta-\gamma)+1}}{\Gamma(k(\beta-\gamma)+2)}.
\end{aligned}$$

It follows from the proof of Theorem 1 that $\mathcal{M}\zeta \in C[0, 1]$. We are going to show that $\mathcal{M}$ is contractive. Clearly,

$$\begin{aligned}
&\mathcal{M}\zeta_1 - \mathcal{M}\zeta_2\\
&= \sum_{k=0}^{\infty}(-1)^k\lambda^k I^{k(\beta-\gamma)+\beta}(\eta(x, \zeta_1(x)) - \eta(x, \zeta_2(x)))\\
&\quad + (\phi(\zeta_1) - \phi(\zeta_2))\sum_{k=0}^{\infty}(-1)^k\lambda^k\frac{x^{k(\beta-\gamma)+1}}{\Gamma(k(\beta-\gamma)+2)}
\end{aligned}$$

$$+I_{x=1}^{\beta-\gamma}(\zeta_1(x)-\zeta_2(x))\sum_{k=0}^{\infty}(-1)^k\lambda^{k+1}\frac{x^{k(\beta-\gamma)+1}}{\Gamma(k(\beta-\gamma)+2)}$$
$$-I_{x=1}^{\beta}(\eta(x,\zeta_1(x))-\eta(x,\zeta_2(x)))$$
$$\cdot\sum_{k=0}^{\infty}(-1)^k\lambda^k\frac{x^{k(\beta-\gamma)+1}}{\Gamma(k(\beta-\gamma)+2)}.$$

Hence,

$$||\mathcal{M}\zeta_1-\mathcal{M}\zeta_2||\le\mathcal{L}_1||\zeta_1-\zeta_2||E_{(\beta-\gamma,\,\beta+1)}(|\lambda|)$$
$$+\mathcal{L}_2||\zeta_1-\zeta_2||E_{(\beta-\gamma,\,2)}(|\lambda|)$$
$$+\frac{|\lambda|}{\Gamma(\beta-\gamma+1))}||\zeta_1-\zeta_2||E_{(\beta-\gamma,\,2)}(|\lambda|)$$
$$+\frac{\mathcal{L}_1}{\Gamma(\beta+1)}||\zeta_1-\zeta_2||E_{(\beta-\gamma,\,2)}(|\lambda|)=S||\zeta_1-\zeta_2||.$$

Since $S<1$, equation (1) has a unique solution using Banach's contractive principle. The proof is complete.

As an application, we have the following example.

Example. The following nonlinear fractional differential equation with the nonlocal boundary condition:

$$_CD^{1.5}\zeta(x)-\frac{1}{2}\ _CD^{0.5}\zeta(x)=\frac{1}{19}\sin((x^2+1)\zeta(x))$$
$$+\arctan(x^3+1),\quad x\in[0,1],\qquad(4)$$
$$\zeta(0)=0,\quad\zeta(1)=\frac{1}{10(1+\zeta^2(1/2))},$$

has a unique solution in $C[0,1]$.

Proof.

$$\eta(x,\zeta)=\frac{1}{19}\sin((x^2+1)\zeta)+\arctan(x^3+1).$$

Then η is a continuous and bounded function on $[0,1]\times\mathcal{R}$, satisfying

$$|\eta(x,\zeta_1)-\eta(x,\zeta_2)|\le\frac{1}{19}|\sin((x^2+1)\zeta_1)-\sin((x^2+1)\zeta_2)|$$
$$\le\frac{2}{19}|\zeta_1-\zeta_2|,$$

which infers that $\mathcal{L}_1=2/19$. On the other hand,

$$\phi(\zeta)=\frac{1}{10(1+\zeta^2(1/2))}$$

satisfies

$$|\phi(\zeta_1)-\phi(\zeta_2)|\le\left|\frac{1}{10(1+\zeta_1^2(1/2))}-\frac{1}{10(1+\zeta_2^2(1/2))}\right|$$
$$\le\frac{1}{10}|\zeta_1(1/2)-\zeta_2(1/2)|\le\frac{1}{10}||\zeta_1-\zeta_2||,$$

by the mean value theorem and noting that

$$\left|\frac{d}{dx}\left(\frac{1}{1+x^2}\right)\right|=\frac{2|x|}{(1+x^2)^2}\le1,\quad x\in\mathcal{R}.$$

So $\mathcal{L}_2=1/10$ and

$$S=\frac{2}{19}E_{(1,\,2.5)}(1/2)$$
$$+\left(\frac{1}{10}+\frac{2/19}{\Gamma(1.5+1)}+\frac{1/2}{\Gamma(1.5-0.5+1)}\right)E_{(1,\,2)}(1/2)$$
$$=\frac{2}{19}E_{(1,\,2.5)}(1/2)+\left(\frac{1}{10}+\frac{2}{19\,\Gamma(2.5)}+1/2\right)$$
$$\cdot E_{(1,\,2)}(1/2)$$
$$\approx\frac{2}{19}*0.926819+\left(\frac{1}{10}+\frac{2}{19\,\Gamma(2.5)}+1/2\right)*1.2974$$
$$\approx0.978734<1.$$

By Theorem 2, equation (4) has a unique solution in $C[0,1]$.

3. EXISTENCE

Using Leray–Schauder's fixed point theorem, we present the following existence theorem.

Theorem 3. Let η be a continuous and bounded function on $[0,1]\times\mathcal{R}$ and $\phi:C[0,1]\to\mathcal{R}$ be a functional satisfying the condition for a nonnegative constant $\mathcal{L}_2$

$$|\phi(\zeta_1)-\phi(\zeta_2)|\le\mathcal{L}_2||\zeta_1-\zeta_2||,$$

for $\zeta_1,\zeta_2\in C[0,1]$. In addition, we assume

$$\mathcal{Q}=1-\left(\mathcal{L}_2+\frac{|\lambda|}{\Gamma(\beta-\gamma+1)}\right)E_{(\beta-\gamma,\,2)}(|\lambda|)>0.$$

Then there exists at least one solution to equation (1) in the space $C[0,1]$.

Proof. Clearly,

$$|\phi(\zeta)|\le|\phi(\zeta)-\phi(0)|+|\phi(0)|\le\mathcal{L}_2||\zeta||+|\phi(0)|<+\infty,$$

if $\zeta\in C[0,1]$.

We define the nonlinear mapping $\mathcal{M}$ over $C[0,1]$ again as

$$\mathcal{M}\zeta=\sum_{k=0}^{\infty}(-1)^k\lambda^kI^{k(\beta-\gamma)+\beta}\eta(x,\zeta(x))$$
$$+\phi(\zeta)\sum_{k=0}^{\infty}(-1)^k\lambda^k\frac{x^{k(\beta-\gamma)+1}}{\Gamma(k(\beta-\gamma)+2)}$$
$$+I_{x=1}^{\beta-\gamma}\zeta(x)\sum_{k=0}^{\infty}(-1)^k\lambda^{k+1}\frac{x^{k(\beta-\gamma)+1}}{\Gamma(k(\beta-\gamma)+2)}$$
$$-I_{x=1}^{\beta}\eta(x,\zeta(x))\sum_{k=0}^{\infty}(-1)^k\lambda^k\frac{x^{k(\beta-\gamma)+1}}{\Gamma(k(\beta-\gamma)+2)}.$$

It follows from the proof of Theorem 1 that

$$||\mathcal{M}\zeta||\le E_{(\beta-\gamma,\,\beta+1)}(|\lambda|)\sup_{(x,y)\in[0,1]\times\mathcal{R}}|\eta(x,y)|$$
$$+|\phi(\zeta)|E_{(\beta-\gamma,\,2)}(|\lambda|)$$
$$+\frac{|\lambda|||\zeta||}{\Gamma(\beta-\gamma+1)}E_{(\beta-\gamma,\,2)}(|\lambda|)$$
$$+\frac{1}{\Gamma(\beta+1)}E_{(\beta-\gamma,\,2)}(|\lambda|)\sup_{(x,y)\in[0,1]\times\mathcal{R}}|\eta(x,y)|<+\infty,$$

which claims that $\mathcal{M}\zeta\in C[0,1]$. We first show that (i) $\mathcal{M}$ is continuous. In fact,

$$\begin{aligned}&||\mathcal{M}\zeta_1 - \mathcal{M}\zeta_2|| \\ &\leq E_{(\beta-\gamma,\, \beta+1)}(|\lambda|) \sup_{x\in[0,1]} |\eta(x,\zeta_1) - \eta(x,\zeta_2)| \\ &+\mathcal{L}_2||\zeta_1-\zeta_2||E_{(\beta-\gamma,\, 2)}(|\lambda|) \\ &+\frac{|\lambda|||\zeta_1-\zeta_2||}{\Gamma(\beta-\gamma+1)}E_{(\beta-\gamma,\, 2)}(|\lambda|) \\ &+\frac{1}{\Gamma(\beta+1)}E_{(\beta-\gamma,\, 2)}(|\lambda|) \sup_{x\in[0,1]} |\eta(x,\zeta_1) - \eta(x,\zeta_2)|.\end{aligned}$$

This implies $\mathcal{M}$ is continuous since η is continuous.

(ii) Furthermore, we prove that $\mathcal{M}$ is a mapping from bounded sets to bounded sets. Let $\mathcal{S}$ be a bounded set in $C[0,1]$. Then for $\zeta \in \mathcal{S}$,

$$|\phi(\zeta)| \leq \mathcal{L}_2||\zeta|| + |\phi(0)| < \mathcal{C},$$

where $\mathcal{C}$ is a positive constant. It follows from the above inequality that $\mathcal{M}\zeta$ is uniformly bounded if $\zeta \in S$, as η is bounded.

(iii) We claim $\mathcal{M}$ is completely continuous from $C[0,1]$ to itself. By the Arzela–Ascoli theorem, we need to show $\mathcal{M}$ is equicontinuous on every bounded set $\mathcal{S}$ of $C[0,1]$. For $0 \leq t_1 < t_2 \leq 1$ and $\zeta \in \mathcal{S}$, we have

$$\begin{aligned}&|(\mathcal{M}\zeta)(t_2) - (\mathcal{M}\zeta)(t_1)| \leq \sum_{k=0}^{\infty} \frac{|\lambda|^k}{\Gamma(k(\beta-\gamma)+\beta)} \\ &\cdot|\int_0^{t_2} (t_2-\tau)^{k(\beta-\gamma)+\beta-1}\eta(\tau,\zeta(\tau))d\tau \\ &-\int_0^{t_1} (t_1-\tau)^{k(\beta-\gamma)+\beta-1}\eta(\tau,\zeta(\tau))d\tau| \\ &+|\phi(\zeta)|\sum_{k=0}^{\infty}\frac{|\lambda^k|}{\Gamma(k(\beta-\gamma)+2)}\left|t_2^{k(\beta-\gamma)+1} - t_1^{k(\beta-\gamma)+1}\right| \\ &+\frac{|\lambda|||\zeta||}{\Gamma(\beta-\gamma+1)}\sum_{k=0}^{\infty}\frac{|\lambda^k|}{\Gamma(k(\beta-\gamma)+2)} \\ &\cdot\left|t_2^{k(\beta-\gamma)+1} - t_1^{k(\beta-\gamma)+1}\right| \\ &+\frac{1}{\Gamma(\beta+1)}\sup_{(x,y)\in[0,1]\times\mathcal{R}}|\eta(x,y)|\sum_{k=0}^{\infty}\frac{|\lambda^k|}{\Gamma(k(\beta-\gamma)+2)} \\ &\left|t_2^{k(\beta-\gamma)+1} - t_1^{k(\beta-\gamma)+1}\right| \\ &= I_1 + I_2 + I_3 + I_4.\end{aligned}$$

As for I_1, we have

$$\begin{aligned}&\int_0^{t_2} (t_2-\tau)^{k(\beta-\gamma)+\beta-1}\eta(\tau,\zeta(\tau))d\tau \\ &= \int_0^{t_1} (t_2-\tau)^{k(\beta-\gamma)+\beta-1}\eta(\tau,\zeta(\tau))d\tau \\ &+ \int_{t_1}^{t_2} (t_2-\tau)^{k(\beta-\gamma)+\beta-1}\eta(\tau,\zeta(\tau))d\tau,\end{aligned}$$

and

$$\begin{aligned}&\int_0^{t_2} (t_2-\tau)^{k(\beta-\gamma)+\beta-1}\eta(\tau,\zeta(\tau))d\tau \\ &-\int_0^{t_1} (t_1-\tau)^{k(\beta-\gamma)+\beta-1}\eta(\tau,\zeta(\tau))d\tau \\ &= \int_0^{t_1}[(t_2-\tau)^{k(\beta-\gamma)+\beta-1} - (t_1-\tau)^{k(\beta-\gamma)+\beta-1}]\eta(\tau,\zeta(\tau))d\tau \\ &+\int_{t_1}^{t_2}(t_2-\tau)^{k(\beta-\gamma)+\beta-1}\eta(\tau,\zeta(\tau))d\tau = I_{12} + I_{22}.\end{aligned}$$

Obviously,

$$\begin{aligned}|I_{12}| &\leq \int_0^{t_1}[(t_2-\tau)^{k(\beta-\gamma)+\beta-1} - (t_1-\tau)^{k(\beta-\gamma)+\beta-1}]d\tau \\ &\cdot \sup_{(x,y)\in[0,1]\times\mathcal{R}}|\eta(x,y)| \\ &= (-\frac{(t_2-t_1)^{k(\beta-\gamma)+\beta}}{k(\beta-\gamma)+\beta} + \frac{t_2^{k(\beta-\gamma)+\beta}}{k(\beta-\gamma)+\beta} \\ &-\frac{t_1^{k(\beta-\gamma)+\beta}}{k(\beta-\gamma)+\beta}) \sup_{(x,y)\in[0,1]\times\mathcal{R}}|\eta(x,y)| \\ &\leq \left(\frac{t_2^{k(\beta-\gamma)+\beta}}{k(\beta-\gamma)+\beta} - \frac{t_1^{k(\beta-\gamma)+\beta}}{k(\beta-\gamma)+\beta}\right)\sup_{(x,y)\in[0,1]\times\mathcal{R}}|\eta(x,y)|.\end{aligned}$$

By the mean value theorem, we deduce

$$\begin{aligned}&\frac{t_2^{k(\beta-\gamma)+\beta}}{k(\beta-\gamma)+\beta} - \frac{t_1^{k(\beta-\gamma)+\beta}}{k(\beta-\gamma)+\beta} \\ &= \theta^{k(\beta-\gamma)+\beta-1}(t_2-t_1) \leq t_2 - t_1,\end{aligned}$$

where $t_1 < \theta < t_2$. In summary,

$$|I_{12}| \leq (t_2-t_1)\sup_{(x,y)\in[0,1]\times\mathcal{R}}|\eta(x,y)|.$$

On the other hand,

$$\begin{aligned}|I_{22}| &\leq (t_2-t_1)\max_{\tau\in[t_1,t_2]}|(t_2-\tau)^{k(\beta-\gamma)+\beta-1}| \\ &\cdot\sup_{(x,y)\in[0,1]\times\mathcal{R}}|\eta(x,y)| \\ &\leq (t_2-t_1)\sup_{(x,y)\in[0,1]\times\mathcal{R}}|\eta(x,y)|.\end{aligned}$$

Regarding I_2, I_3 and I_4, we notice that the factor

$$\left|t_2^{k(\beta-\gamma)+1} - t_1^{k(\beta-\gamma)+1}\right|$$

contains the term $t_2 - t_1$ for all $k \geq 0$. Hence, $\mathcal{M}$ is equicontinuous on every bounded set $\mathcal{S}$ of $C[0,1]$.

(iv) Finally, we will prove that the set for $0 < \delta < 1$

$$Y = \{\zeta \in C[0,1] : \zeta = \delta\mathcal{M}\zeta\}$$

is bounded. Using

$$||\zeta|| \leq ||\mathcal{M}\zeta|| \leq E_{(\beta-\gamma,\, \beta+1)}(|\lambda|)$$

$$\cdot \sup_{(x,y)\in[0,1]\times\mathcal{R}} |\eta(x,y)| + |\phi(\zeta)|E_{(\beta-\gamma,\,2)}(|\lambda|)$$
$$+\frac{|\lambda|||\zeta||}{\Gamma(\beta-\gamma+1)}E_{(\beta-\gamma,\,2)}(|\lambda|)$$
$$+\frac{1}{\Gamma(\beta+1)}E_{(\beta-\gamma,\,2)}(|\lambda|)\sup_{(x,y)\in[0,1]\times\mathcal{R}}|\eta(x,y)|$$
$$\leq E_{(\beta-\gamma,\,\beta+1)}(|\lambda|)\sup_{(x,y)\in[0,1]\times\mathcal{R}}|\eta(x,y)| + (\mathcal{L}_2||\zeta||$$
$$+|\phi(0)|)E_{(\beta-\gamma,\,2)}(|\lambda|)$$
$$+\frac{|\lambda|||\zeta||}{\Gamma(\beta-\gamma+1)}E_{(\beta-\gamma,\,2)}(|\lambda|)$$
$$+\frac{1}{\Gamma(\beta+1)}E_{(\beta-\gamma,\,2)}(|\lambda|)\sup_{(x,y)\in[0,1]\times\mathcal{R}}|\eta(x,y)|,$$

and

$$\mathcal{Q} = 1 - \left(\mathcal{L}_2 + \frac{|\lambda|}{\Gamma(\beta-\gamma+1)}\right)E_{(\beta-\gamma,\,2)}(|\lambda|) > 0,$$

we come to

$$||\zeta|| \leq \frac{1}{\mathcal{Q}}(E_{(\beta-\gamma,\,\beta+1)}(|\lambda|)$$
$$+\frac{1}{\Gamma(\beta+1)}E_{(\beta-\gamma,\,2)}(|\lambda|))\sup_{(x,y)\in[0,1]\times\mathcal{R}}|\eta(x,y)|$$
$$+\frac{1}{\mathcal{Q}}|\phi(0)|E_{(\beta-\gamma,\,2)}(|\lambda|) < \infty,$$

which indicates that Y is bounded. By Leray–Schauder's fixed point theorem, equation (1) has at least one solution in $C[0,1]$. This completes the proof.

Example. The following nonlinear fractional differential equation with the nonlocal boundary condition:

$$_CD^{1.8}\zeta(x) + \frac{1}{4}\ {_CD^{0.7}}\zeta(x)$$
$$= \frac{x^2|\zeta(x)|}{2(1+\zeta^2(x))} + \arctan(\zeta^3(x)),\ \ x\in[0,1], \tag{5}$$
$$\zeta(0) = 0,\ \ \zeta(1) = \frac{1}{10}\sin\zeta(0.8),$$

has at least one solution in $C[0,1]$.

Proof. Clearly,

$$\eta(x,y) = \frac{x^2|y|}{2(1+y^2)} + \arctan y^3$$

is a continuous and bounded function on $[0,1]\times\mathcal{R}$, and

$$\phi(\zeta) = \frac{1}{10}\sin\zeta(0.8)$$

satisfies

$$|\phi(\zeta_1)-\phi(\zeta_2)| \leq \frac{1}{10}|\sin\zeta_1(0.8) - \sin\zeta_2(0.8)|$$
$$\leq \frac{1}{10}|\zeta_1(0.8)-\zeta_2(0.8)|$$
$$\leq \frac{1}{10}||\zeta_1-\zeta_2||,\ \ \zeta_1,\ \zeta_2\in C[0,1],$$

which implies $\mathcal{L}_2 = 1/10$. We compute

$$\mathcal{Q} = 1 - \left(1/10 + \frac{1/4}{\Gamma(1.1+1)}\right)E_{(1.1,\,2)}(1/4)$$
$$\approx 1 - \left(1/10 + \frac{1/4}{\Gamma(1.1+1)}\right) * 1.1224 \approx 0.619677 > 0.$$

So, the equation has at least one solution in $C[0,1]$ using Theorem 3. This completes the proof.

To end off this section, we would like to point out that Theorem 3 does not require that the function η satisfies the Lipschitz condition. Moreover, $S < 1$ in Theorem 2 implies that $\mathcal{Q} > 0$ in Theorem 3. In addition, equation (4) is handled by Theorem 2, rather than Theorem 3, since we need

$$\eta = \frac{1}{19}\sin((x^2+1)\zeta(x)) + \arctan(x^3+1)$$

to be a Lipschitz function to derive the uniqueness. However, equation (5) is different as

$$\eta = \frac{x^2|\zeta(x)|}{2(1+\zeta^2(x))} + \arctan(\zeta^3(x))$$

does not meet the Lipschitz condition, but it is a continuous and bounded function on $[0,1]\times\mathcal{R}$, satisfying the condition in Theorem 3 which shows the existence of solutions.

4. CONCLUSION

We have investigated the uniqueness and existence of solutions for the nonlinear fractional differential equation (1) with functional boundary condition using the two-parameter Mittag-Leffler function, Babenko's approach, Banach's contractive principle and Leray–Schauder's fixed point theorem. In addition, we presented applicable examples making use of the theorems. By using similar techniques, we can study other differential equations including PDEs with nonlocal boundary or initial conditions.

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