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On boundary value problem of the nonlinear fractional partial integro-differential equation via inverse operators

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Abstract

This paper is to obtain sufficient conditions for the uniqueness and existence of solutions to a new nonlinear fractional partial integro-differential equation with boundary conditions. Our analysis relies on an equivalent implicit integral equation in series obtained from an inverse operator, the multivariate Mittag-Leffler function, Leray-Schauder's fixed point theorem as well as Banach's contractive principle. Several illustrative examples are also presented to show applications of the key results derived. Finally, we consider the generalized fractional wave equation in $\mathbb{R}^n$ and deduce the analytic solution for the first time based on the inverse operator method, which leads us a fresh approach to studying some well-known partial differential equations.

Keywords Fractional calculus (primary) · Partial integro-differential equation · Banach's contractive principle · Multivariate Mittag-Leffler function · Inverse operator · Leray-Schauder's fixed point theorem · Generalized fractional wave equation

Mathematics Subject Classification 26A33 (primary) · 35A02 · 35C15 · 45E10

1 Introduction

Let $T, b > 0$ and $\phi_1(x), \phi_2(x) \in C[0, b]$. The objective of this paper is to study the following nonlinear partial integro-differential equation through a well-defined inverse operator and a few notable fixed-point theorems:

$$\begin{cases} \frac{{}_c\partial^\alpha}{\partial t^\alpha} u(t,x) + \sum_{i=1}^{m} a_i \, I_x^{\beta_i} u(t,x) = g(t,x,u(t,x)), \quad 1 < \alpha \leq 2, \ \beta_i \geq 0, \\ u(0,x) = -\phi_1(x), \quad u(T,x) = \phi_2(x), \end{cases} \tag{1.1}$$

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where $(t, x) \in [0, T] \times [0, b]$, all a_i are arbitrary constants for $i = 1, 2, \cdots, m$, $g : [0, T] \times [0, b] \times \mathbb{R} \to \mathbb{R}$ satisfies certain conditions to be given later, I_x^β is the partial Riemann-Liouville fractional integral of order $\beta \in \mathbb{R}^+$ with respect to $x \in [0, b]$ with initial point zero (see [1, 2]),

$$(I_x^\beta u)(t, x) = \frac{1}{\Gamma(\beta)} \int_0^x (x - \tau)^{\beta-1} u(t, \tau) d\tau = \frac{x_+^{\beta-1}}{\Gamma(\beta)} * u(t, x),$$

and $\frac{{}_c\partial^\alpha}{\partial t^\alpha}$ is the partial Caputo fractional derivative of order α with respect to $t \in [0, T]$ [2],

$$\left(\frac{{}_c\partial^\alpha}{\partial t^\alpha} u \right)(t, x) = \frac{1}{\Gamma(2 - \alpha)} \int_0^t (t - s)^{1-\alpha} u_s''(s, x) ds.$$

In particular,

$$(I_x^0 u)(t, x) = \delta(x) * u(t, x) = u(t, x),$$

from [3] by noting that

$$I_x^0 = \frac{x_+^{-1}}{\Gamma(0)} = \delta(x)$$

in Schwartz's distribution theory (see page 117 in [4]).

Equation (1.1) is new and, to the best of our knowledge, has not been previously investigated. Another motivation of considering this equation is to demonstrate how the use of an inverse operator of a bounded integral in a complete space can be used to study the nonlinear fractional integro-differential equation with boundary conditions.

In addition, we construct a new space S and derive the analytic solution for the generalized wave equation in $\mathbb{R}^n$, and obtain several uniform and simple formulas such as solutions (4.5) and (4.7) using inverse operators.

The increasing attempts in applied mathematics to describe real world phenomena often lead to differential or integro-differential equations ([5–7]). This explains a growing interest in the applied mathematics community to integro-differential equations, and in particular, to partial integro-differential equations [8]. They frequently arise and play an important role in many areas of mathematics, physics, engineering, biology, and other sciences. Main challenges in solving these kinds of problems, both numerically and analytically, are due to different types of factors, such as variable coefficients, large range of variables, nonlinearity and non-local phenomena, etc. Yoon et al. [9] considered the following linear partial integro differential equation with a series solution:

$$u_t = \mu u_{xx} + \int_0^t K(t - s) u_{xx}(x, s) ds,$$

where $\mu > 0,\ K(t-s) = (t-s)^{-1/2}$ is the kernel function and the unknown function $u(x,t)$ is sought for $0 \le t \le T,\ 0 \le x \le 1$, with the initial condition

$$u(x,t) = \sin(\pi x), \quad 0 \le x \le 1,$$

and the boundary conditions

$$u(0,t) = u(1,t) = 0, \quad 0 \le t \le T.$$

In 2011, Ouyang [10] studied the following fractional order delay partial differential equation based on Banach's contractive principle, Leray-Schauder's fixed point theorem, Lebesgue dominated convergence theorem as well as an integral equation:

$$\frac{{}_c\partial^\alpha}{\partial t^\alpha}u(t,x) = a(t)\triangle u(t,x) + f(t, u(t,\tau_1(x)), \cdots, u(t,\tau_l(x))), \quad t \in [0, T_0],$$

where $\triangle$ is the Laplacian operator, $0 < \alpha < 1$, l is a positive integer number and the function f is defined as $f(t, u_1, \cdots, u_l) : \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} \to \mathbb{R}$, and $x \in \Omega$ (Ω is a M-dimensional space).

The associated boundary conditions are given by

$$\begin{aligned} u(t,x) &= 0, (t,x) \in [0, T_0] \times \partial\Omega, \\ \frac{\partial}{\partial N}u(t,x) &= 0, (t,x) \in [0, T_0] \times \partial\Omega, \end{aligned}$$

where N is the unit exterior normal vector to $\partial\Omega$, and initial data is

$$u(0,x) = \phi(x), \ \ x \in \Omega.$$

Recently, Zhu and Han [11] considered the following initial boundary value problem of nonlinear time fractional partial integro-differential equations:

$$\begin{cases} \frac{\partial}{\partial t}(u(t,x) + h(u(t,x))) = \int_0^t \frac{(t-s)^{\beta-2}}{\Gamma(\beta-1)}\frac{\partial^2}{\partial x^2}(u(s,x) + h(u(s,x)))ds \\ \qquad\qquad\qquad\qquad\qquad\quad + f(t, u(t,x), \mathcal{G}u(t,x)), \quad t \in [0, b], \\ u(t,0) = u(t,\pi) = 0, \quad t \in [0, b], \\ u(0,x) = \phi(x), \quad x \in [0, \pi], \end{cases}$$

where $\beta \in (1,2)$, $h : \mathbb{R} \to \mathbb{R}$ and $f : [0,b] \times \mathbb{R}^2 \to$ are continuous, $\phi \in L^2[0,\pi]$, and the linear operator $\mathcal{G}$ defined by

$$\mathcal{G}u(t,x) = \int_0^t K(t,s)u(s,x)ds,$$

where $K \in C(D, R_+)$ and

$$D = \{(t, s) \in \mathbb{R}^2 : 0 \le s \le t \le b\}.$$

In this paper, we will use the inverse operator method to deduce implicit integral equations to investigate the uniqueness and existence to equation (1.1). This technique is a useful tool for studying partial integro-differential equations with initial or boundary conditions. To illustrate this in detail, we consider the following nonlinear fractional partial integro-differential equation with boundary conditions and a variable coefficient:

$$\begin{cases} \frac{{}_c\partial^\alpha}{\partial t^\alpha} u(t, x) + a(x)\, I_x^\beta u(t, x) = f(t, x, u(t, x)), & 1 < \alpha \le 2,\ \beta \ge 0, \\ u(0, x) = 0, \quad u(1, x) = \phi(x), & (t, x) \in [0, 1] \times [0, 1], \end{cases} \tag{1.2}$$

where a and ϕ are in $C[0, 1]$, and $f : [0, 1] \times [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

We begin by applying the operator I_t^α to both sides of equation (1.2) to get

$$u(t, x) - u(0, x) - u_t'(0, x)t + I_t^\alpha a(x)\, I_x^\beta u(t, x) = I_t^\alpha f(t, x, u(t, x)).$$

Setting $t = 1$ and using $u(0, x) = 0$, we have

$$\phi(x) - u_t'(0, x) + \frac{a(x)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 (1 - \tau)^{\alpha - 1} \int_0^x (x - s)^{\beta - 1} u(\tau, s) ds d\tau$$
$$= \frac{1}{\Gamma(\alpha)} \int_0^1 (1 - \tau)^{\alpha - 1} f(\tau, x, u(\tau, x)) d\tau.$$

This implies that

$$u_t'(0, x) = \phi(x) + \frac{a(x)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 (1 - \tau)^{\alpha - 1} \int_0^x (x - s)^{\beta - 1} u(\tau, s) ds d\tau$$
$$- \frac{1}{\Gamma(\alpha)} \int_0^1 (1 - \tau)^{\alpha - 1} f(\tau, x, u(\tau, x)) d\tau,$$

and

$$\begin{aligned} \left(1 + I_t^\alpha a(x)\, I_x^\beta\right) u(t, x) &= I_t^\alpha f(t, x, u(t, x)) + \phi(x)t \\ &+ \frac{a(x)t}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 (1 - \tau)^{\alpha - 1} \int_0^x (x - s)^{\beta - 1} u(\tau, s) ds d\tau \\ &- \frac{t}{\Gamma(\alpha)} \int_0^1 (1 - \tau)^{\alpha - 1} f(\tau, x, u(\tau, x)) d\tau. \end{aligned} \tag{1.3}$$

We are going to show that the inverse operator of $1 + I_t^\alpha a(x)\, I_x^\beta$ is

$$V = \sum_{k=0}^{\infty} (-1)^k \left(I_t^\alpha a(x)\, I_x^\beta\right)^k$$

in the Banach space $C([0,1]\times[0,1])$. Indeed, for any $w \in C([0,1]\times[0,1])$ we come to

$$\|Vw\| \le \|w\| \sum_{k=0}^{\infty} \|a\|^k \left\|I_t^{\alpha k}\right\| \left\|I_x^{\beta k}\right\| \le \|w\| \sum_{k=0}^{\infty} \frac{\|a\|^k}{\Gamma(\alpha k+1)} \frac{1}{\Gamma(\beta k+1)} < +\infty,$$

using the fact that the two–parameter Mittag-Leffler function

$$E_{\gamma_1,\gamma_2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\gamma_1 k + \gamma_2)},\ \ z \in \mathbb{C},\ \gamma_1, \gamma_2 > 0,$$

is an entire function in the complex plane $\mathbb{C}$. Furthermore, we prove that

$$V\left(1 + I_t^\alpha a(x)\, I_x^\beta\right) = \left(1 + I_t^\alpha a(x)\, I_x^\beta\right) V = 1 \ \ \text{(identity)}.$$

Clearly,

$$\begin{aligned} V\left(1 + I_t^\alpha a(x)\, I_x^\beta\right) &= V + \sum_{k=0}^{\infty} (-1)^k \left(I_t^\alpha a(x)\, I_x^\beta\right)^{k+1} \\ &= 1 + \sum_{k=1}^{\infty} (-1)^k \left(I_t^\alpha a(x)\, I_x^\beta\right)^{k} + \sum_{k=0}^{\infty} (-1)^k \left(I_t^\alpha a(x)\, I_x^\beta\right)^{k+1} \\ &= 1 + \sum_{k=0}^{\infty} (-1)^{k+1} \left(I_t^\alpha a(x)\, I_x^\beta\right)^{k+1} + \sum_{k=0}^{\infty} (-1)^k \left(I_t^\alpha a(x)\, I_x^\beta\right)^{k+1} = 1. \end{aligned}$$

Similarly,

$$\left(1 + I_t^\alpha a(x)\, I_x^\beta\right) V = 1,$$

the uniqueness follows obviously.

From equation (1.3), we derive

$$\begin{aligned} u(t,x) = \sum_{k=0}^{\infty} (-1)^k \left(I_t^\alpha a(x)\, I_x^\beta\right)^k \Big[I_t^\alpha f(t,x,u(t,x)) + \phi(x)t + \\ + \frac{a(x)t}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 (1-\tau)^{\alpha-1} \int_0^x (x-s)^{\beta-1} u(\tau,s)\,ds\,d\tau \end{aligned}$$

$$-\frac{t}{\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1} f(\tau, x, u(\tau, x))d\tau\Bigg],$$

which is equivalent to equation (1.2).

In addition, if we assume

$$\|f\| = \sup_{(t,x,y)\in[0,1]^2\times\mathbb{R}} |f(t, x, y)| < +\infty,$$

and

$$A = 1 - \frac{\|a\|}{\Gamma(\alpha+1)\Gamma(\beta+1)}\sum_{k=0}^{\infty}\frac{\|a\|^k}{\Gamma(\alpha k+1)}\frac{1}{\Gamma(\beta k+1)} > 0,$$

then

$$\begin{aligned}\|u\| \le \|f\|\sum_{k=0}^{\infty}\frac{\|a\|^k}{\Gamma(\alpha k+\alpha+1)\Gamma(\beta k+1)} + \|\phi\|\sum_{k=0}^{\infty}\frac{\|a\|^k}{\Gamma(\alpha k+1)\Gamma(\beta k+1)}\\ + \frac{\|a\|\,\|u\|}{\Gamma(\alpha+1)\Gamma(\beta+1)}\sum_{k=0}^{\infty}\frac{\|a\|^k}{\Gamma(\alpha k+1)\Gamma(\beta k+1)}\\ + \frac{\|f\|}{\Gamma(\alpha+1)}\sum_{k=0}^{\infty}\frac{\|a\|^k}{\Gamma(\alpha k+1)\Gamma(\beta k+1)}.\end{aligned}$$

This claims that

$$\begin{aligned}\|u\| \le &\frac{\|f\|}{A}\sum_{k=0}^{\infty}\frac{\|a\|^k}{\Gamma(\alpha k+\alpha+1)\Gamma(\beta k+1)} + \frac{\|\phi\|}{A}\sum_{k=0}^{\infty}\frac{\|a\|^k}{\Gamma(\alpha k+1)\Gamma(\beta k+1)}\\ &+ \frac{\|f\|}{A\Gamma(\alpha+1)}\sum_{k=0}^{\infty}\frac{\|a\|^k}{\Gamma(\alpha k+1)\Gamma(\beta k+1)},\end{aligned}$$

which infers that u is uniformly bounded.

To consider the uniqueness, we further suppose that f satisfies the following Lipchitz condition with respect to the third variable for a nonnegative constant L:

$$|f(t, x, u_1) - f(t, x, u_2)| \le L|u_1 - u_2|, \quad u_1, u_2 \in \mathbb{R},$$

and

$$\begin{aligned}B =&L\sum_{k=0}^{\infty}\frac{\|a\|^k}{\Gamma(\alpha k+\alpha+1)\Gamma(\beta k+1)} + \frac{\|a\|}{\Gamma(\alpha+1)\Gamma(\beta+1)}\sum_{k=0}^{\infty}\frac{\|a\|^k}{\Gamma(\alpha k+1)\Gamma(\beta k+1)}\\ &+ \frac{L}{\Gamma(\alpha+1)}\sum_{k=0}^{\infty}\frac{\|a\|^k}{\Gamma(\alpha k+1)\Gamma(\beta k+1)} < 1.\end{aligned}$$

Then equation (1.3) has a unique solution in $C([0,1]\times[0,1])$ from Banach's contractive principle. In fact, we define a nonlinear mapping M over $C([0,1]\times[0,1])$ as

$$(Mu)(t,x) = \sum_{k=0}^{\infty}(-1)^k \left(I_t^{\alpha} a(x)\, I_x^{\beta}\right)^k \Big[I_t^{\alpha} f(t,x,u(t,x)) + \phi(x)t + \\ + \frac{a(x)t}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 (1-\tau)^{\alpha-1} \int_0^x (x-s)^{\beta-1} u(\tau,s)\,ds\,d\tau \\ - \frac{t}{\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} f(\tau,x,u(\tau,x))\,d\tau\Big].$$

Then it follows that $\|Mu\| < +\infty$ by noting that $f(t,x,0)$ is bounded, and

$$|f(t,x,u)| = |f(t,x,u) - f(t,x,0) + f(t,x,0)| \leq L\,\|u\| + |f(t,x,0)| < +\infty,$$

where $u \in C([0,1]\times[0,1])$.

It remains to be shown that M is contractive. Clearly,

$$Mu_1 - Mu_2 = \sum_{k=0}^{\infty}(-1)^k \left(I_t^{\alpha} a(x)\, I_x^{\beta}\right)^k \Big[I_t^{\alpha}(f(t,x,u_1(t,x)) - f(t,x,u_2(t,x))) \\ + \frac{a(x)t}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 (1-\tau)^{\alpha-1} \int_0^x (x-s)^{\beta-1}(u_1(\tau,s) - u_2(\tau,s))\,ds\,d\tau \\ - \frac{t}{\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1}(f(\tau,x,u_1(\tau,x)) - f(\tau,x,u_2(\tau,x)))\,d\tau\Big].$$

Then

$$\|Mu_1 - Mu_2\| \leq L \sum_{k=0}^{\infty} \frac{\|a\|^k}{\Gamma(\alpha k+\alpha+1)\Gamma(\beta k+1)}\,\|u_1 - u_2\| \\ + \frac{\|a\|}{\Gamma(\alpha+1)\Gamma(\beta+1)} \sum_{k=0}^{\infty} \frac{\|a\|^k}{\Gamma(\alpha k+1)\Gamma(\beta k+1)}\,\|u_1 - u_2\| \\ + \frac{L}{\Gamma(\alpha+1)} \sum_{k=0}^{\infty} \frac{\|a\|^k}{\Gamma(\alpha k+1)\Gamma(\beta k+1)}\,\|u_1 - u_2\| = B\,\|u_1 - u_2\|.$$

Since $B < 1$, equation (1.2) has a unique solution from Banach's contractive principle.

In summary, we have the following theorem:

Theorem 1 *Assume that a and ϕ are continuous functions over* $[0,1]$, $f : [0,1]\times[0,1]\times\mathbb{R}\to\mathbb{R}$ *is a continuous function satisfying*

$$|f(t,x,u_1) - f(t,x,u_2)| \leq L|u_1 - u_2|, \quad u_1, u_2 \in \mathbb{R},$$

for a nonnegative constant L, and

$$
\begin{aligned}
B =& L\sum_{k=0}^{\infty}\frac{\|a\|^k}{\Gamma(\alpha k+\alpha+1)\Gamma(\beta k+1)}+\frac{\|a\|}{\Gamma(\alpha+1)\Gamma(\beta+1)}\sum_{k=0}^{\infty}\frac{\|a\|^k}{\Gamma(\alpha k+1)\Gamma(\beta k+1)}\\
&+\frac{L}{\Gamma(\alpha+1)}\sum_{k=0}^{\infty}\frac{\|a\|^k}{\Gamma(\alpha k+1)\Gamma(\beta k+1)}<1.
\end{aligned}
$$

Then there exists a unique solution to equation (1.2).

Example 1 The following nonlinear partial integro-differential equation with a variable coefficient:

$$
\begin{cases}\dfrac{{}_c\partial^{1.5}}{\partial t^{1.5}}u(t,x)+\sin x^2\, I_x^{1.6}u(t,x)=\dfrac{1}{120}|\cos u(t,x)|+\dfrac{t^2}{x^2+1},\\ u(0,x)=0,\quad u(1,x)=x^3,\end{cases} \tag{1.4}
$$

has a unique solution in the space $C([0,1]\times[0,1])$.

Proof Clearly,

$$
f(t,x,y)=\frac{1}{120}|\cos y|+\frac{t^2}{x^2+1},
$$

is a continuous function over $[0,1]\times[0,1]\times\mathbb{R}$, and

$$
|f(t,x,u(t,x))-f(t,x,v(t,x))|\leq\frac{1}{120}|u(t,x)-v(t,x)|.
$$

Thus, $L=1/120$. Furthermore, we have

$$
|\sin x^2|\leq 1,\quad \alpha=1.5,\quad \beta=1.6.
$$

We need to compute

$$
\begin{aligned}
B =& L\sum_{k=0}^{\infty}\frac{\|a\|^k}{\Gamma(\alpha k+\alpha+1)\Gamma(\beta k+1)}+\frac{\|a\|}{\Gamma(\alpha+1)\Gamma(\beta+1)}\sum_{k=0}^{\infty}\frac{\|a\|^k}{\Gamma(\alpha k+1)\Gamma(\beta k+1)}\\
&+\frac{L}{\Gamma(\alpha+1)}\sum_{k=0}^{\infty}\frac{\|a\|^k}{\Gamma(\alpha k+1)\Gamma(\beta k+1)}\\
\leq&\frac{1}{120}\sum_{k=0}^{\infty}\frac{1}{\Gamma(1.5k+1.5+1)\Gamma(1.6k+1)}\\
&+\frac{1}{\Gamma(1.5+1)\Gamma(1.6+1)}\sum_{k=0}^{\infty}\frac{1}{\Gamma(1.5k+1)\Gamma(1.6k+1)}
\end{aligned}
$$

$$+\frac{1}{120\Gamma(1.5+1)}\sum_{k=0}^{\infty}\frac{1}{\Gamma(1.5k+1)\Gamma(1.6k+1)}$$
$$\approx\frac{1}{120}*0.871313+0.526189*1.5479+0.0062877*1.5479=0.83124<1,$$

using online calculators from the site https://www.wolframalpha.com/ (accessed on 05 December 2024). So equation (1.4) has a unique solution in the space $C([0,1]\times[0,1])$ by Theorem 1. □

The multivariate Mittag-Leffler function [12] is defined as follows:

$$E_{(\alpha_1,\cdots,\alpha_m),\beta}(z_1,\cdots,z_m)$$
$$=\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,\cdots,k_m}\frac{z_1^{k_1}\cdots z_m^{k_m}}{\Gamma(\alpha_1k_1+\cdots+\alpha_mk_m+\beta)},$$

where $(z_1,\cdots,z_m)\in\mathbb{C}^m$, $\alpha_i,\beta>0$ for $i=1,2,\cdots,m$, and

$$\binom{k}{k_1,\cdots,k_m}=\frac{k!}{k_1!\cdots k_m!}.$$

The rest of this paper is organized as follows. Section 2 begins converting equation (1.1) to an equivalent implicit integral equation using an inverse operator. Then we obtain sufficient conditions for the uniqueness of solutions by Banach's contractive principle with an example. Section 3 is to study the existence problem to equation (1.1) based on Leray-Schauder's fixed point theorem and equicontinuity with applications. Finally, we find the analytic solution for the generalized fractional wave equation in $\mathbb{R}^n$ by using a new space S and the inverse operator method, which provides a novel technique for seeking solutions of some PEDs involving the Laplacian as well gradient operators in Section 4. At the end, we summarize the entire work in Section 5.

2 Uniqueness

In this section, we will study the uniqueness of solutions to equation (1.1) based on Banach's contractive principle and an implicit integral equation which is equivalent to equation (1.1) in the space $C([0,T]\times[0,b])$.

Theorem 2 *Let a_i be constants for all $i=1,2,\cdots,m$, $g:[0,T]\times[0,b]\times\mathbb{R}\to\mathbb{R}$ be a continuous and bounded function, and $\phi_1(x),\phi_2(x)\in C[0,b]$. Assume that*

$$Q=1-\frac{\chi T^{\alpha}}{\Gamma(\alpha+1)}\,E_{(\beta_1,\cdots,\beta_m),1}\left(T^{\alpha}\chi b^{\beta_1},\cdots,T^{\alpha}\chi b^{\beta_m}\right)\sum_{i=1}^{m}\frac{b^{\beta_i}}{\Gamma(\beta_i+1)}>0,$$

where

$$\chi=\max\{|a_1|,|a_2|,\cdots,|a_m|\}.$$

Then equation (1.1) is equivalent to the following implicit integral equation in the space $C([0,T]\times[0,b])$:

$$\begin{aligned} u(t,x) =& \sum_{k=0}^{\infty}(-1)^k \sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,k_2,\cdots,k_m} a_1^{k_1}\cdots a_m^{k_m} \\ &\cdot I_t^{\alpha k+\alpha} I_x^{\beta_1 k_1+\cdots+\beta_m k_m} g(t,x,u(t,x)) \\ &-\sum_{k=0}^{\infty}(-1)^k \sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,k_2,\cdots,k_m} a_1^{k_1}\cdots a_m^{k_m} \\ &\cdot\frac{t^{\alpha k}}{\Gamma(\alpha k+1)} I_x^{\beta_1 k_1+\cdots+\beta_m k_m}\phi_1(x) \\ &-\sum_{k=0}^{\infty}(-1)^k \sum_{k_1+\cdots+k_m=k}\binom{k}{k_1,k_2,\cdots,k_m} a_1^{k_1}\cdots a_m^{k_m} \\ &\cdot\frac{t^{\alpha k+1}}{\Gamma(\alpha k+2)} I_x^{\beta_1 k_1+\cdots+\beta_m k_m} c(x)t, \end{aligned}$$

where

$$\begin{aligned} c(x) =& \frac{1}{T\Gamma(\alpha)}\int_0^T (T-\tau)^{\alpha-1} g(\tau,x,u(\tau,x))d\tau - \frac{\phi_1(x)+\phi_2(x)}{T} \\ &-\frac{1}{T\Gamma(\alpha)}\sum_{i=1}^{m}\frac{a_i}{\Gamma(\beta_i)}\int_0^T (T-\tau)^{\alpha-1}d\tau\int_0^x (x-s)^{\beta_i-1}u(\tau,s)ds. \end{aligned}$$

In addition,

$$\begin{aligned} \|u\| \le& \left(\frac{\|g\|\,T^{\alpha}}{Q} + \frac{\|g\|\,T^{\alpha}}{Q\Gamma(\alpha+1)} + \frac{2\,\|\phi_1\| + \|\phi_2\|}{Q}\right) \\ &\cdot E_{(\beta_1,\cdots,\beta_m),\,1}\left(T^{\alpha}\chi b^{\beta_1},\cdots,T^{\alpha}\chi b^{\beta_m}\right), \end{aligned}$$

which deduces that u is uniformly bounded.

Proof Clearly,

$$I_t^{\alpha}\,{}_c\frac{\partial^{\alpha}}{\partial t^{\alpha}}u(t,x) = u(t,x) - u(0,x) + c(x)t = u(t,x) + \phi_1(x) + c(x)t,$$

where $c(x)$ is a function of x to be determined by $u(T,x)=\phi_2(x)$ in the following. Applying the operator I_t^{α} to both sides of equation (1.1), we get

$$I_t^{\alpha}\,{}_c\frac{\partial^{\alpha}}{\partial t^{\alpha}}u(t,x) + \sum_{i=1}^{m} I_t^{\alpha} a_i\, I_x^{\beta_i} u(t,x) = I_t^{\alpha} g(t,x,u(t,x)).$$

Setting $t = T$, we come to

$$u(T, x) + \phi_1(x) + c(x)T + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{m} \frac{a_i}{\Gamma(\beta_i)} \int_0^T (T-\tau)^{\alpha-1} d\tau \cdot \int_0^x (x-s)^{\beta_i - 1} u(\tau, s) ds = \frac{1}{\Gamma(\alpha)} \int_0^T (T-\tau)^{\alpha-1} g(\tau, x, u(\tau, x)) d\tau.$$

This implies that

$$c(x) = \frac{1}{T\Gamma(\alpha)} \int_0^T (T-\tau)^{\alpha-1} g(\tau, x, u(\tau, x)) d\tau - \frac{\phi_1(x) + \phi_2(x)}{T} - \frac{1}{T\Gamma(\alpha)} \sum_{i=1}^{m} \frac{a_i}{\Gamma(\beta_i)} \int_0^T (T-\tau)^{\alpha-1} d\tau \int_0^x (x-s)^{\beta_i - 1} u(\tau, s) ds.$$

Hence,

$$\left(1 + \sum_{i=1}^{m} I_t^{\alpha} a_i \, I_x^{\beta_i}\right) u(t, x) = I_t^{\alpha} g(t, x, u(t, x)) - \phi_1(x) - c(x)t. \tag{2.1}$$

Next, we will show that the inverse operator of $1 + \sum_{i=1}^{m} I_t^{\alpha} a_i \, I_x^{\beta_i}$ is

$$U = \sum_{k=0}^{\infty} (-1)^k \left(\sum_{i=1}^{m} I_t^{\alpha} a_i \, I_x^{\beta_i} \right)^k = \sum_{k=0}^{\infty} (-1)^k \sum_{k_1 + \cdots + k_m = k} \binom{k}{k_1, k_2, \cdots, k_m} a_1^{k_1} \cdots a_m^{k_m} I_t^{\alpha k} I_x^{\beta_1 k_1 + \cdots + \beta_m k_m}$$

in the Banach space $C([0, T] \times [0, b])$. Obviously, for any $w \in C([0, T] \times [0, b])$ we have

$$\|Uw\| \leq \|w\| \sum_{k=0}^{\infty} \sum_{k_1 + \cdots + k_m = k} \binom{k}{k_1, k_2, \cdots, k_m} \chi^k \left\| I_t^{\alpha k} \right\| \left\| I_x^{\beta_1 k_1 + \cdots + \beta_m k_m} \right\| \leq \|w\| \sum_{k=0}^{\infty} \sum_{k_1 + \cdots + k_m = k} \binom{k}{k_1, k_2, \cdots, k_m} \chi^k \frac{T^{\alpha k}}{\Gamma(\alpha k + 1)} \frac{b^{\beta_1 k_1 + \cdots + \beta_m k_m}}{\Gamma(\beta_1 k_1 + \cdots + \beta_m k_m + 1)},$$

where

$$\chi = \max\{|a_1|, |a_2|, \cdots, |a_m|\}.$$

Clearly,

$$\frac{T^{\alpha k}}{\Gamma(\alpha k+1)} \leq T^{\alpha k},$$

for all $k = 0, 1, \cdots$. Therefore,

$$\begin{aligned}\|Uw\| &\leq \|w\| \sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_m=k} \binom{k}{k_1, k_2, \cdots, k_m} \chi^k T^{\alpha k} \frac{b^{\beta_1 k_1+\cdots+\beta_m k_m}}{\Gamma(\beta_1 k_1+\cdots+\beta_m k_m+1)} \\ &= \|w\|\, E_{(\beta_1,\cdots,\beta_m),\,1}\left(T^{\alpha}\chi b^{\beta_1}, \cdots, T^{\alpha}\chi b^{\beta_m}\right) < +\infty,\end{aligned}$$

which deduces that U is a continuous mapping from $C([0, T]\times[0, b])$ to itself. Further, we show that

$$U\left(1+\sum_{i=1}^{m} I_t^{\alpha} a_i\, I_x^{\beta_i}\right) = \left(1+\sum_{i=1}^{m} I_t^{\alpha} a_i\, I_x^{\beta_i}\right) U = 1. \tag{2.2}$$

Indeed,

$$\begin{aligned}U\left(1+\sum_{i=1}^{m} I_t^{\alpha} a_i\, I_x^{\beta_i}\right) &= U + \sum_{k=0}^{\infty}(-1)^k \left(\sum_{i=1}^{m} I_t^{\alpha} a_i\, I_x^{\beta_i}\right)^{k+1} \\ &= 1 + \sum_{k=1}^{\infty}(-1)^k \left(\sum_{i=1}^{m} I_t^{\alpha} a_i\, I_x^{\beta_i}\right)^{k} + \sum_{k=0}^{\infty}(-1)^k \left(\sum_{i=1}^{m} I_t^{\alpha} a_i\, I_x^{\beta_i}\right)^{k+1} \\ &= 1 + \sum_{k=0}^{\infty}(-1)^{k+1} \left(\sum_{i=1}^{m} I_t^{\alpha} a_i\, I_x^{\beta_i}\right)^{k+1} + \sum_{k=0}^{\infty}(-1)^k \left(\sum_{i=1}^{m} I_t^{\alpha} a_i\, I_x^{\beta_i}\right)^{k+1} = 1.\end{aligned}$$

Similarly,

$$\left(1+\sum_{i=1}^{m} I_t^{\alpha} a_i\, I_x^{\beta_i}\right) U = 1.$$

Assuming U_0 is another inverse operator, then we have $U = U_0$ by applying U_0 to both sides of equation (2.2).

From equation (2.1), we obtain that

$$\begin{aligned}u(t,x) = &\sum_{k=0}^{\infty}(-1)^k \sum_{k_1+\cdots+k_m=k} \binom{k}{k_1, k_2, \cdots, k_m} a_1^{k_1}\cdots a_m^{k_m} I_t^{\alpha k} \\ &\cdot I_x^{\beta_1 k_1+\cdots+\beta_m k_m}\left[I_t^{\alpha} g(t,x,u(t,x)) - \phi_1(x) - c(x)t\right] \\ &\sum_{k=0}^{\infty}(-1)^k \sum_{k_1+\cdots+k_m=k} \binom{k}{k_1, k_2, \cdots, k_m} a_1^{k_1}\cdots a_m^{k_m} I_t^{\alpha k+\alpha}\end{aligned}$$

$$
\begin{aligned}
&\cdot I_x^{\beta_1 k_1+\cdots+\beta_m k_m} g(t,x,u(t,x)) \\
&-\sum_{k=0}^{\infty}(-1)^k \sum_{k_1+\cdots+k_m=k} \binom{k}{k_1,k_2,\cdots,k_m} a_1^{k_1}\cdots a_m^{k_m} \frac{t^{\alpha k}}{\Gamma(\alpha k+1)} \\
&\cdot I_x^{\beta_1 k_1+\cdots+\beta_m k_m} \phi_1(x) \\
&-\sum_{k=0}^{\infty}(-1)^k \sum_{k_1+\cdots+k_m=k} \binom{k}{k_1,k_2,\cdots,k_m} a_1^{k_1}\cdots a_m^{k_m} \frac{t^{\alpha k+1}}{\Gamma(\alpha k+2)} \\
&\cdot I_x^{\beta_1 k_1+\cdots+\beta_m k_m} c(x).
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\|u\| \le \|g\| &\sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_m=k} \binom{k}{k_1,k_2,\cdots,k_m} \chi^k \left\| I_t^{\alpha k+\alpha} \right\| \left\| I_x^{\beta_1 k_1+\cdots+\beta_m k_m} \right\| \\
&+ \|\phi_1\| \sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_m=k} \binom{k}{k_1,k_2,\cdots,k_m} \chi^k T^{\alpha k} \left\| I_x^{\beta_1 k_1+\cdots+\beta_m k_m} \right\| \\
&+ \|c\| \sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_m=k} \binom{k}{k_1,k_2,\cdots,k_m} \chi^k T^{\alpha k+1} \left\| I_x^{\beta_1 k_1+\cdots+\beta_m k_m} \right\| \\
&\le \|g\|\, T^{\alpha} \sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_m=k} \binom{k}{k_1,k_2,\cdots,k_m} \chi^k T^{\alpha k} \frac{b^{\beta_1 k_1+\cdots+\beta_m k_m}}{\Gamma(\beta_1 k_1+\cdots+\beta_m k_m+1)} \\
&+ \|\phi_1\| \sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_m=k} \binom{k}{k_1,k_2,\cdots,k_m} \chi^k T^{\alpha k} \frac{b^{\beta_1 k_1+\cdots+\beta_m k_m}}{\Gamma(\beta_1 k_1+\cdots+\beta_m k_m+1)} \\
&+ \|c\|\, T \sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_m=k} \binom{k}{k_1,k_2,\cdots,k_m} \chi^k T^{\alpha k} \frac{b^{\beta_1 k_1+\cdots+\beta_m k_m}}{\Gamma(\beta_1 k_1+\cdots+\beta_m k_m+1)} \\
&= \|g\|\, T^{\alpha} E_{(\beta_1,\cdots,\beta_m),\,1}\left(T^{\alpha}\chi b^{\beta_1},\cdots,T^{\alpha}\chi b^{\beta_m}\right) \\
&+ \|\phi_1\|\, E_{(\beta_1,\cdots,\beta_m),\,1}\left(T^{\alpha}\chi b^{\beta_1},\cdots,T^{\alpha}\chi b^{\beta_m}\right) \\
&+ \|c\|\, T\, E_{(\beta_1,\cdots,\beta_m),\,1}\left(T^{\alpha}\chi b^{\beta_1},\cdots,T^{\alpha}\chi b^{\beta_m}\right).
\end{aligned}
$$

Clearly,

$$
\|c\|\, T \le \|g\| \frac{T^{\alpha}}{\Gamma(\alpha+1)} + \|\phi_1\| + \|\phi_2\| + \frac{\chi T^{\alpha}}{\Gamma(\alpha+1)} \sum_{i=1}^{m} \frac{b^{\beta_i}}{\Gamma(\beta_i+1)} \|u\|.
$$

Since

$$
Q = 1 - \frac{\chi T^{\alpha}}{\Gamma(\alpha+1)}\, E_{(\beta_1,\cdots,\beta_m),\,1}\left(T^{\alpha}\chi b^{\beta_1},\cdots,T^{\alpha}\chi b^{\beta_m}\right) \sum_{i=1}^{m} \frac{b^{\beta_i}}{\Gamma(\beta_i+1)} > 0,
$$

we claim that

$$\|u\| \leq \left(\frac{\|g\| \, T^{\alpha}}{Q} + \frac{\|g\| \, T^{\alpha}}{Q\Gamma(\alpha+1)} + \frac{2\,\|\phi_1\| + \|\phi_2\|}{Q} \right) \cdot E_{(\beta_1,\cdots,\beta_m),\,1}\left(T^{\alpha}\chi b^{\beta_1}, \cdots, T^{\alpha}\chi b^{\beta_m}\right),$$

which indicates that u is uniformly bounded. This completes the proof of Theorem 2. □

We are now ready to present the following theorem regarding the uniqueness of solutions to equation (1.1).

Theorem 3 *Let a_i be constants for all $i = 1, 2, \cdots, m$, $g : [0, T] \times [0, b] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the Lipchitz condition for a constant $L \geq 0$,*

$$|g(t, x, y_1) - g(t, x, y_2)| \leq L|y_1 - y_2|, \quad y_1, y_2 \in \mathbb{R},$$

and $\phi_1(x), \phi_2(x) \in C[0, b]$. Further we assume that

$$F = \left(L + \frac{L}{\Gamma(\alpha+1)} + \frac{\chi}{\Gamma(\alpha+1)} \sum_{i=1}^{m} \frac{b^{\beta_i}}{\Gamma(\beta_i+1)} \right) \cdot T^{\alpha} E_{(\beta_1,\cdots,\beta_m),\,1}\left(T^{\alpha}\chi b^{\beta_1}, \cdots, T^{\alpha}\chi b^{\beta_m}\right) < 1.$$

Then equation (1.1) has a unique solution in the space $C([0, T] \times [0, b])$.

Proof Define a nonlinear mapping W over the space $C([0, T] \times [0, b])$ by

$$\begin{aligned}(Wu)(t, x) = &\sum_{k=0}^{\infty} (-1)^k \sum_{k_1+\cdots+k_m=k} \binom{k}{k_1, k_2, \cdots, k_m} a_1^{k_1} \cdots a_m^{k_m} I_t^{\alpha k+\alpha} \\ &\cdot I_x^{\beta_1 k_1+\cdots+\beta_m k_m} g(t, x, u(t, x)) \\ &- \sum_{k=0}^{\infty} (-1)^k \sum_{k_1+\cdots+k_m=k} \binom{k}{k_1, k_2, \cdots, k_m} a_1^{k_1} \cdots a_m^{k_m} \frac{t^{\alpha k}}{\Gamma(\alpha k+1)} \\ &\cdot I_x^{\beta_1 k_1+\cdots+\beta_m k_m} \phi_1(x) \\ &- \sum_{k=0}^{\infty} (-1)^k \sum_{k_1+\cdots+k_m=k} \binom{k}{k_1, k_2, \cdots, k_m} a_1^{k_1} \cdots a_m^{k_m} \frac{t^{\alpha k+1}}{\Gamma(\alpha k+2)} \\ &\cdot I_x^{\beta_1 k_1+\cdots+\beta_m k_m} c(x) t.\end{aligned}$$

It follows from Theorem 2 that $Wu \in C([0, T] \times [0, b])$ by noting that for $u \in C([0, T] \times [0, b])$,

$$|g(t, x, u)| = |g(t, x, u) - g(t, x, 0) + g(t, x, 0)| \leq L|u| + |g(t, x, 0| < +\infty,$$

since $g(t, x, 0)$ is a continuous and bounded function. We only need to show that W is contractive. Indeed,

$$\max_{(t,x)\in[0,T]\times[0,b]} |g(t, x, u(t, x)) - g(t, x, v(t, x))| \leq L \|u - v\|,$$

and

$$\begin{aligned}
\|Wu - Wv\| &\leq LT^{\alpha} E_{(\beta_1,\cdots,\beta_m),1}\left(T^{\alpha} M b^{\beta_1}, \cdots, T^{\alpha} M b^{\beta_m}\right) \|u - v\| \\
&\quad + \frac{LT^{\alpha}}{\Gamma(\alpha+1)} E_{(\beta_1,\cdots,\beta_m),1}\left(T^{\alpha}\chi b^{\beta_1}, \cdots, T^{\alpha}\chi b^{\beta_m}\right) \|u - v\| \\
&\quad + \frac{\chi}{\Gamma(\alpha+1)} T^{\alpha} E_{(\beta_1,\cdots,\beta_m),1}\left(T^{\alpha}\chi b^{\beta_1}, \cdots, T^{\alpha}\chi b^{\beta_m}\right) \sum_{i=1}^{m} \frac{b^{\beta_i}}{\Gamma(\beta_i+1)} \|u - v\| \\
&= \left(L + \frac{L}{\Gamma(\alpha+1)} + \frac{\chi}{\Gamma(\alpha+1)} \sum_{i=1}^{m} \frac{b^{\beta_i}}{\Gamma(\beta_i+1)}\right) \cdot \\
&\quad T^{\alpha} E_{(\beta_1,\cdots,\beta_m),1}\left(T^{\alpha}\chi b^{\beta_1}, \cdots, T^{\alpha}\chi b^{\beta_m}\right) \|u - v\|.
\end{aligned}$$

By the assumption that

$$\begin{aligned}
F = &\left(L + \frac{L}{\Gamma(\alpha+1)} + \frac{\chi}{\Gamma(\alpha+1)} \sum_{i=1}^{m} \frac{b^{\beta_i}}{\Gamma(\beta_i+1)}\right) \\
&\cdot T^{\alpha} E_{(\beta_1,\cdots,\beta_m),1}\left(T^{\alpha}\chi b^{\beta_1}, \cdots, T^{\alpha} M b^{\beta_m}\right) < 1,
\end{aligned}$$

so W is contractive. Equation (1.1) has a unique solution in the space $C([0, T] \times [0, b])$ from Banach's contractive principle. This completes the proof of Theorem 3. □

Example 2 The following nonlinear fractional partial integro-differential equation:

$$\begin{cases}
\frac{{}_c\partial^{1.7}}{\partial t^{1.7}} u(t, x) + \frac{1}{13} I_x^{2.5} u(t, x) - \frac{1}{15} I_x^{2.1} u(t, x) = \frac{1}{17} \arctan|u(t, x)| + t^2 x^3, \\
u(0, x) = \sin(x + 1), \quad u(1, x) = \frac{x}{x + 1},
\end{cases} \tag{2.3}$$

has a unique solution in the space $C([0, 1] \times [0, 1])$.

Proof Since

$$g(t, x, u) = \frac{1}{17} \arctan|u(t, x)| + t^2 x^3$$

is a continuous function over $[0, 1]^2 \times \mathbb{R}$, satisfying the Lipchitz condition:

$$|g(t, x, u_1) - g(t, x, u_2)| = \frac{1}{17} |\arctan|u_1| - \arctan|u_2|| \leq \frac{1}{17} |u_1 - u_2|,$$

using the mean value theorem and

$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2} \leq 1.$$

Clearly,

$$\chi = 1/13, \quad \alpha = 1.7, \quad \beta_1 = 2.5, \quad \beta_2 = 2.1, \quad L = 1/17.$$

We need to evaluate

$$F = \left(L + \frac{L}{\Gamma(\alpha+1)} + \frac{\chi}{\Gamma(\alpha+1)} \sum_{i=1}^{m} \frac{b^{\beta_i}}{\Gamma(\beta_i+1)} \right)$$
$$\cdot T^{\alpha} E_{(\beta_1,\cdots,\beta_m),1}\left(T^{\alpha}\chi b^{\beta_1}, \cdots, T^{\alpha}\chi b^{\beta_m}\right)$$
$$= \left(\frac{1}{17} + \frac{1}{17\Gamma(2.7)} + \frac{1}{13\Gamma(2.7)} \left[\frac{1}{\Gamma(3.5)} + \frac{1}{\Gamma(3.1)} \right] \right) E_{(2.5,2.1),1}(1/13, 1/13).$$

It follows that

$$E_{(2.5,2.1),1}(1/13, 1/13) = \sum_{k=0}^{\infty} \sum_{k_1+k_2=k} \binom{k}{k_1, k_2} \frac{(1/13)^k}{\Gamma(2.5k_1 + 2.1k_2 + 1)}$$
$$\leq \sum_{k=0}^{\infty} \frac{(2/13)^k}{\Gamma(2.1k+1)} \approx 1.07074.$$

Hence,

$$F \leq \left(\frac{1}{17} + \frac{1}{17\Gamma(2.7)} + \frac{1}{13\Gamma(2.7)} \left[\frac{1}{\Gamma(3.5)} + \frac{1}{\Gamma(3.1)} \right] \right) * 1.07074$$
$$= (0.05882353 + 0.0380812 + 0.0497985 * 0.755939) * 1.07074 < 1.$$

Thus equation (2.3) has a unique solution from Theorem 3. □

3 Existence

We are going to present the theorem regarding the existence of equation (1.1) based on Leray-Schauder's fixed point theorem and equicontinuity which are given below.

Theorem 4 *(Leray-Schauder's Fixed Point Theorem (see [13] and [14])) Consider the continuous and compact function W of a Banach space B into itself. The boundedness of*

$$\{x \in B : x = \theta W x \ \ \textit{for some} \ \ 0 \leq \theta \leq 1\}$$

implies that W has a fixed point in B.

Definition 1 Let (X, d) be a metric space, and $\mathcal{F}$ a family of functions from X to X. The family $\mathcal{F}$ is uniformly equicontinuous if for every $\epsilon > 0$, there exists a $\delta > 0$ such that $d(f(x_1), f(x_2)) < \epsilon$ for all $f \in \mathcal{F}$ and all $x_1, x_2 \in X$ such that $d(x_1, x_2) < \delta$, which may depend only on ϵ.

Theorem 5 *Let a_i be constants for all $i = 1, 2, \cdots, m$, $g : [0, T] \times [0, b] \times \mathbb{R} \to \mathbb{R}$ be a continuous and bounded function satisfying the Lipchitz condition for a constant $L \geq 0$:*

$$|g(t, x, y_1) - g(t, x, y_2)| \leq L|y_1 - y_2|, \quad y_1, y_2 \in \mathbb{R}.$$

Further assume that $\phi_1(x), \phi_2(x) \in C[0, b]$ and

$$Q = 1 - \frac{\chi T^\alpha}{\Gamma(\alpha+1)} \; E_{(\beta_1,\cdots,\beta_m),1}\left(T^\alpha \chi b^{\beta_1}, \cdots, T^\alpha \chi b^{\beta_m}\right) \sum_{i=1}^{m} \frac{b^{\beta_i}}{\Gamma(\beta_i+1)} > 0,$$

where

$$\chi = \max\{|a_1|, |a_2|, \cdots, |a_m|\}.$$

Then there is a solution to equation (1.1) in $C([0, T] \times [0, b])$

Proof We again consider the nonlinear mapping W over the space $C([0, T] \times [0, b])$ by

$$\begin{aligned}
(Wu)(t, x) = &\sum_{k=0}^{\infty}(-1)^k \sum_{k_1+\cdots+k_m=k} \binom{k}{k_1, k_2, \cdots, k_m} a_1^{k_1} \cdots a_m^{k_m} I_t^{\alpha k+\alpha} \\
&\cdot I_x^{\beta_1 k_1+\cdots+\beta_m k_m} g(t, x, u(t, x)) \\
&- \sum_{k=0}^{\infty}(-1)^k \sum_{k_1+\cdots+k_m=k} \binom{k}{k_1, k_2, \cdots, k_m} a_1^{k_1} \cdots a_m^{k_m} \frac{t^{\alpha k}}{\Gamma(\alpha k+1)} \\
&\cdot I_x^{\beta_1 k_1+\cdots+\beta_m k_m} \phi_1(x) \\
&- \sum_{k=0}^{\infty}(-1)^k \sum_{k_1+\cdots+k_m=k} \binom{k}{k_1, k_2, \cdots, k_m} a_1^{k_1} \cdots a_m^{k_m} \frac{t^{\alpha k+1}}{\Gamma(\alpha k+2)} \\
&\cdot I_x^{\beta_1 k_1+\cdots+\beta_m k_m} c(x) t.
\end{aligned}$$

It follows from Theorem 3 that $Tu \in C([0, T] \times [0, b])$, and

$$\|Wu - Wv\| \leq F \, \|u - v\| \, ,$$

where

$$F = \left(L + \frac{L}{\Gamma(\alpha+1)} + \frac{\chi}{\Gamma(\alpha+1)} \sum_{i=1}^{m} \frac{b^{\beta_i}}{\Gamma(\beta_i+1)}\right)$$

$$\cdot\, T^{\alpha} E_{(\beta_1,\cdots,\beta_m),\,1}\left(T^{\alpha}\chi b^{\beta_1},\cdots,T^{\alpha}\chi b^{\beta_m}\right) > 0$$

which is not required to be less than 1. These imply that (i) W is a continuous mapping from $C([0,T]\times[0,b])$ to itself.

(ii) Furthermore, we will prove that W is a mapping from bounded sets to bounded sets in $C([0,T]\times[0,b])$. Let $\mathcal{B}$ be a bounded set in $C([0,T]\times[0,b])$. Then there exists a positive constant $\mathcal{C}$ such that

$$\|w\| \le \mathcal{C}$$

for all $w\in\mathcal{B}$. Using the following facts

$$\begin{aligned}\|Wu\| \le{}& \|g\|\,T^{\alpha} E_{(\beta_1,\cdots,\beta_m),\,1}\left(T^{\alpha}\chi b^{\beta_1},\cdots,T^{\alpha}\chi b^{\beta_m}\right)\\ &+\|\phi_1\|\,E_{(\beta_1,\cdots,\beta_m),\,1}\left(T^{\alpha}\chi b^{\beta_1},\cdots,T^{\alpha}\chi b^{\beta_m}\right)\\ &+\|c\|\,T E_{(\beta_1,\cdots,\beta_m),\,1}\left(T^{\alpha}\chi b^{\beta_1},\cdots,T^{\alpha}\chi b^{\beta_m}\right),\end{aligned}$$

and

$$\|c\|\,T \le \|g\|\,\frac{T^{\alpha}}{\Gamma(\alpha+1)} + \|\phi_1\| + \|\phi_2\| + \frac{\chi T^{\alpha}}{\Gamma(\alpha+1)}\sum_{i=1}^{m}\frac{b^{\beta_i}}{\Gamma(\beta_i+1)}\,\|u\|,$$

as well as g is bounded, we claim that Wu is bounded.

(iii) W is equicontinuous on every bounded set $\mathcal{B}$ in $C([0,T]\times[0,b])$. Then W is a compact operator by the Arzela-Ascoli theorem. It follows directly from Definition 1 and

$$\|Wu - Wv\| \le F\,\|u-v\|,$$

using $\mathcal{F}=\{W\}$ and d is the defined norm of the space $C([0,T]\times[0,b])$.

(iv) Finally, we show that the set

$$\{u\in C([0,T]\times[0,b]) : u=\theta Wu \;\text{ for some }\; 0<\theta\le 1\}$$

is bounded. From the proof of Theorem 2, we have

$$\begin{aligned}\|u\| \le \|Wu\| \le{}& \left(\frac{\|g\|\,T^{\alpha}}{Q} + \frac{\|g\|\,T^{\alpha}}{Q\Gamma(\alpha+1)} + \frac{2\,\|\phi_1\| + \|\phi_2\|}{Q}\right)\\ &\cdot E_{(\beta_1,\cdots,\beta_m),\,1}\left(T^{\alpha}\chi b^{\beta_1},\cdots,T^{\alpha}\chi b^{\beta_m}\right),\end{aligned}$$

where

$$Q = 1 - \frac{\chi T^{\alpha}}{\Gamma(\alpha+1)}\,E_{(\beta_1,\cdots,\beta_m),1}\left(T^{\alpha}\chi b^{\beta_1},\cdots,T^{\alpha}\chi b^{\beta_m}\right)\sum_{i=1}^{m}\frac{b^{\beta_i}}{\Gamma(\beta_i+1)} > 0.$$

Hence, u is bounded. This completes the proof. □

Remark 1 We must point out that $F < 1$ in Theorem 3 implies that $Q > 0$ in Theorem 5, but the converse is not true in general.

Example 3 The following nonlinear fractional partial integro-differential equation:

$$\begin{cases} \frac{{}_c\partial^{1.5}}{\partial t^{1.5}} u(t,x) + \frac{1}{10} I_x^{2.2} u(t,x) - \frac{1}{13} I_x^{2.7} u(t,x) = 2\cos|u(t,x)| + t^7 + x^2 + 1, \\ u(0,x) = \sin^2(x+1), \quad u(1,x) = \frac{x^4}{x+1}, \end{cases} \tag{3.1}$$

has a solution in the space $C([0,1]\times[0,1])$.

Proof Clearly,

$$g(t,x,u) = 2\cos|u| + t^7 + x^2 + 1$$

is a continuous and bounded function over $[0,1]^2 \times \mathbb{R}$ with the condition

$$|g(t,x,u_1) - g(t,x,u_2)| = 2|\cos|u_1| - \cos|u_2|| \leq 2|u_1 - u_2|,$$

by the mean value theorem. From Theorem 5, we need to evaluate

$$\begin{aligned} Q_0 &= \frac{\chi T^\alpha}{\Gamma(\alpha+1)} E_{(\beta_1,\cdots,\beta_m),1}\left(T^\alpha \chi b^{\beta_1}, \cdots, T^\alpha \chi b^{\beta_m}\right) \sum_{i=1}^m \frac{b^{\beta_i}}{\Gamma(\beta_i+1)} \\ &= \frac{1}{10\Gamma(2.5)} E_{(2.2,2.7),1}(1/10, 1/10)\left[\frac{1}{\Gamma(3.2)} + \frac{1}{\Gamma(3.7)}\right]. \end{aligned}$$

Clearly,

$$\begin{aligned} E_{(2.2,2.7),1}(1/10,1/10) &= \sum_{k=0}^\infty \sum_{k_1+k_2=k} \binom{k}{k_1, k_2} \frac{(1/10)^k}{\Gamma(2.2k_1 + 2.7k_2 + 1)} \\ &\leq \sum_{k=0}^\infty \frac{(2/10)^k}{\Gamma(2.2k+1)} \approx 1.08341. \end{aligned}$$

Therefore,

$$Q_0 \approx 0.0752253 * 1.08341 * 0.652318 < 1.$$

By Theorem 5, there is a solution to equation (3.1). However, $F > 1$ for this equation so we are not that if solution is unique. □

4 An application of inverse operators

In this section, we are going to find the analytic solution for the following generalized fractional wave equation in $\mathbb{R}^n$:

$$\begin{cases} \frac{{}_c\partial^\alpha}{\partial t^\alpha} u(t,x) = \triangle_{\lambda_1,\cdots,\lambda_n} u(t,x) + g(t,x), \quad 1 < \alpha \leq 2, \\ u(0,x) = \phi_1(x), \quad u'_t(0,x) = \phi_2(x), \quad (t,x) \in \mathbb{R}^+ \times R^n, \end{cases} \tag{4.1}$$

where

$$\triangle_{\lambda_1,\cdots,\lambda_n} = \lambda_1 \frac{\partial^2}{\partial x_1^2} + \cdots + \lambda_n \frac{\partial^2}{\partial x_n^2}, \quad \text{all } \lambda_i \text{ are contants.}$$

In particular, if $\alpha = 2$ and $\lambda_1 = \cdots = \lambda_n = 1$, then equation (4.1) turns out to be the nonhomogeneous wave equation in $\mathbb{R}^n$:

$$\begin{cases} \frac{\partial^2}{\partial t^2} u(t,x) = \triangle u(t,x) + g(t,x), \\ u(0,x) = \phi_1(x), \quad u'_t(0,x) = \phi_2(x), \quad (t,x) \in \mathbb{R}^+ \times R^n, \end{cases} \tag{4.2}$$

where

$$\triangle_{1,\cdots,1} = \triangle = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

is the Laplacian operator.

As far as we know there does not exist any analytic solution to equation (4.1) up to date. To demonstrate the use of inverse operators, we will establish the following theorem finding the analytic solution.

Theorem 6 *Let g, ϕ_1 and ϕ_2 be in the space S given by*

$$S = \Big\{ g \in C(\mathbb{R}^+ \times \mathbb{R}^n) : \exists \text{ a constant } M_g > 0 \text{ such that } \sup_{(t,x)\in\mathbb{R}^+\times\mathbb{R}^n} \left| \frac{\partial^{2s_1}}{\partial x_1^{2s_1}} \cdots \frac{\partial^{2s_n}}{\partial x_n^{2s_n}} g(t,\sigma) \right| \leq M_g^{s_1+\cdots+s_n} \Big\},$$

where $(s_1, \cdots, s_n) \in (\mathbb{N} \cup \{0\})^n$. Then equation (4.1) has a unique solution

$$u(t,x) = \sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_n=k} \binom{k}{k_1,\cdots,k_n} \lambda_1^{k_1} \cdots \lambda_m^{k_m} I_t^{\alpha k+\alpha} \frac{\partial^{2k_1}}{\partial x_1^{2k_1}} \cdots \frac{\partial^{2k_n}}{\partial x_n^{2k_n}} g(t,\sigma)$$
$$+ \sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_n=k} \binom{k}{k_1,\cdots,k_n} \lambda_1^{k_1} \cdots \lambda_m^{k_m} \frac{t^{\alpha k}}{\Gamma(\alpha k+1)} \frac{\partial^{2k_1}}{\partial x_1^{2k_1}} \cdots \frac{\partial^{2k_n}}{\partial x_n^{2k_n}} \phi_1(x)$$

$$+\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_n=k}\binom{k}{k_1,\cdots,k_n}\lambda_1^{k_1}\cdots\lambda_m^{k_m}\frac{t^{\alpha k+1}}{\Gamma(\alpha k+2)}\frac{\partial^{2k_1}}{\partial x_1^{2k_1}}\cdots\frac{\partial^{2k_n}}{\partial x_n^{2k_n}}\phi_2(x).$$

Proof Applying the operator I_t^α to both sides of equation (4.1), we get

$$u(t,x)-\phi_1(x)-\phi_2(x)t-I_t^\alpha\sum_{i=1}^{n}\lambda_i\frac{\partial^2}{\partial x_i^2}u(t,x)=I_t^\alpha g(t,x),$$

which implies that

$$\left(1-I_t^\alpha\sum_{i=1}^{n}\lambda_i\frac{\partial^2}{\partial x_i^2}\right)u(t,x)=I_t^\alpha g(t,x)+\phi_1(x)+\phi_2(x)t. \tag{4.3}$$

We shall show that the inverse operator of $1-I_t^\alpha\sum_{i=1}^{n}\lambda_i\frac{\partial^2}{\partial x_i^2}$ is

$$\begin{aligned}\mathcal{V}=&\sum_{k=0}^{\infty}\left(I_t^\alpha\sum_{i=1}^{n}\lambda_i\frac{\partial^2}{\partial x_i^2}\right)^k\\=&\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_n=k}\binom{k}{k_1,\cdots,k_n}\lambda_1^{k_1}\cdots\lambda_n^{k_n}I_t^{\alpha k}\frac{\partial^{2k_1}}{\partial x_1^{2k_1}}\cdots\frac{\partial^{2k_n}}{\partial x_n^{2k_n}}\end{aligned}$$

in the space S given above. Indeed, for any $g\in S$ we have

$$\begin{aligned}\|\mathcal{V}g\|\leq&\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_n=k}\binom{k}{k_1,\cdots,k_n}|\lambda_1|^{k_1}\cdots|\lambda_n|^{k_n}\\&\cdot\frac{t^{\alpha k_1+\cdots+\alpha k_n}}{\Gamma(\alpha k_1+\cdots+\alpha k_n+1)}\sup_{(t,x)\in\mathbb{R}^+\times\mathbb{R}^n}\left|\frac{\partial^{2k_1}}{\partial x_1^{2k_1}}\cdots\frac{\partial^{2k_n}}{\partial x_n^{2k_n}}g(t,\sigma)\right|,\\\leq&\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_n=k}\binom{k}{k_1,\cdots,k_n}|\lambda_1|^{k_1}\cdots|\lambda_n|^{k_n}\\&\cdot\frac{t^{\alpha k_1+\cdots+\alpha k_n}}{\Gamma(\alpha k_1+\cdots+\alpha k_n+1)}M_g^{k_1+\cdots+k_n}\\=&E_{(\alpha,\alpha,\cdots,\alpha)\,1}\left(|\lambda_1|t^\alpha M_g,\cdots,|\lambda_n|t^\alpha M_g\right)<+\infty,\end{aligned}$$

which claims that $\mathcal{V}$ is a well-defined operator over S under the norm of $C(\mathbb{R}^+\times R^n)$.
In addition, we show that

$$\mathcal{V}\left(1-I_t^\alpha\sum_{i=1}^{n}\lambda_i\frac{\partial^2}{\partial x_i^2}\right)=\left(1-I_t^\alpha\sum_{i=1}^{n}\lambda_i\frac{\partial^2}{\partial x_i^2}\right)\mathcal{V}=1.$$

In fact,

$$\mathcal{V}\left(1 - I_t^{\alpha}\sum_{i=1}^{n}\lambda_i\frac{\partial^2}{\partial x_i^2}\right) = \mathcal{V} - \sum_{k=0}^{\infty}\left(I_t^{\alpha}\sum_{i=1}^{n}\lambda_i\frac{\partial^2}{\partial x_i^2}\right)^{k+1}$$
$$= 1 + \sum_{k=0}^{\infty}\left(I_t^{\alpha}\sum_{i=1}^{n}\lambda_i\frac{\partial^2}{\partial x_i^2}\right)^{k+1} - \sum_{k=0}^{\infty}\left(I_t^{\alpha}\sum_{i=1}^{n}\lambda_i\frac{\partial^2}{\partial x_i^2}\right)^{k+1} = 1.$$

Similarly,

$$\left(1 - I_t^{\alpha}\sum_{i=1}^{n}\lambda_i\frac{\partial^2}{\partial x_i^2}\right)\mathcal{V} = 1,$$

and the uniqueness follows easily.

From equation (4.3), we derive that

$$\begin{aligned} u(t,x) = &\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_n=k}\binom{k}{k_1,\cdots,k_n}\lambda_1^{k_1}\cdots\lambda_m^{k_m}I_t^{\alpha k+\alpha} \\ &\cdot\frac{\partial^{2k_1}}{\partial x_1^{2k_1}}\cdots\frac{\partial^{2k_n}}{\partial x_n^{2k_n}}g(t,\sigma) \\ &+\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_n=k}\binom{k}{k_1,\cdots,k_n}\lambda_1^{k_1}\cdots\lambda_m^{k_m}\frac{t^{\alpha k}}{\Gamma(\alpha k+1)}\frac{\partial^{2k_1}}{\partial x_1^{2k_1}}\cdots\frac{\partial^{2k_n}}{\partial x_n^{2k_n}}\phi_1(x) \\ &+\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_n=k}\binom{k}{k_1,\cdots,k_n}\lambda_1^{k_1}\cdots\lambda_m^{k_m}\frac{t^{\alpha k+1}}{\Gamma(\alpha k+2)}\frac{\partial^{2k_1}}{\partial x_1^{2k_1}}\cdots\frac{\partial^{2k_n}}{\partial x_n^{2k_n}}\phi_2(x). \end{aligned}$$

This completes the proof. □

Using

$$\left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right)^k = \sum_{k_1+\cdots+k_n=k}\binom{k}{k_1,\cdots,k_n}\frac{\partial^{2k_1}}{\partial x_1^{2k_1}}\cdots\frac{\partial^{2k_n}}{\partial x_n^{2k_n}},$$

we claim the fractional wave equation in $\mathbb{R}^n$

$$\begin{cases} \frac{{}_c\partial^{\alpha}}{\partial t^{\alpha}}u(t,x) = \triangle u(t,x) + g(t,x), \quad 1 < \alpha \leq 2, \\ u(0,x) = \phi_1(x), \quad u_t'(0,x) = \phi_2(x), \quad (t,x) \in \mathbb{R}^+ \times R^n \end{cases} \tag{4.4}$$

has the solution

$$u(t,x)=\sum_{k=0}^{\infty} I_t^{\alpha k+\alpha}\triangle^k g(t,x)+\sum_{k=0}^{\infty}\frac{t^{\alpha k}}{\Gamma(\alpha k+1)}\triangle^k\phi_1(x)+\sum_{k=0}^{\infty}\frac{t^{\alpha k+1}}{\Gamma(\alpha k+2)}\triangle^k\phi_2(x). \tag{4.5}$$

In particular, the wave equation in $\mathbb{R}^n$

$$\begin{cases}\frac{\partial^2}{\partial t^2}u(t,x)=\triangle u(t,x)+g(t,x), \quad 1<\alpha\leq 2,\\ u(0,x)=\phi_1(x), \quad u_t'(0,x)=\phi_2(x), \quad (t,x)\in\mathbb{R}^+\times R^n\end{cases} \tag{4.6}$$

has the solution

$$u(t,x)=\sum_{k=0}^{\infty} I_t^{2k+2}\triangle^k g(t,x)+\sum_{k=0}^{\infty}\frac{t^{2k}}{(2k)!}\triangle^k\phi_1(x)+\sum_{k=0}^{\infty}\frac{t^{2k+1}}{(2k+1)!}\triangle^k\phi_2(x). \tag{4.7}$$

Example 4 The following fractional wave equation in $\mathbb{R}^{100}$:

$$\begin{cases}\frac{{}_c\partial^{1.5}}{\partial t^{1.5}}u(t,x)=\triangle u(t,x)+tx_1x_{15}, \quad 1<\alpha\leq 2,\\ u(0,x)=x_{100}, \quad u_t'(0,x)=x_2, \quad (t,x)\in\mathbb{R}\times\mathbb{R}^{100}\end{cases} \tag{4.8}$$

has the solution

$$u(t,x)=\frac{t^{2.5}}{\Gamma(3.5)}x_1x_{15}+x_{100}+tx_2.$$

Proof From solution (4.5), we have

$$\begin{aligned}u(t,x)&=\sum_{k=0}^{\infty} I_t^{\alpha k+\alpha}\triangle^k g(t,x)+\sum_{k=0}^{\infty}\frac{t^{\alpha k}}{\Gamma(\alpha k+1)}\triangle^k\phi_1(x)\\ &\quad+\sum_{k=0}^{\infty}\frac{t^{\alpha k+1}}{\Gamma(\alpha k+2)}\triangle^k\phi_2(x)\\ &=\sum_{k=0}^{\infty} I_t^{1.5k+1.5}\triangle^k tx_1x_{15}+\sum_{k=0}^{\infty}\frac{t^{1.5k}}{\Gamma(1.5k+1)}\triangle^k x_{100}+\sum_{k=0}^{\infty}\frac{t^{1.5k+1}}{\Gamma(1.5k+2)}\triangle^k x_2\\ &=\frac{t^{2.5}}{\Gamma(3.5)}x_1x_{15}+x_{100}+tx_2.\end{aligned}$$

□

Remark 2 This solution is much simpler than using the classical formula which involves complicated integrals. Moreover, the solution given in (4.7) is equivalent to the classical results, such as d'Alembert and Kirchoff's formulas, but in a neater and simpler format.

5 Conclusion

We have studied the uniqueness and existence of solutions to the nonlinear fractional partial integro-differential equation (1.1) with boundary condition using its implicit integral equation, the inverse operator, the multivariate Mittag-Leffler function, Banach's contractive principle as well as Leray-Schauder's fixed point theorem, with several illustrative examples. Finally, we presented the analytic solution to the generalized fractional wave equation in $\mathbb{R}^n$ for the first time based on a new space S and the inverse operator method, which provides a new method for studying some PDEs with the Laplacian and gradient operators.

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Declarations

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