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The existence of a unique solution and stability results with numerical solutions for the fractional hybrid integro-differential equations with Dirichlet boundary conditions

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Abstract

In this paper, we investigate the fractional hybrid integro-differential equations with Dirichlet boundary conditions. We first prove the existence of a unique solution for the equation using a fixed point technique. Our main goal is to obtain the best approximation using optimal controllers. After studying the stability, we present the reproducing kernel Hilbert space numerical method to obtain approximate solutions to the equation. We finally conclude with numerical results.

Mathematics Subject Classification: 46E22; 34A08; 35R11; 35Q92; 92C32

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1 Introduction

Today, many complex life problems are modeled with fractional-order differential equations and fractional-order integro-differential equations. Potential theory, robotics, electromagnetism, signal processing, thermal acoustic engineering, viscous fluid dynamics, and others are among important fields of applications [1–8]. In the literature, researchers used fixed point theory to study the existence and stability of solutions for fractional-order differential equations; we refer the readers to [9–18] for more detail.

Studies on hybrid differential equations and hybrid integro-differential equations are ongoing; for example, the existence of a unique solution of a classical Cauchy problem of hybrid differential equations was considered in the literature, as well as Cauchy problems of hybrid differential equations with the Caputo–Liouville derivative, a nonlocal boundary value problem of hybrid fractional integro-differential inclusions was investigated, and an initial value problem of nonlinear hybrid fractional integro-differential equations was examined in the literature; see [19–23] for more information.

In this paper, we consider the following hybrid proportional fractional integro-differential equations with Dirichlet boundary conditions (HPFIDE):

$$\begin{cases} {}_{m}^{C}(\Xi_D)_{0^+}^{\varsigma_1,\varsigma_2}\left(\frac{\phi(\zeta)-\sum_{r=1}^{r=m}\Pi_{I_{0^+}}^{\varsigma_{3r}\varsigma_2}\mathcal{W}_r(\zeta,\phi(\zeta))}{\mathcal{Y}(\zeta,\phi(\zeta))}\right)=\mathcal{K}(\zeta,\phi(\zeta)), \quad \zeta\in[0,\mathrm{F}], \\ \phi(0)=\phi_0, \phi(\mathrm{F})=\phi_{\mathrm{F}}, \end{cases} \tag{1.1}$$

where ${}_{m}^{C}(\Xi_D)_{0^+}^{\varsigma_1,\varsigma_2}$ represents the Caputo–Liouville proportional fractional derivative (C-LPFD) of order ς_1, $(\Pi_I)_{0^+}^{\varsigma_{3r},\varsigma_2}$ denotes the Riemann–Liouville proportional fractional integral (R-LPFD) of order ς_{3_r} for $\varsigma_2\in(0,1]$, $1<\varsigma_1\le 2$, $\varsigma_{3_r}>0$ ($r=1,\dots,m$), $m\in\mathbb{N}^*$, $\phi_0,\phi_{\mathrm{F}}\in\mathbb{R}$, $\mathcal{W}_r:[0,\mathrm{F}]\times\mathbb{R}\longrightarrow\mathbb{R}$, $\mathcal{Y}:[0,\mathrm{F}]\times\mathbb{R}\longrightarrow\mathbb{R}\backslash\{0\}$, and $\mathcal{K}:[0,\mathrm{F}]\times\mathbb{R}\longrightarrow\mathbb{R}$.

We note the following:

- In the case $\varsigma_2=1$, HPFIDE (1.1) becomes a combined integro-differential equation with the Caputo–Liouville fractional derivative and the Riemann–Liouville fractional integral.
- For all $(\zeta,\phi(\zeta))\in[0,\mathrm{F}]\times\mathbb{R}$ and $r=1,\dots,m$, in the case that $\mathcal{Y}(\zeta,\phi(\zeta))=1$ and $\mathcal{W}_r(\zeta,\phi(\zeta))=0$, HPFIDE (1.1) reduces to the Caputo–Liouville proportional fractional boundary value problem (C-LPFBVP)

$$\begin{cases} {}_{m}^{C}(\Xi_D)_{0^+}^{\varsigma_1,\varsigma_2}(\phi(\zeta))=\mathcal{K}(\zeta,\phi(\zeta)), \quad \zeta\in[0,\mathrm{F}], \\ \phi(0)=\phi_0, \phi(\mathrm{F})=\phi_{\mathrm{F}}. \end{cases} \tag{1.2}$$

- For all $(\zeta,\phi(\zeta))\in[0,\mathrm{F}]\times\mathbb{R}$ and $r=1,\dots,m$, in the case that $\mathcal{Y}(\zeta,\phi(\zeta))=1$ and $\varsigma_2=1$, HPFIDE (1.1) reduces to the C-LPFBVP

$$\begin{cases} {}^{C}(\Xi_D)_{0^+}^{\varsigma_1,\varsigma_2}(\phi(\zeta))=\mathcal{K}(\zeta,\phi(\zeta)), \quad \zeta\in[0,\mathrm{F}], \\ \phi(0)=\phi_0, \phi(\mathrm{F})=\phi_{\mathrm{F}}. \end{cases} \tag{1.3}$$

The method of variational iteration, the Adomian decomposition method, the Laplace transform, the finite difference method, the Fourier transform, the collocation method, etc. were considered among the techniques of solving fractional differential equations; see [24–31]. For example, fractional differential equations with periodic conditions were investigated in [32–34]. In the early twentieth century, the reproducing kernel method was first used in boundary value problems. In 1907, Zarmba was the first to introduce the kernel of certain functions and to express their reproducing properties. The reproducing kernel functions of a Hilbert space have been introduced in the form of very simple polynomials since 1980. Many researchers use the reproducing kernel Hilbert space method to approximate solutions to various problems. The reproducing kernel method was applied to boundary value problems in [35, 36]. In particular, this method has also been considered for fractional differential equations such as the fractional gas dynamics, fractional advection–diffusion equation, fractional diffusion and advection–dispersion equations, fractional Duffing–van der Pol oscillator equations, and so on [37–40]. Other new applications of this method can be mentioned in machine learning [41–43]. Our motivation is to use the new fractional operators to obtain numerical solutions for the considered problems using the reproducing kernel method. In addition, using the fixed point technique, we study the stability of the given problem in a completely theoretical way.

The paper is organized as follows. In Sect. 2, we introduce the basic concepts of fractional calculus and the theorems and lemmas needed to describe and examine equation (1.1). Also, we present the fixed point technique to obtain a suitable approximation. In addition, we introduce the functions and spaces required for numerical solutions of equation (1.1) using the reproducing kernel method. In Sect. 3, by implementing the fixed point technique we investigate the stability of equation (1.1) and establish the existence of a suitable approximation. In Sect. 4, by introducing the reproducing kernel method, we present the steps to numerically solve equation (1.1). In Sect. 5, we consider a numerical example of equation (1.1), including some graphs and tables. A conclusion section is given at the end.

2 Preliminaries

We start this section with definitions of integral and derivative of fractional order and explanations of gamma function used in fractional calculations [43].

Definition 2.1 ([44, 45]) For $Re(\varsigma_1) > 0$, the upper incomplete gamma function, the lower incomplete gamma function, the upper regularized incomplete gamma function, and lower regularized incomplete gamma function are defined as follows, respectively:

- $\Gamma_1(\varsigma_1,\zeta) = \int_\zeta^{+\infty} h^{\varsigma_1-1}\exp(-h)dh$, $\zeta \geq 0$;
- $\Gamma_2(\varsigma_1,\zeta) = \int_0^\zeta h^{\varsigma_1-1}\exp(-h)dh$, $\zeta \geq 0$;
- $\Gamma_3(\varsigma_1,\zeta) = \frac{\Gamma_1(\varsigma_1,\zeta)}{\Gamma(\varsigma_1)}$;
- $\Gamma_4(\varsigma_1,\zeta) = 1 - \Gamma_3(\varsigma_1,\zeta) = \frac{\Gamma_2(\varsigma_1,\zeta)}{\Gamma(\varsigma_1)}$.

Definition 2.2 ([44, 45]) For a function $\mathcal{Z} \in L^1([0,\mathrm{F}],\mathbb{R})$, the left fractional Riemann–Liouville integral (L-FRLI) and the left generalized proportional Riemann–Liouville fractional integral (L-GPRLFI) are defined as follows, respectively:

- $(\Pi_I)_{p^+}^{0}\mathcal{Z}(\zeta) = \mathcal{Z}(\zeta)$;
- $(\Pi_I)_{p^+}^{\varsigma_1}\mathcal{Z}(\zeta) = \frac{1}{\Gamma(\varsigma_1)}\int_p^\zeta (\zeta-\mu)^{\varsigma_1-1}\mathcal{Z}(\mu)d\mu$;
- $(\Pi_I)_{p^+}^{0,\varsigma_2}\mathcal{Z}(\zeta) = \mathcal{Z}(\zeta)$;
- $(\Pi_I)_{p^+}^{\varsigma_1,\varsigma_2}\mathcal{Z}(\zeta) = \frac{1}{\varsigma_2^{\varsigma_1}\Gamma(\varsigma_1)}\int_p^\zeta (\zeta-\mu)^{\varsigma_1-1}\exp(\frac{\varsigma_2-1}{\varsigma_2}(\zeta-\mu))\mathcal{Z}(\mu)d\mu$;

where $\zeta \in [p,q]$, $\varsigma_2 \in (0,1]$, and $\varsigma_1 \geq 0$.

Definition 2.3 For a function $\mathcal{Z} \in C^e([0,\mathrm{F}],\mathbb{R})$, the left fractional Caputo–Liouville derivative (L-FCLD) and the left generalized proportional Caputo–Liouville fractional derivative (L-GPCLFD) are defined as follows, respectively:

- ${}_m^C(\Xi_D)_{p^+}^{0}\mathcal{Z}(\zeta) = \mathcal{Z}(\zeta)$;
- ${}^C(\Xi_D)_{p^+}^{\varsigma_1}\mathcal{Z}(\zeta) = (\Pi_I)_{p^+}^{e-\varsigma_1}\left(D^e\mathcal{Z}\right)(\zeta) = \frac{1}{\Gamma(n-\varsigma_1)}\int_p^\zeta (\zeta-\mu)^{n-\varsigma_1-1}D^e\mathcal{Z}(\mu)$;
- ${}_m^C(\Xi_D)_{p^+}^{0,\varsigma_2}\mathcal{Z}(\zeta) = \mathcal{Z}(\zeta)$;
-

$$
\begin{aligned}
&{}_m^C(\Xi_D)_{p^+}^{\varsigma_1,\varsigma_2}\mathcal{Z}(\zeta) = (\Pi_I)_{p^+}^{e-\varsigma_1,\varsigma_2}\left((\Xi_D)^{n,\varsigma_2\varsigma_2}\mathcal{Z}\right)(\zeta)\\
&= \frac{1}{\varsigma_2^{\varsigma_1}\Gamma(e-\varsigma_1)}\int_p^\zeta (\zeta-\mu)^{e-\varsigma_1-1}\exp(\frac{e-1}{\varsigma_2}(\zeta-\mu))\left((\Xi_D)^{e,\varsigma_2}\mathcal{Z}\right)(\mu)d\mu,
\end{aligned}
$$

where $\zeta \in [p,q]$, $e-1 < \varsigma_1 \leq e$, $e \in \mathbb{N}$, $\varsigma_2 \in (0,1]$, and $\varsigma_1 \geq 0$. Also, for the last item, we have $\left((\Xi_D)^{1,\varsigma_2}\mathcal{Z}\right)(\zeta) = ((\Xi_D)^{\varsigma_2}\mathcal{Z})(\zeta) = (1-\varsigma_2)\mathcal{Z}(\zeta) + \varsigma_2\mathcal{Z}'(\zeta)$, $\left((\Xi_D)^{e,\varsigma_2}\mathcal{Z}\right)(\zeta) = \mathcal{Z}(\zeta)$ for $e = 0$, and $\left((\Xi_D)^{e,\varsigma_2}\mathcal{Z}\right)(\zeta) = (\underbrace{(\Xi_D)^{\varsigma_2}D^{\varsigma_2}\cdots(\Xi_D)^{\varsigma_2}}_{e\text{ times}}\mathcal{Z})(\zeta)$ for $e \geq 1$.

For L-GPRLFI and L-GPCLFD, we have the following properties:

- $(\Pi_I)_{p^+}^{\varsigma_1,\varsigma_2}\left((\Pi_I)_{p^+}^{\varsigma_3,\varsigma_2}\mathcal{Z}\right)(\zeta) = (\Pi_I)_{p^+}^{\varsigma_3,\varsigma_2}\left((\Pi_I)_{p^+}^{\varsigma_1\varsigma_2\varsigma_2}\mathcal{Z}\right)(\zeta) = (\Pi_I)_{p^+}^{\varsigma_1+\varsigma_3,\varsigma_2}\mathcal{Z}(\zeta)$;
- ${}_m^C(\Xi_D)_{p^+}^{\varsigma_1,\varsigma_2}\left((\Pi_I)_{p^+}^{\varsigma_1\varsigma_2\varsigma_2}\mathcal{Z}\right)(\zeta) = \mathcal{Z}(\zeta)$;
- $(\Pi_I)_{p^+}^{\varsigma_1,\tau}\left({}_m^C(\Xi_D)_{p^+}^{\varsigma_1,\varsigma_2}\mathcal{Z}\right)(\zeta) = \mathcal{Z}(\zeta) - \sum_{i=0}^{p-1} d_i(\zeta-p)^i \exp(\frac{\varsigma_2-1}{\varsigma_2}(\zeta-p))$,

where, $\varsigma_2 \in (0,1]$, $\varsigma_3 > 0$, $\varsigma_1 > 0$ with $e-1 < \varsigma_1 \le e$, $\mathcal{Z} \in L^1([0,\mathrm{F}],\mathbb{R})$, $\mathcal{Z} \in C^e([0,\mathrm{F}],\mathbb{R}))$, and $d_i = \frac{\left((\Xi_D)^{i,\varsigma_2}\cdot\mathcal{Z}\right)(p)}{\varsigma_2^i i!}$. Also, for $1 < \varsigma_1 \le 2$ and $p = 0$,

- $(\Pi_I)_{0^+}^{\varsigma_1,\varsigma_2}\left({}_m^C(\Xi_D)_{0^+}^{\varsigma_1,\varsigma_2}\mathcal{Z}\right)(\zeta) = \mathcal{Z}(\zeta) - d_0 e^{\frac{\varsigma_2-1}{\varsigma_2}\zeta} - d_1\zeta\exp(\frac{\varsigma_2-1}{\varsigma_2}\zeta)$.

By considering the lower regularized incomplete gamma function Γ_4, for $\varsigma \in \mathbb{R}^+$, $\varsigma_1 \in \mathbb{C}$, $\mathrm{Re}(\varsigma_1) > 0$, $\varsigma_2 \in (0,1]$, and $g > 0$, we have

- For $\zeta \in [p,q]$, $\Gamma_4(\varsigma_1, x(\zeta-a))$ is a nondecreasing function;
- For all $\zeta \ge p$, $\Gamma_4(\varsigma_1, x(\zeta-p)) \in [0,1]$;
- $\max_{\zeta\in[p,q]} \Gamma_4(\varsigma_1, x(\zeta-p)) = \Gamma_4(\varsigma_1, x(\zeta-p))|_{\zeta=q} = \Gamma_4(\varsigma_1, x(q-p))$;
- $\min_{\zeta\in[p,q]} \Gamma_4(\varsigma_1, x(\zeta-p)) = \Gamma_4(\varsigma_1, x(\zeta-p))|_{\zeta=p} = 0$;
- For $\varsigma_2 \in (0,1)$, $\left((\Pi_I)_{p^+}^{g,\varsigma_2}1\right)(\zeta) = \frac{\Gamma_4\left(g,\frac{1-\varsigma_2}{\varsigma_2}(\zeta-p)\right)}{(1-\varsigma_2)^g}$;
- For $\varsigma_2 = 1$, $\left((\Pi_I)_{p^+}^{g,\varsigma_2}1\right)(\zeta) = \left((\Pi_I)_{p^+}^{g}\right)(\zeta) = \frac{(\zeta-p)^\mu}{\Gamma_1(g+1)}$;
- $\lim_{\varsigma_2\to1} \frac{\Gamma_4\left(g,\frac{1-\varsigma_2}{\varsigma_2}(\zeta-p)\right)}{(1-\varsigma_2)^g} = \left((\Pi_I)_{p^+}^{g}1\right)(\zeta) = \frac{(\zeta-p)^g}{\Gamma_1(g+1)}$;
- $\max_{g\in[p,q]}\left(\lim_{\varsigma_2\to1} \frac{\Gamma_4\left(g,\frac{1-\varsigma_2}{\varsigma_2}(\zeta-p)\right)}{(1-\varsigma_2)^g}\right) = \frac{(q-p)^g}{\Gamma_1(g+1)}$;

Proposition 2.4 ([43]) *If Γ_4 is the lower regularized incomplete gamma function, then we have*

(1) *For $\varsigma_2 \in (0,1)$, ζ_1, $\zeta_2 \in [p,q]$ $(\zeta_1 \le \zeta_2)$, and $g > 0$,*

$$\int_{\zeta_1}^{\zeta_2} (q-\mu)^{g-1}\exp(\frac{\varsigma_2-1}{\varsigma_2}(q-\mu))d\mu$$
$$= \frac{\varsigma_2^g\Gamma(g)}{(1-\varsigma_2)^g}\left[\Gamma_4\left(g,\frac{1-\varsigma_2}{\varsigma_2}(q-\zeta_1)\right) - \Gamma_4\left(g,\frac{1-\varsigma_2}{\varsigma_2}(q-\zeta_2)\right)\right];$$

(2) *For $\varsigma_2 \in (0,1]$, $\varsigma_1 > 0$, and $p \le \mu \le \zeta_1 < \zeta_2 \le q$,*

$$\lim_{\zeta_2\to\zeta_1}\int_p^{\zeta_1}\left|(\zeta_2-\mu)^{\varsigma_1-1}\exp(\frac{\varsigma_2-1}{\varsigma_2}(\zeta_2-\mu)) - (\zeta_1-\mu)^{\varsigma_1-1}\exp(\frac{-1}{\varsigma_2}(\zeta_1-\mu))\right|d\mu$$
$$= 0.$$

The next theorem states the fixed point technique, which is our main method for establishing stability [11, 12].

Theorem 2.5 *Consider the generalized complete metric space $(\mathcal{A}, \Delta_{CON})$ (GCMS). Assume that a function $\Upsilon : \mathcal{A} \to \mathcal{A}$ with Lipschitz coefficient $0 \le \alpha < 1$ satisfies in the following inequality*:

$$\Delta_{CON}(\Upsilon\phi, \Upsilon\psi) \le \alpha\ \Delta_{CON}(\psi,\phi)$$

for $\phi, \psi \in \mathcal{A}$.

Let $\phi \in \mathcal{A}$. For any $\mathbf{b}, \mathbf{b}_0 \in \mathbb{N}$ with $\mathbf{b} \geq \mathbf{b}_0$, we have two options: $\Delta_{CON}\left(\Upsilon^{\mathbf{b}}\phi, \Upsilon^{\mathbf{b}+1}\phi\right) = \infty$, or $\Delta_{CON}\left(\Upsilon^{\mathbf{e}}\phi, \Upsilon^{\mathbf{b}+1}\phi\right) < +\infty$. If $\Delta_{CON}\left(\Upsilon^{\mathbf{b}}\phi, \Psi^{\mathbf{b}+1}\phi\right) < +\infty$, then the sequence $\left\{\Psi^{\mathbf{b}}\phi\right\}$ converges to the unique fixed point ψ^* of Υ, which The fixed point ψ^* is in the set $\mathcal{N}^* = \left\{\psi \in \mathcal{A} \mid \Delta_{CON}\left(\Upsilon^{\mathbf{b}_0}\phi, \psi\right) < +\infty\right\}$. For $\psi \in \mathcal{A}$, we have $(1-\alpha)\Delta_{CON}(\psi, \psi^*) \leq \Delta_{CON}(\psi, \Upsilon\psi)$.

Definition 2.6 ([11–13]) We say that (1.1) is stable if for the function $\phi(\zeta)$ and the control function $\Omega(\zeta)$,

$$\left| {}^{C}_{m}\Xi_{D_{0^+}^{\varsigma_1,\varsigma_2}} \left(\frac{\phi(\zeta) - \sum_{r=1}^{r=m} \Pi_{I_{0^+}^{\varsigma_{3r}\varsigma_2}} \mathcal{W}_r(\zeta, \phi(\zeta))}{\mathcal{Y}(\zeta, \phi(\zeta))} \right) - \mathcal{K}(\zeta, \phi(\zeta)) \right| \leq \delta\Omega(\zeta) \tag{2.1}$$

for $\delta > 0$, $\zeta \in \mathrm{F}$, and there exists a solution $\psi(\zeta)$ of Equation (1.1) such that

$$|\psi(\zeta) - \phi(\zeta)| \leq \omega\, \delta\Omega(\zeta)$$

for $\zeta \in \mathrm{F}$, $\omega > 0$.

Definition 2.7 ([46]) Consider a space $\mathcal{A}$ and a Hilbert space H and assume that $\Theta_{[rk]}(\zeta, \eta)$ is a function in H. Then, we say that the function $\Theta_{[rk]}(\zeta, \eta)$ is a reproduction kernel function and the Hilbert space H is a reproducing kernel space, if we have
(RKF-1) $\Theta_{[rk]}(\zeta, \eta) \in H$ for $\eta \in \mathcal{A}$;
(RKF-2) $\left\langle \mathcal{Y}(\zeta), \Theta_{[rk]}(\zeta, \eta) \right\rangle_H = \mathcal{Y}(\eta)$,
where $\langle \cdot, \cdot \rangle_H$ is the inner product in H.

Definition 2.8 ([47]) Consider the set $L^2[0, \mathrm{F}]$ of Lebesgue square-integrable functions on $[0, \mathrm{F}]$. The Hilbert space $\mathrm{U}_2^1[0, \mathrm{F})$ is the space of functions defined as

$$\mathrm{U}_2^1[0, \mathrm{F}] = \{\psi(\zeta) \mid \psi \text{ is absolutely continuous, } \psi' \in L^2[0, \mathrm{F}]\},$$

in which the inner product and norm are defined, respectively, as
(IP-U_2^1) $\langle \psi(\zeta), z(\zeta) \rangle_{\mathrm{U}_2^1} = \psi(0)z(0) + \int_0^{\mathrm{F}} \psi'(\zeta) z'(\zeta) \mathrm{d}\zeta$;
(N-U_2^1) $\|\psi(\zeta)\|_{\mathrm{U}_2^1} = \langle \psi(\zeta), \psi(\zeta) \rangle_{\mathrm{U}_2^1}^{\frac{1}{2}}$.

Definition 2.9 ([47]) The Hilbert space $\mathrm{U}_2^2[0, \mathrm{F}]$ is the space of functions defined as

$$\mathrm{U}_2^2[0, \mathrm{F}] = \left\{\psi(\zeta) \mid \psi(\zeta)' \text{ is absolutely continuous, } \psi'' \in L^2[0, \mathrm{F}], \psi(0) = \psi(\mathrm{F})\right\},$$

where the inner product and norm are given, respectively, as
(IP-U_2^2) $\langle \psi(\zeta), z(\zeta) \rangle_{\mathrm{U}_2^2} = \sum_{r=0}^{1} \psi^{(r)}(0) z^{(r)}(0) + \int_0^{\mathrm{F}} \psi^{(2)}(\zeta) z^{(2)}(\zeta) \mathrm{d}\zeta$;
(N-U_2^2) $\|\psi(\zeta)\|_{\mathrm{U}_2^2} = \langle \psi(\zeta), \psi(\zeta) \rangle_{\mathrm{U}_2^2}^{1/2}$.

Lemma 2.10 *The reproducing kernel function $\Theta_{[rk]}(\zeta, \eta)$ of the space $\mathrm{U}_2^1[0, \mathrm{F}]$ is*

$$\Theta_{[rk]}(\zeta, \eta) = \begin{cases} \zeta + 1, & \zeta \leq \eta, \\ \eta + 1, & \zeta > \eta. \end{cases} \tag{2.2}$$

Proof Considering (IP-U_2^1), we have

$$\langle \psi(\zeta), \Theta_{[rk]}(\zeta,\eta)\rangle_{U_2^2} = \psi(0)\Theta_{[rk]\eta}(0) + \int_0^F \psi'(\zeta)\Theta_{[rk]\eta}'(\zeta)d\zeta, \tag{2.3}$$

so

$$\begin{aligned}\langle \psi(\zeta), \Theta_{[rk]}(\zeta,\eta)\rangle_{U_2^2} &= \psi(0)\Theta_{[rk]\eta}(0) + \psi(F)\Theta_{[rk]\eta}'(F)\\ &\quad - \psi(0)\Theta'_{[rk]\eta}(0) - \int_0^F \psi(\zeta)\Theta''_{[rk]\eta}(\zeta)d\zeta.\end{aligned} \tag{2.4}$$

Now let $\Theta_{[rk]\eta}(0) - \Theta'_{[rk]\eta}(0) = 0$ and $\Theta'_{[rk]\eta}(F) = 0$. Then $-\Theta''_{[rk]\eta}(\zeta) = \delta(\zeta - \eta)$, where δ denotes the Dirac delta function. When $\zeta \neq \eta$, we have $\Theta''_{[rk]\eta}(\zeta) = 0$. Then

$$\Theta_{[rk]}(\zeta,\eta) = \begin{cases} \tau_1(\eta) + \tau_2(\eta)\zeta, & \zeta \le \eta, \\ \tau_3(\eta) + \tau_4(\eta)\zeta, & \zeta > \eta. \end{cases} \tag{2.5}$$

Now using the properties of δ, we have $\Theta_{[rk]\eta^+}(\zeta) = \Theta_{[rk]\eta^-}(\zeta)$ and $\Theta'_{[rk]\eta^+}(\zeta) = \Theta'_{[rk]\eta^-}(\zeta) = -1$. Then by (2.5) and the obtained properties we obtain the coefficients of τ_{1r}, τ_{2r}, τ_{1r}, and τ_{4r} for $r = 1,2$, i.e., $\tau_1(\zeta) = 1$, $\tau_2(\zeta) = 1$, $\tau_3(\zeta) = 1 + \zeta$, $\tau_4(\zeta) = 0$. □

Lemma 2.11 *The reproducing kernel function* $\Lambda_{[rk]}(\zeta,\eta)$ *of the space* $U_2^2[0,F]$ *is*

$$\Lambda_{[rk]}(\zeta,\eta) = \begin{cases} \zeta\eta + \frac{1}{2}\zeta^2\eta - \frac{1}{6}\zeta^3, & \zeta \le \eta, \\ \eta\zeta + \frac{1}{2}\zeta\eta^2 - \frac{1}{6}\eta^3, & \zeta > \eta. \end{cases}$$

Proof Similarly to the previous theorem, considering (IP-U_2^2), we obtain

$$\begin{aligned}&\langle \psi(\zeta), \Lambda_{[rk]}(\zeta,\eta)\rangle_{U_2^2}\\ &\quad= \psi(0)\Lambda_{[rk]\eta}(0) + \psi'(0)\Lambda'_{[rk]\eta}(0) + \psi'(F)\Lambda''_{[rk]\eta}(F) - \psi'(0)\Lambda''_{[rk]\eta}(0)\\ &\qquad - \psi(F)\Lambda^3_{[rk]\eta}(F) + \psi(0)\Lambda^3_{[rk]\eta}(0) + \int_0^F \psi(\zeta)\Lambda^4_{[rk]\eta}(F)d\zeta.\end{aligned} \tag{2.6}$$

Now let $\Lambda'_{[rk]\eta}(0) - \Lambda''_{[rk]\eta}(0) = 0$, $\Lambda_{[rk]\eta}(0) = 0$, $\Lambda''_{[rk]\eta}(F) = 0$, $\Lambda^3_{[rk]\eta}(F) = 0$. Then $\Lambda^4_{[rk]\eta}(\zeta) = \delta(\zeta - \eta)$. When $\zeta \neq \eta$, we have $\Lambda^4_{[rk]\eta}(\zeta) = 0$. Then

$$\Lambda_{[rk]}(\zeta,\eta) = \begin{cases} \tau_1(\eta) + \tau_2(\eta)\zeta + \tau_5(\eta)\zeta^2 + \tau_6(\eta)\zeta^3, & \zeta \le \eta, \\ \tau_3(\eta) + \tau_4(\eta)\zeta + \tau_7(\eta)\zeta^2 + \tau_8(\eta)\zeta^3, & \zeta > \eta. \end{cases} \tag{2.7}$$

Now using the properties of δ, we have $\Lambda_{[rk]\eta^+}(\zeta) = \Lambda_{[rk]\eta^-}(\zeta)$, $\Lambda'_{[rk]\eta^+}(\zeta) = \Lambda'_{[rk]\eta^-}(\zeta)$, $\Lambda''_{[rk]\eta^+}(\zeta) = \Lambda''_{[rk]\eta^-}(\zeta)$, $\Lambda^3_{[rk]\eta^+}(\zeta) - \Lambda^3_{[rk]\eta^-}(\zeta) = 1$. Then by (2.7) and the obtained properties we obtain the coefficients of τ_r and τ_r for $r = 1,2,3,4$, i.e., $\tau_1(\zeta) = 0$, $\tau_2(\zeta) = \zeta$, $\tau_5(\zeta) = \frac{\zeta}{2}$, $\tau_6(\zeta) = \frac{-1}{6}$, $\tau_3(\zeta) = \frac{-\zeta^3}{6}$, $\tau_4(\zeta) = \frac{1}{2}\zeta(\zeta + 2)$, $\tau_7(\zeta) = 0$, $\tau_8(\zeta) = 0$. □

3 Stability results for HPFIDE (1.1)

We introduce a mapping that will be used throughout our stability results.

Definition 3.1 Considering the set $\mathcal{N} = \{\phi : \mathrm{F} \to \mathbb{R}, \phi \text{ is differentiable}\}$, the operator $\Delta_{CON} : \mathcal{N} \times \mathcal{N} \to [0, +\infty]$ is defined as

$$\Delta_{CON}(\phi, \psi) = \inf\{\mathcal{E} \in [0, \infty] : |\phi(\zeta) - \psi(\zeta)| \le \mathcal{E}\Omega(\zeta),\ \zeta \in \mathrm{F}\}. \tag{3.1}$$

Lemma 3.2 *The set $\mathcal{N}$ along with the operator defined in Definition 3.1 is a generalized complete metric space (GCMS).*

Proof We assume that $\Delta_{CON}(\phi, \psi) > \Delta_{CON}(\phi, \psi) + \Delta_{CON}(\psi, \psi)$ for $\phi, \psi \in \mathcal{N}$. Then we have $|\phi(\zeta_0) - \psi(\zeta_0)| > \Delta_{CON}(\phi, \psi) + \Delta_{CON}(\psi, \psi)\Omega(\zeta)$. Now, with respect to Δ_{CON}, we arrive at a contradiction, i.e., we have $|\phi(\zeta_0) - \psi(\zeta_0)| > |\phi(\zeta_0) - \psi(\zeta_0)| + |\psi(\zeta_0) - \psi(\zeta_0)|$. Consider a Cauchy sequence $\{\phi_n\}$ on $(\mathcal{N}, \Delta_{CON})$. For $\delta > 0$, we can find $N_\delta \in \mathbb{N}$ such that $|\phi_n(\zeta) - \phi_m(\zeta)| < \delta\Omega(\zeta)$ for all $m, n \ge N_\delta$ and $\zeta \in \mathrm{F}$. Now taking into account the continuity property of the control function $\Omega(\zeta)$ on $[0, \mathrm{F}]$ (which is a compact set), we conclude that the sequence $\{\phi_n\}$ is uniformly convergent to $\phi \in \mathcal{N}$. Then, for $\delta > 0$, we can find $N_\delta \in \mathbb{N}$ such that $|\phi_n(\zeta) - \phi(\zeta)| < \delta\Omega(\zeta)$ for $n \ge N_\delta$ and $\zeta \in \mathcal{N}$. Let $m \to \infty$, and then $\Delta_{CON}(\phi_n, \phi) \le \delta$. We conclude that $(\mathcal{N}, \Delta_{CON})$ is a GCMS. □

Theorem 3.3 *Suppose we have positive constants $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$, $\mathcal{M}_4$, $\mathcal{M}_5$, ϱ such that*

$$\begin{aligned} 0 &< \mathcal{M}_1 \frac{\psi_0}{\mathcal{Y}(0, \psi_0)} \left(1 - \frac{\zeta}{\mathrm{F}}\right) \exp(\frac{\varsigma_2 - 1}{\varsigma_2}\zeta) + \mathcal{M}_1 \frac{\psi_{\mathrm{F}} \frac{\zeta}{\mathrm{F}} \exp(\frac{\varsigma_3 - 1}{\varsigma_2}\zeta)}{\mathcal{Y}(\mathrm{F}, \psi_{\mathrm{F}}) \exp(\frac{\varsigma_3 - 1}{\varsigma_2}\mathrm{F})} \\ &+ \frac{\zeta \exp(\frac{\varsigma_2 - 1}{\varsigma_2}\zeta)}{\mathcal{Y}(\mathrm{F}, \psi_{\mathrm{F}}) \mathrm{F} \exp(\frac{\varsigma_3 - 1}{\varsigma_2}\mathrm{F})} \sum_{r=1}^{r=m} \frac{1}{\varsigma_2^{\varsigma_{3r}} \Gamma\left(\varsigma_{3r}\right)} \mathcal{M}_2 + \mathcal{M}_3 \frac{\zeta \exp(\frac{\varsigma_3 - 1}{\varsigma_2}\zeta)}{\mathrm{F} \exp(\frac{\varsigma_2 - 1}{\varsigma_2}\mathrm{F})} \frac{1}{\varsigma_2^{\varsigma_1} \Gamma(\varsigma_1)} \\ &+ \mathcal{M}_4 \frac{1}{\varsigma_2^{\varsigma_1} \Gamma(\varsigma_1)} + \sum_{r=1}^{m} \frac{1}{\varsigma_2^{\varsigma_{3r}} \Gamma\left(\varsigma_{3r}\right)} \int_0^\zeta (\zeta - \mu)^{\varsigma_{3r} - 1} \exp(\frac{\varsigma_2 - 1}{\varsigma_2}(\zeta - \mu)) \mathcal{M}_5 \varrho < 1. \end{aligned} \tag{3.2}$$

Suppose that for continuous functions $\mathcal{Y} : [0, \mathrm{F}] \times \mathbb{R} \to \mathbb{R} - \{0\}$, $\mathcal{W}_r : [0, \mathrm{F}] \times \mathbb{R} \to \mathbb{R}$, $\mathcal{K} : [0, \mathrm{F}] \times \mathbb{R} \to \mathbb{R}$, and $\Omega : \mathbb{R} \longrightarrow \mathbb{R}$, we have

$$|\mathcal{Y}(\zeta, \phi(\zeta)) - \mathcal{Y}(\zeta, \psi(\zeta))| \le \mathcal{M}_1 |\phi(\zeta) - \psi(\zeta)|, \tag{3.3}$$

$$\begin{aligned} &\Bigg|\mathcal{Y}(\zeta, \psi(\zeta)) \int_0^{\mathrm{F}} (\mathrm{F} - \mu)^{\varsigma_{3r} - 1} \exp(\frac{\varsigma_3 - 1}{\varsigma_2}(\mathrm{F} - \mu)) \mathcal{W}_r(\mu, \psi(\mu)) d\mu \\ &- \mathcal{Y}(\zeta, \phi(\zeta)) \int_0^{\mathrm{F}} (\mathrm{F} - \mu)^{\varsigma_{3r} - 1} \exp(\frac{\varsigma_3 - 1}{\varsigma_2}(\mathrm{F} - \mu)) \mathcal{W}_r(\mu, \phi(\mu)) d\mu\Bigg| \\ &\le \mathcal{M}_2 |\phi(\zeta) - \psi(\zeta)|, \end{aligned} \tag{3.4}$$

$$\begin{aligned} &\Bigg|\mathcal{Y}(\zeta, \psi(\zeta)) \int_0^{\mathrm{F}} (\mathrm{F} - \mu)^{\varsigma_1 - 1} \exp(\frac{\varsigma_2 - 1}{\varsigma_2}(\mathrm{F} - \mu)) \mathcal{K}(\mu, \psi(\mu)) d\mu - \\ &- \mathcal{Y}(\zeta, \phi(\zeta)) \int_0^{\mathrm{F}} (\mathrm{F} - \mu)^{\varsigma_1 - 1} \exp(\frac{\varsigma_2 - 1}{\varsigma_2}(\mathrm{F} - \mu)) \mathcal{K}(\mu, \phi(\mu)) d\mu\Bigg| \\ &\le \mathcal{M}_3 |\phi(\zeta) - \psi(\zeta)|, \end{aligned} \tag{3.5}$$

$$\begin{aligned}&\Big|\mathcal{Y}(\zeta,\phi(\zeta))\int_0^{\zeta}(\zeta-\mu)^{\varsigma_1-1}\exp(\frac{\varsigma_3-1}{\varsigma_2}(\zeta-\mu))\mathcal{K}(\mu,\phi(\mu))d\mu \\ &-\mathcal{Y}(\zeta,\psi(\zeta))\int_0^{\zeta}(\zeta-\mu)^{\varsigma_1-1}\exp(\frac{\varsigma_3-1}{\varsigma_2}(\zeta-\mu))\mathcal{K}(\mu,\psi(\mu))d\mu\Big| \\ &\le \mathcal{M}_4|\phi(\zeta)-\psi(\zeta)|,\end{aligned} \tag{3.6}$$

$$\frac{1}{{\varsigma_2}^{\varsigma_1}\Gamma(\varsigma_1)}\int_0^{\zeta}(\zeta-\mu)^{\varsigma_1-1}\exp(\varsigma_2-1(\zeta-\mu))\Omega(\zeta)d\mu \le \varrho\Omega(\zeta). \tag{3.7}$$

Assume that $\phi:[0,\mathrm{F}]\to\mathbb{R}$ *is a function such that for* $\delta>0$ *and* $\zeta\in[0,\mathrm{F}]$,

$$\left|{}^{C}_{m}\Xi_{D_{0^+}}^{\varsigma_1,\varsigma_2}\left(\frac{\phi(\zeta)-\sum_{r=1}^{r=m}\Pi_{I_{0^+}}^{\varsigma_{3r}\varsigma_2}\mathcal{W}_r(\zeta,\phi(\zeta))}{\mathcal{Y}(\zeta,\phi(\zeta))}\right)-\mathcal{K}(\zeta,\phi(\zeta))\right|\le\delta\Omega(\zeta), \tag{3.8}$$

$$\begin{aligned}&\Big|\mathcal{K}(\zeta,\phi(\zeta)) \\ &\quad-\Bigg(\mathcal{Y}(\zeta,\phi(\zeta))\Bigg[\frac{\phi_0}{\mathcal{Y}(0,\phi_0)}\left(1-\frac{\zeta}{\mathrm{F}}\right)\exp(\frac{\varsigma_2-1}{\varsigma_2}\zeta)+\frac{\phi_{\mathrm{F}}\frac{\zeta}{\mathrm{F}}\exp(\frac{\varsigma_3-1}{\varsigma_2}\zeta)}{\mathcal{Y}(\mathrm{F},\phi_{\mathrm{F}})\exp(\frac{\varsigma_3-1}{\varsigma_2}\mathrm{F})} \\ &\quad-\frac{\zeta\exp(\frac{\varsigma_2-1}{\varsigma_2}\zeta)}{\mathcal{Y}(\mathrm{F},\phi_{\mathrm{F}})\,\mathrm{F}\exp(\frac{\varsigma_3-1}{\varsigma_2}\mathrm{F})} \\ &\quad\times\sum_{r=1}^{r=m}\frac{1}{\varsigma_2^{\varsigma_{3r}}\Gamma\left(\varsigma_{3r}\right)}\int_0^{\mathrm{F}}(\mathrm{F}-\mu)^{\varsigma_{3r}-1}\exp(\frac{\varsigma_3-1}{\varsigma_2}(\mathrm{F}-\mu))\mathcal{W}_r(\mu,\phi(\mu))d\mu \\ &\quad-\frac{\zeta\exp(\frac{\varsigma_3-1}{\varsigma_2}\zeta)}{\mathrm{F}\exp(\frac{\varsigma_2-1}{\varsigma_2}\mathrm{F})}\frac{1}{\varsigma_2^{\varsigma_1}\Gamma(\varsigma_1)}\int_0^{\mathrm{F}}(\mathrm{F}-\mu)^{\varsigma_1-1}\exp(\frac{\varsigma_2-1}{\varsigma_2}(\mathrm{F}-\mu)\mathcal{K}(\zeta,\phi(\zeta))d\mu \\ &\quad+\frac{1}{\varsigma_2^{\varsigma_1}\Gamma(\varsigma_1)}\int_0^{\zeta}(\zeta-\mu)^{\varsigma_1-1}\exp(\frac{\varsigma_3-1}{\varsigma_2}(\zeta-\mu))\mathcal{K}(\zeta,\phi(\zeta))d\mu\Bigg] \\ &\quad+\sum_{r=1}^{m}\frac{1}{\varsigma_2^{\varsigma_{3r}}\Gamma\left(\varsigma_{3r}\right)}\int_0^{\zeta}(\zeta-\mu)^{\varsigma_{3r}-1}\exp(\frac{\varsigma_2-1}{\varsigma_2}(\zeta-\mu))\mathcal{W}_r(\mu,\phi(\mu))d\mu\Bigg)\Big| \\ &\le\delta\frac{1}{\varsigma_2^{\varsigma_1}\Gamma(\varsigma_1)}\int_a^{\zeta}(\zeta-\mu)^{\varsigma_1-1}\exp(\varsigma_2-1(\zeta-\mu))\Omega(\zeta)d\mu.\end{aligned} \tag{3.9}$$

Then HPFIDE (1.1) *has a unique solution* $\psi:[0,\mathrm{F}]\to\mathbb{R}$ *such that* $|\psi(\zeta)-\phi(\zeta)|\le\frac{\delta\varrho}{1-\upsilon}\Omega(\zeta)$, *where* υ *is the number given between* 0 *and* 1 *in* (3.2).

Proof We define the mapping $\Psi:\mathcal{N}\to\mathcal{N}$ as

$$\begin{aligned}\Psi(\phi(\zeta))=\mathcal{Y}(\zeta,\phi(\zeta))&\Bigg[\frac{\phi_0}{\mathcal{Y}(0,\phi_0)}\left(1-\frac{\zeta}{\mathrm{F}}\right)\exp(\frac{\varsigma_2-1}{\varsigma_2}\zeta)+\frac{\phi_{\mathrm{F}}\frac{\zeta}{\mathrm{F}}\exp(\frac{\varsigma_3-1}{\varsigma_2}\zeta)}{\mathcal{Y}(\mathrm{F},\phi_{\mathrm{F}})\exp(\frac{\varsigma_3-1}{\varsigma_2}\mathrm{F})} \\ &-\frac{\zeta\exp(\frac{\varsigma_2-1}{\varsigma_2}\zeta)}{\mathcal{Y}(\mathrm{F},\phi_{\mathrm{F}})\,\mathrm{F}\exp(\frac{\varsigma_3-1}{\varsigma_2}\mathrm{F})} \\ &\times\sum_{r=1}^{r=m}\frac{1}{\varsigma_2^{\varsigma_{3r}}\Gamma\left(\varsigma_{3r}\right)}\int_0^{\mathrm{F}}(\mathrm{F}-\mu)^{\varsigma_{3r}-1}\exp(\frac{\varsigma_3-1}{\varsigma_2}(\mathrm{F}-\mu))\mathcal{W}_r(\mu,\phi(\mu))d\mu\end{aligned}$$

$$
\begin{aligned}
&- \frac{\zeta \exp(\frac{\varsigma_3-1}{\varsigma_2}\zeta)}{\mathrm{F}\exp(\frac{\varsigma_2-1}{\varsigma_2}\mathrm{F})} \frac{1}{\varsigma_2^{\varsigma_1}\Gamma(\varsigma_1)} \int_0^{\mathrm{F}} (\mathrm{F}-\mu)^{\varsigma_1-1} \exp(\frac{\varsigma_2-1}{\varsigma_2}(\mathrm{F}-\mu))\mathcal{K}(\mu,\phi(\mu))d\mu \\
&+ \frac{1}{\varsigma_2^{\varsigma_1}\Gamma(\varsigma_1)} \int_0^{\zeta} (\zeta-\mu)^{\varsigma_1-1} \exp(\frac{\varsigma_3-1}{\varsigma_2}(\zeta-\mu))\mathcal{K}(\mu,\phi(\mu))d\mu \Bigg] \\
&+ \sum_{r=1}^{m} \frac{1}{\varsigma_2^{\varsigma_{3r}}\Gamma\left(\varsigma_{3r}\right)} \int_0^{\zeta} (\zeta-\mu)^{\varsigma_{3r}-1} \exp(\frac{\varsigma_2-1}{\varsigma_2}(\zeta-\mu))\mathcal{W}_r(\mu,\phi(\mu))d\mu.
\end{aligned}
\tag{3.10}
$$

We show that this mapping is a contraction. For this purpose, we assume that $\Delta_{CON}(\phi, \psi) \le \mathcal{E}_{\phi\psi}$ for $\phi, \psi \in \mathcal{N}$, the coefficient $\mathcal{E}_{\phi\psi} \in [0,\infty]$, and $|\phi(\zeta) - \psi(\zeta)| < \mathcal{E}_{\phi\psi}\Omega(\zeta)$ for $\zeta \in [0,\mathrm{F}]$. Using (3.3), (3.4), (3.5), (3.6), and (3.7), we get

$$
\begin{aligned}
&|\Psi\phi(\zeta) - \Psi\psi(\zeta)| \\
&= \Bigg| \mathcal{Y}(\zeta,\phi(\zeta)) \Bigg[\frac{\phi_0}{\mathcal{Y}(0,\phi_0)} \left(1 - \frac{\zeta}{\mathrm{F}}\right) \exp(\frac{\varsigma_2-1}{\varsigma_2}\zeta) + \frac{\phi_{\mathrm{F}} \frac{\zeta}{\mathrm{F}} \exp(\frac{\varsigma_3-1}{\varsigma_2}\zeta)}{\mathcal{Y}(\mathrm{F},\phi_{\mathrm{F}}) \exp(\frac{\varsigma_3-1}{\varsigma_2}\mathrm{F})} \\
&- \frac{\zeta \exp(\frac{\varsigma_2-1}{\varsigma_2}\zeta)}{\mathcal{Y}(\mathrm{F},\phi_{\mathrm{F}})\, \mathrm{F}\exp(\frac{\varsigma_3-1}{\varsigma_2}\mathrm{F})} \\
&\times \sum_{r=1}^{r=m} \frac{1}{\varsigma_2^{\varsigma_{3r}}\Gamma\left(\varsigma_{3r}\right)} \int_0^{\mathrm{F}} (\mathrm{F}-\mu)^{\varsigma_{3r}-1} \exp(\frac{\varsigma_3-1}{\varsigma_2}(\mathrm{F}-\mu))\mathcal{W}_r(\mu,\phi(\mu))d\mu \\
&- \frac{\zeta \exp(\frac{\varsigma_3-1}{\varsigma_2}\zeta)}{\mathrm{F}e^{\frac{\varsigma_2-1}{\varsigma_2}\mathrm{F}}} \frac{1}{\varsigma_2^{\varsigma_1}\Gamma(\varsigma_1)} \int_0^{\mathrm{F}} (\mathrm{F}-\mu)^{\varsigma_1-1} \exp(\frac{\varsigma_2-1}{\varsigma_2}(\mathrm{F}-\mu))\mathcal{K}(\mu,\phi(\mu))d\mu \\
&+ \frac{1}{\varsigma_2^{\varsigma_1}\Gamma(\varsigma_1)} \int_0^{\zeta} (\zeta-\mu)^{\varsigma_1-1} \exp(\frac{\varsigma_3-1}{\varsigma_2}(\zeta-\mu))\mathcal{K}(\mu,\phi(\mu))d\mu \Bigg] \\
&+ \sum_{r=1}^{m} \frac{1}{\varsigma_2^{\varsigma_{3r}}\Gamma\left(\varsigma_{3r}\right)} \int_0^{\zeta} (\zeta-\mu)^{\varsigma_{3r}-1} \exp(\frac{\varsigma_2-1}{\varsigma_2}(\zeta-\mu))\mathcal{W}_r(\mu,\phi(\mu))d\mu \\
&- \Bigg(\mathcal{Y}(\zeta,\psi(\zeta)) \Bigg[\frac{\psi_0}{\mathcal{Y}(0,\psi_0)} \left(1 - \frac{\zeta}{\mathrm{F}}\right) \exp(\frac{\varsigma_2-1}{\varsigma_2}\zeta) + \frac{\psi_{\mathrm{F}} \frac{\zeta}{\mathrm{F}} \exp(\frac{\varsigma_3-1}{\varsigma_2}\zeta)}{\mathcal{Y}(\mathrm{F},\psi_{\mathrm{F}}) \exp(\frac{\varsigma_3-1}{\varsigma_2}\mathrm{F})} \\
&- \frac{\zeta \exp(\frac{\varsigma_2-1}{\varsigma_2}\zeta)}{\mathcal{Y}(\mathrm{F},\psi_{\mathrm{F}})\, \mathrm{F}\exp(\frac{\varsigma_3-1}{\varsigma_2}\mathrm{F})} \\
&\times \sum_{r=1}^{r=m} \frac{1}{\varsigma_2^{\varsigma_{3r}}\Gamma\left(\varsigma_{3r}\right)} \int_0^{\mathrm{F}} (\mathrm{F}-\mu)^{\varsigma_{3r}-1} \exp(\frac{\varsigma_3-1}{\varsigma_2}(\mathrm{F}-\mu))\mathcal{W}_r(\mu,\psi(\mu))d\mu \\
&- \frac{\zeta \exp(\frac{\varsigma_3-1}{\varsigma_2}\zeta)}{\mathrm{F}\exp(\frac{\varsigma_2-1}{\varsigma_2}\mathrm{F})} \frac{1}{\varsigma_2^{\varsigma_1}\Gamma(\varsigma_1)} \int_0^{\mathrm{F}} (\mathrm{F}-\mu)^{\varsigma_1-1} \exp(\frac{\varsigma_2-1}{\varsigma_2}(\mathrm{F}-\mu))\mathcal{K}(\mu,\psi(\mu))d\mu \\
&+ \frac{1}{\varsigma_2^{\varsigma_1}\Gamma(\varsigma_1)} \int_0^{\zeta} (\zeta-\mu)^{\varsigma_1-1} \exp(\frac{\varsigma_3-1}{\varsigma_2}(\zeta-\mu))\mathcal{K}(\mu,\psi(\mu))d\mu \Bigg] \\
&+ \sum_{r=1}^{m} \frac{1}{\varsigma_2^{\varsigma_{3r}}\Gamma\left(\varsigma_{3r}\right)} \int_0^{\zeta} (\zeta-\mu)^{\varsigma_{3r}-1} \exp(\frac{\varsigma_2-1}{\varsigma_2}(\zeta-\mu))\mathcal{W}_r(\mu,\psi(\mu))d\mu \Bigg) \Bigg| \\
&\le \frac{\psi_0}{\mathcal{Y}(0,\psi_0)} \left(1 - \frac{\zeta}{\mathrm{F}}\right) \exp(\frac{\varsigma_2-1}{\varsigma_2}\zeta) \left(\Big| \mathcal{Y}(\zeta,\phi(\zeta)) - \mathcal{Y}(\zeta,\psi(\zeta)) \Big| \right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\psi_{\mathrm{F}}\frac{\zeta}{\mathrm{F}}\exp(\frac{\varsigma_3-1}{\varsigma_2}\zeta)}{\mathcal{Y}(\mathrm{F},\psi_{\mathrm{F}})\exp(\frac{\varsigma_3-1}{\varsigma_2}\mathrm{F})}\left(\Big|\mathcal{Y}(\zeta,\phi(\zeta))-\mathcal{Y}(\zeta,\psi(\zeta))\Big|\right)\\
&+\frac{\zeta\exp(\frac{\varsigma_2-1}{\varsigma_2}\zeta)}{\mathcal{Y}(\mathrm{F},\psi_{\mathrm{F}})\,\mathrm{F}\exp(\frac{\varsigma_3-1}{\varsigma_2}\mathrm{F})}\\
&\times\sum_{r=1}^{r=m}\frac{1}{\varsigma_2^{\varsigma_{3r}}\Gamma\left(\varsigma_{3r}\right)}\\
&\times\left(\Big|\mathcal{Y}(\zeta,\psi(\zeta))\int_0^{\mathrm{F}}(\mathrm{F}-\mu)^{\varsigma_{3r}-1}\exp(\frac{\varsigma_3-1}{\varsigma_2}(\mathrm{F}-\mu))\mathcal{W}_r(\mu,\psi(\mu))d\mu\right.\\
&\left.-\mathcal{Y}(\zeta,\phi(\zeta))\int_0^{\mathrm{F}}(\mathrm{F}-\mu)^{\varsigma_{3r}-1}\exp(\frac{\varsigma_3-1}{\varsigma_2}(\mathrm{F}-\mu))\mathcal{W}_r(\mu,\phi(\mu))d\mu\Big|\right)\\
&+\frac{\zeta\exp(\frac{\varsigma_3-1}{\varsigma_2}\zeta)}{\mathrm{F}\exp(\frac{\varsigma_2-1}{\varsigma_2}\mathrm{F})}\frac{1}{\varsigma_2^{\varsigma_1}\Gamma(\varsigma_1)}\\
&\times\left(\mathcal{Y}(\mu,\psi(\mu))\Big|\int_0^{\mathrm{F}}(\mathrm{F}-\mu)^{\varsigma_1-1}\exp(\frac{\varsigma_2-1}{\varsigma_2}(\mathrm{F}-\mu))\mathcal{K}(\mu,\psi(\mu))d\mu\right.\\
&\left.-\mathcal{Y}(\zeta,\phi(\zeta))\int_0^{\mathrm{F}}(\mathrm{F}-\mu)^{\varsigma_1-1}\exp(\frac{\varsigma_2-1}{\varsigma_2}(\mathrm{F}-\mu))\mathcal{K}(\mu,\phi(\mu))d\mu\Big|\right)\\
&+\frac{1}{\varsigma_2^{\varsigma_1}\Gamma(\varsigma_1)}\left(\Big|\mathcal{Y}(\zeta,\phi(\zeta))\int_0^{\zeta}(\zeta-\mu)^{\varsigma_1-1}\exp(\frac{\varsigma_3-1}{\varsigma_2}(\zeta-\mu))\mathcal{K}(\mu,\phi(\mu))d\mu\right.\\
&\left.-\mathcal{Y}(\zeta,\psi(\zeta))\int_0^{\zeta}(\zeta-\mu)^{\varsigma_1-1}\exp(\frac{\varsigma_3-1}{\varsigma_2}(\zeta-\mu))\mathcal{K}(\mu,\psi(\mu))d\mu\Big|\right)\\
&+\sum_{r=1}^{m}\frac{1}{\varsigma_2^{\varsigma_{3r}}\Gamma\left(\varsigma_{3r}\right)}\\
&\times\int_0^{\zeta}(\zeta-\mu)^{\varsigma_{3r}-1}\exp(\frac{\varsigma_2-1}{\varsigma_2}(\zeta-\mu))\Big|\mathcal{W}_r(\mu,\phi(\mu))-\mathcal{W}_r(\mu,\psi(\mu))\Big|d\mu\\
\le&\frac{\psi_0}{\mathcal{Y}(0,\psi_0)}\left(1-\frac{\zeta}{\mathrm{F}}\right)\exp(\frac{\varsigma_2-1}{\varsigma_2}\zeta)\left(\mathcal{M}_1\Big|\phi(\zeta)-\psi(\zeta)\Big|\right)\\
&+\frac{\psi_{\mathrm{F}}\frac{\zeta}{\mathrm{F}}\exp(\frac{\varsigma_3-1}{\varsigma_2}\zeta)}{\mathcal{Y}(\mathrm{F},\psi_{\mathrm{F}})\exp(\frac{\varsigma_3-1}{\varsigma_2}\mathrm{F})}\left(\mathcal{M}_1\Big|\phi(\zeta)-\psi(\zeta)\Big|\right)\\
&+\frac{\zeta\exp(\frac{\varsigma_2-1}{\varsigma_2}\zeta)}{\mathcal{Y}(\mathrm{F},\psi_{\mathrm{F}})\,\mathrm{F}\exp(\frac{\varsigma_3-1}{\varsigma_2}\mathrm{F})}\sum_{r=1}^{r=m}\frac{1}{\varsigma_2^{\varsigma_{3r}}\Gamma\left(\varsigma_{3r}\right)}\left(\mathcal{M}_2\Big|\phi(\zeta)-\psi(\zeta)\Big|\right)\\
&+\frac{\zeta\exp(\frac{\varsigma_3-1}{\varsigma_2}\zeta)}{\mathrm{F}\exp(\frac{\varsigma_2-1}{\varsigma_2}\mathrm{F})}\frac{1}{\varsigma_2^{\varsigma_1}\Gamma(\varsigma_1)}\left(\mathcal{M}_3\Big|\phi(\zeta)-\psi(\zeta)\Big|\right)\\
&+\frac{1}{\varsigma_2^{\varsigma_1}\Gamma(\varsigma_1)}\left(\mathcal{M}_4\Big|\phi(\zeta)-\psi(\zeta)\Big|\right)\\
&+\sum_{r=1}^{m}\frac{1}{\varsigma_2^{\varsigma_{3r}}\Gamma\left(\varsigma_{3r}\right)}\int_0^{\zeta}(\zeta-\mu)^{\varsigma_{3r}-1}\exp(\frac{\varsigma_2-1}{\varsigma_2}(\zeta-\mu))\left(\mathcal{M}_5\Big|\phi(\zeta)-\psi(\zeta)\Big|\right)d\mu\\
\le&\frac{\psi_0}{\mathcal{Y}(0,\psi_0)}\left(1-\frac{\zeta}{\mathrm{F}}\right)\exp(\frac{\varsigma_2-1}{\varsigma_2}\zeta)\left(\mathcal{M}_1\mathcal{E}_{\phi\psi}\Omega(\zeta)\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\psi_{\mathrm{F}}\frac{\zeta}{\mathrm{F}}\exp(\frac{\varsigma_3-1}{\varsigma_2}\zeta)}{\mathcal{Y}(\mathrm{F},\psi_{\mathrm{F}})\exp(\frac{\varsigma_3-1}{\varsigma_2}\mathrm{F})}\Big(\mathcal{M}_1\mathcal{E}_{\phi\psi}\Omega(\zeta)\Big)\\
&+\frac{\zeta\exp(\frac{\varsigma_2-1}{\varsigma_2}\zeta)}{\mathcal{Y}(\mathrm{F},\psi_{\mathrm{F}})\,\mathrm{F}\exp(\frac{\varsigma_3-1}{\varsigma_2}\mathrm{F})}\sum_{r=1}^{r=m}\frac{1}{\varsigma_2^{\varsigma_{3r}}\Gamma\left(\varsigma_{3r}\right)}\Big(\mathcal{M}_2\mathcal{E}_{\phi\psi}\Omega(\zeta)\Big)\\
&+\frac{\zeta\exp(\frac{\varsigma_3-1}{\varsigma_2}\zeta)}{\mathrm{F}\exp(\frac{\varsigma_2-1}{\varsigma_2}\mathrm{F})}\frac{1}{\varsigma_2^{\varsigma_1}\Gamma(\varsigma_1)}\Big(\mathcal{M}_3\mathcal{E}_{\phi\psi}\Omega(\zeta)\Big)\\
&+\frac{1}{\varsigma_2^{\varsigma_1}\Gamma(\varsigma_1)}\Big(\mathcal{M}_4\mathcal{E}_{\phi\psi}\Omega(\zeta)\Big)\\
&+\sum_{r=1}^{m}\frac{1}{\varsigma_2^{\varsigma_{3r}}\Gamma\left(\varsigma_{3r}\right)}\int_0^{\zeta}(\zeta-\mu)^{\varsigma_{3r}-1}\exp(\frac{\varsigma_2-1}{\varsigma_2}(\zeta-\mu))\Big(\mathcal{M}_5\mathcal{E}_{\phi\psi}\Omega(\zeta)\Big)d\mu\\
\le&\Bigg(\mathcal{M}_1\frac{\psi_0}{\mathcal{Y}(0,\psi_0)}\left(1-\frac{\zeta}{\mathrm{F}}\right)\exp(\frac{\varsigma_2-1}{\varsigma_2}\zeta)+\mathcal{M}_1\frac{\psi_{\mathrm{F}}\frac{\zeta}{\mathrm{F}}\exp(\frac{\varsigma_3-1}{\varsigma_2}\zeta)}{\mathcal{Y}(\mathrm{F},\psi_{\mathrm{F}})\exp(\frac{\varsigma_3-1}{\varsigma_2}\mathrm{F})}\\
&+\frac{\zeta\exp(\frac{\varsigma_2-1}{\varsigma_2}\zeta)}{\mathcal{Y}(\mathrm{F},\psi_{\mathrm{F}})\,\mathrm{F}\exp(\frac{\varsigma_3-1}{\varsigma_2}\mathrm{F})}\sum_{r=1}^{r=m}\frac{1}{\varsigma_2^{\varsigma_{3r}}\Gamma\left(\varsigma_{3r}\right)}\mathcal{M}_2+\mathcal{M}_3\frac{\zeta\exp(\frac{\varsigma_3-1}{\varsigma_2}\zeta)}{\mathrm{F}\exp(\frac{\varsigma_2-1}{\varsigma_2}\mathrm{F})}\frac{1}{\varsigma_2^{\varsigma_1}\Gamma(\varsigma_1)}\\
&+\mathcal{M}_4\frac{1}{\varsigma_2^{\varsigma_1}\Gamma(\varsigma_1)}\\
&+\sum_{r=1}^{m}\frac{1}{\varsigma_2^{\varsigma_{3r}}\Gamma\left(\varsigma_{3r}\right)}\int_0^{\zeta}(\zeta-\mu)^{\varsigma_{3r}-1}\exp(\frac{\varsigma_2-1}{\varsigma_2}(\zeta-\mu))\mathcal{M}_5\varrho\Bigg)\mathcal{E}_{\phi\psi}\Omega(\zeta).
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
&\Delta_{CON}(\Psi\phi,\Psi\psi)\\
&\le\Bigg[\mathcal{M}_1\frac{\psi_0}{\mathcal{Y}(0,\psi_0)}\left(1-\frac{\zeta}{\mathrm{F}}\right)\exp(\frac{\varsigma_2-1}{\varsigma_2}\zeta)+\mathcal{M}_1\frac{\psi_{\mathrm{F}}\frac{\zeta}{\mathrm{F}}\exp(\frac{\varsigma_3-1}{\varsigma_2}\zeta)}{\mathcal{Y}(\mathrm{F},\psi_{\mathrm{F}})\exp(\frac{\varsigma_3-1}{\varsigma_2}\mathrm{F})}\\
&\quad+\frac{\zeta\exp(\frac{\varsigma_2-1}{\varsigma_2}\zeta)}{\mathcal{Y}(\mathrm{F},\psi_{\mathrm{F}})\,\mathrm{F}\exp(\frac{\varsigma_3-1}{\varsigma_2}\mathrm{F})}\sum_{r=1}^{r=m}\frac{1}{\varsigma_2^{\varsigma_{3r}}\Gamma\left(\varsigma_{3r}\right)}\mathcal{M}_2+\mathcal{M}_3\frac{\zeta\exp(\frac{\varsigma_3-1}{\varsigma_2}\zeta)}{\mathrm{F}\exp(\frac{\varsigma_2-1}{\varsigma_2}\mathrm{F})}\frac{1}{\varsigma_2^{\varsigma_1}\Gamma(\varsigma_1)}\\
&\quad+\mathcal{M}_4\frac{1}{\varsigma_2^{\varsigma_1}\Gamma(\varsigma_1)}\\
&\quad+\sum_{r=1}^{m}\frac{1}{\varsigma_2^{\varsigma_{3r}}\Gamma\left(\varsigma_{3r}\right)}\int_0^{\zeta}(\zeta-\mu)^{\varsigma_{3r}-1}\exp(\frac{\varsigma_2-1}{\varsigma_2}(\zeta-\mu))\mathcal{M}_5\varrho\Bigg]\Delta_{CON}(\phi,\psi).
\end{aligned}
\tag{3.11}
$$

Since

$$
\begin{aligned}
&0<\mathcal{M}_1\frac{\psi_0}{\mathcal{Y}(0,\psi_0)}\left(1-\frac{\zeta}{\mathrm{F}}\right)\exp(\frac{\varsigma_2-1}{\varsigma_2}\zeta)+\mathcal{M}_1\frac{\psi_{\mathrm{F}}\frac{\zeta}{\mathrm{F}}\exp(\frac{\varsigma_3-1}{\varsigma_2}\zeta)}{\mathcal{Y}(\mathrm{F},\psi_{\mathrm{F}})\exp(\frac{\varsigma_3-1}{\varsigma_2}\mathrm{F})}\\
&+\frac{\zeta\exp(\frac{\varsigma_2-1}{\varsigma_2}\zeta)}{\mathcal{Y}(\mathrm{F},\psi_{\mathrm{F}})\,\mathrm{F}\exp(\frac{\varsigma_3-1}{\varsigma_2}\mathrm{F})}\sum_{r=1}^{r=m}\frac{1}{\varsigma_2^{\varsigma_{3r}}\Gamma\left(\varsigma_{3r}\right)}\mathcal{M}_2+\mathcal{M}_3\frac{\zeta\exp(\frac{\varsigma_3-1}{\varsigma_2}\zeta)}{\mathrm{F}\exp(\frac{\varsigma_2-1}{\varsigma_2}\mathrm{F})}\frac{1}{\varsigma_2^{\varsigma_1}\Gamma(\varsigma_1)}\\
&+\mathcal{M}_4\frac{1}{\varsigma_2^{\varsigma_1}\Gamma(\varsigma_1)}+\sum_{r=1}^{m}\frac{1}{\varsigma_2^{\varsigma_{3r}}\Gamma\left(\varsigma_{3r}\right)}\int_0^{\zeta}(\zeta-\mu)^{\varsigma_{3r}-1}\exp(\frac{\varsigma_2-1}{\varsigma_2}(\zeta-\mu))\mathcal{M}_5\varrho<1,
\end{aligned}
\tag{3.12}
$$

Λ is a contraction. Now we show that $\Delta_{CON}(\Psi\psi,\psi)<\infty$. For $\psi\in\mathcal{N}$, we get

$$\begin{aligned}
&|\Psi(\psi(\zeta))-\psi(\zeta)|\\
&=\Bigg|\mathcal{Y}(\zeta,\phi(\zeta))\Bigg[\frac{\phi_0}{\mathcal{Y}(0,\phi_0)}\left(1-\frac{\zeta}{\mathrm{F}}\right)\exp(\frac{\varsigma_2-1}{\varsigma_2}\zeta)+\frac{\phi_{\mathrm{F}}\frac{\zeta}{\mathrm{F}}\exp(\frac{\varsigma_3-1}{\varsigma_2}\zeta)}{\mathcal{Y}(\mathrm{F},\phi_{\mathrm{F}})\exp(\frac{\varsigma_3-1}{\varsigma_2}\mathrm{F})}\\
&\quad-\frac{\zeta\exp(\frac{\varsigma_2-1}{\varsigma_2}\zeta)}{\mathcal{Y}(\mathrm{F},\phi_{\mathrm{F}})\,\mathrm{F}\exp(\frac{\varsigma_3-1}{\varsigma_2}\mathrm{F})}\\
&\quad\times\sum_{r=1}^{r=m}\frac{1}{\varsigma_2^{\varsigma_{3r}}\Gamma\left(\varsigma_{3r}\right)}\int_0^{\mathrm{F}}(\mathrm{F}-\mu)^{\varsigma_{3r}-1}\exp(\frac{\varsigma_3-1}{\varsigma_2}(\mathrm{F}-\mu))\mathcal{W}_r(\mu,\phi(\mu))d\mu\\
&\quad-\frac{\zeta\exp(\frac{\varsigma_3-1}{\varsigma_2}\zeta)}{\mathrm{F}\exp(\frac{\varsigma_2-1}{\varsigma_2}\mathrm{F})}\frac{1}{\varsigma_2^{\varsigma_1}\Gamma(\varsigma_1)}\int_0^{\mathrm{F}}(\mathrm{F}-\mu)^{\varsigma_1-1}\exp(\frac{\varsigma_2-1}{\varsigma_2}(\mathrm{F}-\mu))\mathcal{K}(\zeta,\phi(\zeta))d\mu\\
&\quad+\frac{1}{\varsigma_2^{\varsigma_1}\Gamma(\varsigma_1)}\int_0^{\zeta}(\zeta-\mu)^{\varsigma_1-1}\exp(\frac{\varsigma_3-1}{\varsigma_2}(\zeta-\mu))\mathcal{K}(\zeta,\phi(\zeta))d\mu\Bigg]\\
&\quad+\sum_{r=1}^{m}\frac{1}{\varsigma_2^{\varsigma_{3r}}\Gamma\left(\varsigma_{3r}\right)}\int_0^{\zeta}(\zeta-\mu)^{\varsigma_{3r}-1}\exp(\frac{\varsigma_2-1}{\varsigma_2}(\zeta-\mu))\mathcal{W}_r(\mu,\phi(\mu))d\mu-\phi(\zeta)\Bigg|\\
&\le\delta\frac{1}{\varsigma_2^{\varsigma_1}\Gamma(\varsigma_1)}\int_p^{\zeta}(\zeta-\mu)^{\varsigma_1-1}\exp(\varsigma_2-1(\zeta-\mu))\Omega(\zeta)d\mu\\
&\le\delta\varrho\Omega(\zeta).
\end{aligned}$$

Consequently, $\Delta_{CON}(\Psi\psi,\psi)\le\delta\varrho<\infty$, $\varrho<1$. Therefore by Theorem 2.5 the sequence $\{\Psi^{\mathbf{b}}\phi\}$ converges to a fixed point ϕ of Ψ, which is in the set $\mathcal{N}^*=\{\psi\in\mathcal{N}:\Delta_{CON}(\Psi\psi,\phi)<\infty\}$. In other words, $\Psi\phi=\phi$, or

$$\begin{aligned}
\Psi(\phi(\zeta))&=\mathcal{Y}(\zeta,\phi(\zeta))\\
&\quad\times\Bigg[\frac{\phi_0}{\mathcal{Y}(0,\phi_0)}\left(1-\frac{\zeta}{\mathrm{F}}\right)\exp(\frac{\varsigma_2-1}{\varsigma_2}\zeta)+\frac{\phi_{\mathrm{F}}\frac{\zeta}{\mathrm{F}}\exp(\frac{\varsigma_3-1}{\varsigma_2}\zeta)}{\mathcal{Y}(\mathrm{F},\phi_{\mathrm{F}})\exp(\frac{\varsigma_3-1}{\varsigma_2}\mathrm{F})}\\
&\quad-\frac{\zeta\exp(\frac{\varsigma_2-1}{\varsigma_2}\zeta)}{\mathcal{Y}(\mathrm{F},\phi_{\mathrm{F}})\,\mathrm{F}\exp(\frac{\varsigma_3-1}{\varsigma_2}\mathrm{F})}\\
&\quad\times\sum_{r=1}^{r=m}\frac{1}{\varsigma_2^{\varsigma_{3r}}\Gamma\left(\varsigma_{3r}\right)}\int_0^{\mathrm{F}}(\mathrm{F}-\mu)^{\varsigma_{3r}-1}\exp(\frac{\varsigma_3-1}{\varsigma_2}(\mathrm{F}-\mu))\mathcal{W}_r(\mu,\phi(\mu))d\mu\\
&\quad-\frac{\zeta\exp(\frac{\varsigma_3-1}{\varsigma_2}\zeta)}{\mathrm{F}\exp(\frac{\varsigma_2-1}{\varsigma_2}\mathrm{F})}\frac{1}{\varsigma_2^{\varsigma_1}\Gamma(\varsigma_1)}\int_0^{\mathrm{F}}(\mathrm{F}-\mu)^{\varsigma_1-1}\exp(\frac{\varsigma_2-1}{\varsigma_2}(\mathrm{F}-\mu))\mathcal{K}(\zeta,\phi(\zeta))d\mu\\
&\quad+\frac{1}{\varsigma_2^{\varsigma_1}\Gamma(\varsigma_1)}\int_0^{\zeta}(\zeta-\mu)^{\varsigma_1-1}\exp(\frac{\varsigma_3-1}{\varsigma_2}(\zeta-\mu))\mathcal{K}(\zeta,\phi(\zeta))d\mu\Bigg]\\
&\quad+\sum_{r=1}^{m}\frac{1}{\varsigma_2^{\varsigma_{3r}}\Gamma\left(\varsigma_{3r}\right)}\int_0^{\zeta}(\zeta-\mu)^{\varsigma_{3r}-1}\exp(\frac{\varsigma_2-1}{\varsigma_2}(\zeta-\mu))\mathcal{W}_r(\mu,\phi(\mu))d\mu.
\end{aligned}\tag{3.13}$$

Also, we have

$${}_{m}^{C}\Xi_{D_{0^+}}^{\varsigma_1,\varsigma_2}\left(\frac{\phi(\zeta)-\sum_{r=1}^{r=m}\Pi_{I_{0^+}}^{\varsigma_{3r}\varsigma_2}\mathcal{W}_r(\zeta,\phi(\zeta))}{\mathcal{Y}(\zeta,\phi(\zeta))}\right)=\mathcal{K}(\zeta,\phi(\zeta)),\tag{3.14}$$

and, as a result,

$$\Delta_{CON}(\psi,\phi) \leq \frac{1}{1-\upsilon}\Delta_{CON}(\Psi\psi,\psi) \leq \frac{\delta\varrho}{1-\upsilon}\Omega(\zeta).$$

We now prove the uniqueness of the solution. Consider the function $\mathcal{L}$ and

$$ {}^{C}_{m}\Xi_{D_{0^+}}^{\varsigma_1,\varsigma_2}\left(\frac{\mathcal{L}(\zeta)-\sum_{r=1}^{r=m}\Pi_{I_{0^+}}^{\varsigma_{3r}\varsigma_2}\mathcal{W}_r(\zeta,\mathcal{L}(\zeta))}{\mathcal{Y}(\zeta,\mathcal{L}(\zeta))}\right)=\mathcal{K}(\zeta,\mathcal{L}(\zeta)). \tag{3.15}$$

According to (3.15), $\Psi\mathcal{L}=\mathcal{L}$. We show that $\Delta_{CON}(\Psi\psi,\mathcal{L})<\infty$. Using (3.15) and assuming that $\psi\in\mathcal{N}$ and $\Delta_{CON}(\psi,\mathcal{L})<\frac{\delta\varrho}{1-\upsilon}$, we have

$$\begin{aligned}
&|\Psi(\psi(\zeta))-\mathcal{L}(\zeta)| \\
&=\Bigg|\mathcal{Y}(\zeta,\phi(\zeta))\Bigg[\frac{\phi_0}{\mathcal{Y}(0,\phi_0)}\left(1-\frac{\zeta}{\mathrm{F}}\right)\exp(\frac{\varsigma_2-1}{\varsigma_2}\zeta)+\frac{\phi_{\mathrm{F}}\frac{\zeta}{\mathrm{F}}\exp(\frac{\varsigma_3-1}{\varsigma_2}\zeta)}{\mathcal{Y}(\mathrm{F},\phi_{\mathrm{F}})\exp(\frac{\varsigma_3-1}{\varsigma_2}\mathrm{F})} \\
&-\frac{\zeta\exp(\frac{\varsigma_2-1}{\varsigma_2}\zeta)}{\mathcal{Y}(\mathrm{F},\phi_{\mathrm{F}})\,\mathrm{F}\exp(\frac{\varsigma_3-1}{\varsigma_2}\mathrm{F})} \\
&\times\sum_{r=1}^{r=m}\frac{1}{\varsigma_2^{\varsigma_{3r}}\Gamma\left(\varsigma_{3r}\right)}\int_0^{\mathrm{F}}(\mathrm{F}-\mu)^{\varsigma_{3r}-1}\exp(\frac{\varsigma_3-1}{\varsigma_2}(\mathrm{F}-\mu))\mathcal{W}_r(\mu,\phi(\mu))d\mu \\
&-\frac{\zeta\exp(\frac{\varsigma_3-1}{\varsigma_2}\zeta)}{\mathrm{F}\exp(\frac{\varsigma_2-1}{\varsigma_2}\mathrm{F})}\frac{1}{\varsigma_2^{\varsigma_1}\Gamma(\varsigma_1)}\int_0^{\mathrm{F}}(\mathrm{F}-\mu)^{\varsigma_1-1}\exp(\frac{\varsigma_2-1}{\varsigma_2}(\mathrm{F}-\mu))\mathcal{K}(\zeta,\phi(\zeta))d\mu \\
&+\frac{1}{\varsigma_2^{\varsigma_1}\Gamma(\varsigma_1)}\int_0^{\zeta}(\zeta-\mu)^{\varsigma_1-1}\exp(\frac{\varsigma_3-1}{\varsigma_2}(\zeta-\mu))\mathcal{K}(\zeta,\phi(\zeta))d\mu\Bigg] \\
&+\sum_{r=1}^{m}\frac{1}{\varsigma_2^{\varsigma_{3r}}\Gamma\left(\varsigma_{3r}\right)}\int_0^{\zeta}(\zeta-\mu)^{\varsigma_{3r}-1}\exp(\frac{\varsigma_2-1}{\varsigma_2}(\zeta-\mu))\mathcal{W}_r(\mu,\phi(\mu))d\mu \\
&-\Bigg(\mathcal{Y}(\zeta,\mathcal{L}(\zeta))\Bigg[\frac{\phi_0}{\mathcal{Y}(0,\mathcal{L}_0)}\left(1-\frac{\zeta}{\mathrm{F}}\right)\exp(\frac{\varsigma_2-1}{\varsigma_2}\zeta)+\frac{\mathcal{L}_T\frac{\zeta}{\mathrm{F}}\exp(\frac{\varsigma_3-1}{\varsigma_2}\zeta)}{\mathcal{Y}(\mathrm{F},\mathcal{L}_T)\exp(\frac{\varsigma_3-1}{\varsigma_2}\mathrm{F})} \\
&-\frac{\zeta\exp(\frac{\varsigma_2-1}{\varsigma_2}\zeta)}{\mathcal{Y}(\mathrm{F},\mathcal{L}_T)\,\mathrm{F}\exp(\frac{\varsigma_3-1}{\varsigma_2}\mathrm{F})} \\
&\times\sum_{r=1}^{r=m}\frac{1}{\varsigma_2^{\varsigma_{3r}}\Gamma\left(\varsigma_{3r}\right)}\int_0^{\mathrm{F}}(\mathrm{F}-\mu)^{\varsigma_{3r}-1}\exp(\frac{\varsigma_3-1}{\varsigma_2}(\mathrm{F}-\mu))\mathcal{W}_r(\mu,\mathcal{L}(\mu))d\mu \\
&-\frac{\zeta\exp(\frac{\varsigma_3-1}{\varsigma_2}\zeta)}{\mathrm{F}\exp(\frac{\varsigma_2-1}{\varsigma_2}\mathrm{F})}\frac{1}{\varsigma_2^{\varsigma_1}\Gamma(\varsigma_1)}\int_0^{\mathrm{F}}(\mathrm{F}-\mu)^{\varsigma_1-1}\exp(\frac{\varsigma_2-1}{\varsigma_2}(\mathrm{F}-\mu))\mathcal{K}(\zeta,\mathcal{L}(\zeta))d\mu \\
&+\frac{1}{\varsigma_2^{\varsigma_1}\Gamma(\varsigma_1)}\int_0^{\zeta}(\zeta-\mu)^{\varsigma_1-1}\exp(\frac{\varsigma_3-1}{\varsigma_2}(\zeta-\mu))\mathcal{K}(\zeta,\mathcal{L}(\zeta))d\mu\Bigg] \\
&+\sum_{r=1}^{m}\frac{1}{\varsigma_2^{\varsigma_{3r}}\Gamma\left(\varsigma_{3r}\right)}\int_0^{\zeta}(\zeta-\mu)^{\varsigma_{3r}-1}\exp(\frac{\varsigma_2-1}{\varsigma_2}(\zeta-\mu))\mathcal{W}_r(\mu,\mathcal{L}(\mu))d\mu\Bigg)\Bigg| \\
&\leq\frac{\psi_0}{\mathcal{Y}(0,\psi_0)}\left(1-\frac{\zeta}{\mathrm{F}}\right)\exp(\frac{\varsigma_2-1}{\varsigma_2}\zeta)\left(\Big|\mathcal{Y}(\zeta,\phi(\zeta))-\mathcal{Y}(\zeta,\mathcal{L}(\zeta))\Big|\right)
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\psi_{\mathrm{F}} \frac{\zeta}{\mathrm{F}} \exp(\frac{\varsigma_3 - 1}{\varsigma_2}\zeta)}{\mathcal{Y}(\mathrm{F}, \psi_{\mathrm{F}}) \exp(\frac{\varsigma_3 - 1}{\varsigma_2}\mathrm{F})} \left(\left| \mathcal{Y}(\zeta, \phi(\zeta)) - \mathcal{Y}(\zeta, \mathcal{L}(\zeta)) \right| \right) \\
&+ \frac{\zeta \exp(\frac{\varsigma_2 - 1}{\varsigma_2}\zeta)}{\mathcal{Y}(\mathrm{F}, \psi_{\mathrm{F}}) \mathrm{F} \exp(\frac{\varsigma_3 - 1}{\varsigma_2}\mathrm{F})} \sum_{r=1}^{r=m} \frac{1}{\varsigma_2^{\varsigma_{3r}} \Gamma\left(\varsigma_{3r}\right)} \\
&\times \left(\left| \mathcal{Y}(\zeta, \mathcal{L}(\zeta)) \int_0^{\mathrm{F}} (\mathrm{F} - \mu)^{\varsigma_{3r} - 1} \exp(\frac{\varsigma_3 - 1}{\varsigma_2}(\mathrm{F} - \mu)) \mathcal{W}_r(\mu, \mathcal{L}(\mu)) d\mu \right.\right. \\
&\left.\left. - \mathcal{Y}(\zeta, \phi(\zeta)) \int_0^{\mathrm{F}} (\mathrm{F} - \mu)^{\varsigma_{3r} - 1} \exp(\frac{\varsigma_3 - 1}{\varsigma_2}(\mathrm{F} - \mu)) \mathcal{W}_r(\mu, \phi(\mu)) d\mu \right| \right) \\
&+ \frac{\zeta \exp(\frac{\varsigma_3 - 1}{\varsigma_2}\zeta)}{\mathrm{F} \exp(\frac{\varsigma_2 - 1}{\varsigma_2}\mathrm{F})} \frac{1}{\varsigma_2^{\varsigma_1} \Gamma(\varsigma_1)} \\
&\times \left(\mathcal{Y}(\mu, \mathcal{L}(\mu)) \left| \int_0^{\mathrm{F}} (\mathrm{F} - \mu)^{\varsigma_1 - 1} \exp(\frac{\varsigma_2 - 1}{\varsigma_2}(\mathrm{F} - \mu)) \mathcal{K}(\mu, \mathcal{L}(\mu)) d\mu \right.\right. \\
&\left.\left. - \mathcal{Y}(\zeta, \phi(\zeta)) \int_0^{\mathrm{F}} (\mathrm{F} - \mu)^{\varsigma_1 - 1} \exp(\frac{\varsigma_2 - 1}{\varsigma_2}(\mathrm{F} - \mu)) \mathcal{K}(\mu, \phi(\mu)) d\mu \right| \right) \\
&+ \frac{1}{\varsigma_2^{\varsigma_1} \Gamma(\varsigma_1)} \left(\left| \mathcal{Y}(\zeta, \phi(\zeta)) \int_0^{\zeta} (\zeta - \mu)^{\varsigma_1 - 1} \exp(\frac{\varsigma_3 - 1}{\varsigma_2}(\zeta - \mu)) \mathcal{K}(\mu, \phi(\mu)) d\mu \right.\right. \\
&\left.\left. - \mathcal{Y}(\zeta, \mathcal{L}(\zeta)) \int_0^{\zeta} (\zeta - \mu)^{\varsigma_1 - 1} \exp(\frac{\varsigma_3 - 1}{\varsigma_2}(\zeta - \mu)) \mathcal{K}(\mu, \mathcal{L}(\mu)) d\mu \right| \right) \\
&+ \sum_{r=1}^{m} \frac{1}{\varsigma_2^{\varsigma_{3r}} \Gamma\left(\varsigma_{3r}\right)} \\
&\times \int_0^{\zeta} (\zeta - \mu)^{\varsigma_{3r} - 1} \exp(\frac{\varsigma_2 - 1}{\varsigma_2}(\zeta - \mu)) \left| \mathcal{W}_r(\mu, \phi(\mu)) - \mathcal{W}_r(\mu, \mathcal{L}(\mu)) \right| d\mu \\
\leq{}& \frac{\psi_0}{\mathcal{Y}(0, \psi_0)} \left(1 - \frac{\zeta}{\mathrm{F}}\right) \exp(\frac{\varsigma_2 - 1}{\varsigma_2}\zeta) \left(\mathcal{M}_1 \left| \phi(\zeta) - \mathcal{L}(\zeta) \right| \right) \\
&+ \frac{\psi_{\mathrm{F}} \frac{\zeta}{\mathrm{F}} \exp(\frac{\varsigma_3 - 1}{\varsigma_2}\zeta)}{\mathcal{Y}(\mathrm{F}, \psi_{\mathrm{F}}) \exp(\frac{\varsigma_3 - 1}{\varsigma_2}\mathrm{F})} \left(\mathcal{M}_1 \left| \phi(\zeta) - \mathcal{L}(\zeta) \right| \right) \\
&+ \frac{\zeta \exp(\frac{\varsigma_2 - 1}{\varsigma_2}\zeta)}{\mathcal{Y}(\mathrm{F}, \psi_{\mathrm{F}}) \mathrm{F} \exp(\frac{\varsigma_3 - 1}{\varsigma_2}\mathrm{F})} \sum_{r=1}^{r=m} \frac{1}{\varsigma_2^{\varsigma_{3r}} \Gamma\left(\varsigma_{3r}\right)} \left(\mathcal{M}_2 \left| \phi(\zeta) - \mathcal{L}(\zeta) \right| \right) \\
&+ \frac{\zeta \exp(\frac{\varsigma_3 - 1}{\varsigma_2}\zeta)}{\mathrm{F} \exp(\frac{\varsigma_2 - 1}{\varsigma_2}\mathrm{F})} \frac{1}{\varsigma_2^{\varsigma_1} \Gamma(\varsigma_1)} \left(\mathcal{M}_3 \left| \phi(\zeta) - \mathcal{L}(\zeta) \right| \right) \\
&+ \frac{1}{\varsigma_2^{\varsigma_1} \Gamma(\varsigma_1)} \left(\mathcal{M}_4 \left| \phi(\zeta) - \mathcal{L}(\zeta) \right| \right) \\
&+ \sum_{r=1}^{m} \frac{1}{\varsigma_2^{\varsigma_{3r}} \Gamma\left(\varsigma_{3r}\right)} \int_0^{\zeta} (\zeta - \mu)^{\varsigma_{3r} - 1} \exp(\frac{\varsigma_2 - 1}{\varsigma_2}(\zeta - \mu)) \left(\mathcal{M}_5 \left| \phi(\zeta) - \mathcal{L}(\zeta) \right| \right) d\mu \\
\leq{}& \frac{\psi_0}{\mathcal{Y}(0, \psi_0)} \left(1 - \frac{\zeta}{\mathrm{F}}\right) \exp(\frac{\varsigma_2 - 1}{\varsigma_2}\zeta) \left(\mathcal{M}_1 \frac{\delta \varrho}{1 - \upsilon} \Omega(\zeta) \right) \\
&+ \frac{\psi_{\mathrm{F}} \frac{\zeta}{\mathrm{F}} \exp(\frac{\varsigma_3 - 1}{\varsigma_2}\zeta)}{\mathcal{Y}(\mathrm{F}, \psi_{\mathrm{F}}) \exp(\frac{\varsigma_3 - 1}{\varsigma_2}\mathrm{F})} \left(\mathcal{M}_1 \frac{\delta \varrho}{1 - \upsilon} \Omega(\zeta) \right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{\zeta \exp(\frac{\varsigma_2-1}{\varsigma_2}\zeta)}{\mathcal{Y}(\mathrm{F},\psi_{\mathrm{F}})\,\mathrm{F}\exp(\frac{\varsigma_3-1}{\varsigma_2}\mathrm{F})}\sum_{r=1}^{r=m}\frac{1}{\varsigma_2^{\varsigma_{3r}}\Gamma(\varsigma_{3r})}\left(\mathcal{M}_2\frac{\delta\varrho}{1-\upsilon}\Omega(\zeta)\right) \\
&+ \frac{\zeta \exp(\frac{\varsigma_3-1}{\varsigma_2}\zeta)}{\mathrm{F}\exp(\frac{\varsigma_2-1}{\varsigma_2}\mathrm{F})}\frac{1}{\varsigma_2^{\varsigma_1}\Gamma(\varsigma_1)}\left(\mathcal{M}_3\frac{\delta\varrho}{1-\upsilon}\Omega(\zeta)\right) \\
&+ \frac{1}{\varsigma_2^{\varsigma_1}\Gamma(\varsigma_1)}\left(\mathcal{M}_4\frac{\delta\varrho}{1-\upsilon}\Omega(\zeta)\right) \\
&+ \sum_{r=1}^{m}\frac{1}{\varsigma_2^{\varsigma_{3r}}\Gamma(\varsigma_{3r})}\int_0^{\zeta}(\zeta-\mu)^{\varsigma_{3r}-1}\exp(\frac{\varsigma_2-1}{\varsigma_2}(\zeta-\mu))\left(\mathcal{M}_5\frac{\delta\varrho}{1-\upsilon}\Omega(\zeta)\right)d\mu \\
\le{}& \left(\mathcal{M}_1\frac{\psi_0}{\mathcal{Y}(0,\psi_0)}\left(1-\frac{\zeta}{\mathrm{F}}\right)\exp(\frac{\varsigma_2-1}{\varsigma_2}\zeta)+\mathcal{M}_1\frac{\psi_{\mathrm{F}}\frac{\zeta}{\mathrm{F}}\exp(\frac{\varsigma_3-1}{\varsigma_2}\zeta)}{\mathcal{Y}(\mathrm{F},\psi_{\mathrm{F}})\exp(\frac{\varsigma_3-1}{\varsigma_2}\mathrm{F})}\right. \\
&+ \frac{\zeta \exp(\frac{\varsigma_2-1}{\varsigma_2}\zeta)}{\mathcal{Y}(\mathrm{F},\psi_{\mathrm{F}})\,\mathrm{F}\exp(\frac{\varsigma_3-1}{\varsigma_2}\mathrm{F})}\sum_{r=1}^{r=m}\frac{1}{\varsigma_2^{\varsigma_{3r}}\Gamma(\varsigma_{3r})}\mathcal{M}_2+\mathcal{M}_3\frac{\zeta \exp(\frac{\varsigma_3-1}{\varsigma_2}\zeta)}{\mathrm{F}\exp(\frac{\varsigma_2-1}{\varsigma_2}\mathrm{F})}\frac{1}{\varsigma_2^{\varsigma_1}\Gamma(\varsigma_1)} \\
&+ \mathcal{M}_4\frac{1}{\varsigma_2^{\varsigma_1}\Gamma(\varsigma_1)} \\
&+ \left.\sum_{r=1}^{m}\frac{1}{\varsigma_2^{\varsigma_{3r}}\Gamma(\varsigma_{3r})}\int_0^{\zeta}(\zeta-\mu)^{\varsigma_{3r}-1}\exp(\frac{\varsigma_2-1}{\varsigma_2}(\zeta-\mu))\mathcal{M}_5\varrho\right)\frac{\delta\varrho}{1-\upsilon}\Omega(\zeta),
\end{aligned}
$$

and so

$$
\begin{aligned}
&\Delta_{CON}(\Psi\psi,\mathcal{L}) \\
&\le \left(\mathcal{M}_1\frac{\psi_0}{\mathcal{Y}(0,\psi_0)}\left(1-\frac{\zeta}{\mathrm{F}}\right)\exp(\frac{\varsigma_2-1}{\varsigma_2}\zeta)+\mathcal{M}_1\frac{\psi_{\mathrm{F}}\frac{\zeta}{\mathrm{F}}\exp(\frac{\varsigma_3-1}{\varsigma_2}\zeta)}{\mathcal{Y}(\mathrm{F},\psi_{\mathrm{F}})\exp(\frac{\varsigma_3-1}{\varsigma_2}\mathrm{F})}\right. \\
&\quad+ \frac{\zeta \exp(\frac{\varsigma_2-1}{\varsigma_2}\zeta)}{\mathcal{Y}(\mathrm{F},\psi_{\mathrm{F}})\,\mathrm{F}\exp(\frac{\varsigma_3-1}{\varsigma_2}\mathrm{F})}\sum_{r=1}^{r=m}\frac{1}{\varsigma_2^{\varsigma_{3r}}\Gamma(\varsigma_{3r})}\mathcal{M}_2+\mathcal{M}_3\frac{\zeta \exp(\frac{\varsigma_3-1}{\varsigma_2}\zeta)}{\mathrm{F}\exp(\frac{\varsigma_2-1}{\varsigma_2}\mathrm{F})}\frac{1}{\varsigma_2^{\varsigma_1}\Gamma(\varsigma_1)} \\
&\quad+ \mathcal{M}_4\frac{1}{\varsigma_2^{\varsigma_1}\Gamma(\varsigma_1)} \\
&\quad+ \left.\sum_{r=1}^{m}\frac{1}{\varsigma_2^{\varsigma_{3r}}\Gamma(\varsigma_{3r})}\int_0^{\zeta}(\zeta-\mu)^{\varsigma_{3r}-1}\exp(\frac{\varsigma_2-1}{\varsigma_2}(\zeta-\mu))\mathcal{M}_5\varrho\right)\frac{\delta\varrho}{1-\upsilon}<\infty.
\end{aligned}
$$

This completes the proof. □

4 Implementation of reproducing kernel method for HPFIDE (1.1)

In this section, we apply the reproducing kernel method of equation (1.1) and obtain the solution.

We define the operator $\mathcal{J}:\mathrm{U}_2^2[0,\mathrm{F}]\to\mathrm{U}_2^1[0,\mathrm{F}]$ as

$$
\mathcal{J}\phi(\zeta) = {}_m^C\Xi_{D_{0^+}}^{\varsigma_1,\varsigma_2}\left(\frac{\phi(\zeta)-\sum_{r=1}^{r=m}\Pi_{I_{0^+}}^{\varsigma_{3r}\varsigma_2}\mathcal{W}_r(\zeta,\phi(\zeta))}{\mathcal{Y}(\zeta,\phi(\zeta))}\right). \tag{4.1}
$$

Then, according to the above differential operator, we have the system

$$
\begin{cases}\mathcal{J}\phi(\zeta)=\mathcal{K}(\zeta,\phi(\zeta)), & \zeta\in[0,\mathrm{F}],\\ \phi(0)=\phi_0,\phi(\mathrm{F})=\phi_{\mathrm{F}}.\end{cases} \tag{4.2}
$$

Theorem 4.1 *The operator $\mathcal{J} : \mathrm{U}_2^2[0,\mathrm{F}] \to \mathrm{U}_2^1[0,\mathrm{F}]$ defined in (4.1) is a linear bounded operator.*

Proof Step 1. First, we show that $\mathcal{J}$ is a linear operator. For this purpose, we assume that $\psi(\zeta), \phi(\zeta) \in \mathrm{U}_2^2[0,\mathrm{F}]$. Then we have

$$\begin{aligned}
&\mathcal{J}(\psi+\phi)(\zeta)\\
&= {}^{C}_{m}\Xi_{D_{0^+}}^{\varsigma_1,\varsigma_2}\left(\frac{(\psi(\zeta)+\phi(\zeta)) - \sum_{r=1}^{r=m}\Pi_{I_{0^+}}^{\varsigma_{3r}\varsigma_2}\mathcal{W}_r(\zeta,(\psi(\zeta)+\phi(\zeta)))}{\mathcal{Y}(\zeta,(\psi(\zeta)+\phi(\zeta)))}\right)\\
&= \frac{1}{\varsigma_2^{\varsigma_1}\Gamma(n-\varsigma_1)}\int_0^\zeta (\zeta-\mu)^{n-\varsigma_1-1}\exp(\frac{n-1}{\varsigma_2}(\zeta-\mu))\\
&\quad\times\left(\Xi_D^{n,\varsigma_2}\left(\frac{(\psi(\mu)+\phi(\mu)) - \sum_{r=1}^{r=m}\Pi_{I_{0^+}}^{\varsigma_{3r}\varsigma_2}\mathcal{W}_r(\mu,(\psi(\mu)+\phi(\mu)))}{\mathcal{Y}(\mu,(\psi(\mu)+\phi(\mu)))}\right)\right)(\mu)d\mu\\
&= {}^{C}_{m}\Xi_{D_{0^+}}^{\varsigma_1,\varsigma_2}\left(\frac{(\psi(\zeta)) - \sum_{r=1}^{r=m}\Pi_{I_{0^+}}^{\varsigma_{3r}\varsigma_2}\mathcal{W}_r(\zeta,\psi(\zeta))}{\mathcal{Y}(\zeta,\psi(\zeta))}\right)\\
&\quad+ {}^{C}_{m}\Xi_{D_{0^+}}^{\varsigma_1,\varsigma_2}\left(\frac{\phi(\zeta) - \sum_{r=1}^{r=m}\Pi_{I_{0^+}}^{\varsigma_{3r}\varsigma_2}\mathcal{W}_r(\zeta,\phi(\zeta))}{\mathcal{Y}(\zeta,\phi(\zeta))}\right)\\
&= \mathcal{J}\psi(\zeta) + \mathcal{J}\phi(\zeta).
\end{aligned}$$

To complete the proof of linearity, we assume that $\psi(\zeta) \in \mathrm{U}_2^2[0,\mathrm{F}]$ and $s \in \mathbb{R}$. Then we have

$$\begin{aligned}
\mathcal{J}(s\psi)(\zeta) &= {}^{C}_{m}\Xi_{D_{0^+}}^{\varsigma_1,\varsigma_2}\left(\frac{(s\psi(\zeta)) - \sum_{r=1}^{r=m}\Pi_{I_{0^+}}^{\varsigma_{3r}\varsigma_2}\mathcal{W}_r(\zeta,(s\psi(\zeta)))}{\mathcal{Y}(\zeta,(s\psi(\zeta)))}\right)\\
&= \frac{1}{\varsigma_2^{\varsigma_1}\Gamma(n-\varsigma_1)}\int_0^\zeta (\zeta-\mu)^{n-\varsigma_1-1}\exp(\frac{n-1}{\varsigma_2}(\zeta-\mu))\\
&\quad\times\left(\Xi_D^{n,\varsigma_2}\left(\frac{(s\psi(\mu)) - \sum_{r=1}^{r=m}\Pi_{I_{0^+}}^{\varsigma_{3r}\varsigma_2}\mathcal{W}_r(\mu,(s\psi(\mu)))}{\mathcal{Y}(\mu,(s\psi(\mu)))}\right)\right)(\mu)d\mu\\
&= s\ {}^{C}_{m}\Xi_{D_{0^+}}^{\varsigma_1,\varsigma_2}\left(\frac{(\psi(\zeta)) - \sum_{r=1}^{r=m}\Pi_{I_{0^+}}^{\varsigma_{3r}\varsigma_2}\mathcal{W}_r(\zeta,(\psi(\zeta)))}{\mathcal{Y}(\zeta,(\psi(\zeta)))}\right) = s\ \mathcal{J}(\psi)(\zeta).
\end{aligned}$$

Step 2. For the boundedness, we consider (IP-U_2^1) and (N-U_2^1):

$$\|\mathcal{J}\psi(\zeta)\|_{\mathrm{U}_2^1}^2 = \langle\mathcal{J}\psi(\zeta),\mathcal{J}\psi(\zeta)\rangle_{\mathrm{U}_2^1} = [(\mathcal{J}\psi)(0)]^2 + \int_0^{\mathrm{F}}\left[(\mathcal{J}\psi)'(\zeta)\right]^2\,\mathrm{d}\zeta. \tag{4.3}$$

On the other hand, according to (RKF-2), we have

- $\psi(\zeta) = \left\langle\psi(\eta),\Lambda_{[rk]}(\eta,\zeta)\right\rangle_{\mathrm{U}_2^2[0,1]}$;
- $(\mathcal{J}\psi)(\zeta) = \left\langle\psi(\eta),\mathcal{J}\Lambda_{[rk]}(\eta,\zeta)\right\rangle_{\mathrm{U}_2^2[0,1]}$;
- $(\mathcal{J}\psi)'(\zeta) = \left\langle\psi(\eta),\left(\mathcal{J}\Lambda'_{[rk]}(\eta,\zeta)\right)\right\rangle_{\mathrm{U}_2^2[0,1]}$.

Using Schwarz's inequality and considering positive constants $\mathcal{B}_1,\mathcal{B}_2 > 0$ and the continuity of $\Lambda_{[rk]}(\zeta,\eta)$, we have

$$|(\mathcal{J}\psi)(\zeta)| = \left|\left\langle\psi(\eta),\mathcal{J}\Lambda_{[rk]\zeta}(\eta)\right\rangle_{\mathrm{U}_2^2}\right| \le \left\|\mathcal{J}\Lambda_{[rk]\zeta}\right\|_{\mathrm{U}_2^1}\|\psi(\zeta)\|_{\mathrm{U}_2^2} = \mathcal{B}_1\|\psi(\zeta)\|_{\mathrm{U}_2^2}$$

and, similarly, $\left|(\mathcal{J}\psi)'(\zeta)\right| \leq \mathcal{B}_2\|\psi\|_{\mathrm{U}_2^2}$. Therefore

$$\|(\mathcal{J}\psi(\zeta))\|^2_{\mathrm{U}_2^1} \leq \mathcal{B}_1^2\|\psi(\zeta)\|^2_{\mathrm{U}_2^2} + \int_0^{\mathrm{F}} \mathcal{B}_2^2\|\psi(\zeta)\|^2_{\mathrm{U}_2^2} d\zeta = \left(\mathcal{B}_1^2 + \mathrm{F}\mathcal{B}_2^2\right)\|\psi(\zeta)\|^2_{\mathrm{U}_2^2}. \qquad \square$$

Now, to complete the reproducing kernel Hilbert space method, we make an orthogonal system of $\mathrm{U}_2^2[0,\mathrm{F}]$. For this purpose, we consider the following:

(O-1) $\mathcal{Q}_{1r}(\zeta) = \Lambda_{[rk]\zeta_r}(\zeta)$;

(O-2) $\mathcal{Q}_{2r}(\zeta) = \mathcal{J}^*\mathcal{Q}_{1r}(\zeta)$;

where $\{\zeta_r\}_{r=1}^{\infty}$ is a dense sequence in $[0,\mathrm{F}]$, $\mathcal{J}^\star : \mathrm{U}_2^1[0,\mathrm{F}] \longrightarrow \mathrm{U}_2^2[0,\mathrm{F}]$ is the conjugate operator of $\mathcal{J}$, and $\{\mathcal{Q}_{2r}(\cdot)\}_{r=1}^{\infty}$ is a complete system of $\mathrm{U}_2^2[0,\mathrm{F}]$. Then we have

$$\begin{aligned}\mathcal{Q}_{2r}(\zeta) &= \mathcal{J}^\star\mathcal{Q}_{1r}(\zeta) = \left\langle \mathcal{J}^\star\mathcal{Q}_{1r}(\zeta), \Lambda_{[rk]\zeta}(\eta)\right\rangle_{\mathrm{U}_2^2} = \left\langle \mathcal{Q}_{1r}(\eta), \mathcal{J}\Lambda_{[rk]\zeta}(\eta)\right\rangle_{\mathrm{U}_2^1} \\ &= \left\langle \Theta_{[rk]\zeta_r}(\eta), \mathcal{J}\Lambda_{[rk]\zeta}(\eta)\right\rangle_{\mathrm{U}_2^2} = L_y\Lambda_{[rk]\zeta}(\eta)\Big|_{\eta=\zeta_r}.\end{aligned}$$

By the Gram–Schmidt process on $\mathrm{U}_2^2[0,\mathrm{F}]$ we obtain an orthogonal basis in $\{\mathcal{Q}_{2r}(\zeta)\}_{r=1}^{\infty}$, which is shown as follows:

$$\bar{\mathcal{Q}}_{2r}(\zeta) = \sum_{i=1}^{r} \vartheta_{ji}\mathcal{Q}_{2i}(\zeta), \tag{4.4}$$

where the orthogonalization coefficients of ϑ_{jk} are as follows:

(OC-1) $\vartheta_{11} = \frac{1}{\|\mathcal{Q}_{21}\|}$;

(OC-2) $\vartheta_{rr} = \frac{1}{\sqrt{\|\mathcal{Q}_{2r}'\|^2 - \sum_{i=1}^{r-1} f_{rk}^2}}$;

(OC-3) $\vartheta_{rj} = \frac{-\sum_{i=j}^{r-1} f_{ri}\vartheta_{ij}}{\sqrt{\|\mathcal{Q}_{2r}\|^2 - \sum_{i=1}^{r-1} f_{ri}^2}}$

such that $f_{ri} = \left\langle \mathcal{Q}_{2r}, \bar{\mathcal{Q}}_{2i}\right\rangle_{\mathrm{U}_2^2}$.

In the following, we present several lemmas regarding exact and approximate solutions obtained by reproducing kernel Hilbert space method [40].

Lemma 4.2 *Let* $\psi(\zeta)$ *be an exact solution of* (4.2). *Then, for any dense set* $\{\zeta_r\}_{r=1}^{\infty}$ *in* $\mathrm{U}_2^2[0,\mathrm{F}]$, *we have*

$$\psi(\zeta) = \sum_{r=1}^{\infty}\sum_{i=1}^{r} \vartheta_{ri}\mathcal{K}\left(\zeta_k, \psi\left(\zeta_k\right)\right)\bar{\mathcal{Q}}_{2r}(\zeta). \tag{4.5}$$

If we substitute a finite sum into (4.5) *instead of an infinite sum, we obtain an approximate solution, which is written as*

$$\psi_n(\zeta) = \sum_{r=1}^{n}\sum_{i=1}^{r} \vartheta_{ri}\mathcal{K}\left(\zeta_k, \psi\left(\zeta_k\right)\right)\bar{\mathcal{Q}}_{2r}(\zeta). \tag{4.6}$$

Lemma 4.3 *If* $\psi \in \mathrm{U}_2^2[0,\mathrm{F}]$, *then there exists a constant* $C > 0$ *such that*

$$\left|\psi^{(r)}(\zeta)\right| \leq C\|\psi\|_{\mathrm{U}_2^2}, \quad r = 0, 1.$$

Lemma 4.4 *The approximate solutions* $\psi_n(\zeta)$ *and* $\psi_n'(\zeta)$ *uniformly converge to the exact solutions* $\psi(\zeta)$ *and* $\psi'(\zeta)$, *respectively.*

5 EXAMPLE

In this section, we analyze qualitatively and numerically an HPFID equation. First, we investigate the optimal stability of the equation using specific controller functions. Before discussing the stability, we will introduce some special functions and control functions. After studying optimal stability and obtaining the best approximation, we find the numerical solutions of the equation by considering the reproducing kernel Hilbert space method [48].

Remark 5.1 The specific functions used in the optimal control function are as follows:

- If for any $(\xi_1,\dots,\xi_{\mathrm{m}}),(\varrho_1,\dots,\varrho_{\mathrm{m}})\in\mathbb{R}^{\mathrm{m}}$, and $\iota\in\{1,\dots,\mathrm{m}\}$ and an idempotent function $\mathrm{z}^{(\mathrm{m})}:\mathbb{R}^{\mathrm{m}}\longrightarrow\mathbb{R}$, we have $\xi_\iota\le\varrho_\iota\Longrightarrow\mathrm{z}^{(\mathrm{m})}(\xi_1,\dots,\xi_{\mathrm{m}})\le\mathrm{z}^{(\mathrm{m})}(\varrho_1,\dots,\varrho_{\mathrm{m}})$, then the m-ary $\mathrm{z}^{(\mathrm{m})}$ is a generalized aggregation function for $\mathrm{m}\in\mathbb{N}$. For $\mathrm{m}=1$ and each $\xi\in\mathbb{R}$, we have $\mathrm{z}^{(1)}(\xi)=\xi$, and for the convenience of writing, we can remove m (m indicates the number of function variables).

  The arithmetic mean function, the projection function, the order statistic function, the median function, and the minimum and maximum functions are among the important functions of aggregation type. In [48–50] the authors showed that a control function constructed from the minimum aggregation function is an optimal controller. The minimum (MIN) is the smallest generalized aggregation function, and it is defined as

$$MIN(\xi)=OS_1(\xi)=\min\{\xi_1,\dots,\xi_\jmath\}=\bigwedge_{\iota=1}^{\mathrm{m}}\xi_\iota. \tag{5.1}$$

  Consider the generalized aggregate function for the values of

$$\mathcal{R}=\left(\mathbb{W}_{E_{\lambda,\mu}},\mathbb{W}_{W_{\lambda,\mu}},\mathbb{W}_{{}_2F_1},\mathbb{W}_{H^{\sigma_3,\sigma_1}_{\sigma_2,\sigma_4}},\mathbb{W}_{\exp}\right). \tag{5.2}$$

  In the following, we introduce special functions used in the Ω function [48–50].

- For $\lambda,\mu\in\mathbb{C}$ such that $Re(\lambda),Re(\mu)>0$, the Mittag-Leffler functions are defined as follows

$$\mathbb{W}_{\mathrm{U}_\lambda}(\xi)=\sum_{\jmath=0}^{\infty}\frac{\xi^\jmath}{\Gamma(\jmath\lambda+1)},\qquad \mathbb{W}_{E_{\lambda,\mu}}(\xi)=\sum_{\jmath=0}^{\infty}\frac{\xi^\jmath}{\Gamma(\jmath\lambda+\mu)},$$

  where $\Gamma(\cdot)$ is the gamma function, and $\mathbb{W}_{\mathrm{U}_\lambda}$ and $\mathbb{W}_{E_{\lambda,\mu}}$ are the one- and two-parameter Mittag-Leffler functions, respectively.
- For $\lambda>-1$, $\mu>0$ and $\xi\in\mathbb{R}$, the Wright function is defined as

$$\mathbb{W}_{W_{\lambda,\mu}}(\xi)=\sum_{\jmath=0}^{\infty}\frac{\xi^\jmath}{\jmath!\Gamma(\lambda\jmath+\mu)},$$

  so that it is of order $1/(1+\sigma)$.
- H-Fox function for $0\le\sigma_1\le\sigma_2$, $1\le\sigma_3\le\sigma_4$, $\{\mathrm{b}_\iota,\mathrm{c}_\iota\}\in\mathbb{C}$, and $\{\mathrm{d}_\iota,\mathrm{e}_\iota\}\in\mathbb{R}^+$ is defined as

$$\mathbb{W}_{H^{\sigma_3,\sigma_1}_{\sigma_2,\sigma_4}}(\xi)=\mathbb{W}_{H^{\sigma_3,\sigma_1}_{\sigma_2,\sigma_4}}\left[\xi\left|\begin{matrix}(\mathrm{b}_\iota,\mathrm{d}_\iota)_{\iota=1,\dots,\sigma_2}\\ (\mathrm{c}_\iota,\mathrm{e}_\iota)_{\iota=1,\dots,\sigma_2}\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_{\mathcal{A}}H^{\sigma_3,\sigma_1}_{\sigma_2,\sigma_4}(\varsigma)\xi^{\varsigma}d\mathrm{t}, \tag{5.3}$$

where $\mathcal{A} \in \mathbb{C}$ is a path that is deleted, $\mathcal{R}_1(\varsigma) = \prod_{\iota=1}^{\sigma_1} \Gamma(\mathrm{c}_\iota - \varrho_\iota \varsigma)$,
$\mathcal{R}_2(\varsigma) = \prod_{\iota=1}^{\sigma_3} \Gamma(1 - \mathrm{b}_\iota + \xi_\iota \varsigma)$, $\mathcal{R}_3(\varsigma) = \prod_{\iota=\sigma_3+1}^{\sigma_3} \Gamma(1 - \mathrm{c}_\iota + \varrho_\iota \varsigma)$,
$\mathcal{R}_4(\varsigma) = \prod_{\iota=\sigma_1+1}^{\sigma_2} \Gamma(\mathrm{b}_\iota - \xi_\iota \varsigma)$, and $\xi^\varsigma = \exp\{\varsigma(\log|\xi| + i \arg \xi)\}$. For these functions, $\sigma_1 = 0$ if and only if $\mathcal{R}_2(\varsigma) = 1$, $\sigma_3 = \sigma_4$ if and only if $\mathcal{R}_3(\varsigma) = 1$, and $\sigma_1 = \sigma_2$ if and only if $\sigma_4(\varsigma) = 1$. Also, $H^{\sigma_3,\sigma_1}_{\sigma_2,\sigma_4}(\varsigma) = \frac{\mathcal{R}_1(\varsigma)\mathcal{R}_2(\varsigma)}{\mathcal{R}_3(\varsigma)\mathcal{R}_4(\varsigma)}$.

- For $p, q, r > 0$, the Gaussian hypergeometric function $\mathbb{W}_{2F_1} : \mathbb{R}^3 \times \mathcal{S}_3 \longrightarrow \mathcal{S}_2$ is defined as

$$\mathbb{W}_{2F_1}(p,q,r;\xi) = \sum_{\jmath=0}^{\infty} \frac{(p)_\jmath (q)_\jmath}{(r)_\jmath} \frac{\xi^\jmath}{\jmath!} = \frac{\Gamma(r)}{\Gamma(p)\Gamma(q)} \sum_{\jmath=0}^{\infty} \frac{\Gamma(p+\jmath)\Gamma(q+\jmath)}{\Gamma(r+\jmath)} \frac{\xi^\jmath}{\jmath!}.$$

Example 5.2 We consider the HPFIDE system

$$\begin{cases} {}^{C}_{m}\Xi_{D_{0^+}}^{\frac{5}{6},\frac{5}{8}} \left(\frac{\phi(\zeta) - \sum_{r=1}^{r=4} \Pi_{I_{0^+}}^{\frac{3r-2}{8},\frac{5}{8}} \frac{\cos\phi(\zeta)\zeta)}{\zeta+150r}}{\frac{1}{20}\sqrt{4+|\phi(\zeta)|^2}} \right) = \frac{\epsilon(2+\zeta)\sin|\phi(\zeta)|}{4+|\phi(\zeta)|}, \\ \phi(0) = \frac{1}{100}, \phi(1) = \frac{1}{200}, \end{cases} \tag{5.4}$$

where $\zeta \in [0,\mathrm{F}]$, $\mathrm{F} = 1$, $\varsigma_1 = \frac{5}{6}$, $\varsigma_2 = \frac{5}{8}$, $\varsigma_{3_r} = \frac{3r-2}{5}$, $\phi_0 = \frac{1}{100}$, $\phi_1 = \frac{1}{200}$, $\varrho = \frac{1}{4}$, $\mathcal{W}_r(\zeta,\phi(\zeta)) = \frac{\cos|\phi(\zeta)|}{\zeta+150r}$ $(r = 1,\dots,4)$, $\mathcal{K}(\zeta,\phi(\zeta)) = \frac{\epsilon(2+\zeta)\sin|\phi(\zeta)|}{4+|\phi(\zeta)|}$, and $\mathcal{Y}(\zeta,\phi(\zeta)) = \frac{1}{20}\sqrt{4+|\phi(\zeta)|^2}$. For continuous functions $\mathcal{Y} : [0,\mathrm{F}] \times \mathbb{R} \to \mathbb{R} - \{0\}$, $\mathcal{W}_r : [0,1] \times \mathbb{R} \to \mathbb{R}$, $\mathcal{K} : [0,1] \times \mathbb{R} \to \mathbb{R}$, and $\Omega : \mathbb{R} \longrightarrow \mathbb{R}$, we have the following inequalities:

$$|\mathcal{Y}(\zeta,\phi(\zeta)) - \mathcal{Y}(\zeta,\psi(\zeta))| \le \frac{1}{20}|\phi(\zeta) - \psi(\zeta)|, \tag{5.5}$$

$$\begin{aligned} &\Big| \frac{1}{20}\sqrt{4+|\psi(\zeta)|^2} \int_0^1 (1-\mu)^{\frac{3r-2}{5}-1} \exp(\frac{-3}{5}(1-\mu)) \frac{\cos|\psi(\mu)|}{\mu+150r} d\mu \\ &- \frac{1}{20}\sqrt{4+|\phi(\zeta)|^2} \int_0^1 (1-\mu)^{\frac{3r-2}{5}-1} \exp(\frac{-3}{5}(1-\mu)) \frac{\cos|\phi(\mu)|}{\mu+150r} d\mu \Big| \\ &\le \frac{1}{40}|\phi(\zeta) - \psi(\zeta)|, \end{aligned} \tag{5.6}$$

$$\begin{aligned} &\Big| \frac{1}{20}\sqrt{4+|\phi(\zeta)|^2} \int_0^1 (1-\mu)^{-\frac{1}{6}} \exp(\frac{-3}{5}(1-\mu)) \frac{\epsilon(2+\mu)\sin|\psi(\mu)|}{4+|\psi(\mu)|} d\mu \\ &- \frac{1}{20}\sqrt{4+|\phi(\zeta)|^2} \int_0^1 (1-\mu)^{\varsigma_1-1} \exp(\frac{-3}{5}(1-\mu)) \frac{\epsilon(2+\mu)\sin|\phi(\mu)|}{4+|\phi(\mu)|} d\mu \Big| \\ &\le \frac{1}{30}|\phi(\zeta) - \psi(\zeta)|, \end{aligned} \tag{5.7}$$

$$\begin{aligned} &\Big| \frac{1}{20}\sqrt{4+|\phi(\zeta)|^2} \int_0^\zeta (\zeta-\mu)^{-\frac{-1}{6}} \exp(\frac{-3}{5}(\zeta-\mu)) \frac{\epsilon(2+\mu)\sin|\phi(\mu)|}{4+|\phi(\mu)|} d\mu \\ &- \frac{1}{20}\sqrt{4+|\psi(\zeta)|^2} \int_0^\zeta (\zeta-\mu)^{-\frac{-1}{6}} \exp(\frac{-3}{5}(\zeta-\mu)) \frac{\epsilon(2+\mu)\sin|\psi(\mu)|}{4+|\psi(\mu)|} d\mu \Big| \\ &\le \frac{1}{50}|\phi(\zeta) - \psi(\zeta)|, \end{aligned} \tag{5.8}$$

$$\begin{aligned} &\frac{1}{\frac{5}{8}^{\frac{5}{6}}\Gamma(\frac{5}{6})} \int_0^\zeta (\zeta-\mu)^{\frac{-1}{6}} \exp(\varsigma_2 - 1(\zeta-\mu)) \Big(MIN_{OPT}(\mathcal{R}) \Big) d\mu \\ &\quad \le \frac{1}{4} \Big(MIN_{OPT}(\mathcal{R}) \Big), \end{aligned} \tag{5.9}$$

where $\mathcal{M}_1 = \frac{1}{20}$, $\mathcal{M}_2 = \frac{1}{40}$, $\mathcal{M}_3 = \frac{1}{30}$, $\mathcal{M}_4 = \frac{1}{50}$, $\mathcal{M}_5 = \frac{1}{60}$. Assume that $\phi : [0,1] \to \mathbb{R}$ is a function such that for $\delta > 0$ and $\zeta \in [0,1]$,

$$\left| {}^{C}_{m}\Xi_{D_{0^+}}^{\frac{5}{6},\frac{5}{8}} \left(\frac{\phi(\zeta) - \sum_{r=1}^{r=4} \Pi_{I_{0^+}}^{\frac{3r-2}{8},\frac{5}{8}} \frac{\cos\phi(\zeta)\zeta)}{\zeta+150r}}{\frac{1}{20}\sqrt{4+|\phi(\zeta)|^2}} \right) - \frac{\epsilon(2+\zeta)\sin|\phi(\zeta)|}{4+|\phi(\zeta)|} \right| \le \frac{1}{8} \left(MIN_{OPT}(\mathcal{R}) \right), \tag{5.10}$$

$$\begin{aligned} &\Bigg| \mathcal{K}(\zeta,\phi(\zeta)) \\ &- \Bigg(\frac{1}{20}\sqrt{4+|\phi(\zeta)|^2} \Bigg[\frac{\phi_0}{\mathcal{Y}(0,\phi_0)}(1-\zeta)\exp(\frac{-3}{5}\zeta) + \frac{\phi_{\mathrm{F}}\zeta\exp(\frac{-3}{5}\zeta)}{\mathcal{Y}(\mathrm{F},\phi_{\mathrm{F}})\exp(\frac{-3}{5})} \\ &- \frac{\zeta\exp(\frac{-3}{5}\zeta)}{\mathcal{Y}(\mathrm{F},\phi_{\mathrm{F}})\exp(\frac{-3}{5})} \\ &\times \sum_{r=1}^{r=m} \frac{1}{\frac{5}{8}^{\frac{3i-2}{8}}\Gamma\left(\frac{3i-2}{8}\right)} \int_0^1 (1-\mu)^{\frac{3i-2}{8}-1}\exp(\frac{-3}{5}(1-\mu))\frac{\cos\phi(\mu)\mu)}{\mu+150r}d\mu \\ &- \frac{\zeta\exp(\frac{-3}{5}\zeta)}{\exp(\frac{-3}{5})}\frac{1}{\frac{5}{8}^{\frac{5}{6}}\Gamma(\frac{5}{6})}\int_0^1 (1-\mu)^{-\frac{1}{6}}\exp(\frac{-3}{5}(1-\mu))\frac{\epsilon(2+\mu)\sin|\phi(\mu)|}{4+|\phi(\mu)|}d\mu \\ &+ \frac{1}{\frac{5}{8}^{\frac{5}{6}}\Gamma(\frac{5}{6})}\int_0^\zeta (\zeta-\mu)^{-\frac{1}{6}}\exp(\frac{-3}{5}(\zeta-\mu))\frac{\epsilon(2+\zeta)\sin|\phi(\zeta)|}{4+|\phi(\zeta)|}d\mu \Bigg] \\ &+ \sum_{r=1}^{m} \frac{1}{\varsigma_2^{\frac{3i-2}{8}}\Gamma\left(\frac{3i-2}{8}\right)} \int_0^\zeta (\zeta-\mu)^{\frac{3i-2}{8}-1}\exp(\frac{\varsigma_2-1}{\varsigma_2}(\zeta-\mu))\mathcal{W}_r(\mu,\phi(\mu))d\mu \Bigg) \Bigg| \\ &\le \frac{1}{8}\frac{1}{\frac{5}{8}^{\frac{5}{6}}\Gamma(\frac{5}{6})}\int_0^\zeta (\zeta-\mu)^{-\frac{1}{6}}\exp(-3(\zeta-\mu))\left(MIN_{OPT}(\mathcal{R}) \right)d\mu. \end{aligned} \tag{5.11}$$

Then HPFIDE (5.4) has a unique solution $\psi : [0,1] \to \mathbb{R}$ such that

$$|\psi(\zeta) - \phi(\zeta)| \le \frac{1}{8-8\upsilon} \left(MIN_{OPT}(\mathcal{R}) \right),$$

where

$$\begin{aligned} \upsilon &= \mathcal{M}_1 \frac{\psi_0}{\mathcal{Y}(0,\psi_0)}\left(1-\frac{\zeta}{\mathrm{F}}\right)\exp(\frac{\varsigma_2-1}{\varsigma_2}\zeta) + \mathcal{M}_1\frac{\psi_{\mathrm{F}}\frac{\zeta}{\mathrm{F}}\exp(\frac{\varsigma_3-1}{\varsigma_2}\zeta)}{\mathcal{Y}(\mathrm{F},\psi_{\mathrm{F}})\exp(\frac{\varsigma_3-1}{\varsigma_2}\mathrm{F})} \\ &+ \frac{\zeta\exp(\frac{\varsigma_2-1}{\varsigma_2}\zeta)}{\mathcal{Y}(\mathrm{F},\psi_{\mathrm{F}})\mathrm{F}\exp(\frac{\varsigma_3-1}{\varsigma_2}\mathrm{F})}\sum_{r=1}^{r=m}\frac{1}{\varsigma_2^{\varsigma_{3r}}\Gamma\left(\varsigma_{3r}\right)}\mathcal{M}_2 + \mathcal{M}_3\frac{\zeta\exp(\frac{\varsigma_3-1}{\varsigma_2}\zeta)}{\mathrm{F}\exp(\frac{\varsigma_2-1}{\varsigma_2}\mathrm{F})}\frac{1}{\varsigma_2^{\varsigma_1}\Gamma(\varsigma_1)} \\ &+ \mathcal{M}_4\frac{1}{\varsigma_2^{\varsigma_1}\Gamma(\varsigma_1)} + \sum_{r=1}^{m}\frac{1}{\varsigma_2^{\varsigma_{3r}}\Gamma\left(\varsigma_{3r}\right)}\int_0^\zeta (\zeta-\mu)^{\varsigma_{3r}-1}\exp(\frac{\varsigma_2-1}{\varsigma_2}(\zeta-\mu))\mathcal{M}_5\varrho d\mu, \end{aligned}$$

Table 1 Exact solution, approximate solution, and absolute error for Example 5.2 using 15 points

ζ	Exact solution	Approximate solution	Approximate solution
0.1	0.01024838364	0.01164548364	0.00139710000
0.2	0.02132564272	0.0257943685	0.00446872578
0.3	0.17599125413	0.19635721478	0.0203659607
0.4	0.19921237451	0.21025681527	0.0110444408
0.5	0.205598631164	0.2115635268	0.0059648956
0.6	0.257000186351	0.289000369573	0.0320001832
0.7	0.317221400000	0.365437500000	0.0482161000
0.8	0.388924536100	0.397001573410	0.0080770373
0.9	0.455245160370	0.470763894721	0.0155187343
1	0.491100000275	0.505178569316	0.0140785690

Table 2 Exact solution, approximate solution, and absolute error for Example 5.2 using 60 points

ζ	Exact solution	Approximate solution	Approximate solution
0.1	0.03247084555	0.04658193455	0.01411108900
0.2	0.05452367100	0.06614758213	0.01162391113
0.3	0.06971289364	0.075223741587	0.00551084795
0.4	0.07141258369	0.079051478216	0.00763889453
0.5	0.08030479612	0.087475111357	0.00717031524
0.6	0.08568713549	0.089581746920	0.00389461143
0.7	0.08442100030	0.092458697125	0.00803769682
0.8	0.08976395721	0.099478963581	0.00971500637
0.9	0.10214007362	0.104596327100	0.0024562535
1	0.109971463827	0.127854126903	0.0178826631

and $\upsilon = 0.657124821 < 1$. Now we will implement the reproducing kernel Hilbert space method for (5.4). We have

$$\mathcal{J} = {}_m^C\Xi_{D_{0^+}}^{\frac{5}{6},\frac{5}{8}} \left(\frac{\phi(\zeta) - \sum_{r=1}^{r=4} \Pi_{I_{0^+}}^{\frac{3r-2}{8},\frac{5}{8}} \frac{\cos\phi(\zeta)\zeta)}{\zeta+150r}}{\frac{1}{20}\sqrt{4+|\phi(\zeta)|^2}} \right), \tag{5.12}$$

$$\mathcal{K}(\zeta,\phi(\zeta)) = \frac{\epsilon(2+\zeta)\sin|\phi(\zeta)|}{4+|\phi(\zeta)|}, \tag{5.13}$$

and

$$\mathcal{Q}_{2r}(\zeta) = \mathcal{J}\Lambda_{[rk]\eta}(\zeta)|_{\eta=\zeta_r} = {}_m^C\Xi_{D_{0^+}}^{\frac{5}{6},\frac{5}{8}} \left(\frac{\Lambda_{\zeta_r}(\zeta) - \sum_{r=1}^{r=4} \Pi_{I_{0^+}}^{\frac{3r-2}{8},\frac{5}{8}} \frac{\cos(\Lambda_{\zeta_r}(\zeta))\zeta)}{\zeta+150r}}{\frac{1}{20}\sqrt{4+|\Lambda_{\zeta_r}(\zeta)|^2}} \right). \tag{5.14}$$

Here, in the interval $[0,1]$, we consider 15, 60, and 95 points, respectively, and using the introduced method, we obtain approximate numerical solutions. For the solutions obtained in each of these points, we calculate the absolute errors. Tables 1–3 show all numerical results obtained for exact solutions, approximate solutions, and absolute errors.

6 Conclusions

In this work, we have considered an HPFID equation to investigate the existence of a unique solution and obtain a suitable approximation for this equation. The stability and existence of a unique solution were established using the controller. We considered the reproducing kernel Hilbert space method to obtain approximate solutions of the equation.

Table 3 Exact solution, approximate solution, and absolute error for Example 5.2 using 95 points

ζ	Exact solution	Approximate solution	Approximate solution
0.1	0.06657142301	0.07375472970	0.00718330669
0.2	0.06914523781	0.07991753624	0.01077229843
0.3	0.10156789413	0.11157368910	0.0100057950
0.4	0.11943692817	0.15067813281	0.0312412046
0.5	0.13652147369	0.18302016870	0.0464986950
0.6	0.15111200748	0.21314285364	0.0620308461
0.7	0.19423147569	0.24856931407	0.0543378384
0.8	0.22110036710	0.28136874501	0.0602683779
0.9	0.280005320144	0.30100268234	0.0209973622
1	0.310781267382	0.35615732815	0.0453760608

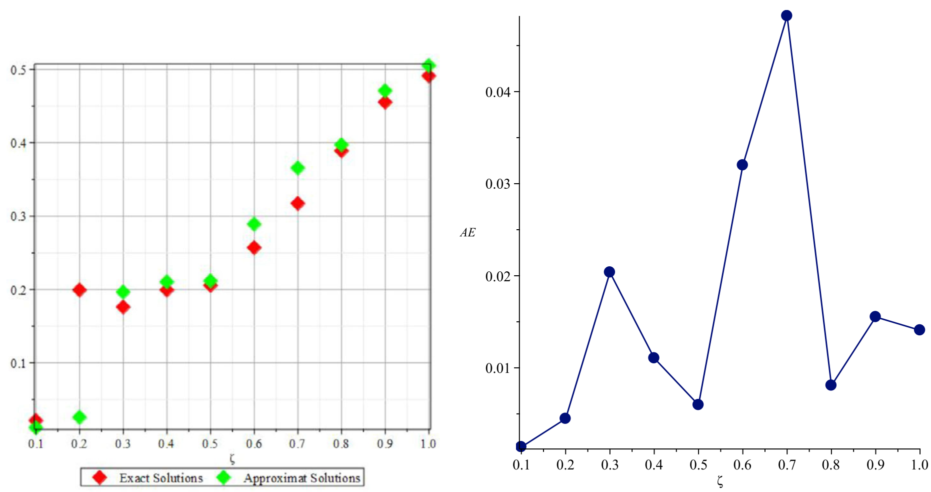

Figure 1 Diagrams of approximate and exact solutions and absolute error of Example 5.2 by selecting 15 points

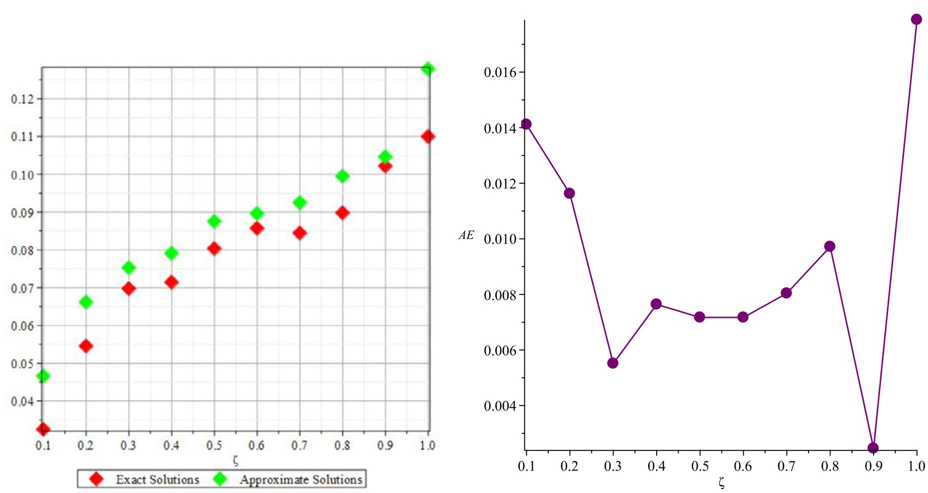

Figure 2 Diagrams of approximate and exact solutions and absolute error of Example 5.2 by selecting 60 points

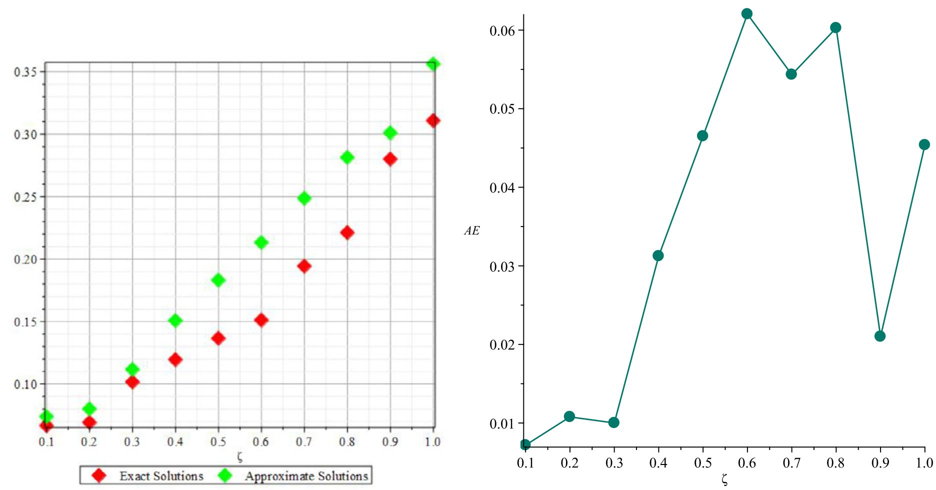

Figure 3 Diagrams of approximate and exact solutions and absolute error of Example 5.2 by selecting 95 points

Also, we implemented the algorithm in a numerical example and obtained approximate solutions for the equation. We have also selected a suitable controller by considering specific functions and derived the best approximation using the optimal control function. All the obtained numerical results are presented in Tables 1–3, and the graphic representations of the calculations are shown in Figs. 1–3.

Author contributions
Z.E., methodology, writing–original draft preparation. R.S., supervision, and project administration. J.V. and C.L., methodology, project administration, supervision, and editing–original draft preparation. T.A., project administration, supervision, and editing–original draft preparation. All authors read and approved the final manuscript.

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Data Availability
No datasets were generated or analysed during the current study.

Declarations

Competing interests
The authors declare no competing interests.

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