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Applications of Inverse Operators to a Fractional Partial Integro-Differential Equation and Several Well-Known Differential Equations

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Abstract: This paper mainly consists of two parts: (i) We study the uniqueness, existence, and stability of a new fractional nonlinear partial integro-differential equation in $\mathbb{R}^n$ with three-point conditions and variable coefficients in a Banach space using inverse operators containing multi-variable functions, a generalized Mittag-Leffler function, as well as a few popular fixed-point theorems. These studies have good applications in general since uniqueness, existence and stability are key and important topics in many fields. Several examples are presented to demonstrate applications of results obtained by computing approximate values of the generalized Mittag-Leffler functions. (ii) We use the inverse operator method and newly established spaces to find analytic solutions to a number of notable partial differential equations, such as a multi-term time-fractional convection problem and a generalized time-fractional diffusion-wave equation in $\mathbb{R}^n$ with initial conditions only, which have never been previously considered according to the best of our knowledge. In particular, we deduce the uniform solution to the non-homogeneous wave equation in n dimensions for all $n \geq 1$, which coincides with classical results such as d'Alembert and Kirchoff's formulas but is much easier in the computation of finding solutions without any complicated integrals on balls or spheres.

Keywords: fractional nonlinear partial integro-differential equation; uniqueness and existence; stability; fixed-point theory; generalized Mittag-Leffler function; inverse operator method; time-fractional convection problem; time-fractional diffusion-wave equation

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1. Introduction and Preliminaries

Fractional differential equations (FDEs) play crucial roles in simulating real-world systems due to their unique ability to model complex phenomena that involve memory and hereditary properties, which are often not adequately captured by classical integer-order differential equations. In addition, fractional derivatives or integrals are non-local operators, meaning that they consider the influence of the function over an interval rather than at a point. This property is beneficial for modeling systems with non-local interactions, such as those found in continuum mechanics, electromagnetism, and population dynamics.

The gamma function, denoted by $\Gamma(z)$, is a generalization of the factorial function to complex numbers. It is defined as

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad \mathrm{Re}(z) > 0.$$

Let $(t, \sigma) \in [0,1] \times [0,1]^n$. We define the fractional partial integral operator $I_1^{\rho_1} \dots I_n^{\rho_n}$ of a function, $M(t,\sigma)$, for $\rho_1, \dots, \rho_n \geq 0$ [1,2] as

$$I_1^{\rho_1} \dots I_n^{\rho_n} M(t,\sigma) = \frac{1}{\Gamma(\rho_1) \dots \Gamma(\rho_n)} \cdot \int_0^{\sigma_1} \dots \int_0^{\sigma_n} (\sigma_1 - \tau_1)^{\rho_1 - 1} \dots (\sigma_n - \tau_n)^{\rho_n - 1} M(t, \tau_1, \dots, \tau_n) d\tau_n \dots d\tau_1,$$

where M is a continuous function from $[0,1] \times [0,1]^n$ to $\mathbb{R}$ and $\sigma = (\sigma_1, \dots, \sigma_n)$.

In particular, from [3], we have

$$I_1^0 \dots I_n^0 M(t,\sigma) = M(t,\sigma).$$

The partial Liouville–Caputo fractional derivative ${}_c\partial^\rho / \partial t^\rho$ of the order $m - 1 < \rho \leq m$ ($m \in \mathbb{N}$) with respect to t is defined in [1] as

$$\left(\frac{{}_c\partial^\rho}{\partial t^\rho} M \right)(t,\sigma) = \frac{1}{\Gamma(m-\rho)} \int_0^t (t-\tau)^{m-\rho-1} M_t^{(m)}(\tau,\sigma) d\tau.$$

In particular, for $2 < \rho \leq 3$,

$$\left(\frac{{}_c\partial^\rho}{\partial t^\rho} M \right)(t,\sigma) = \frac{1}{\Gamma(3-\rho)} \int_0^t (t-\tau)^{2-\rho} M_t^{(3)}(\tau,\sigma) d\tau.$$

Recently, Sadek et al. [4] investigated the foundational iterative processes of fractional calculus, focusing on Θ-conformable fractional derivatives, and defined several novel fractional operators as well as associated function spaces, which are useful in dealing with differential equations involving the Θ-conformable derivatives.

$C([0,1] \times [0,1]^n)$ is the Banach space of all continuous functions from $[0,1] \times [0,1]^n$ into $\mathbb{R}$ with the norm

$$\|M\| = \max_{(t,\sigma) \in [0,1] \times [0,1]^n} |M(t,\sigma)| < +\infty.$$

Let $a_2 \in C[0,1]$, a_3, $a_4 \in C([0,1]^n)$ and $a_1 \in C([0,1] \times [0,1]^n)$. In this paper, we will first study the uniqueness, existence, and stability for the following new equation with three-point conditions and variable coefficients for all $\rho_i \geq 0$ ($i = 1, 2, \dots, n \in \mathbb{N}$) and $2 < \rho \leq 3$:

$$\begin{cases} \frac{{}_c\partial^\rho}{\partial t^\rho} M(t,\sigma) + a_1(t,\sigma)\, I_1^{\rho_1} \dots I_n^{\rho_n} M(t,\sigma) \\ \qquad = N(t, \sigma, M(t,\sigma)), \quad (t,\sigma) \in [0,1] \times [0,1]^n, \\ M(0,\sigma) = \int_0^1 a_2(t) M(t,\sigma) dt, \quad M(1/2, \sigma) = a_3(\sigma), \quad M(1,\sigma) = a_4(\sigma), \end{cases} \tag{1}$$

where $N : [0,1] \times [0,1]^n \times \mathbb{R} \to \mathbb{R}$ is a given function with certain conditions.

It follows that for $M \in C([0,1] \times [0,1]^n)$,

$$\left\| I_t^\rho a_1(t,\sigma)\, I_1^{\rho_1} \dots I_n^{\rho_n} M \right\| \leq \|a_1\| \|M\| \left\| I_t^\rho \right\| \left\| I_1^{\rho_1} \right\| \dots \left\| I_n^{\rho_n} \right\| \leq \|a_1\| \|M\| \frac{1}{\Gamma(\rho+1)} \frac{1}{\Gamma(\rho_1+1)} \cdots \frac{1}{\Gamma(\rho_n+1)},$$

which implies that

$$\left\| I_t^{\rho} a_1 \ I_1^{\rho_1} \ldots I_n^{\rho_n} \right\| \leq \|a_1\| \frac{1}{\Gamma(\rho+1)} \frac{1}{\Gamma(\rho_1+1)} \cdots \frac{1}{\Gamma(\rho_n+1)}.$$

Hence,

$$\left\| \left(I_t^{\rho} a_1 \ I_1^{\rho_1} \ldots I_n^{\rho_n} \right)^k \right\| \leq \|a_1\|^k \frac{1}{\Gamma(k\rho+1)} \frac{1}{\Gamma(k\rho_1+1)} \cdots \frac{1}{\Gamma(k\rho_n+1)},$$

for all integers $k \geq 0$.

We are going to derive an equivalent implicit integral equation of Equation (1) by an inverse operator over $C([0,1] \times [0,1]^n)$ and then present sufficient conditions for the uniqueness, existence, and stability using several fixed-point theorems and a newly established generalized Mittag-Leffler function given below.

The generalized Mittag-Leffler function is defined by

$$E_{(\rho,\rho_1,\ldots,\rho_n),\ (\beta,\beta_1,\ldots,\beta_n)}(\theta) = \sum_{r=0}^{\infty} \frac{\theta^r}{\Gamma(\rho r+\beta)\Gamma(\rho_1 r+\beta_1)\ldots\Gamma(\rho_n r+\beta_n)},$$

where $\theta \in \mathbb{C},\ \rho > 0,\ \beta > 0,\ \rho_i \geq 0,\ \beta_i > 0$ for $i = 1,2,\ldots,n$. In particular,

$$E_{(\rho,0,\ldots,0),\ (\beta,1,\ldots,1)}(\theta) = E_{\alpha,\beta}(\theta) = \sum_{r=0}^{\infty} \frac{\theta^r}{\Gamma(\rho r+\beta)},$$

which is the well-known two-parameter Mittag-Leffler function [5]. We should point out that $E_{(\rho,\rho_1,\ldots,\rho_n),\ (\beta,\beta_1,\ldots,\beta_n)}(\theta)$ clearly converges since there is a positive integer, r, such that

$$\rho_i r + \beta_i \geq 2,$$

for all $i = 1,2,\ldots,n$ if $\rho_i > 0$.

To consider Equation (1), we first demonstrate use of the inverse operator method to convert the following equation with an integral boundary condition (nonlocal) to an equivalent implicit integral equation for $0 < \rho \leq 1$ and $\rho_1 \geq 0$:

$$\begin{cases} \frac{{}_c\partial^{\rho}}{\partial t^{\rho}} M(t,\sigma) + a_1(t,\sigma)\ I_{\sigma}^{\rho_1} M(t,\sigma) = N(t,\sigma,M(t,\sigma)), \quad (t,\sigma) \in [0,1] \times [0,1], \\ M(1,\sigma) = \int_0^1 a_2(t) M(t,\sigma) dt, \quad a_2 \in C[0,1], \end{cases} \tag{2}$$

where a_1 and N are given continuous functions with

$$\|N\| = \sup_{(t,\sigma,y)\in[0,1]\times[0,1]\times\mathbb{R}} |N(t,\sigma,y)| < +\infty.$$

The motivation of employing the inverse operator method and the Mittag-Leffler functions in the current work is that, as far as we know, there are no existing integral transforms or other approaches that can change Equation (2) to an equivalent integral equation. To study the uniqueness and existence by fixed-point theory, we need an equivalent integral equation to define a nonlinear mapping.

Theorem 1. *Let $0 < \rho \leq 1$, $\rho_1 \geq 0$, and a_1, N, a_2 be given continuous functions. Furthermore, we assume that N satisfies the following Lipschitz condition with respect to the third variable,*

$$|N(t,\sigma,M_1) - g(t,\sigma,M_2)| \leq \mathcal{L}|M_1 - M_2|, \quad M_1, M_2 \in \mathbb{R},$$

for a non-negative constant, $\mathcal{L}$, and

$$\mathcal{Q} = \mathcal{L}E_{(\rho,\rho_1),(\rho+1,1)}(\|a_1\|) + \left(\|a_2\| + \frac{\|a_1\|}{\Gamma(\rho+1)\Gamma(\rho_1+1)} + \frac{\mathcal{L}}{\Gamma(\rho+1)}\right)E_{(\rho,\rho_1),(1,1)}(\|a_1\|) < 1.$$

Then, there is a unique solution in $C([0,1]\times[0,1])$ *to Equation (2).*

Proof. Applying I_t^ρ to Equation (2), we obtain

$$M(t,\sigma) - M(0,\sigma) + I_t^\rho a_1(t,\sigma)\ I_\sigma^{\rho_1} M(t,\sigma) = I_t^\rho N(t,\sigma,M(t,\sigma)),$$

by noting that $0 < \rho \leq 1$. Setting $t = 1$ yields

$$M(0,\sigma) = \int_0^1 a_2(t)M(t,\sigma)dt + I_{t=1}^\rho a_1(t,\sigma)\ I_\sigma^{\rho_1} M(t,\sigma) - I_{t=1}^\rho N(t,\sigma,M(t,\sigma)).$$

Hence,

$$\begin{aligned}\left(1 + I_t^\rho a_1(t,\sigma)\ I_\sigma^{\rho_1}\right)M(t,\sigma) &= I_t^\rho N(t,\sigma,M(t,\sigma))\\ &+ \int_0^1 a_2(t)M(t,\sigma)dt + I_{t=1}^\rho a_1(t,\sigma)\ I_\sigma^{\rho_1} M(t,\sigma) - I_{t=1}^\rho N(t,\sigma,M(t,\sigma)).\end{aligned}$$

We shall show that the operator $1 + I_t^\rho a_1(t,x)\ I_\sigma^{\rho_1}$ has a unique inverse operator,

$$v_{a_1} = \sum_{r=0}^\infty (-1)^r \left(I_t^\rho a_1(t,\sigma)\ I_\sigma^{\rho_1}\right)^r$$

in the space $C([0,1]\times[0,1])$. Indeed, for any $M \in C([0,1]\times[0,1])$, we have

$$\begin{aligned}\|v_{a_1}M\| &= \left\|\sum_{r=0}^\infty (-1)^r \left(I_t^\rho a_1(t,\sigma)\ I_\sigma^{\gamma_1}\right)^r M\right\| \leq \sum_{r=0}^\infty \left\|\left(I_t^\rho a_1(t,\sigma)\ I_\sigma^{\rho_1}\right)^r\right\| \|M\|\\ &\leq \|M\| \sum_{r=0}^\infty \|a_1\|^r \frac{1}{\Gamma(r\rho+1)}\frac{1}{\Gamma(r\rho_1+1)}\\ &= \|M\| E_{(\rho,\rho_1),(1,1)}(\|a_1\|) < +\infty,\end{aligned}$$

which claims that the operator v_{a_1} is well defined and continuous over the space $C([0,1]\times[0,1])$, and the series is uniformly convergent. Moreover,

$$v_{a_1}\left(1 + I_t^\rho a_1(t,\sigma)\ I_\sigma^{\rho_1}\right) = \left(1 + I_t^\rho a_1(t,\sigma)\ I_\sigma^{\rho_1}\right)v_{a_1} = 1 \ \ \text{(identity operator)}.$$

In fact,

$$\begin{aligned}v_{a_1}\left(1 + I_t^\rho a_1(t,\sigma)\ I_\sigma^{\rho_1}\right) &= \sum_{r=0}^\infty (-1)^r \left(I_t^\rho a_1(t,\sigma)\ I_\sigma^{\rho_1}\right)^r + \sum_{r=0}^\infty (-1)^r \left(I_t^\rho a_1(t,\sigma)\ I_\sigma^{\rho_1}\right)^{r+1}\\ &= 1 + \sum_{r=1}^\infty (-1)^r \left(I_t^\rho a_1(t,\sigma)\ I_\sigma^{\rho_1}\right)^r + \sum_{r=0}^\infty (-1)^r \left(I_t^\rho a_1(t,\sigma)\ I_\sigma^{\rho_1}\right)^{r+1} = 1.\end{aligned}$$

Similarly,

$$\left(1 + I_t^\rho a_1(t,\sigma)\ I_\sigma^{\rho_1}\right)v_{a_1} = 1.$$

We assume that v'_{a_1} is another operator such that

$$v'_{a_1}\left(1 + I_t^\rho a_1(t,\sigma)\ I_\sigma^{\rho_1}\right) = \left(1 + I_t^\rho a_1(t,\sigma)\ I_\sigma^{\rho_1}\right)v'_{a_1} = 1.$$

Then, $v_{a_1} = v'_{a_1}$ by applying v_{a_1} to the above. Therefore,

$$M(t,\sigma) = \sum_{r=0}^{\infty}(-1)^r \left(I_t^{\rho} a_1(t,\sigma)\ I_{\sigma}^{\rho_1}\right)^r I_t^{\rho} N(t,\sigma,M(t,\sigma)) + \sum_{r=0}^{\infty}(-1)^r \left(I_t^{\rho} a_1(t,\sigma)\ I_{\sigma}^{\rho_1}\right)^r$$
$$\cdot \left[\int_0^1 a_2(t)M(t,\sigma)dt + I_{t=1}^{\rho} a_1(t,\sigma)\ I_{\sigma}^{\rho_1} M(t,\sigma) - I_{t=1}^{\rho} N(t,\sigma,M(t,\sigma))\right], \quad (3)$$

and

$$\|M\| \le \|N\| E_{(\rho,\rho_1),(\rho+1,1)}(\|a_1\|)$$
$$+ \left(\|a_2\|\|M\| + \frac{\|a_1\|\|M\|}{\Gamma(\rho+1)\Gamma(\rho_1+1)} + \frac{\|N\|}{\Gamma(\rho+1)}\right) E_{(\rho,\rho_1),(1,1)}(\|a_1\|),$$

where

$$\|a_1\| = \sup_{(t,\sigma)\in[0,1]\times[0,1]} |a_1(t,\sigma)| < +\infty.$$

If

$$W = 1 - \left(\|a_2\| + \frac{\|a_1\|}{\Gamma(\rho+1)\Gamma(\rho_1+1)}\right) E_{(\rho,\rho_1),(1,1)}(\|a_1\|) > 0,$$

then

$$\|M\| \le \frac{\|N\|}{W} E_{(\rho,\rho_1),(\rho+1,1)}(\|a_1\|) + \frac{\|N\|}{W\Gamma(\rho+1)} E_{(\rho,\rho_1),(1,1)}(\|a_1\|)$$

is uniformly bounded.

We further assume that N satisfies the following Lipschitz condition with respect to the third variable,

$$|N(t,\sigma,M_1) - g(t,\sigma,M_2)| \le \mathcal{L}|M_1 - M_2|, \quad M_1, M_2 \in \mathbb{R},$$

for a nonnegative constant, $\mathcal{L}$, and

$$\mathcal{Q} = \mathcal{L} E_{(\rho,\rho_1),(\rho+1,1)}(\|a_1\|) + \left(\|a_2\| + \frac{\|a_1\|}{\Gamma(\rho+1)\Gamma(\rho_1+1)} + \frac{\mathcal{L}}{\Gamma(\rho+1)}\right) E_{(\rho,\rho_1),(1,1)}(\|a_1\|) < 1.$$

Then, there is a unique solution in $C([0,1]\times[0,1])$ to Equation (2). To prove this, we define a nonlinear mapping, $\mathcal{T}$, over $C([0,1]\times[0,1])$ by Equation (3)

$$(\mathcal{T}M)(t,\sigma) = \sum_{r=0}^{\infty}(-1)^r \left(I_t^{\rho} a_1(t,\sigma)\ I_{\sigma}^{\rho_1}\right)^r I_t^{\rho} N(t,\sigma,M(t,\sigma)) + \sum_{r=0}^{\infty}(-1)^r \left(I_t^{\rho} a_1(t,\sigma)\ I_{\sigma}^{\rho_1}\right)^r$$
$$\cdot \left[\int_0^1 a_2(t)M(t,\sigma)dt + I_{t=1}^{\rho} a_1(t,\sigma)\ I_{\sigma}^{\rho_1} M(t,\sigma) - I_{t=1}^{\rho} N(t,\sigma,M(t,\sigma))\right]$$
$$= \sum_{r=0}^{\infty}(-1)^r \left(I_t^{\rho} a_1(t,\sigma)\ I_{\sigma}^{\rho_1}\right)^r I_t^{\rho} N(t,\sigma,M(t,\sigma)) + \sum_{r=0}^{\infty}(-1)^r \left(I_t^{\rho} a_1(t,\sigma)\ I_{\sigma}^{\rho_1}\right)^r \int_0^1 a_2(t)M(t,\sigma)dt$$
$$+ \sum_{r=0}^{\infty}(-1)^r \left(I_t^{\rho} a_1(t,\sigma)\ I_{\sigma}^{\rho_1}\right)^r I_{t=1}^{\rho} a_1(t,\sigma)\ I_{\sigma}^{\rho_1} M(t,\sigma)$$
$$- \sum_{r=0}^{\infty}(-1)^r \left(I_t^{\rho} a_1(t,\sigma)\ I_{\sigma}^{\rho_1}\right)^r I_{t=1}^{\rho} N(t,\sigma,M(t,\sigma)).$$

It follows from the above that $\|\mathcal{T}M\| < +\infty$. Thus, we only need to show that $\mathcal{T}$ is contractive. Clearly, for $M_1, M_2 \in C([0,1] \times [0,1])$,

$$\begin{aligned}
\mathcal{T}M_1 - \mathcal{T}M_2 &= \sum_{r=0}^{\infty} (-1)^r \left(I_t^{\rho} a_1(t,\sigma)\ I_{\sigma}^{\rho_1} \right)^r I_t^{\rho} (N(t,\sigma,M_1(t,\sigma)) - N(t,\sigma,M_2(t,\sigma)) \\
&+ \sum_{r=0}^{\infty} (-1)^r \left(I_t^{\rho} a_1(t,\sigma)\ I_{\sigma}^{\rho_1} \right)^r \left[\int_0^1 a_2(t)(M_1(t,\sigma) - M_2(t,\sigma))dt \right. \\
&\left. + I_{t=1}^{\rho} a_1(t,\sigma)\ I_{\sigma}^{\rho_1} (M_1(t,\sigma) - M_2(t,\sigma)) - I_{t=1}^{\rho} (N(t,\sigma,M_1(t,\sigma)) - N(t,\sigma,M_2(t,\sigma))) \right].
\end{aligned}$$

So,

$$\begin{aligned}
&\|\mathcal{T}M_1 - \mathcal{T}M_2\| \\
&\leq \left(\mathcal{L} E_{(\rho,\rho_1),(\rho+1,1)}(\|a_1\|) + \left(\|a_2\| + \frac{\|a_1\|}{\Gamma(\rho+1)\Gamma(\rho_1+1)} + \frac{\mathcal{L}}{\Gamma(\rho+1)} \right) E_{(\rho,\rho_1),(1,1)}(\|a_1\|) \right) \\
&\cdot \|M_1 - M_2\| = \mathcal{Q} \|M_1 - M_2\|.
\end{aligned}$$

Since $\mathcal{Q} < 1$, there exists a unique solution to Equation (2) in $C([0,1] \times [0,1])$ by Banach's contractive principle. This completes the proof. □

Example 1. *The equation with the integral boundary condition*

$$\begin{cases}
\dfrac{{}_c\partial^{0.6}}{\partial t^{0.6}} M(t,\sigma) + \dfrac{1}{13(1+t+\sigma^2)}\ I_{\sigma}^{2.1} M(t,\sigma) \\
\qquad\qquad\qquad = \dfrac{1}{21} \cos(t\sigma M(t,\sigma)), \quad (t,\sigma) \in [0,1] \times [0,1], \\
M(1,\sigma) = \displaystyle\int_0^1 \sin(t^2/11) M(t,\sigma) dt,
\end{cases} \tag{4}$$

has a unique solution in $C([0,1] \times [0,1])$.

Proof. Evidently,

$$N(t,\sigma,M) = \frac{1}{21} \cos(t\sigma M)$$

is a continuous and bounded function with

$$|N(t,\sigma,M_1) - N(t,\sigma,M_2)| \leq \frac{1}{21} |\cos(t\sigma M_1) - \cos(t\sigma M_2)| \leq \frac{1}{21} |M_1 - M_2|,$$

which infers that $\mathcal{L} = 1/21$. In addition,

$$\rho = 0.6, \quad \rho_1 = 2.1, \quad \|a_1\| = 1/13, \quad \|a_2\| \leq 1/11.$$

We need to evaluate the value of

$$\begin{aligned}
\mathcal{Q} &= \mathcal{L} E_{(\rho,\rho_1),(\rho+1,1)}(\|a_1\|) + \left(\|a_2\| + \frac{\|a_1\|}{\Gamma(\rho+1)\Gamma(\rho_1+1)} + \frac{\mathcal{L}}{\Gamma(\rho+1)} \right) E_{(\rho,\rho_1),(1,1)}(\|a_1\|) \\
&\leq \frac{1}{21} E_{(0.6,2.1),(1.6,1)}(1/13) + \left(\frac{1}{11} + \frac{1}{13\Gamma(1.6)\Gamma(3.1)} + \frac{1}{21\Gamma(1.6)} \right) E_{(0.6,2.1),(1,1)}(1/13) \\
&\approx \frac{1}{21} * 1.15105 + 0.183378 * 1.03934 \approx 0.24540399528 < 1,
\end{aligned}$$

by noting that

$$E_{(0.6,2.1),(1.6,1)}(1/13) = \sum_{r=0}^{\infty} \frac{(1/13)^r}{\Gamma(0.6r+1.6)\Gamma(2.1r+1)} \approx 1.15105, \quad \text{and,}$$
$$E_{(0.6,2.1),(1,1)}(1/13) = \sum_{r=0}^{\infty} \frac{(1/13)^r}{\Gamma(0.6r+1)\Gamma(2.1r+1)} \approx 1.03934,$$

using online calculators on 11 December 2024 from the site https://www.wolframalpha.com. So, Equation (4) has a unique solution in $C([0,1] \times [0,1])$ from Theorem 1. □

In addition, we provide applications of the inverse operator method to finding analytic solutions to some well-known partial differential equations, such as the following multi-term time-fractional convection problem in $\mathbb{R}^n$ for $0 < \rho \leq 1$, for the first time:

$$\begin{cases} \frac{{}_c\partial^\rho}{\partial t^\rho} M(t,\sigma) + \sum_{i=1}^{m} \beta_i \frac{{}_c\partial^{\rho_j}}{\partial t^{\rho_j}} M(t,\sigma) + \sum_{j=1}^{n} \lambda_j(\sigma_j) \frac{\partial}{\partial \sigma_j} M(t,\sigma) \\ = f_1(t,\sigma), \quad (t,\sigma) \in [0,1] \times [0,1]^n, \\ M(0,\sigma) = f_2(\sigma), \end{cases} \tag{5}$$

We provide the same for the generalized time-fractional diffusion-wave equation for $1 < \rho \leq 2$,

$$\begin{cases} \frac{{}_c\partial^\rho}{\partial t^\rho} M(t,\sigma) + \sum_{j=1}^{m} \lambda_j \frac{{}_c\partial^{\rho_j}}{\partial t^{\rho_j}} M(t,\sigma) = \triangle^l M(t,\sigma) + g(t,\sigma), \\ M(0,\sigma) = \theta(\sigma), \quad M_t'(0,\sigma) = \beta(\sigma), \quad (t,\sigma) \in [0,1] \times [0,1]^n, \quad m, \ l \in \mathbb{N}, \end{cases} \tag{6}$$

based on the multivariate Mittag-Leffler function and several newly constructed spaces. As far as we know, there are no analytic solutions to the above two equations to date, although there are some investigations on the convection–diffusion equations of a fractional order, particularly in numerical studies [6–9].

Especially for all $\lambda_j = 0$ and $n = l = 1$, Equation (6) turns out be

$$\begin{cases} \frac{{}_c\partial^\rho}{\partial t^\rho} M(t,\sigma) = \frac{\partial^2}{\partial \sigma^2} M(t,\sigma) + g(t,\sigma), \\ M(0,\sigma) = \theta(\sigma), \quad M_t'(0,\sigma) = \beta(\sigma), \end{cases}$$

which is a non-homogeneous fractional wave equation in one dimension.

Fractional differential equations have played important roles in constructing mathematical models, and they have also been extensively studied and used in various research fields, particularly in materials, economics, mechanics, dynamic systems, environmental science, signal and image processing, control theory, physics, and chemistry [10–15]. In fact, fractional-order models are more suitable, in comparison to integer-order settings, in modeling many biological phenomena due to their non-local nature and the presence of memory functions [16]. There are many interesting works on fractional partial differential equations, especially on uniqueness, existence, and stability analysis since they are important studies in many pure and applied areas. In 2024, Li [17] investigated the uniqueness and stability for the following equation for $\rho_{ij} \geq 0$ $(i = 1,2,\ldots,n, j = 1,2,\ldots,l \in \mathbb{N})$

using Babenko's approach (the inverse operator method) and the generalized multivariate Mittag-Leffler function:

$$\begin{cases} {}_{c}\dfrac{\partial^{\rho}}{\partial t^{\rho}}M(t,\sigma) + \displaystyle\sum_{j=1}^{l} a_j(\sigma) I_1^{\rho_{1j}} \ldots I_n^{\rho_{nj}} M(t,\sigma) = \phi_1(t,\sigma,M(t,\sigma)), \;\; 2 < \rho \leq 3, \\ M(0,\sigma) = \phi_2(\sigma), \;\; M(1,\sigma) = \phi_3(\sigma), \;\; M_t'(1,\sigma) = \phi_4(\sigma), \end{cases}$$

where $(t,\sigma) \in [0,1] \times [0,1]^n$, $a_j, \phi_k \in C([0,1]^n)$ for $k = 2,3,4$, and $\phi_1 : [0,1] \times [0,1]^n \times \mathbb{R} \to \mathbb{R}$, with some conditions.

Kumar et al. [18] presented applications of fractional partial differential equations in biology. Lu et al. [19] explored deep learning methods for solving fractional partial differential equations.

Very recently, Li [20] studied the uniqueness and existence of solutions for the following nonlinear partial integro-differential equation through a well-defined inverse operator and a few fixed-point theorems.

$$\begin{cases} {}_{c}\dfrac{\partial^{\alpha}}{\partial t^{\alpha}}u(t,x) + \displaystyle\sum_{i=1}^{m} a_i\, I_x^{\beta_i} u(t,x) = g(t,x,u(t,x)), \;\; 1 < \alpha \leq 2, \;\; \beta_i \geq 0, \\ u(0,x) = -\phi_1(x), \;\; u(T,x) = \phi_2(x), \quad \phi_1, \phi_2 \in C[0,b], \;\; b > 0, \end{cases}$$

where $(t,x) \in [0,T] \times [0,b]$ with $T > 0$, all a_i are arbitrary constants for $i = 1,2,\ldots,m$, $g : [0,T] \times [0,b] \times \mathbb{R} \to \mathbb{R}$ satisfies certain conditions, and the analytic solution for the following generalized fractional wave equation in $\mathbb{R}^n$ is derived:

$$\begin{cases} {}_{c}\dfrac{\partial^{\alpha}}{\partial t^{\alpha}}u(t,x) = \triangle_{\lambda_1,\ldots,\lambda_n} u(t,x) + g(t,x), \;\; 1 < \alpha \leq 2, \\ u(0,x) = \phi_1(x), \;\; u_t'(0,x) = \phi_2(x), \;\; (t,x) \in \mathbb{R} \times \mathbb{R}^n, \end{cases}$$

where

$$\triangle_{\lambda_1,\ldots,\lambda_n} = \lambda_1 \frac{\partial^2}{\partial x_1^2} + \ldots + \lambda_n \frac{\partial^2}{\partial x_n^2}, \;\; \text{all } \lambda_i \text{ are arbitrary contants.}$$

The key contributions of this paper are listed as follows.

- We study the uniqueness, existence, and stability for the new Equation (1) using several notable fixed-point theorems, an equivalent implicit integral equation from inverse operators, and the equicontinuity concept. Clearly, there are more studies focusing on ordinary fractional differential equations and far fewer on FPDEs.
- We derive a new analytic solution to the generalized multi-term time-fractional convection problem (5) by the multivariate Mittag-Leffler function, an inverse operator, and a subspace space, S, with several illustrative examples showing applications of our main results.
- We obtain a unique series solution in terms of the Laplacian operators, for the first time, to the generalized time-fractional diffusion-wave Equation (6), and further, we establish the uniform solution to the non-homogeneous wave equation in n dimensions for all $n \geq 1$, which is consistent with all classical consequences but without any complicated integrals in computation.

In the following, we shall derive sufficient conditions for the uniqueness, existence, and stability of Equation (1) by an inverse operator, the generalized two-parameter Mittag-Leffler function, Banach's contractive principle, and Leray–Schauder's fixed-point theorem, with illustrative examples demonstrating applications of our main results in part (i) containing Sections 2 and 3. To present applications of the inverse operator method, we will find well-defined series solutions to a partial integro-differential equation and the two important partial differential Equations (5) and (6) by introducing some new spaces in part (ii) (Section 4), which are the key contributions of this paper.

2. Uniqueness and Stability

Stability is an essential concept in differential equations since it guarantees that a small perturbation from a model caused by errors will have a correspondingly slight effect on the solution, so that the equation describing the model will predict the future outcomes accurately. We begin defining a stability of Equation (1) as follows.

Definition 1. *Problem (1) is stable if there exists a constant $\mathcal{K} > 0$, such that for all $\epsilon > 0$ and for each fixed solution, $M(t,\sigma) \in C([0,1] \times [0,1]^n)$, of*

$$\begin{cases} \left\| \frac{{}^c\partial^\rho}{\partial t^\rho} M(t,\sigma) + a_1(t,\sigma)\ I_1^{\rho_1} \dots I_n^{\rho_n} M(t,\sigma) - N(t,\sigma,M(t,\sigma)) \right\| < \epsilon, \\ M(0,\sigma) = \int_0^1 a_2(t) M(t,\sigma)dt, \quad M(1/2,\sigma) = a_3(\sigma), \quad M(1,\sigma) = a_4(\sigma), \end{cases}$$

then there exists a solution, $M_0(t,\sigma) \in C([0,1] \times [0,1]^n)$, of Equation (1), satisfying

$$\|M - M_0\| \leq \mathcal{K}\epsilon,$$

where $\mathcal{K}$ is a stability constant and is independent of ϵ.

Theorem 2. *Let $2 < \rho \leq 3$, $a_2 \in C[0,1]$, $a_3, a_4 \in C([0,1]^n)$, $a_1 \in C([0,1] \times [0,1]^n)$ and $N : [0,1] \times [0,1]^n \times \mathbb{R} \to \mathbb{R}$ be continuous and bounded. Then, for all $\rho_i \geq 0$ $(i = 1,2,\dots,n)$, Equation (1) is equivalent to the following implicit integral equation in the space $C([0,1] \times [0,1]^n)$:*

$$\begin{aligned} M(t,\sigma) = &\sum_{s=0}^\infty (-1)^s \left(I_t^\rho a_1(t,\sigma) I_1^{\rho_1} \dots I_n^{\rho_n} \right)^s I_t^\rho N(t,\sigma,M(t,\sigma)) \\ &+ \sum_{s=0}^\infty (-1)^s \left(I_t^\rho a_1(t,\sigma) I_1^{\rho_1} \dots I_n^{\rho_n} \right)^s (4t - 4t^2) a_3(\sigma) \\ &+ \sum_{s=0}^\infty (-1)^s \left(I_t^\rho a_1(t,\sigma) I_1^{\rho_1} \dots I_n^{\rho_n} \right)^s (2t^2 - t) a_4(\sigma) \\ &+ \sum_{s=0}^\infty (-1)^s \left(I_t^\rho a_1(t,\sigma) I_1^{\rho_1} \dots I_n^{\rho_n} \right)^s (2t^2 - 3t + 1) \int_0^1 a_2(t) M(t,\sigma)dt \\ &+ \sum_{s=0}^\infty (-1)^s \left(I_t^\rho a_1(t,\sigma) I_1^{\rho_1} \dots I_n^{\rho_n} \right)^s (4t - 4t^2) I_{t=1/2}^\rho a_1(t,\sigma) I_1^{\rho_1} \dots I_n^{\rho_n} M(t,\sigma) \\ &+ \sum_{s=0}^\infty (-1)^s \left(I_t^\rho a_1(t,\sigma) I_1^{\rho_1} \dots I_n^{\rho_n} \right)^s (2t^2 - t) I_{t=1}^\rho a_1(t,\sigma) I_1^{\rho_1} \dots I_n^{\rho_n} M(t,\sigma) \\ &+ \sum_{s=0}^\infty (-1)^s \left(I_t^\rho a_1(t,\sigma) I_1^{\rho_1} \dots I_n^{\rho_n} \right)^s (4t^2 - 4t) I_{t=1/2}^\rho N(t,\sigma,M(t,\sigma)) \\ &+ \sum_{s=0}^\infty (-1)^s \left(I_t^\rho a_1(t,\sigma) I_1^{\rho_1} \dots I_n^{\rho_n} \right)^s (t - 2t^2) I_{t=1}^\rho N(t,\sigma,M(t,\sigma)). \end{aligned}$$

In addition, if

$$\begin{aligned} G = &1 - \|a_2\| E_{(\rho,\rho_1,\dots,\rho_n),\ (1,1,\dots,1)}(\|a_1\|) - \frac{(1/2)^\rho \|a_1\|}{\Gamma(\rho+1)} E_{(\rho,\rho_1,\dots,\rho_n),\ (1,\rho_1+1,\dots,\rho_n+1)}(\|a_1\|) \\ &- \|a_1\| E_{(\rho,\rho_1,\dots,\rho_n),\ (\rho+1,\rho_1+1,\dots,\rho_n+1)}(\|a_1\|) > 0, \end{aligned}$$

then

$$\|M\| \leq \frac{\|N\|}{G} E_{(\rho,\rho_1,\ldots,\rho_n),\,(\rho+1,1,\ldots,1)}(\|a_1\|) + \frac{\|a_3\|+\|a_4\|}{G} E_{(\rho,\rho_1,\ldots,\rho_n),\,(1,1,\ldots,1)}(\|a_1\|) + \frac{(1/2)^\rho \|N\|}{G\,\Gamma(\rho+1)} E_{(\rho,\rho_1,\ldots,\rho_n),\,(1,1,\ldots,1)}(\|a_1\|) + \frac{\|N\|}{G} E_{(\rho,\rho_1,\ldots,\rho_n),\,(\rho+1,1,\ldots,1)}(\|a_1\|),$$

which is uniformly bounded.

Proof. We apply I^ρ to Equation (1) and obtain

$$M(t,\sigma) - M(0,\sigma) - M'_t(0,\sigma)t - M''_t(0,\sigma)\frac{t^2}{2} + I_t^\rho a_1(t,\sigma) I_1^{\rho_1}\ldots I_n^{\rho_n} M(t,\sigma) = I_t^\rho N(t,\sigma,M(t,\sigma)),$$

which implies that

$$M(t,\sigma) - \int_0^1 a_2(t)M(t,\sigma)dt - M'_t(0,\sigma)t - M''_t(0,\sigma)\frac{t^2}{2} + I_t^\rho a_1(t,\sigma) I_1^{\rho_1}\ldots I_n^{\rho_n} M(t,\sigma) = I_t^\rho N(t,\sigma,M(t,\sigma)).$$

Setting $t=1$, we obtain

$$a_4(\sigma) - \int_0^1 a_2(t)M(t,\sigma)dt - M'_t(0,\sigma) - \frac{1}{2}M''_t(0,\sigma) + I_{t=1}^\rho a_1(t,\sigma) I_1^{\rho_1}\ldots I_n^{\rho_n} M(t,\sigma) = I_{t=1}^\rho N(t,\sigma,M(t,\sigma)),$$

by $M(1,\sigma) = a_4(\sigma)$, and $t = 1/2$ deduces

$$a_3(\sigma) - \int_0^1 a_2(t)M(t,\sigma)dt - \frac{1}{2}M'_t(0,\sigma) - \frac{1}{8}M''_t(0,\sigma) + I_{t=1/2}^\rho a_1(t,\sigma) I_1^{\rho_1}\ldots I_n^{\rho_n} M(t,\sigma) = I_{t=1/2}^\rho N(t,\sigma,M(t,\sigma)).$$

Thus,

$$\begin{aligned} M'_t(0,\sigma) =& 4a_3(\sigma) - a_4(\sigma) - 3\int_0^1 a_2(t)M(t,\sigma)dt + 4I_{t=1/2}^\rho a_1(t,\sigma) I_1^{\rho_1}\ldots I_n^{\rho_n} M(t,\sigma) \\ & - I_{t=1}^\rho a_1(t,\sigma) I_1^{\rho_1}\ldots I_n^{\rho_n} M(t,\sigma) \\ & + I_{t=1}^\alpha f(t,x,u(t,x)) - 4I_{t=1/2}^\alpha f(t,x,u(t,x)), \quad \text{and} \\ M''_t(0,\sigma) =& -8a_3(\sigma) + 4a_4(\sigma) + 4\int_0^1 a_2(t)M(t,\sigma)dt - 8I_{t=1/2}^\rho a_1(t,\sigma) I_1^{\rho_1}\ldots I_n^{\rho_n} M(t,\sigma) \\ & + 4I_{t=1}^\rho a_1(t,\sigma) I_1^{\rho_1}\ldots I_n^{\rho_n} M(t,\sigma) \\ & + 8I_{t=1/2}^\rho N(t,\sigma,M(t,\sigma)) - 4I_{t=1}^\rho N(t,\sigma,M(t,\sigma)). \end{aligned}$$

Hence,

$$\begin{aligned} &\left(1 + I_t^\rho a_1(t,\sigma) I_1^{\rho_1}\ldots I_n^{\rho_n}\right) M(t,\sigma) = I_t^\rho N(t,\sigma,M(t,\sigma)) \\ &+ \int_0^1 a_2(t)M(t,\sigma)dt + M'_t(0,\sigma)t + M''_t(0,\sigma)\frac{t^2}{2} \\ &= I_t^\rho N(t,\sigma,M(t,\sigma)) + (4t-4t^2)a_3(\sigma) + (2t^2-t)a_4(\sigma) + (2t^2-3t+1)\int_0^1 a_2(t)M(t,\sigma)dt \\ &+ (4t-4t^2) I_{t=1/2}^\rho a_1(t,\sigma) I_1^{\rho_1}\ldots I_n^{\rho_n} M(t,\sigma) + (2t^2-t) I_{t=1}^\rho a_1(t,\sigma) I_1^{\rho_1}\ldots I_n^{\rho_n} M(t,\sigma) \\ &+ (4t^2-4t) I_{t=1/2}^\rho N(t,\sigma,M(t,\sigma)) + (t-2t^2) I_{t=1}^\rho N(t,\sigma,M(t,\sigma)). \end{aligned}$$

Following Section 1, we can prove that the inverse operator of $1 + I_t^{\rho} a_1(t,x) I_1^{\rho_1} \ldots I_n^{\rho_n}$ is

$$\mathcal{V}_{a_1} = \sum_{s=0}^{\infty} (-1)^s \left(I_t^{\rho} a_1(t,\sigma) I_1^{\rho_1} \ldots I_n^{\rho_n} \right)^s$$

in the space $C([0,1] \times [0,1]^n)$. Indeed, for any $\phi \in C([0,1] \times [0,1]^n)$, we claim that

$$\begin{aligned} \|\mathcal{V}_{a_1}\phi\| \leq & \|\phi\| \sum_{s=0}^{\infty} \|a_1\|^s \frac{1}{\Gamma(\rho s+1)\Gamma(\rho_1 s+1)\ldots\Gamma(\rho_n s+1)} \\ & = \|\phi\| E_{(\rho,\rho_1,\ldots,\rho_n),(1,1,\ldots,1)}(\|a_1\|) < +\infty. \end{aligned}$$

Furthermore,

$$\left(1 + I_t^{\rho} a_1(t,\sigma) I_1^{\rho_1} \ldots I_n^{\rho_n}\right) \mathcal{V}_{a_1} = \mathcal{V}_{a_1} \left(1 + I_t^{\rho} a_1(t,\sigma) I_1^{\rho_1} \ldots I_n^{\rho_n}\right) = 1.$$

Clearly,

$$\begin{aligned} &\left(1 + I_t^{\rho} a_1(t,\sigma) I_1^{\rho_1} \ldots I_n^{\rho_n}\right) \mathcal{V}_{a_1} = \sum_{s=0}^{\infty} (-1)^s \left(I_t^{\rho} a_1(t,\sigma) I_1^{\rho_1} \ldots I_n^{\rho_n} \right)^s \\ &+ \sum_{s=0}^{\infty} (-1)^s \left(I_t^{\rho} a_1(t,\sigma) I_1^{\rho_1} \ldots I_n^{\rho_n} \right)^{s+1} \\ &= 1 + \sum_{s=1}^{\infty} (-1)^s \left(I_t^{\rho} a_1(t,\sigma) I_1^{\rho_1} \ldots I_n^{\rho_n} \right)^s + \sum_{s=0}^{\infty} (-1)^s \left(I_t^{\rho} a_1(t,\sigma) I_1^{\rho_1} \ldots I_n^{\rho_n} \right)^{s+1} \\ &= 1, \end{aligned}$$

and the uniqueness of the operator follows similarly.

Therefore,

$$\begin{aligned} M(t,\sigma) = & \sum_{s=0}^{\infty} (-1)^s \left(I_t^{\rho} a_1(t,\sigma) I_1^{\rho_1} \ldots I_n^{\rho_n} \right)^s I_t^{\rho} N(t,\sigma,M(t,\sigma)) \quad (= I_1) \\ & + \sum_{s=0}^{\infty} (-1)^s \left(I_t^{\rho} a_1(t,\sigma) I_1^{\rho_1} \ldots I_n^{\rho_n} \right)^s (4t - 4t^2) a_3(\sigma) \quad (= I_2) \\ & + \sum_{s=0}^{\infty} (-1)^s \left(I_t^{\rho} a_1(t,\sigma) I_1^{\rho_1} \ldots I_n^{\rho_n} \right)^s (2t^2 - t) a_4(\sigma) \quad (= I_3) \\ & + \sum_{s=0}^{\infty} (-1)^s \left(I_t^{\rho} a_1(t,\sigma) I_1^{\rho_1} \ldots I_n^{\rho_n} \right)^s (2t^2 - 3t + 1) \int_0^1 a_2(t) M(t,\sigma) dt \quad (= I_4) \\ & + \sum_{s=0}^{\infty} (-1)^s \left(I_t^{\rho} a_1(t,\sigma) I_1^{\rho_1} \ldots I_n^{\rho_n} \right)^s (4t - 4t^2) I_{t=1/2}^{\rho} a_1(t,\sigma) I_1^{\rho_1} \ldots I_n^{\rho_n} M(t,\sigma) (= I_5) \\ & + \sum_{s=0}^{\infty} (-1)^s \left(I_t^{\rho} a_1(t,\sigma) I_1^{\rho_1} \ldots I_n^{\rho_n} \right)^s (2t^2 - t) I_{t=1}^{\rho} a_1(t,x) I_1^{\rho_1} \ldots I_n^{\rho_n} M(t,\sigma) \quad (= I_6) \\ & + \sum_{s=0}^{\infty} (-1)^s \left(I_t^{\rho} a_1(t,\sigma) I_1^{\rho_1} \ldots I_n^{\rho_n} \right)^s (4t^2 - 4t) I_{t=1/2}^{\rho} N(t,\sigma,M(t,\sigma)) \quad (= I_7) \\ & + \sum_{s=0}^{\infty} (-1)^s \left(I_t^{\rho} a_1(t,\sigma) I_1^{\rho_1} \ldots I_n^{\rho_n} \right)^s (t - 2t^2) I_{t=1}^{\rho} N(t,\sigma,M(t,\sigma)) \quad (= I_8) \\ & = I_1 + \ldots + I_8. \end{aligned}$$

Since

$$\max_{t\in[0,1]} |4t - 4t^2| = 1, \quad \max_{t\in[0,1]} |2t^2 - t| = 1, \quad \max_{t\in[0,1]} |2t^2 - 3t + 1| = 1,$$

we derive

$$\begin{aligned}
\|M\| \leq & \|N\| \sum_{s=0}^{\infty} \|a_1\|^s \frac{1}{\Gamma(s\rho+\rho+1)} \frac{1}{\Gamma(s\rho_1+1)} \cdots \frac{1}{\Gamma(s\rho_n+1)} \quad (\text{from } I_1) \\
& + (\|a_3\|+\|a_4\|) \sum_{s=0}^{\infty} \|a_1\|^s \frac{1}{\Gamma(s\rho+1)} \frac{1}{\Gamma(s\rho_1+1)} \cdots \frac{1}{\Gamma(s\rho_n+1)} \quad (\text{from } I_2, I_3) \\
& + \|a_2\|\|M\| \sum_{s=0}^{\infty} \|a_1\|^s \frac{1}{\Gamma(s\rho+1)} \frac{1}{\Gamma(s\rho_1+1)} \cdots \frac{1}{\Gamma(s\rho_n+1)} \quad (\text{from } I_4) \\
& + \frac{(1/2)^\rho \|a_1\|\|M\|}{\Gamma(\rho+1)} \sum_{s=0}^{\infty} \frac{\|a_1\|^s}{\Gamma(s\rho+1)} \frac{1}{\Gamma(s\rho_1+\rho_1+1)} \cdots \frac{1}{\Gamma(s\rho_n+\rho_n+1)} \quad (\text{from } I_5) \\
& + \|a_1\|\|M\| \sum_{s=0}^{\infty} \frac{\|a_1\|^s}{\Gamma(s\rho+\rho+1)} \frac{1}{\Gamma(s\rho_1+\rho_1+1)} \cdots \frac{1}{\Gamma(s\rho_n+\rho_n+1)} \quad (\text{from } I_6) \\
& + \frac{(1/2)^\rho \|N\|}{\Gamma(\rho+1)} \sum_{s=0}^{\infty} \frac{\|a_1\|^s}{\Gamma(s\rho+1)} \frac{1}{\Gamma(s\rho_1+1)} \cdots \frac{1}{\Gamma(s\rho_n+1)} \quad (\text{from } I_7) \\
& + \|N\| \sum_{s=0}^{\infty} \frac{\|a_1\|^s}{\Gamma(s\rho+\rho+1)} \frac{1}{\Gamma(s\rho_1+1)} \cdots \frac{1}{\Gamma(s\rho_n+1)} \quad (\text{from } I_8) \\
= & \|N\| E_{(\rho,\rho_1,\ldots,\rho_n),\,(\rho+1,1,\ldots,1)}(\|a_1\|) + (\|a_3\|+\|a_4\|) E_{(\rho,\rho_1,\ldots,\rho_n),\,(1,1,\ldots,1)}(\|a_1\|) \\
& + \|a_2\|\|M\| E_{(\rho,\rho_1,\ldots,\rho_n),\,(1,1,\ldots,1)}(\|a_1\|) \\
& + \frac{(1/2)^\rho \|a_1\|\|M\|}{\Gamma(\rho+1)} E_{(\rho,\rho_1,\ldots,\rho_n),\,(1,\rho_1+1,\ldots,\rho_n+1)}(\|a_1\|) \\
& + \|a_1\|\|M\| E_{(\rho,\rho_1,\ldots,\rho_n),\,(\rho+1,\rho_1+1,\ldots,\rho_n+1)}(\|a_1\|) \\
& + \frac{(1/2)^\rho \|N\|}{\Gamma(\rho+1)} E_{(\rho,\rho_1,\ldots,\rho_n),\,(1,1,\ldots,1)}(\|a_1\|) + \|N\| E_{(\rho,\rho_1,\ldots,\rho_n),\,(\rho+1,1,\ldots,1)}(\|a_1\|).
\end{aligned}$$

Since

$$\begin{aligned}
G = & 1 - \|a_2\| E_{(\rho,\rho_1,\ldots,\rho_n),\,(1,1,\ldots,1)}(\|a_1\|) - \frac{(1/2)^\rho \|a_1\|}{\Gamma(\rho+1)} E_{(\rho,\rho_1,\ldots,\rho_n),\,(1,\rho_1+1,\ldots,\rho_n+1)}(\|a_1\|) \\
& - \|a_1\| E_{(\rho,\rho_1,\ldots,\rho_n),\,(\rho+1,\rho_1+1,\ldots,\rho_n+1)}(\|a_1\|) > 0,
\end{aligned}$$

we obtain

$$\begin{aligned}
\|M\| \leq & \frac{\|N\|}{G} E_{(\rho,\rho_1,\ldots,\rho_n),\,(\rho+1,1,\ldots,1)}(\|a_1\|) + \frac{\|a_3\|+\|a_4\|}{G} E_{(\rho,\rho_1,\ldots,\rho_n),\,(1,1,\ldots,1)}(\|a_1\|) \\
& + \frac{(1/2)^\rho \|N\|}{G\,\Gamma(\rho+1)} E_{(\rho,\rho_1,\ldots,\rho_n),\,(1,1,\ldots,1)}(\|a_1\|) + \frac{\|N\|}{G} E_{(\rho,\rho_1,\ldots,\rho_n),\,(\rho+1,1,\ldots,1)}(\|a_1\|),
\end{aligned}$$

which is uniformly bounded. □

The following is our key theorem regarding the uniqueness and stability of Equation (1).

Theorem 3. *Let* $2 < \rho \leq 3$, $a_2 \in C[0,1]$, $a_3, a_4 \in C([0,1]^n)$, $a_1 \in C([0,1] \times [0,1]^n)$, *and* $N : [0,1] \times [0,1]^n \times \mathbb{R} \to \mathbb{R}$ *be a continuous and bounded function satisfying the Lipschitz condition*

$$|N(t,\sigma,M_1) - N(t,\sigma,M_2)| \leq \mathcal{C}|M_1 - M_2|, \quad M_1, M_2 \in \mathbb{R},$$

for a non-negative constant, $\mathcal{C}$. Furthermore, we assume that

$$
\begin{aligned}
F =& 2\mathcal{C}E_{(\rho,\rho_1,\ldots,\rho_n),\,(\rho+1,1,\ldots,1)}(\|a_1\|) + \left(\|a_2\| + \frac{(1/2)^{\rho}\mathcal{C}}{\Gamma(\rho+1)}\right)E_{(\rho,\rho_1,\ldots,\rho_n),\,(1,1,\ldots,1)}(\|a_1\|) \\
&+ \frac{(1/2)^{\rho}\|a_1\|}{\Gamma(\rho+1)}E_{(\rho,\rho_1,\ldots,\rho_n),\,(1,\rho_1+1,\ldots,\rho_n+1)}(\|a_1\|) \\
&+ \|a_1\|E_{(\rho,\rho_1,\ldots,\rho_n),\,(\rho+1,\rho_1+1,\ldots,\rho_n+1)}(\|a_1\|) < 1.
\end{aligned}
$$

Then, Equation (1) has a unique solution in $C([0,1]\times[0,1]^n)$ and is stable.

Proof. To prove the uniqueness, we start defining a nonlinear mapping, $\mathbb{W}$, over $C([0,1]\times[0,1]^n)$ as

$$
\begin{aligned}
(\mathbb{W}M)(t,\sigma) =& \sum_{s=0}^{\infty}(-1)^s\left(I_t^{\rho}a_1(t,\sigma)I_1^{\rho_1}\ldots I_n^{\rho_n}\right)^s I_t^{\rho}N(t,\sigma,M(t,\sigma)) \\
&+ \sum_{s=0}^{\infty}(-1)^s\left(I_t^{\rho}a_1(t,\sigma)I_1^{\rho_1}\ldots I_n^{\rho_n}\right)^s(4t-4t^2)a_3(\sigma) \\
&+ \sum_{s=0}^{\infty}(-1)^s\left(I_t^{\rho}a_1(t,\sigma)I_1^{\rho_1}\ldots I_n^{\rho_n}\right)^s(2t^2-t)a_4(\sigma) \\
&+ \sum_{s=0}^{\infty}(-1)^s\left(I_t^{\rho}a_1(t,\sigma)I_1^{\rho_1}\ldots I_n^{\rho_n}\right)^s(2t^2-3t+1)\int_0^1 a_2(t)M(t,\sigma)dt \\
&+ \sum_{s=0}^{\infty}(-1)^s\left(I_t^{\rho}a_1(t,\sigma)I_1^{\rho_1}\ldots I_n^{\rho_n}\right)^s(4t-4t^2)I_{t=1/2}^{\rho}a_1(t,\sigma)I_1^{\rho_1}\ldots I_n^{\rho_n}M(t,\sigma) \\
&+ \sum_{s=0}^{\infty}(-1)^s\left(I_t^{\rho}a_1(t,\sigma)I_1^{\rho_1}\ldots I_n^{\rho_n}\right)^s(2t^2-t)I_{t=1}^{\rho}a_1(t,\sigma)I_1^{\rho_1}\ldots I_n^{\rho_n}M(t,\sigma) \\
&+ \sum_{s=0}^{\infty}(-1)^s\left(I_t^{\rho}a_1(t,\sigma)I_1^{\rho_1}\ldots I_n^{\rho_n}\right)^s(4t^2-4t)I_{t=1/2}^{\rho}N(t,\sigma,M(t,\sigma)) \\
&+ \sum_{s=0}^{\infty}(-1)^s\left(I_t^{\rho}a_1(t,\sigma)I_1^{\rho_1}\ldots I_n^{\rho_n}\right)^s(t-2t^2)I_{t=1}^{\rho}N(t,\sigma,M(t,\sigma)).
\end{aligned}
$$

It follows from the proof of Theorem 2 that $\mathbb{W}$ is a mapping from $C([0,1]\times[0,1]^n)$ to itself. We only need to prove that $\mathbb{W}$ is contractive. For any $M_1, M_2 \in C([0,1]\times[0,1]^n)$, we have

$$
\begin{aligned}
&\mathbb{W}M_1 - \mathbb{W}M_2 \\
&= \sum_{s=0}^{\infty}(-1)^s\left(I_t^{\rho}a_1(t,\sigma)I_1^{\rho_1}\ldots I_n^{\rho_n}\right)^s I_t^{\rho}(N(t,\sigma,M_1(t,\sigma)) - N(t,\sigma,M_2(t,\sigma))) \\
&+ \sum_{s=0}^{\infty}(-1)^s\left(I_t^{\rho}a_1(t,\sigma)I_1^{\rho_1}\ldots I_n^{\rho_n}\right)^s(2t^2-3t+1)\int_0^1 a_2(t)(M_1(t,\sigma)-M_2(t,\sigma))dt \\
&+ \sum_{s=0}^{\infty}(-1)^s\left(I_t^{\rho}a_1(t,\sigma)I_1^{\rho_1}\ldots I_n^{\rho_n}\right)^s(4t-4t^2)I_{t=1/2}^{\rho}a_1(t,\sigma)I_1^{\rho_1}\ldots I_n^{\rho_n}(M_1(t,\sigma)-M_2(t,\sigma)) \\
&+ \sum_{s=0}^{\infty}(-1)^s\left(I_t^{\rho}a_1(t,\sigma)I_1^{\rho_1}\ldots I_n^{\rho_n}\right)^s(2t^2-t)I_{t=1}^{\rho}a_1(t,\sigma)I_1^{\rho_1}\ldots I_n^{\rho_n}(M_1(t,\sigma)-M_2(t,\sigma)) \\
&+ \sum_{s=0}^{\infty}(-1)^s\left(I_t^{\rho}a_1(t,\sigma)I_1^{\rho_1}\ldots I_n^{\rho_n}\right)^s(4t^2-4t)I_{t=1/2}^{\rho}(N(t,\sigma,M_1(t,\sigma)) - N(t,\sigma,M_2(t,\sigma))) \\
&+ \sum_{s=0}^{\infty}(-1)^s\left(I_t^{\rho}a_1(t,\sigma)I_1^{\rho_1}\ldots I_n^{\rho_n}\right)^s(t-2t^2)I_{t=1}^{\rho}(N(t,\sigma,M_1(t,\sigma)) - N(t,\sigma,M_2(t,\sigma))).
\end{aligned}
$$

Then,

$$
\begin{aligned}
&\|\mathbb{W}M_1 - \mathbb{W}M_2\| \\
&\leq \Big[\mathcal{C}E_{(\rho,\rho_1,\ldots,\rho_n),\,(\rho+1,1,\ldots,1)}(\|a_1\|) + \|a_2\|E_{(\rho,\rho_1,\ldots,\rho_n),\,(1,1,\ldots,1)}(\|a_1\|) \\
&+ \frac{(1/2)^\rho\|a_1\|}{\Gamma(\rho+1)}E_{(\rho,\rho_1,\ldots,\rho_n),\,(1,\rho_1+1,\ldots,\rho_n+1)}(\|a_1\|) \\
&+ \|a_1\|E_{(\rho,\rho_1,\ldots,\rho_n),\,(\rho+1,\rho_1+1,\ldots,\rho_n+1)}(\|a_1\|) \\
&+ \frac{(1/2)^\rho\mathcal{C}}{\Gamma(\rho+1)}E_{(\rho,\rho_1,\ldots,\rho_n),\,(1,1,\ldots,1)}(\|a_1\|) + \mathcal{C}E_{(\rho,\rho_1,\ldots,\rho_n),\,(\rho+1,1,\ldots,1)}(\|a_1\|)\Big]\|M_1 - M_2\| \\
&= F\|M_1 - M_2\|.
\end{aligned}
$$

Since $F < 1$, Equation (1) has a unique solution in $C([0,1]\times[0,1]^n)$ from Banach's contractive principle.

To show the stability, we let

$$z(t,\sigma) = \frac{{}_c\partial^\rho}{\partial t^\rho}M(t,\sigma) + a_1(t,\sigma)\ I_1^{\rho_1}\ldots I_n^{\rho_n}M(t,\sigma) - N(t,\sigma,M(t,\sigma)),$$

with the conditions

$$M(0,\sigma) = \int_0^1 a_2(t)M(t,\sigma)dt, \quad M(1/2,\sigma) = a_3(x), \quad M(1,\sigma) = a_4(\sigma).$$

It follows from Definition 1 that

$$\|z\| < \epsilon,$$

and

$$\frac{{}_c\partial^\rho}{\partial t^\rho}M(t,\sigma) + a_1(t,\sigma)\ I_1^{\rho_1}\ldots I_n^{\rho_n}M(t,\sigma) = N(t,\sigma,M(t,\sigma)) + z(t,\sigma). \tag{7}$$

Since

$$|N(t,\sigma,y_1) + z(t,\sigma) - N(t,\sigma,y_2) - z(t,\sigma)| \leq \mathcal{C}|y_1 - y_2|,$$

we claim that Equation (7) has a unique solution by the first part of Theorem 3 and

$$
\begin{aligned}
M(t,\sigma) = &\sum_{s=0}^{\infty}(-1)^s\left(I_t^\rho a_1(t,\sigma)I_1^{\rho_1}\ldots I_n^{\rho_n}\right)^s I_t^\rho(N(t,\sigma,M(t,\sigma)) + z(t,\sigma)) \\
&+ \sum_{s=0}^{\infty}(-1)^s\left(I_t^\rho a_1(t,\sigma)I_1^{\rho_1}\ldots I_n^{\rho_n}\right)^s(4t-4t^2)a_3(\sigma) \\
&+ \sum_{s=0}^{\infty}(-1)^s\left(I_t^\rho a_1(t,\sigma)I_1^{\rho_1}\ldots I_n^{\rho_n}\right)^s(2t^2-t)a_4(\sigma) \\
&+ \sum_{s=0}^{\infty}(-1)^s\left(I_t^\rho a_1(t,\sigma)I_1^{\rho_1}\ldots I_n^{\rho_n}\right)^s(2t^2-3t+1)\int_0^1 a_2(t)M(t,\sigma)dt \\
&+ \sum_{s=0}^{\infty}(-1)^s\left(I_t^\rho a_1(t,\sigma)I_1^{\rho_1}\ldots I_n^{\rho_n}\right)^s(4t-4t^2)I_{t=1/2}^\rho a_1(t,\sigma)I_1^{\rho_1}\ldots I_n^{\rho_n}M(t,\sigma) \\
&+ \sum_{s=0}^{\infty}(-1)^s\left(I_t^\rho a_1(t,\sigma)I_1^{\rho_1}\ldots I_n^{\rho_n}\right)^s(2t^2-t)I_{t=1}^\rho a_1(t,\sigma)I_1^{\rho_1}\ldots I_n^{\rho_n}M(t,\sigma) \\
&+ \sum_{s=0}^{\infty}(-1)^s\left(I_t^\rho a_1(t,\sigma)I_1^{\rho_1}\ldots I_n^{\rho_n}\right)^s(4t^2-4t)I_{t=1/2}^\rho(N(t,\sigma,M(t,\sigma)) + z(t,\sigma)) \\
&+ \sum_{s=0}^{\infty}(-1)^s\left(I_t^\rho a_1(t,\sigma)I_1^{\rho_1}\ldots I_n^{\rho_n}\right)^s(t-2t^2)I_{t=1}^\rho(N(t,\sigma,M(t,\sigma)) + z(t,\sigma)).
\end{aligned}
$$

On the other hand, there exists a unique $M_0(t,\sigma) \in C([0,1]\times[0,1]^n)$ of Equation (1) from the above uniqueness proof, such that

$$
\begin{aligned}
M_0(t,\sigma) =& \sum_{s=0}^{\infty}(-1)^s\left(I_t^{\rho}a_1(t,\sigma)I_1^{\rho_1}\dots I_n^{\rho_n}\right)^s I_t^{\rho}N(t,\sigma,M_0(t,\sigma)) \\
&+\sum_{s=0}^{\infty}(-1)^s\left(I_t^{\rho}a_1(t,\sigma)I_1^{\rho_1}\dots I_n^{\rho_n}\right)^s(4t-4t^2)a_3(\sigma) \\
&+\sum_{s=0}^{\infty}(-1)^s\left(I_t^{\rho}a_1(t,\sigma)I_1^{\rho_1}\dots I_n^{\rho_n}\right)^s(2t^2-t)a_4(\sigma) \\
&+\sum_{s=0}^{\infty}(-1)^s\left(I_t^{\rho}a_1(t,\sigma)I_1^{\rho_1}\dots I_n^{\rho_n}\right)^s(2t^2-3t+1)\int_0^1 a_2(t)M_0(t,\sigma)dt \\
&+\sum_{s=0}^{\infty}(-1)^s\left(I_t^{\rho}a_1(t,\sigma)I_1^{\rho_1}\dots I_n^{\rho_n}\right)^s(4t-4t^2)I_{t=1/2}^{\rho}a_1(t,\sigma)I_1^{\rho_1}\dots I_n^{\rho_n}M_0(t,\sigma) \\
&+\sum_{s=0}^{\infty}(-1)^s\left(I_t^{\rho}a_1(t,\sigma)I_1^{\rho_1}\dots I_n^{\rho_n}\right)^s(2t^2-t)I_{t=1}^{\rho}a_1(t,\sigma)I_1^{\rho_1}\dots I_n^{\rho_n}M_0(t,\sigma) \\
&+\sum_{s=0}^{\infty}(-1)^s\left(I_t^{\rho}a_1(t,\sigma)I_1^{\rho_1}\dots I_n^{\rho_n}\right)^s(4t^2-4t)I_{t=1/2}^{\rho}N(t,\sigma,M_0(t,\sigma)) \\
&+\sum_{s=0}^{\infty}(-1)^s\left(I_t^{\rho}a_1(t,\sigma)I_1^{\rho_1}\dots I_n^{\rho_n}\right)^s(t-2t^2)I_{t=1}^{\rho}N(t,\sigma,M_0(t,\sigma)).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\|M-M_0\| \leq& F\|M-M_0\| + E_{(\rho,\rho_1,\dots,\rho_n),\,(\rho+1,1,\dots,1)}(\|a_1\|)\,\epsilon \\
&+\frac{(1/2)^{\rho}}{\Gamma(\rho+1)}E_{(\rho,\rho_1,\dots,\rho_n),\,(1,1,\dots,1)}(\|a_1\|)\,\epsilon + E_{(\rho,\rho_1,\dots,\rho_n),\,(\rho+1,1,\dots,1)}(\|a_1\|)\,\epsilon,
\end{aligned}
$$

which implies that

$$\|M-M_0\| \leq \mathcal{K}\,\epsilon,$$

where

$$
\begin{aligned}
\mathcal{K} =& \frac{1}{1-F}\Big\{E_{(\rho,\rho_1,\dots,\rho_n),\,(\rho+1,1,\dots,1)}(\|a_1\|) + \frac{(1/2)^{\rho}}{\Gamma(\rho+1)}E_{(\rho,\rho_1,\dots,\rho_n),\,(1,1,\dots,1)}(\|a_1\|) \\
&+ E_{(\rho,\rho_1,\dots,\rho_n),\,(\rho+1,1,\dots,1)}(\|a_1\|)\Big\},
\end{aligned}
$$

is a stability constant, which is independent of ϵ. □

Remark 1. *(i) Along the same line, we are able to study the uniqueness for the following partial integro-differential equation for $\lambda \in C([0,1]^n)$:*

$$
\begin{cases}
\dfrac{{}^c\partial^{\rho}}{\partial t^{\rho}}M(t,\sigma) + a_1(t,\sigma)\; I_1^{\rho_1}\dots I_n^{\rho_n}M(t,\sigma) = N(t,\sigma,u(t,\sigma)), & (t,\sigma)\in[0,1]\times[0,1]^n, \\
M(0,\sigma) = \lambda(\sigma), \quad M(t_0,\sigma) = a_3(x), \quad M(1,\sigma) = a_4(\sigma), & 0 < t_0 < 1.
\end{cases}
$$

(ii) In particular if $\rho = 3$ and $\rho_i = 0$ for $i = 1,2,\dots,n$, then Equation (1) turns out to be the following:

$$
\begin{cases}
\dfrac{\partial^3}{\partial t^3}M(t,\sigma) + a_1(t,\sigma)\; M(t,\sigma) = N(t,\sigma,M(t,\sigma)), \quad (t,\sigma)\in[0,1]\times[0,1]^n, \\
M(0,\sigma) = \displaystyle\int_0^1 a_2(t)M(t,\sigma)dt, \quad M(1/2,\sigma) = a_3(\sigma), \quad M(1,\sigma) = a_4(\sigma).
\end{cases}
$$

Example 2. *The following equation with three-point conditions and a variable coefficient,*

$$\begin{cases} \frac{{}_c\partial^{2.5}}{\partial t^{2.5}} M(t,\sigma) + \frac{1}{13(t^2+\sigma_1^3+1)} \ I_1^{0.7} I_2^{1.4} M(t,\sigma) = \frac{1}{17(M^2(t,\sigma)+1)} + \sin t\sigma_2, \\ (t,\sigma) \in [0,1]\times[0,1]^2 \\ M(0,\sigma) = \frac{1}{31}\int_0^1 \cos t \ M(t,\sigma)dt, \quad M(1/2,\sigma) = \frac{1}{21(|\sigma|^2+1)}, \\ M(1,\sigma) = \frac{1}{7(\sigma_1^2+1)}, \end{cases} \tag{8}$$

where $\sigma = (\sigma_1, \sigma_2)$ *and* $|\sigma|^2 = \sigma_1^2 + \sigma_2^2$, *has a unique solution in* $C([0,1]\times[0,1]^2)$ *and is stable.*

Proof. From the equation, we can see that

$$\rho = 2.5, \ \ \rho_1 = 0.7, \ \ \rho_2 = 1.4, \ \ \|a_1\| = 1/13, \ \ \|a_2\| = 1/31, \ \ \|a_3\| = 1/21, \ \ \|a_4\| = 1/7,$$

and

$$N(t,\sigma,y) = \frac{1}{17(y^2+1)} + \sin t\sigma_2$$

is a continuous and bounded function, with

$$|N(t,\sigma,y_1) - N(t,\sigma,y_2)| \leq \frac{1}{17}|y_1 - y_2|,$$

by using the mean value theorem and noting that

$$\left|\frac{d}{dy}\frac{1}{y^2+1}\right| = \frac{2|y|}{(1+y^2)^2} \leq 1.$$

So, $\mathcal{C} = 1/17$. We need to find the value of

$$\begin{aligned} F =& 2\mathcal{C} E_{(\rho,\rho_1,\ldots,\rho_n),\,(\rho+1,1,\ldots,1)}(\|a_1\|) + \left(\|a_2\| + \frac{(1/2)^\rho \mathcal{C}}{\Gamma(\rho+1)}\right) E_{(\rho,\rho_1,\ldots,\rho_n),\,(1,1,\ldots,1)}(\|a_1\|) \\ &+ \frac{(1/2)^\rho \|a_1\|}{\Gamma(\rho+1)} E_{(\rho,\rho_1,\ldots,\rho_n),\,(1,\rho_1+1,\ldots,\rho_n+1)}(\|a_1\|) \\ &+ \|a_1\| E_{(\rho,\rho_1,\ldots,\rho_n),\,(\rho+1,\rho_1+1,\ldots,\rho_n+1)}(\|a_1\|) \\ =& \frac{2}{17} E_{(2.5,0.7,1.4),(3.5,1,1)}(1/13) + \left(\frac{1}{31} + \frac{(1/2)^{2.5}}{17\Gamma(3.5)}\right) E_{(2.5,0.7,1.4),(1,1,1)}(1/13) \\ &+ \frac{(1/2)^{2.5}}{13\Gamma(3.5)} E_{(2.5,0.7,1.4),(1,1.7,2.4)}(1/13) + \frac{1}{13} E_{(2.5,0.7,1.4),(3.5,1.7,2.4)}(1/13) \\ \approx& \frac{2}{17} * 0.30146912 + 0.0353870 * 1.0205157523 + 0.00409172 * 0.88995845 \\ &+ \frac{1}{13} * 0.266704780 \approx 0.095737159 < 1, \end{aligned}$$

where

$$E_{(2.5,0.7,1.4),(3.5,1,1)}(1/13) = \sum_{k=0}^{\infty} \frac{(1/13)^k}{\Gamma(2.5k+3.5)\Gamma(0.7k+1)\Gamma(1.4k+1)} \approx 0.30146912,$$

$$E_{(2.5,0.7,1.4),(1,1,1)}(1/13) = \sum_{k=0}^{\infty} \frac{(1/13)^k}{\Gamma(2.5k+1)\Gamma(0.7k+1)\Gamma(1.4k+1)} \approx 1.0205157523,$$

$$E_{(2.5,0.7,1.4),(1,1.7,2.4)}(1/13) = \sum_{k=0}^{\infty} \frac{(1/13)^k}{\Gamma(2.5k+1)\Gamma(0.7k+1.7)\Gamma(1.4k+2.4)} \approx 0.88995845,$$

$$E_{(2.5,0.7,1.4),(3.5,1.7,2.4)}(1/13) = \sum_{k=0}^{\infty} \frac{(1/13)^k}{\Gamma(2.5k+1)\Gamma(0.7k+1.7)\Gamma(1.4k+2.4)} \approx 0.266704780.$$

By Theorem 3, Equation (8) has a unique solution in $C([0,1] \times [0,1]^2)$ and is stable. We finish the proof. $\square$

3. Existence

We are now ready to present the theorem regarding the existence of Equation (1) based on Leray–Schauder's fixed-point theorem and the equicontinuity given below.

Definition 2. *Let (X, d) be a metric space and $\mathcal{F}$ be a family of functions from X to X. The family $\mathcal{F}$ is uniformly equicontinuous if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $d(f(x_1), f(x_2)) < \epsilon$ for all $f \in \mathcal{F}$ and all $x_1, x_2 \in X$ such that $d(x_1, x_2) < \delta$, which may depend only on ϵ.*

Theorem 4. *Let $2 < \rho \leq 3$, $a_2 \in C[0,1]$, $a_3, a_4 \in C([0,1]^n)$, $a_1 \in C([0,1] \times [0,1]^n)$, and $N : [0,1] \times [0,1]^n \times \mathbb{R} \to \mathbb{R}$ be a continuous and bounded function satisfying the condition*

$$|N(t,\sigma,y_1) - N(t,\sigma,y_2)| \leq \mathcal{C}|y_1 - y_2|, \quad y_1, y_2 \in \mathbb{R},$$

for a non-negative constant, $\mathcal{C}$. Furthermore, we assume that

$$\begin{aligned} G =& 1 - \|a_2\| E_{(\rho,\rho_1,\ldots,\rho_n),\,(1,1,\ldots,1)}(\|a_1\|) - \frac{(1/2)^\rho \|a_1\|}{\Gamma(\rho+1)} E_{(\rho,\rho_1,\ldots,\rho_n),\,(1,\rho_1+1,\ldots,\rho_n+1)}(\|a_1\|) \\ &- \|a_1\| E_{(\rho,\rho_1,\ldots,\rho_n),\,(\rho+1,\rho_1+1,\ldots,\rho_n+1)}(\|a_1\|) > 0. \end{aligned}$$

Then, Equation (1) has at least one solution in $C([0,1] \times [0,1]^n)$.

Proof. We use $\mathbb{W}$ over $C([0,1] \times [0,1]^n)$ again, given by

$$\begin{aligned} (\mathbb{W}M)(t,\sigma) = &\sum_{s=0}^{\infty} (-1)^s \left(I_t^\rho a_1(t,\sigma) I_1^{\rho_1} \ldots I_n^{\rho_n} \right)^s I_t^\rho N(t,\sigma,M(t,\sigma)) \\ &+ \sum_{s=0}^{\infty} (-1)^s \left(I_t^\rho a_1(t,\sigma) I_1^{\rho_1} \ldots I_n^{\rho_n} \right)^s (4t - 4t^2) a_3(\sigma) \\ &+ \sum_{s=0}^{\infty} (-1)^s \left(I_t^\rho a_1(t,\sigma) I_1^{\rho_1} \ldots I_n^{\rho_n} \right)^s (2t^2 - t) a_4(\sigma) \\ &+ \sum_{s=0}^{\infty} (-1)^s \left(I_t^\rho a_1(t,\sigma) I_1^{\rho_1} \ldots I_n^{\rho_n} \right)^s (2t^2 - 3t + 1) \int_0^1 a_2(t) M(t,\sigma) dt \\ &+ \sum_{s=0}^{\infty} (-1)^s \left(I_t^\rho a_1(t,\sigma) I_1^{\rho_1} \ldots I_n^{\rho_n} \right)^s (4t - 4t^2) I_{t=1/2}^\rho a_1(t,\sigma) I_1^{\rho_1} \ldots I_n^{\rho_n} M(t,\sigma) \\ &+ \sum_{s=0}^{\infty} (-1)^s \left(I_t^\rho a_1(t,\sigma) I_1^{\rho_1} \ldots I_n^{\rho_n} \right)^s (2t^2 - t) I_{t=1}^\rho a_1(t,\sigma) I_1^{\rho_1} \ldots I_n^{\rho_n} M(t,\sigma) \\ &+ \sum_{s=0}^{\infty} (-1)^s \left(I_t^\rho a_1(t,\sigma) I_1^{\rho_1} \ldots I_n^{\rho_n} \right)^s (4t^2 - 4t) I_{t=1/2}^\rho N(t,\sigma,M(t,\sigma)) \\ &+ \sum_{s=0}^{\infty} (-1)^s \left(I_t^\rho a_1(t,\sigma) I_1^{\rho_1} \ldots I_n^{\rho_n} \right)^s (t - 2t^2) I_{t=1}^\rho N(t,\sigma,M(t,\sigma)). \end{aligned}$$

It follows from Theorem 2 that

$$\begin{aligned}\|\mathbb{W}M\| \le\ & \|N\|E_{(\rho,\rho_1,\dots,\rho_n),\,(\rho+1,1,\dots,1)}(\|a_1\|) + (\|a_3\| + \|a_4\|)E_{(\rho,\rho_1,\dots,\rho_n),\,(1,1,\dots,1)}(\|a_1\|) \\ & + \|a_2\|\|M\|E_{(\rho,\rho_1,\dots,\rho_n),\,(1,1,\dots,1)}(\|a_1\|) \\ & + \frac{(1/2)^\rho\|a_1\|\|M\|}{\Gamma(\rho+1)}E_{(\rho,\rho_1,\dots,\rho_n),\,(1,\rho_1+1,\dots,\rho_n+1)}(\|a_1\|) \\ & + \|a_1\|\|M\|E_{(\rho,\rho_1,\dots,\rho_n),\,(\rho+1,\rho_1+1,\dots,\rho_n+1)}(\|a_1\|) \\ & + \frac{(1/2)^\rho\|N\|}{\Gamma(\rho+1)}E_{(\rho,\rho_1,\dots,\rho_n),\,(1,1,\dots,1)}(\|a_1\|) + \|N\|E_{(\rho,\rho_1,\dots,\rho_n),\,(\rho+1,1,\dots,1)}(\|a_1\|) < +\infty, \end{aligned} \tag{9}$$

where

$$\|N\| = \sup_{(t,\sigma,y)\in[0,1]\times[0,1]^n\times\mathbb{R}} |N(t,\sigma,y)| < +\infty.$$

We first claim that (i) $\mathbb{W}$ is a continuous mapping from $C([0,1]\times[0,1]^n)$ to itself from the contraction in Theorem 3.

(ii) Furthermore, we are going to prove that $\mathbb{W}$ is a mapping from bounded sets to bounded sets in $C([0,1]\times[0,1]^n)$. Let $\mathcal{B}$ be a bounded set in $C([0,1]\times[0,1]^n)$. Then, there exists a positive constant, $\mathcal{C}$, such that

$$\|M\| \le \mathcal{C}$$

for all $M \in \mathcal{B}$. Using Inequality (9), we claim that $\mathbb{W}\mathcal{B}$ is uniformly bounded.

(iii) $\mathbb{W}$ is equicontinuous on every bounded set $\mathcal{B}$ in $C([0,1]\times[0,1]^n)$. Then, $\mathbb{W}$ is a compact operator by the Arzela–Ascoli theorem. From Theorem 3, we have for $M_1, M_2 \in \mathcal{B}$ that

$$\|\mathbb{W}M_1 - \mathbb{W}M_2\| \le F\|M_1 - M_2\|,$$

where the constant F is

$$\begin{aligned} F =\ & 2\mathcal{C}E_{(\rho,\rho_1,\dots,\rho_n),\,(\rho+1,1,\dots,1)}(\|a_1\|) + \left(\|a_2\| + \frac{(1/2)^\rho\mathcal{C}}{\Gamma(\rho+1)}\right)E_{(\rho,\rho_1,\dots,\rho_n),\,(1,1,\dots,1)}(\|a_1\|) \\ & + \frac{(1/2)^\rho\|a_1\|}{\Gamma(\rho+1)}E_{(\rho,\rho_1,\dots,\rho_n),\,(1,\rho_1+1,\dots,\rho_n+1)}(\|a_1\|) \\ & + \|a_1\|E_{(\rho,\rho_1,\dots,\rho_n),\,(\rho+1,\rho_1+1,\dots,\rho_n+1)}(\|a_1\|) > 0, \end{aligned}$$

which is not required to be less than one here. By Definition 2, we infer that $\mathbb{W}$ is equicontinuous by using $\mathcal{F} = \{\mathbb{W}\}$ and d is the defined norm of the space $C([0,1]\times[0,1]^n)$.

(iv) Finally, we show that the set

$$\{M \in C([0,1]\times[0,1]^n) : M = \theta\mathbb{W}M \ \text{ for some } \ 0 < \theta \le 1\}$$

is bounded. Indeed,

$$\begin{aligned} & \|M\| \le \|\mathbb{W}M\| \le \\ & \|N\|E_{(\rho,\rho_1,\dots,\rho_n),\,(\rho+1,1,\dots,1)}(\|a_1\|) + (\|a_3\| + \|a_4\|)E_{(\rho,\rho_1,\dots,\rho_n),\,(1,1,\dots,1)}(\|a_1\|) \\ & + \|a_2\|\|M\|E_{(\rho,\rho_1,\dots,\rho_n),\,(1,1,\dots,1)}(\|a_1\|) \\ & + \frac{(1/2)^\rho\|a_1\|\|M\|}{\Gamma(\rho+1)}E_{(\rho,\rho_1,\dots,\rho_n),\,(1,\rho_1+1,\dots,\rho_n+1)}(\|a_1\|) \\ & + \|a_1\|\|M\|E_{(\rho,\rho_1,\dots,\rho_n),\,(\rho+1,\rho_1+1,\dots,\rho_n+1)}(\|a_1\|) \\ & + \frac{(1/2)^\rho\|N\|}{\Gamma(\rho+1)}E_{(\rho,\rho_1,\dots,\rho_n),\,(1,1,\dots,1)}(\|a_1\|) + \|N\|E_{(\rho,\rho_1,\dots,\rho_n),\,(\rho+1,1,\dots,1)}(\|a_1\|), \end{aligned}$$

which claims that

$$
\begin{aligned}
G\|M\| \leq\ & \|N\| E_{(\rho,\rho_1,\ldots,\rho_n),\,(\rho+1,1,\ldots,1)}(\|a_1\|) + (\|a_2\| + \|a_3\|) E_{(\rho,\rho_1,\ldots,\rho_n),\,(1,1,\ldots,1)}(\|a_1\|) \\
& + \frac{(1/2)^\rho \|N\|}{\Gamma(\rho+1)} E_{(\rho,\rho_1,\ldots,\rho_n),\,(1,1,\ldots,1)}(\|a_1\|) + \|N\| E_{(\rho,\rho_1,\ldots,\rho_n),\,(\rho+1,1,\ldots,1)}(\|a_1\|),
\end{aligned}
$$

where

$$
\begin{aligned}
G =& 1 - \|a_2\| E_{(\rho,\rho_1,\ldots,\rho_n),\,(1,1,\ldots,1)}(\|a_1\|) - \frac{(1/2)^\rho \|a_1\|}{\Gamma(\rho+1)} E_{(\rho,\rho_1,\ldots,\rho_n),\,(1,\rho_1+1,\ldots,\rho_n+1)}(\|a_1\|) \\
& - \|a_1\| E_{(\rho,\rho_1,\ldots,\rho_n),\,(\rho+1,\rho_1+1,\ldots,\rho_n+1)}(\|a_1\|) > 0.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|M\| \leq & \frac{\|N\|}{G} E_{(\rho,\rho_1,\ldots,\rho_n),\,(\rho+1,1,\ldots,1)}(\|a_1\|) + \frac{\|a_3\| + \|a_4\|}{G} E_{(\rho,\rho_1,\ldots,\rho_n),\,(1,1,\ldots,1)}(\|a_1\|) \\
& + \frac{(1/2)^\rho \|N\|}{G\,\Gamma(\rho+1)} E_{(\rho,\rho_1,\ldots,\rho_n),\,(1,1,\ldots,1)}(\|a_1\|) + \frac{\|N\|}{G} E_{(\rho,\rho_1,\ldots,\rho_n),\,(\rho+1,1,\ldots,1)}(\|a_1\|),
\end{aligned}
$$

which is bounded. Hence, Equation (1) has at least one solution in $C([0,1] \times [0,1]^n)$ from Leray–Schauder's fixed-point theorem. □

Remark 2. *(i) We should add that $F < 1$ in Theorem 3 implies that $G > 0$ in Theorem 4. However, the converse is not true. In other words, the uniqueness theorem requires a stronger condition overall.*

(ii) There may be another possible approach to showing that $\mathbb{W}$ is equicontinuous by considering the difference

$$
|(\mathbb{W}M)(t_1, \sigma_0) - (\mathbb{W}M)(t_2, \sigma_{00})|,
$$

where

$$
|(t_1, \sigma_0) - (t_2, \sigma_{00})| < \delta.
$$

However, it seems challenging due to the multiple variables and partial fractional integrals.

Example 3. *The following equation with three-point conditions and a variable coefficient,*

$$
\begin{cases}
\frac{{}_c\partial^{2.1}}{\partial t^{2.1}} M(t,\sigma) + \frac{1}{10(\sigma_1^2+1)}\ I_1^{0.5} I_2^{1.1} M(t,\sigma) = 200 \arctan|M(t,\sigma)| + \cos(t|\sigma|^{1/2}), \\
(t,\sigma) \in [0,1] \times [0,1]^3 \\
M(0,\sigma) = \frac{1}{15}\int_0^1 t^2\, M(t,\sigma) dt, \quad M(1/2,\sigma) = \frac{1}{11(\sigma_3^2+2)}, \quad M(1,\sigma) = \frac{3}{6(\sigma_1^2+\sigma_2^2+1)},
\end{cases}
\tag{10}
$$

where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ and $|\sigma|^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2$, has at least one solution in $C([0,1] \times [0,1]^3)$.

Proof. From the equation, we have

$\rho = 2.1,\ \rho_1 = 0.5,\ \rho_2 = 1.1, \rho_3 = 0,\ \|a_1\| = 1/10,\ \|a_2\| = 1/15,\ \|a_3\| = 1/22,\ \|a_4\| = 1/2,$

and

$$
N(t,\sigma,y) = 200 \arctan|y| + \cos(t|\sigma|^{1/2})
$$

is a continuous and bounded function over $[0,1] \times [0,1]^3$, with the condition

$|N(t,\sigma,y_1) - N(t,\sigma,y_2)| \leq 200|\arctan|y_1| - \arctan|y_2|| \leq 200||y_1| - |y_2|| \leq 200|y_1 - y_2|,$

which claims that $\mathcal{C} = 200$.

We need to evaluate the value of

$$\begin{aligned} G &= 1 - \|a_2\| E_{(\rho,\rho_1,\ldots,\rho_n),\,(1,1,\ldots,1)}(\|a_1\|) - \frac{(1/2)^\rho \|a_1\|}{\Gamma(\rho+1)} E_{(\rho,\rho_1,\ldots,\rho_n),\,(1,\rho_1+1,\ldots,\rho_n+1)}(\|a_1\|) \\ &\quad - \|a_1\| E_{(\rho,\rho_1,\ldots,\rho_n),\,(\rho+1,\rho_1+1,\ldots,\rho_n+1)}(\|a_1\|) \\ &= 1 - \frac{1}{15} E_{(2.1,0.5,1.1,0),\,(1,1,1,1)}(1/10) - \frac{(1/10)^{2.1}}{10\Gamma(3.1)} E_{(2.1,0.5,1.1,0),\,(1,1.5,2.1,1)}(1/10) \\ &\quad - \frac{1}{10} E_{(2.1,0.5,1.1,0),\,(3.1,1.5,2.1,1)}(1/10) \\ &\approx 1 - \frac{1}{15} * 1.04919 - \frac{(1/10)^{2.1}}{10\Gamma(3.1)} * 1.09705 - \frac{1}{10} * 0.491914 \\ &\approx 0.8804661 > 0, \end{aligned}$$

using

$$\begin{aligned} E_{(2.1,0.5,1.1,0),\,(1,1,1,1)}(1/10) &= \sum_{k=0}^{\infty} \frac{(1/10)^k}{\Gamma(2.1k+1)\,\Gamma(0.5k+1)\,\Gamma(1.1k+1)} \approx 1.04919, \\ E_{(2.1,0.5,1.1,0),\,(1,1.5,2.1,1)}(1/10) &= \sum_{k=0}^{\infty} \frac{(1/10)^k}{\Gamma(2.1k+1)\,\Gamma(0.5k+1.5)\,\Gamma(1.1k+2.1)} \approx 1.09705, \\ E_{(2.1,0.5,1.1,0),\,(3.1,1.5,2.1,1)}(1/10) &= \sum_{k=0}^{\infty} \frac{(1/10)^k}{\Gamma(2.1k+3.1)\,\Gamma(0.5k+1.5)\,\Gamma(1.1k+2.1)} \\ &\approx 0.491914. \end{aligned}$$

By Theorem 4, Equation (10) has at least one solution in the space $C([0,1] \times [0,1]^3)$. □

Remark 3. *(i) As a note, we would like to point out that $F > 1$ in Theorem 3 since*

$$\begin{aligned} F =& 2\mathcal{C} E_{(\rho,\rho_1,\ldots,\rho_n),\,(\rho+1,1,\ldots,1)}(\|a_1\|) + \left(\|a_2\| + \frac{(1/2)^\rho \mathcal{C}}{\Gamma(\rho+1)}\right) E_{(\rho,\rho_1,\ldots,\rho_n),\,(1,1,\ldots,1)}(\|a_1\|) \\ &+ \frac{(1/2)^\rho \|a_1\|}{\Gamma(\rho+1)} E_{(\rho,\rho_1,\ldots,\rho_n),\,(1,\rho_1+1,\ldots,\rho_n+1)}(\|a_1\|) \\ &+ \|a_1\| E_{(\rho,\rho_1,\ldots,\rho_n),\,(\rho+1,\rho_1+1,\ldots,\rho_n+1)}(\|a_1\|) \\ &> \frac{(1/2)^\rho \mathcal{C}}{\Gamma(\rho+1)} E_{(\rho,\rho_1,\ldots,\rho_n),\,(1,1,\ldots,1)}(\|a_1\|) = \frac{(1/2)^{2.1} * 200}{\Gamma(3.1)} E_{(2.1,0.5,1.1,0),\,(1,1,1,1)}(1/10), \\ &\approx \frac{(1/2)^{2.1} * 200}{\Gamma(3.1)} * 1.04919 \approx 22.2725 > 1, \end{aligned}$$

for Equation (10). Hence, we are not sure if it has a unique solution in the space $C([0,1] \times [0,1]^3)$.

(ii) Generally speaking, the Lipschitz constant $\mathcal{C}$ in Theorem 3 is small to make $F < 1$. However, $\mathcal{C}$ in Theorem 4 has no restriction provided it is non-negative.

4. Applications of Inverse Operators

The inverse operator method is also powerful in finding series solutions to certain partial differential or integro-differential equations. We are going to present the following three examples to demonstrate this.

4.1. A Partial Integro-Differential Equation

Theorem 5. *The equation below with the initial conditions for all* $\rho_{ij} \geq 0$ *(*$i = 1, 2, \ldots, n \in \mathbb{N}$ *and* $j = 1, 2, \ldots, m \in \mathbb{N}$*) and* $\lambda_j \in \mathbb{R}$

$$\begin{cases} {}_c\dfrac{\partial^\rho}{\partial t^\rho} M(t,\sigma) + \displaystyle\sum_{j=1}^{m} \lambda_j \, I_1^{\rho_{1j}} \ldots I_n^{\rho_{nj}} M(t,\sigma) = f(t,\sigma), \quad (t,\sigma) \in [0,1] \times [0,1]^n, \\ M(0,\sigma) = \theta(\sigma), \quad M_t'(0,\sigma) = \beta(\sigma), \quad M_t''(0,\sigma) = \gamma(\sigma), \quad 2 < \rho \leq 3, \end{cases} \tag{11}$$

has a unique solution,

$$\begin{aligned} M(t,\sigma) &= \sum_{s=0}^{\infty} (-1)^s I_t^{\rho s + \rho} \sum_{s_1 + s_2 + \ldots + s_m = s} \binom{s}{s_1, s_2, \ldots, s_m} \lambda_1^{s_1} \ldots \lambda_m^{s_m} \\ &\quad I_1^{\rho_{11} s_1 + \ldots + \rho_{1m} s_m} \ldots I_n^{\rho_{n1} s_1 + \ldots + \rho_{nm} s_m} f(t,\sigma) \\ &\quad + \sum_{s=0}^{\infty} (-1)^s \frac{t^{\rho s + \rho}}{\Gamma(\rho s + \rho + 1)} \sum_{s_1 + s_2 + \ldots + s_m = s} \binom{s}{s_1, s_2, \ldots, s_m} \lambda_1^{s_1} \ldots \lambda_m^{s_m} \\ &\quad I_1^{\rho_{11} s_1 + \ldots + \rho_{1m} s_m} \ldots I_n^{\rho_{n1} s_1 + \ldots + \rho_{nm} s_m} \theta(\sigma) \\ &\quad + \sum_{s=0}^{\infty} (-1)^s \frac{t^{\rho s + \rho + 1}}{\Gamma(\rho s + \rho + 2)} \sum_{s_1 + s_2 + \ldots + s_m = s} \binom{s}{s_1, s_2, \ldots, s_m} \lambda_1^{s_1} \ldots \lambda_m^{s_m} \\ &\quad I_1^{\rho_{11} s_1 + \ldots + \rho_{1m} s_m} \ldots I_n^{\rho_{n1} s_1 + \ldots + \rho_{nm} s_m} \beta(\sigma) \\ &\quad + \sum_{s=0}^{\infty} (-1)^s \frac{t^{\rho s + \rho + 2}}{\Gamma(\rho s + \rho + 3)} \sum_{s_1 + s_2 + \ldots + s_m = s} \binom{s}{s_1, s_2, \ldots, s_m} \lambda_1^{s_1} \ldots \lambda_m^{s_m} \\ &\quad I_1^{\rho_{11} s_1 + \ldots + \rho_{1m} s_m} \ldots I_n^{\rho_{n1} s_1 + \ldots + \rho_{nm} s_m} \gamma(\sigma), \end{aligned}$$

where $f \in C([0,1] \times [0,1]^n)$*,* θ*,* β*, and* γ *are all in* $C([0,1]^n)$*.*

Proof. Applying I_t^ρ to Equation (11), we have

$$M(t,\sigma) - M(0,\sigma) - M_t'(0,\sigma) t - M_t''(0,\sigma) \frac{t^2}{2} + \sum_{j=1}^{m} \lambda_j \, I_t^\rho I_1^{\rho_{1j}} \ldots I_n^{\rho_{nj}} M(t,\sigma) = I_t^\rho f(t,\sigma),$$

which implies that

$$\left(1 + \sum_{j=1}^{m} \lambda_j \, I_t^\rho I_1^{\rho_{1j}} \ldots I_n^{\rho_{nj}}\right) M(t,\sigma) = I_t^\rho f(t,\sigma) + \theta(\sigma) + \beta(\sigma) t + \frac{t^2 \gamma(\sigma)}{2}. \tag{12}$$

We claim that the inverse operator of

$$1 + \sum_{j=1}^{m} \lambda_j \, I_t^\rho I_1^{\rho_{1j}} \ldots I_n^{\rho_{nj}}$$

is

$$\begin{aligned} V &= \left(1 + \sum_{j=1}^{m} \lambda_j \, I_t^\rho I_1^{\rho_{1j}} \ldots I_n^{\rho_{nj}}\right)^{-1} = \sum_{s=0}^{\infty} (-1)^s \left(\sum_{j=1}^{m} \lambda_j \, I_t^\rho I_1^{\rho_{1j}} \ldots I_n^{\rho_{nj}}\right)^s \\ &= \sum_{s=0}^{\infty} (-1)^s \sum_{s_1 + s_2 + \ldots + s_m = s} \binom{s}{s_1, s_2, \ldots, s_m} \left(\lambda_1 I_t^\rho I_1^{\rho_{11}} \ldots I_n^{\rho_{n1}}\right)^{s_1} \ldots \left(\lambda_m I_t^\rho I_1^{\rho_{1m}} \ldots I_n^{\rho_{nm}}\right)^{s_m} \\ &= \sum_{s=0}^{\infty} (-1)^s \sum_{s_1 + s_2 + \ldots + s_m = s} \binom{s}{s_1, s_2, \ldots, s_m} \lambda_1^{s_1} \ldots \lambda_m^{s_m} I_t^{\rho s} I_1^{\rho_{11} s_1 + \ldots + \rho_{1m} s_m} \ldots I_n^{\rho_{n1} s_1 + \ldots + \rho_{nm} s_m} \end{aligned}$$

in the space $C([0,1] \times [0,1]^n)$, where

$$\binom{s}{s_1, s_2, \ldots, s_m} = \frac{s!}{s_1! \ldots s_m!}.$$

Indeed, we have that for any $M \in C([0,1] \times [0,1]^n)$,

$$\begin{aligned}
\|VM\| &= \left\| \left(1 + \sum_{j=1}^{m} \lambda_j \, I_t^{\rho} I_1^{\rho_{1j}} \ldots I_n^{\rho_{nj}}\right)^{-1} M \right\| \\
&\leq \|M\| \sum_{s=0}^{\infty} \sum_{s_1+s_2+\ldots+s_m=s} \binom{s}{s_1, s_2, \ldots, s_m} |\lambda_1|^{s_1} \ldots |\lambda_m|^{s_m} \\
&\quad \cdot \left\| I_t^{\rho s} \right\| \left\| I_1^{\rho_{11}s_1+\ldots+\rho_{1m}s_m} \right\| \ldots \left\| I_n^{\rho_{n1}s_1+\ldots+\rho_{nm}s_m} \right\| \\
&\leq \|M\| \sum_{s=0}^{\infty} \sum_{s_1+s_2+\ldots+s_m=k} \binom{s}{s_1, s_2, \ldots, s_m} |\lambda_1|^{s_1} \ldots |\lambda_m|^{s_m} \\
&\quad \cdot \frac{1}{\Gamma(\rho s+1)} \frac{1}{\Gamma(\rho_{11}s_1 + \ldots + \rho_{1m}s_m + 1)} \cdots \frac{1}{\Gamma(\rho_{n1}s_1 + \ldots + \rho_{nm}s_m + 1)}.
\end{aligned}$$

Clearly, there exists a positive constant c such that

$$\begin{aligned}
&\Gamma(\rho_{11}s_1 + \ldots + \rho_{1m}s_m + 1) \geq c, \\
&\ldots, \\
&\Gamma(\rho_{n1}s_1 + \ldots + \rho_{nm}s_m + 1) \geq c,
\end{aligned}$$

for all $\rho_{ij} \geq 0$ and $s_i \geq 0$ for $i = 1, \ldots, m$.

This implies that

$$\begin{aligned}
&\sum_{s=0}^{\infty} \sum_{s_1+s_2+\ldots+s_m=s} \binom{s}{s_1, s_2, \ldots, s_m} |\lambda_1|^{s_1} \ldots |\lambda_m|^{s_m} \\
&\quad \cdot \frac{1}{\Gamma(\alpha s+1)} \frac{1}{\Gamma(\rho_{11}s_1 + \ldots + \rho_{1m}s_m + 1)} \cdots \frac{1}{\Gamma(\rho_{n1}s_1 + \ldots + \rho_{nm}s_m + 1)} \\
&\leq \frac{1}{c^n} \sum_{s=0}^{\infty} \sum_{s_1+s_2+\ldots+s_m=s} \binom{s}{s_1, s_2, \ldots, s_m} |\lambda_1|^{s_1} \ldots |\lambda_m|^{s_m} \frac{1}{\Gamma(\rho s+1)} \\
&= \frac{1}{c^n} E_{(\rho,\rho,\ldots,\rho),1}(|\lambda_1|, \ldots, |\lambda_m|) < +\infty,
\end{aligned}$$

where for $\alpha_i, \beta > 0$, and $z_i \in \mathbb{C}$,

$$\begin{aligned}
&E_{(\alpha_1,\alpha_2,\ldots,\alpha_m),\,\beta}(z_1, \ldots, z_m) \\
&= \sum_{s=0}^{\infty} \sum_{s_1+s_2+\ldots+s_m=s} \binom{s}{s_1, s_2, \ldots, s_m} \frac{z_1^{s_1} \ldots z_m^{s_m}}{\Gamma(\alpha_1 s_1 + \ldots + \alpha_m s_m + \beta)}
\end{aligned}$$

is the well-known multivariate Mittag-Leffler function [17], which is an entire function on complex plane $\mathbb{C}$.

Furthermore, we will show that

$$V\left(1 + \sum_{j=1}^{m} \lambda_j \, I_t^{\rho} I_1^{\rho_{1j}} \ldots I_n^{\rho_{nj}}\right) = \left(1 + \sum_{j=1}^{m} \lambda_j \, I_t^{\rho} I_1^{\rho_{1j}} \ldots I_n^{\rho_{nj}}\right) V = 1 \;\; \text{(identity operator)}.$$

In fact,

$$
\begin{aligned}
& V\left(1+\sum_{j=1}^{m} \lambda_j\, I_t^{\rho} I_1^{\rho_{1j}} \ldots I_n^{\rho_{nj}}\right) \\
& = 1+\sum_{s=1}^{\infty}(-1)^s\left(\sum_{j=1}^{m} \lambda_j\, I_t^{\rho} I_1^{\rho_{1j}} \ldots I_n^{\rho_{nj}}\right)^s+\sum_{s=0}^{\infty}(-1)^s\left(\sum_{j=1}^{m} \lambda_j\, I_t^{\rho} I_1^{\rho_{1j}} \ldots I_n^{\rho_{nj}}\right)^{s+1} \\
& = 1+\sum_{s=0}^{\infty}(-1)^{s+1}\left(\sum_{j=1}^{m} \lambda_j\, I_t^{\rho} I_1^{\rho_{1j}} \ldots I_n^{\rho_{nj}}\right)^{s+1}+\sum_{s=0}^{\infty}(-1)^s\left(\sum_{j=1}^{m} \lambda_j\, I_t^{\rho} I_1^{\rho_{1j}} \ldots I_n^{\rho_{nj}}\right)^{s+1} \\
& = 1.
\end{aligned}
$$

Similarly,

$$
\left(1+\sum_{j=1}^{m} \lambda_j\, I_t^{\rho} I_1^{\rho_{1j}} \ldots I_n^{\rho_{nj}}\right) V = 1,
$$

and V is unique.

From Equation (12), we come to

$$
\begin{aligned}
M(t,\sigma) = & \left(1+\sum_{j=1}^{m} \lambda_j\, I_t^{\rho} I_1^{\rho_{1j}} \ldots I_n^{\rho_{nj}}\right)^{-1}\left(I_t^{\rho} f(t,\sigma)+\theta(\sigma)+\beta(\sigma)t+\frac{t^2\gamma(\sigma)}{2}\right) \\
= & \sum_{s=0}^{\infty}(-1)^s \sum_{s_1+s_2+\ldots+s_m=s}\binom{s}{s_1,s_2,\ldots,s_m}\lambda_1^{s_1}\ldots\lambda_m^{s_m} I_t^{\rho s+\rho} \\
& I_1^{\rho_{11}s_1+\ldots+\rho_{1m}s_m} \ldots I_n^{\rho_{n1}s_1+\ldots+\rho_{nm}s_m} f(t,\sigma) \\
& +\sum_{s=0}^{\infty}(-1)^s \sum_{s_1+s_2+\ldots+s_m=s}\binom{s}{s_1,s_2,\ldots,s_m}\lambda_1^{s_1}\ldots\lambda_m^{s_m} I_t^{\rho s+\rho} \\
& I_1^{\rho_{11}s_1+\ldots+\rho_{1m}s_m} \ldots I_n^{\rho_{n1}s_1+\ldots+\rho_{nm}s_m} \theta(\sigma) \\
& +\sum_{s=0}^{\infty}(-1)^s \sum_{s_1+s_2+\ldots+s_m=s}\binom{s}{s_1,s_2,\ldots,s_m}\lambda_1^{s_1}\ldots\lambda_m^{s_m} I_t^{\rho s+\rho} t \\
& I_1^{\rho_{11}s_1+\ldots+\rho_{1m}s_m} \ldots I_n^{\rho_{n1}s_1+\ldots+\rho_{nm}s_m} \beta(\sigma) \\
& +\sum_{s=0}^{\infty}(-1)^s \sum_{s_1+s_2+\ldots+s_m=s}\binom{s}{s_1,s_2,\ldots,s_m}\lambda_1^{s_1}\ldots\lambda_m^{s_m} I_t^{\rho s+\rho} \frac{t^2}{2} \\
& I_1^{\rho_{11}s_1+\ldots+\rho_{1m}s_m} \ldots I_n^{\rho_{n1}s_1+\ldots+\rho_{nm}s_m} \gamma(\sigma).
\end{aligned}
$$

Using

$$
I_t^{\rho s+\rho} 1 = \frac{t^{\rho s+\rho}}{\Gamma(\rho s+\rho+1)}, \quad I_t^{\rho s+\rho} t = \frac{t^{\rho s+\rho+1}}{\Gamma(\rho s+\rho+2)}, \quad I_t^{\rho s+\rho} t^2/2 = \frac{t^{\rho s+\rho+2}}{\Gamma(\rho s+\rho+3)},
$$

we find that

$$
\begin{aligned}
M(t,\sigma) &= \left(1 + \sum_{j=1}^{m} \lambda_j\, I_t^{\rho} I_1^{\rho_{1j}} \dots I_n^{\rho_{nj}}\right)^{-1} \left(I_t^{\rho} f(t,\sigma) + \theta(\sigma) + \beta(\sigma)t + \frac{t^2\gamma(\sigma)}{2}\right) \\
&= \sum_{s=0}^{\infty} (-1)^s \sum_{s_1+s_2+\dots+s_m=s} \binom{s}{s_1, s_2, \dots, s_m} \lambda_1^{s_1} \dots \lambda_m^{s_m} I_t^{\rho s+\rho} \\
&\quad I_1^{\rho_{11}s_1+\dots+\rho_{1m}s_m} \dots I_n^{\rho_{n1}s_1+\dots+\rho_{nm}s_m} f(t,\sigma) \\
&\quad + \sum_{s=0}^{\infty} (-1)^s \frac{t^{\rho s+\rho}}{\Gamma(\rho s+\rho+1)} \sum_{s_1+s_2+\dots+s_m=s} \binom{s}{s_1, s_2, \dots, s_m} \lambda_1^{s_1} \dots \lambda_m^{s_m} \\
&\quad I_1^{\rho_{11}s_1+\dots+\rho_{1m}s_m} \dots I_n^{\rho_{n1}s_1+\dots+\rho_{nm}s_m} \theta(\sigma) \\
&\quad + \sum_{s=0}^{\infty} (-1)^s \frac{t^{\rho s+\rho+1}}{\Gamma(\rho s+\rho+2)} \sum_{s_1+s_2+\dots+s_m=s} \binom{s}{s_1, s_2, \dots, s_m} \lambda_1^{s_1} \dots \lambda_m^{s_m} \\
&\quad I_1^{\rho_{11}s_1+\dots+\rho_{1m}s_m} \dots I_n^{\rho_{n1}s_1+\dots+\rho_{nm}s_m} \beta(\sigma) \\
&\quad + \sum_{s=0}^{\infty} (-1)^s \frac{t^{\rho s+\rho+2}}{\Gamma(\rho s+\rho+3)} \sum_{s_1+s_2+\dots+s_m=s} \binom{s}{s_1, s_2, \dots, s_m} \lambda_1^{s_1} \dots \lambda_m^{s_m} \\
&\quad I_1^{\rho_{11}s_1+\dots+\rho_{1m}s_m} \dots I_n^{\rho_{n1}s_1+\dots+\rho_{nm}s_m} \gamma(\sigma)
\end{aligned}
$$

in the space $C([0,1] \times [0,1]^n)$. Moreover,

$$
\begin{aligned}
\|M\| \leq\ & \|f\| \sum_{s=0}^{\infty} \frac{1}{\Gamma(\rho s+\rho+1)} \sum_{s_1+s_2+\dots+s_m=s} \binom{s}{s_1, s_2, \dots, s_m} |\lambda_1|^{s_1} \dots |\lambda_m|^{s_m} \\
& \cdot \frac{1}{\Gamma(\rho_{11}s_1+\dots+\rho_{1m}s_m+1)} \cdots \frac{1}{\Gamma(\rho_{n1}s_1+\dots+\rho_{nm}s_m+1)} \\
& + \|\theta\| \sum_{s=0}^{\infty} \frac{1}{\Gamma(\rho s+\rho+1)} \sum_{s_1+s_2+\dots+s_m=s} \binom{s}{s_1, s_2, \dots, s_m} |\lambda_1|^{s_1} \dots |\lambda_m|^{s_m} \\
& \cdot \frac{1}{\Gamma(\rho_{11}s_1+\dots+\rho_{1m}s_m+1)} \cdots \frac{1}{\Gamma(\rho_{n1}s_1+\dots+\rho_{nm}s_m+1)} \\
& + \|\beta\| \sum_{s=0}^{\infty} \frac{1}{\Gamma(\rho s+\rho+2)} \sum_{s_1+s_2+\dots+s_m=s} \binom{s}{s_1, s_2, \dots, s_m} |\lambda_1|^{s_1} \dots |\lambda_m|^{s_m} \\
& \cdot \frac{1}{\Gamma(\rho_{11}s_1+\dots+\rho_{1m}s_m+1)} \cdots \frac{1}{\Gamma(\rho_{n1}s_1+\dots+\rho_{nm}s_m+1)} \\
& + \|\gamma\| \sum_{s=0}^{\infty} \frac{1}{\Gamma(\rho s+\rho+3)} \sum_{s_1+s_2+\dots+s_m=s} \binom{s}{s_1, s_2, \dots, s_m} |\lambda_1|^{s_1} \dots |\lambda_m|^{s_m} \\
& \cdot \frac{1}{\Gamma(\rho_{11}s_1+\dots+\rho_{1m}s_m+1)} \cdots \frac{1}{\Gamma(\rho_{n1}s_1+\dots+\rho_{nm}s_m+1)} < +\infty.
\end{aligned}
$$

The uniqueness follows immediately from the fact that the equation

$$
\begin{cases}
\dfrac{{}_c\partial^{\rho}}{\partial t^{\rho}} M(t,\sigma) + \displaystyle\sum_{j=1}^{m} \lambda_j\, I_1^{\rho_{1j}} \dots I_n^{\rho_{nj}} M(t,\sigma) = 0, \quad (t,\sigma) \in [0,1] \times [0,1]^n, \\
M(0,\sigma) = 0, \quad M_t'(0,\sigma) = 0, \quad M_t''(0,\sigma) = 0, \quad 2 < \rho \leq 3,
\end{cases}
$$

only has solution zero. We complete the proof. □

In particular, the following partial integro-differential equation,

$$
\begin{cases}
\dfrac{{}_c\partial^{\rho}}{\partial t^{\rho}} M(t,\sigma) + \displaystyle\sum_{j=1}^{m} \lambda_j\, I_1^{\rho_{1j}} \dots I_n^{\rho_{nj}} M(t,\sigma) = f(t,\sigma), \quad (t,\sigma) \in [0,1] \times [0,1]^n, \\
M(0,\sigma) = 0, \quad M_t'(0,\sigma) = 0, \quad M_t''(0,\sigma) = 0, \quad 2 < \rho \leq 3,
\end{cases}
$$

has a unique solution,

$$M(t,\sigma) = \sum_{s=0}^{\infty}(-1)^s \sum_{s_1+s_2+\ldots+s_m=s} \binom{s}{s_1, s_2, \ldots, s_m} \lambda_1^{s_1} \ldots \lambda_m^{s_m} I_t^{\rho s+\rho}$$
$$I_1^{\rho_{11}s_1+\ldots+\rho_{1m}s_m} \ldots I_n^{\rho_{n1}s_1+\ldots+\rho_{nm}s_m} f(t,\sigma).$$

Remark 4. *Using Banach's contractive principle, we are able to study the uniqueness for the following nonlinear equation with the initial conditions for all* $\rho_{ij} \geq 0$ *(*$i = 1, 2, \ldots, n \in \mathbb{N}$ *and* $j = 1, 2, \ldots, m \in \mathbb{N}$*) and* $\lambda_j \in \mathbb{R}$.

$$\begin{cases} \frac{{}_c\partial^\rho}{\partial t^\rho} M(t,\sigma) + \sum_{j=1}^{m} \lambda_j \, I_1^{\rho_{1j}} \ldots I_n^{\rho_{nj}} M(t,\sigma) = f(t,\sigma, M(t,\sigma)), \quad (t,\sigma) \in [0,1] \times [0,1]^n, \\ M(0,\sigma) = \theta(\sigma), \quad M_t'(0,\sigma) = \beta(\sigma), \quad M_t''(0,\sigma) = \gamma(\sigma), \quad 2 < \rho \leq 3, \end{cases}$$

where $f \in C([0,1] \times [0,1]^n \times \mathbb{R})$, $\theta,\ \beta,$ *and* γ *are all in* $C([0,1]^n)$ *or the boundary value problem:*

$$\begin{cases} \frac{{}_c\partial^\rho}{\partial t^\rho} M(t,\sigma) + \sum_{j=1}^{m} \lambda_j \, I_1^{\rho_{1j}} \ldots I_n^{\rho_{nj}} M(t,\sigma) = f(t,\sigma, M(t,\sigma)), \quad (t,\sigma) \in [0,1] \times [0,1]^n, \\ M(0,\sigma) = \theta(\sigma), \quad M(1,\sigma) = \beta(\sigma), \quad M_t'(1,\sigma) = \gamma(\sigma), \quad 2 < \rho \leq 3. \end{cases}$$

4.2. A Multi-Term Time-Fractional Convection Problem

Theorem 6. *Let* $m,\ n \in \mathbb{N},\ 0 < \rho_1 < \rho_2 < \ldots < \rho_m < \rho \leq 1$, $\beta_i \in \mathbb{R}$ *for* $i = 1, 2, \ldots, m$. *In addition, we assume that* λ_j *is a function of* σ_j *only in* $C[0,1]$. *Then, the multi-term time-fractional convection Equation (5) has a unique solution,*

$$M(t,\sigma) = \sum_{s=0}^{\infty}(-1)^s \sum_{s_1+\ldots+s_{m+n}=s} \binom{s}{s_1, \ldots, s_{m+n}} \beta_1^{s_1} \ldots \beta_m^{s_m}$$
$$\cdot\, I_t^{(\rho-\rho_1)s_1+\ldots+(\rho-\rho_m)s_m+\rho s_{m+1}+\ldots+\rho s_{m+n}+\rho} \left(\lambda_1(\sigma_1)\frac{\partial}{\partial \sigma_1}\right)^{s_{m+1}} \ldots \left(\lambda_n(\sigma_n)\frac{\partial}{\partial \sigma_n}\right)^{s_{m+n}} f_1(t,\sigma)$$
$$+ \sum_{s=0}^{\infty}(-1)^s \sum_{s_1+\ldots+s_{m+n}=s} \binom{s}{s_1, \ldots, s_{m+n}} \beta_1^{s_1} \ldots \beta_m^{s_m}$$
$$\cdot\, \frac{t^{(\rho-\rho_1)s_1+\ldots+(\rho-\rho_m)s_m+\rho s_{m+1}+\ldots+\rho s_{m+n}}}{\Gamma((\rho-\rho_1)s_1+\ldots+(\rho-\rho_m)s_m+\rho s_{m+1}+\ldots+\rho s_{m+n}+1)}$$
$$\cdot \left(\lambda_1(\sigma_1)\frac{\partial}{\partial \sigma_1}\right)^{s_{m+1}} \ldots \left(\lambda_n(\sigma_n)\frac{\partial}{\partial \sigma_n}\right)^{s_{m+n}} f_2(\sigma)$$
$$+ \sum_{i=1}^{m} \beta_i \sum_{s=0}^{\infty}(-1)^s \sum_{s_1+\ldots+s_{m+n}=s} \binom{s}{s_1, \ldots, s_{m+n}} \beta_1^{s_1} \ldots \beta_m^{s_m}$$
$$\cdot\, \frac{t^{(\rho-\rho_1)s_1+\ldots+(\rho-\rho_m)s_m+\rho s_{m+1}+\ldots+\rho s_{m+n}+\rho-\rho_i}}{\Gamma((\rho-\rho_1)s_1+\ldots+(\rho-\rho_m)s_m+\rho s_{m+1}+\ldots+\rho s_{m+n}+\rho-\rho_i+1)}$$
$$\cdot \left(\lambda_1(\sigma_1)\frac{\partial}{\partial \sigma_1}\right)^{s_{m+1}} \ldots \left(\lambda_n(\sigma_n)\frac{\partial}{\partial \sigma_n}\right)^{s_{m+n}} f_2(\sigma)$$

in the space $C([0,1] \times [0,1]^n)$, *if* $f_1, f_2 \in S$, *which is given by*

$$S = \Big\{ f(t,\sigma) \in C([0,1] \times [0,1]^n) : \ \exists \text{ a constant } M_{f,\lambda_1,\ldots,\lambda_n} > 0 \text{ such that}$$
$$\sup_{(t,\sigma)\in[0,1]\times[0,1]^n} \left| \left(\lambda_1(\sigma_1)\frac{\partial}{\partial \sigma_1}\right)^{s_1} \ldots \left(\lambda_n(\sigma_n)\frac{\partial}{\partial \sigma_n}\right)^{s_n} f(t,\sigma) \right| \leq M_{f,\lambda_1,\ldots,\lambda_n}^{s_1+\ldots+s_n} \Big\},$$

where $(s_1, \ldots, s_n) \in (\mathbb{N} \cup \{0\})^n$.

Proof. Applying I_t^{ρ} to Equation (5), we obtain

$$M(t,\sigma) - M(0,\sigma) + \sum_{i=1}^{m} \beta_i \, I_t^{\rho-\rho_i} I_t^{\rho_i} \frac{{}^c\partial^{\rho_j}}{\partial t^{\rho_j}} M(t,\sigma) + \sum_{j=1}^{n} I_t^{\rho} \lambda_j(\sigma_j) \frac{\partial}{\partial \sigma_j} M(t,\sigma) = I_t^{\rho} f_1(t,\sigma),$$

and

$$\sum_{i=1}^{m} \beta_i \, I_t^{\rho-\rho_i} I_t^{\rho_i} \frac{{}^c\partial^{\rho_j}}{\partial t^{\rho_j}} M(t,\sigma) = \sum_{i=1}^{m} \beta_i \, I_t^{\rho-\rho_i} M(t,\sigma) - \sum_{i=1}^{m} \beta_i \, \frac{t^{\rho-\rho_i}}{\Gamma(\rho-\rho_i+1)} f_2(\sigma).$$

Hence,

$$\begin{aligned} &\left(1 + \sum_{i=1}^{m} \beta_i \, I_t^{\rho-\rho_i} + \sum_{j=1}^{n} I_t^{\rho} \lambda_j(\sigma_j) \frac{\partial}{\partial \sigma_j}\right) M(t,\sigma) \\ &= I_t^{\rho} f_1(t,\sigma) + f_2(\sigma) + \sum_{i=1}^{m} \beta_i \, \frac{t^{\rho-\rho_i}}{\Gamma(\rho-\rho_i+1)} f_2(\sigma). \end{aligned} \tag{13}$$

To find the inverse operator of

$$1 + \sum_{i=1}^{m} \beta_i \, I_t^{\rho-\rho_i} + \sum_{j=1}^{n} I_t^{\rho} \lambda_j(\sigma_j) \frac{\partial}{\partial \sigma_j},$$

we begin to define the operator V as

$$\begin{aligned} V &= \sum_{s=0}^{\infty} (-1)^s \left(\sum_{i=1}^{m} \beta_i \, I_t^{\rho-\rho_i} + \sum_{j=1}^{n} I_t^{\rho} \lambda_j(\sigma_j) \frac{\partial}{\partial \sigma_j} \right)^s \\ &= \sum_{s=0}^{\infty} (-1)^s \sum_{s_1+\ldots+s_{m+n}=s} \binom{s}{s_1,\ldots,s_{m+n}} \beta_1^{s_1} \ldots \beta_m^{s_m} \\ &\quad \cdot I_t^{(\rho-\rho_1)s_1+\ldots+(\rho-\rho_m)s_m+\rho s_{m+1}+\ldots+\rho s_{m+n}} \left(\lambda_1(\sigma_1)\frac{\partial}{\partial \sigma_1}\right)^{s_{m+1}} \ldots \left(\lambda_n(\sigma_n)\frac{\partial}{\partial \sigma_n}\right)^{s_{m+n}}. \end{aligned}$$

Then, V is well defined on S. Indeed, for any function $f(t,x) \in S$, we have

$$\begin{aligned} \|Vf\| &\leq \sum_{s=0}^{\infty} \sum_{s_1+\ldots+s_{m+n}=s} \binom{s}{s_1,\ldots,s_{m+n}} |\beta_1|^{s_1} \ldots |\beta_m|^{s_m} \\ &\quad \cdot \left\| I_t^{(\rho-\rho_1)s_1+\ldots+(\rho-\rho_m)s_m+\rho s_{m+1}+\ldots+\rho s_{m+n}} \right\| \\ &\quad \cdot \sup_{(t,\sigma)\in[0,1]\times[0,1]^n} \left| \left(\lambda_1(\sigma_1)\frac{\partial}{\partial \sigma_1}\right)^{s_{m+1}} \ldots \left(\lambda_n(\sigma_n)\frac{\partial}{\partial \sigma_n}\right)^{s_{m+n}} f(t,\sigma) \right| \\ &\leq \sum_{s=0}^{\infty} \sum_{s_1+\ldots+s_{m+n}=s} \binom{s}{s_1,\ldots,s_{m+n}} |\beta_1|^{s_1} \ldots |\beta_m|^{s_m} M_{f,\lambda_1,\ldots,\lambda_n}^{s_{m+1}} \ldots M_{f,\lambda_1,\ldots,\lambda_n}^{s_{m+n}} \\ &\quad \cdot \frac{1}{\Gamma((\rho-\rho_1)s_1+\ldots+(\rho-\rho_m)s_m+\rho s_{m+1}+\ldots+\rho s_{m+n}+1)} \\ &= E_{(\rho-\rho_1,\ldots,\rho-\rho_m,\rho,\ldots,\rho),1}\left(|\beta_1|,\ldots,|\beta_m|, M_{f,\lambda_1,\ldots,\lambda_n},\ldots,M_{f,\lambda_1,\ldots,\lambda_n}\right) < +\infty. \end{aligned}$$

Moreover, V is an inverse operator since

$$\begin{aligned} &V\left(1 + \sum_{i=1}^{m} \beta_i \, I_t^{\rho-\rho_i} + \sum_{j=1}^{n} I_t^{\rho} \lambda_j(\sigma_j) \frac{\partial}{\partial \sigma_j}\right) \\ &= \left(1 + \sum_{i=1}^{m} \beta_i \, I_t^{\rho-\rho_i} + \sum_{j=1}^{n} I_t^{\rho} \lambda_j(\sigma_j) \frac{\partial}{\partial \sigma_j}\right) V = 1. \end{aligned}$$

In fact,

$$V\left(1+\sum_{i=1}^{m}\ \beta_i\ I_t^{\rho-\rho_i}+\sum_{j=1}^{n}I_t^{\rho}\lambda_j(\sigma_j)\frac{\partial}{\partial\sigma_j}\right)$$
$$=V+\sum_{s=0}^{\infty}(-1)^s\left(\sum_{i=1}^{m}\ \beta_i\ I_t^{\rho-\rho_i}+\sum_{j=1}^{n}I_t^{\rho}\lambda_j(\sigma_j)\frac{\partial}{\partial\sigma_j}\right)^{s+1}$$
$$=1+\sum_{s=1}^{\infty}(-1)^s\left(\sum_{i=1}^{m}\ \beta_i\ I_t^{\rho-\rho_i}+\sum_{j=1}^{n}I_t^{\rho}\lambda_j(\sigma_j)\frac{\partial}{\partial\sigma_j}\right)^{s}$$
$$+\sum_{s=0}^{\infty}(-1)^s\left(\sum_{i=1}^{m}\ \beta_i\ I_t^{\rho-\rho_i}+\sum_{j=1}^{n}I_t^{\rho}\lambda_j(\sigma_j)\frac{\partial}{\partial\sigma_j}\right)^{s+1}$$
$$=1.$$

Similarly,

$$\left(1+\sum_{i=1}^{m}\ \beta_i\ I_t^{\rho-\rho_i}+\sum_{j=1}^{n}I_t^{\rho}\lambda_j(\sigma_j)\frac{\partial}{\partial\sigma_j}\right)V=1,$$

and V is unique. We note that if $f_2\in S$, then

$$\sum_{i=1}^{m}\ \beta_i\ \frac{t^{\rho-\rho_i}}{\Gamma(\rho-\rho_i+1)}f_2(\sigma)\in S,$$

which implies from Equation (13) that

$$M(t,\sigma)=V\left(I_t^{\rho}f_1(t,\sigma)+f_2(\sigma)+\sum_{i=1}^{m}\ \beta_i\ \frac{t^{\rho-\rho_i}}{\Gamma(\rho-\rho_i+1)}f_2(\sigma)\right)$$
$$=\sum_{s=0}^{\infty}(-1)^s\sum_{s_1+\ldots+s_{m+n}=s}\binom{s}{s_1,\ldots,s_{m+n}}\beta_1^{s_1}\ldots\beta_m^{s_m}$$
$$\cdot I_t^{(\rho-\rho_1)s_1+\ldots+(\rho-\rho_m)s_m+\rho s_{m+1}+\ldots+\rho s_{m+n}+\rho}\left(\lambda_1(\sigma_1)\frac{\partial}{\partial\sigma_1}\right)^{s_{m+1}}\ldots\left(\lambda_n(\sigma_n)\frac{\partial}{\partial\sigma_n}\right)^{s_{m+n}}f_1(t,\sigma)$$
$$+\sum_{s=0}^{\infty}(-1)^s\sum_{s_1+\ldots+s_{m+n}=s}\binom{s}{s_1,\ldots,s_{m+n}}\beta_1^{s_1}\ldots\beta_m^{s_m}$$
$$\cdot\frac{t^{(\rho-\rho_1)s_1+\ldots+(\rho-\rho_m)s_m+\rho s_{m+1}+\ldots+\rho s_{m+n}}}{\Gamma((\rho-\rho_1)s_1+\ldots+(\rho-\rho_m)s_m+\rho s_{m+1}+\ldots+\rho s_{m+n}+1)}$$
$$\cdot\left(\lambda_1(\sigma_1)\frac{\partial}{\partial\sigma_1}\right)^{s_{m+1}}\ldots\left(\lambda_n(\sigma_n)\frac{\partial}{\partial\sigma_n}\right)^{s_{m+n}}f_2(\sigma)$$
$$+\sum_{i=1}^{m}\beta_i\sum_{s=0}^{\infty}(-1)^s\sum_{s_1+\ldots+s_{m+n}=s}\binom{s}{s_1,\ldots,s_{m+n}}\beta_1^{s_1}\ldots\beta_m^{s_m}$$
$$\cdot\frac{t^{(\rho-\rho_1)s_1+\ldots+(\rho-\rho_m)s_m+\rho s_{m+1}+\ldots+\rho s_{m+n}+\rho-\rho_i}}{\Gamma((\rho-\rho_1)s_1+\ldots+(\rho-\rho_m)s_m+\rho s_{m+1}+\ldots+\rho s_{m+n}+\rho-\rho_i+1)}$$
$$\cdot\left(\lambda_1(\sigma_1)\frac{\partial}{\partial\sigma_1}\right)^{s_{m+1}}\ldots\left(\lambda_n(\sigma_n)\frac{\partial}{\partial\sigma_n}\right)^{s_{m+n}}f_2(\sigma).$$

The uniqueness follows from the fact that the equation

$$\begin{cases}\frac{{}_c\partial^{\rho}}{\partial t^{\rho}}M(t,\sigma)+\sum_{i=1}^{m}\ \beta_i\ \frac{{}_c\partial^{\rho_j}}{\partial t^{\rho_j}}M(t,\sigma)+\sum_{j=1}^{n}\lambda_j(\sigma_j)\frac{\partial}{\partial\sigma_j}M(t,\sigma)\\ =0,\quad (t,\sigma)\in[0,1]\times[0,1]^n,\\ M(0,\sigma)=0,\end{cases}$$

only has solution zero. This completes the proof. □

In particular, if $\rho = 1$, $\beta_1 = \ldots = \beta_m = 0$, and $\lambda_1(x_1) = \ldots = \lambda_n(x_n) = 1$, then Equation (5) turns out to be

$$\begin{cases} \frac{\partial}{\partial t}M(t,\sigma) + \nabla M(t,\sigma) = f_1(t,\sigma), \quad (t,\sigma) \in [0,1] \times [0,1]^n, \\ M(0,\sigma) = f_2(\sigma), \end{cases} \tag{14}$$

which has the solution from Theorem 6 (derived for the first time):

$$M(t,\sigma) = \sum_{s=0}^{\infty}(-1)^s I_t^{s+1}\nabla^s f_1(t,\sigma) + \sum_{s=0}^{\infty}(-1)^s \frac{t^s}{s!}\nabla^s f_2(\sigma). \tag{15}$$

Thus, the following equation,

$$\begin{cases} \frac{\partial}{\partial t}M(t,\sigma) + \nabla M(t,\sigma) = t(\sigma_1 + \sigma_2 + \ldots + \sigma_n), \quad (t,\sigma) \in [0,1] \times [0,1]^n, \\ M(0,\sigma) = \sin\sigma_1, \end{cases}$$

has the solution

$$\begin{aligned} M(t,\sigma) &= \sum_{s=0}^{\infty}(-1)^s I_t^{s+1} t\nabla^s(\sigma_1 + \sigma_2 + \ldots + \sigma_n) + \sum_{s=0}^{\infty}(-1)^s \frac{t^s}{s!}\nabla^s \sin\sigma_1 \\ &= I_t t(\sigma_1 + \sigma_2 + \ldots + \sigma_n) - I_t^2 t\nabla(\sigma_1 + \sigma_2 + \ldots + \sigma_n) \\ &\quad + \sum_{s=0}^{\infty}(-1)^s \frac{t^s}{s!}\sin(\sigma_1 + s\pi/2) \\ &= \frac{t^2}{2}(\sigma_1 + \sigma_2 + \ldots + \sigma_n) - \frac{nt^3}{6} + \sum_{s=0}^{\infty}(-1)^s \frac{t^s}{s!}\sin(\sigma_1 + s\pi/2). \end{aligned}$$

Example 4. *The following equation for* $0 < \rho_1 < \rho_2 < \ldots < \rho_m < \rho \leq 1$, $\beta_i \in \mathbb{R}$, *and* $m, n \in \mathbb{N}$,

$$\begin{cases} {}_c\frac{\partial^\rho}{\partial t^\rho}M(t,\sigma) + \sum_{i=1}^{m} \beta_i \, {}_c\frac{\partial^{\rho_j}}{\partial t^{\rho_j}}M(t,\sigma) + \sum_{j=1}^{n} \sigma_j \frac{\partial}{\partial \sigma_j}M(t,\sigma) \\ = t\sigma_1^2 \ldots \sigma_n^{n+1}, \quad (t,\sigma) \in [0,1] \times [0,1]^n, \\ M(0,\sigma) = 1, \end{cases}$$

has a unique solution,

$$\begin{aligned} M(t,\sigma) &= \sum_{s=0}^{\infty}(-1)^s \sum_{s_1+\ldots+s_{m+n}=s} \binom{s}{s_1,\ldots,s_{m+n}} \beta_1^{s_1}\ldots\beta_m^{s_m} 2^{s_{m+1}}\ldots(n+1)^{s_{m+n}} \\ &\frac{t^{(\rho-\rho_1)s_1+\ldots+(\rho-\rho_m)s_m+\rho s_{m+1}+\ldots+\rho s_{m+n}+\rho+1}\sigma_1^2\ldots\sigma_n^{n+1}}{\Gamma((\rho-\rho_1)s_1+\ldots+(\rho-\rho_m)s_m+\rho s_{m+1}+\ldots+\rho s_{m+n}+\rho+2)} \\ &+ \sum_{s=0}^{\infty}(-1)^s \sum_{s_1+\ldots+s_m=s} \binom{s}{s_1,\ldots,s_m} \beta_1^{s_1}\ldots\beta_m^{s_m} \\ &\cdot \frac{t^{(\rho-\rho_1)s_1+\ldots+(\rho-\rho_m)s_m}}{\Gamma((\rho-\rho_1)s_1+\ldots+(\rho-\rho_m)s_m+1)} \\ &+ \sum_{i=1}^{m}\beta_i \sum_{s=0}^{\infty}(-1)^s \sum_{s_1+\ldots+s_m=s} \binom{s}{s_1,\ldots,s_m} \beta_1^{s_1}\ldots\beta_m^{s_m} \\ &\cdot \frac{t^{(\rho-\rho_1)s_1+\ldots+(\rho-\rho_m)s_m+\rho-\rho_i}}{\Gamma((\rho-\rho_1)s_1+\ldots+(\rho-\rho_m)s_m+\rho-\rho_i+1)}. \end{aligned}$$

Proof. From Theorem 6, we have

$$
\begin{aligned}
M(t,\sigma) &= \sum_{s=0}^{\infty}(-1)^s \sum_{s_1+\ldots+s_{m+n}=s} \binom{s}{s_1,\ldots,s_{m+n}} \beta_1^{s_1}\ldots\beta_m^{s_m} \\
&\cdot I_t^{(\rho-\rho_1)s_1+\ldots+(\rho-\rho_m)s_m+\rho s_{m+1}+\ldots+\rho s_{m+n}+\rho} \left(\sigma_1\frac{\partial}{\partial\sigma_1}\right)^{s_{m+1}}\ldots\left(\sigma_n\frac{\partial}{\partial\sigma_n}\right)^{s_{m+n}} t\sigma_1^2\ldots\sigma_n^{n+1} \\
&+ \sum_{s=0}^{\infty}(-1)^s \sum_{s_1+\ldots+s_{m+n}=s} \binom{s}{s_1,\ldots,s_{m+n}} \beta_1^{s_1}\ldots\beta_m^{s_m} \\
&\cdot \frac{t^{(\rho-\rho_1)s_1+\ldots+(\rho-\rho_m)s_m+\rho s_{m+1}+\ldots+\rho s_{m+n}}}{\Gamma((\rho-\rho_1)s_1+\ldots+(\rho-\rho_m)s_m+\rho s_{m+1}+\ldots+\rho s_{m+n}+1)} \\
&\cdot \left(\sigma_1\frac{\partial}{\partial\sigma_1}\right)^{s_{m+1}}\ldots\left(\sigma_n\frac{\partial}{\partial\sigma_n}\right)^{s_{m+n}} 1 \\
&+ \sum_{i=1}^{m}\beta_i \sum_{s=0}^{\infty}(-1)^s \sum_{s_1+\ldots+s_{m+n}=s} \binom{s}{s_1,\ldots,s_{m+n}} \beta_1^{s_1}\ldots\beta_m^{s_m} \\
&\cdot \frac{t^{(\rho-\rho_1)s_1+\ldots+(\rho-\rho_m)s_m+\rho s_{m+1}+\ldots+\rho s_{m+n}+\rho-\rho_i}}{\Gamma((\rho-\rho_1)s_1+\ldots+(\rho-\rho_m)s_m+\rho s_{m+1}+\ldots+\rho s_{m+n}+\rho-\rho_i+1)} \\
&\cdot \left(\sigma_1\frac{\partial}{\partial\sigma_1}\right)^{s_{m+1}}\ldots\left(\sigma_n\frac{\partial}{\partial\sigma_n}\right)^{s_{m+n}} 1.
\end{aligned}
$$

Using

$$
\begin{aligned}
& I_t^{(\rho-\rho_1)s_1+\ldots+(\rho-\rho_m)s_m+\rho s_{m+1}+\ldots+\rho s_{m+n}+\rho} t \\
&= \frac{t^{(\rho-\rho_1)s_1+\ldots+(\rho-\rho_m)s_m+\rho s_{m+1}+\ldots+\rho s_{m+n}+\rho+1}}{\Gamma((\rho-\rho_1)s_1+\ldots+(\rho-\rho_m)s_m+\rho s_{m+1}+\ldots+\rho s_{m+n}+\rho+2)}, \\
& \left(\sigma_1\frac{\partial}{\partial\sigma_1}\right)^{s_{m+1}} \sigma_1^2 = 2^{s_{m+1}}\sigma_1^2, \\
& \ldots, \\
& \left(\sigma_n\frac{\partial}{\partial\sigma_n}\right)^{s_{m+n}} \sigma_n^{n+1} = (n+1)^{s_{m+n}}\sigma_n^{n+1},
\end{aligned}
$$

we obtain

$$
\begin{aligned}
M(t,\sigma) &= \sum_{s=0}^{\infty}(-1)^s \sum_{s_1+\ldots+s_{m+n}=s} \binom{s}{s_1,\ldots,s_{m+n}} \beta_1^{s_1}\ldots\beta_m^{s_m} 2^{s_{m+1}}\ldots(n+1)^{s_{m+n}} \\
&\frac{t^{(\rho-\rho_1)s_1+\ldots+(\rho-\rho_m)s_m+\rho s_{m+1}+\ldots+\rho s_{m+n}+\rho+1}\sigma_1^2\ldots\sigma_n^{n+1}}{\Gamma((\rho-\rho_1)s_1+\ldots+(\rho-\rho_m)s_m+\rho s_{m+1}+\ldots+\rho s_{m+n}+\rho+2)} \\
&+ \sum_{s=0}^{\infty}(-1)^s \sum_{s_1+\ldots+s_m=s} \binom{s}{s_1,\ldots,s_m} \beta_1^{s_1}\ldots\beta_m^{s_m} \\
&\cdot \frac{t^{(\rho-\rho_1)s_1+\ldots+(\rho-\rho_m)s_m}}{\Gamma((\rho-\rho_1)s_1+\ldots+(\rho-\rho_m)s_m+1)} \\
&+ \sum_{i=1}^{m}\beta_i \sum_{s=0}^{\infty}(-1)^s \sum_{s_1+\ldots+s_m=s} \binom{s}{s_1,\ldots,s_m} \beta_1^{s_1}\ldots\beta_m^{s_m} \\
&\cdot \frac{t^{(\rho-\rho_1)s_1+\ldots+(\rho-\rho_m)s_m+\rho-\rho_i}}{\Gamma((\rho-\rho_1)s_1+\ldots+(\rho-\rho_m)s_m+\rho-\rho_i+1)}.
\end{aligned}
$$

We complete the proof. □

4.3. A Generalized Time-Fractional Diffusion-Wave Equation

Theorem 7. *Let $\triangle = \sum_{i=1}^{n} \frac{\partial^2}{\partial \sigma_i^2}$. Assume that all λ_j for $j = 1, 2, \ldots, m$ are arbitrary constants, $1 < \rho_1 < \rho_2 < \ldots < \rho_m < \rho \leq 2$, and all θ, β, and g are in S_0 given by*

$$S_0 = \Big\{ g \in C([0,1] \times [0,1]^n) : \exists \text{ a constant } M_g > 0 \text{ such that } \sup_{(t,\sigma)\in[0,1]\times[0,1]^n} \left| \frac{\partial^{2s_1}}{\partial \sigma_1^{2s_1}} \cdots \frac{\partial^{2s_n}}{\partial \sigma_n^{2s_n}} g(t,\sigma) \right| \leq M_g^{s_1+\ldots+s_n} \Big\},$$

where $(s_1, s_2, \ldots, s_n) \in (\mathbb{N} \cup \{0\})^n$. Then, Equation (6) has a unique solution:

$$\begin{aligned}
M(t,\sigma) =& \\
&\sum_{s=0}^{\infty} (-1)^s \sum_{s_1+\ldots+s_{m+1}=s} \binom{s}{s_1, s_2, \ldots, s_{m+1}} \lambda_1^{s_1} \ldots \lambda_m^{s_m} (-1)^{s_{m+1}} \\
&\cdot I_t^{(\rho-\rho_1)s_1+\ldots+(\rho-\rho_m)s_m+\rho s_{m+1}+\rho} \triangle^{l s_{m+1}} g(t,\sigma) \\
&+ \sum_{s=0}^{\infty} (-1)^s \sum_{s_1+\ldots+s_{m+1}=s} \binom{s}{s_1, s_2, \ldots, s_{m+1}} \lambda_1^{s_1} \ldots \lambda_m^{s_m} (-1)^{s_{m+1}} \\
&\cdot \frac{t^{(\rho-\rho_1)s_1+\ldots+(\rho-\rho_m)s_m+\rho s_{m+1}}}{\Gamma((\rho-\rho_1)s_1+\ldots+(\rho-\rho_m)s_m+\rho s_{m+1}+1)} \triangle^{l s_{m+1}} \theta(\sigma) \\
&+ \sum_{s=0}^{\infty} (-1)^s \sum_{s_1+\ldots+s_{m+1}=s} \binom{s}{s_1, s_2, \ldots, s_{m+1}} \lambda_1^{s_1} \ldots \lambda_m^{s_m} (-1)^{s_{m+1}} \\
&\cdot \frac{t^{(\rho-\rho_1)s_1+\ldots+(\rho-\rho_m)s_m+\rho s_{m+1}+1}}{\Gamma((\rho-\rho_1)s_1+\ldots+(\rho-\rho_m)s_m+\rho s_{m+1}+2)} \triangle^{l s_{m+1}} \beta(\sigma) \\
&+ \sum_{j=1}^{m} \frac{\lambda_j}{\Gamma(\rho-\rho_j+1)} \sum_{s=0}^{\infty} (-1)^s \sum_{s_1+\ldots+s_{m+1}=s} \binom{s}{s_1, s_2, \ldots, s_{m+1}} \lambda_1^{s_1} \ldots \lambda_m^{s_m} (-1)^{s_{m+1}} \\
&\cdot \frac{t^{(\rho-\rho_1)s_1+\ldots+(\rho-\rho_m)s_m+\rho s_{m+1}+\rho-\rho_j}}{\Gamma((\rho-\rho_1)s_1+\ldots+(\rho-\rho_m)s_m+\rho s_{m+1}+\rho-\rho_j+1)} \triangle^{l s_{m+1}} \theta(\sigma) \\
&+ \sum_{j=1}^{m} \frac{\lambda_j}{\Gamma(\rho-\rho_j+2)} \sum_{s=0}^{\infty} (-1)^s \sum_{s_1+\ldots+s_{m+1}=s} \binom{s}{s_1, s_2, \ldots, s_{m+1}} \lambda_1^{s_1} \ldots \lambda_m^{s_m} (-1)^{s_{m+1}} \\
&\cdot \frac{t^{(\rho-\rho_1)s_1+\ldots+(\rho-\rho_m)s_m+\rho s_{m+1}+\rho-\rho_j+1}}{\Gamma((\rho-\rho_1)s_1+\ldots+(\rho-\rho_m)s_m+\rho s_{m+1}+\rho-\rho_j+2)} \triangle^{l s_{m+1}} \beta(\sigma).
\end{aligned}$$

Proof. Applying I_t^{ρ} to Equation (6), we arrive at

$$M(t,\sigma) - M(0,\sigma) - M_t'(0,\sigma)t + \sum_{j=1}^{m} \lambda_j \, I_t^{\rho-\rho_j} I_t^{\rho_j} \, {}_c\frac{\partial^{\rho_j}}{\partial t^{\rho_j}} M(t,\sigma) = I_t^{\rho} \triangle^l M(t,\sigma) + I_t^{\alpha} g(t,\sigma),$$

which implies that

$$M(t,\sigma) + \sum_{j=1}^{m} \lambda_j \, I_t^{\rho-\rho_j} [M(t,\sigma) - \theta(\sigma) - \beta(\sigma)t] - I_t^{\rho} \triangle^l M(t,\sigma) = I_t^{\rho} g(t,\sigma) + \theta(\sigma) + \beta(\sigma)t.$$

Hence,

$$
\begin{aligned}
&\left(1+\sum_{j=1}^{m}\lambda_j\, I_t^{\rho-\rho_j}-I_t^{\rho}\triangle^{l}\right)M(t,\sigma)\\
&=I_t^{\rho}g(t,\sigma)+\theta(\sigma)+\beta(\sigma)t+\sum_{j=1}^{m}\lambda_j\, I_t^{\rho-\rho_j}(\theta(\sigma)+\beta(\sigma)t)\\
&=I_t^{\rho}g(t,\sigma)+\theta(\sigma)+\beta(\sigma)t+\theta(\sigma)\sum_{j=1}^{m}\lambda_j\,\frac{t^{\rho-\rho_j}}{\Gamma(\rho-\rho_j+1)}+\beta(\sigma)\sum_{j=1}^{m}\lambda_j\,\frac{t^{\rho-\rho_j+1}}{\Gamma(\rho-\rho_j+2)}. \quad (16)
\end{aligned}
$$

We claim that the inverse operator of

$$
1+\sum_{j=1}^{m}\lambda_j\, I_t^{\rho-\rho_j}-I_t^{\rho}\triangle^{l}
$$

is

$$
\begin{aligned}
\mathbb{V}&=\sum_{s=0}^{\infty}(-1)^s\left(\sum_{j=1}^{m}\lambda_j\, I_t^{\rho-\rho_j}-I_t^{\rho}\triangle^{l}\right)^{s}\\
&=\sum_{s=0}^{\infty}(-1)^s\sum_{s_1+\ldots+s_{m+1}=s}\binom{s}{s_1,s_2,\ldots,s_{m+1}}\left(\lambda_1 I_t^{\rho-\rho_1}\right)^{s_1}\ldots\left(\lambda_m I_t^{\rho-\rho_m}\right)^{s_m}\left(-I_t^{\rho}\triangle^{l}\right)^{s_{m+1}}\\
&=\sum_{s=0}^{\infty}(-1)^s\sum_{s_1+\ldots+s_{m+1}=s}\binom{s}{s_1,s_2,\ldots,s_{m+1}}\lambda_1^{s_1}\ldots\lambda_m^{s_m}(-1)^{s_{m+1}}\\
&\quad\cdot I_t^{(\rho-\rho_1)s_1+\ldots+(\rho-\rho_m)s_m+\rho s_{m+1}}\triangle^{l s_{m+1}}.
\end{aligned}
$$

Using

$$
\begin{aligned}
&\triangle^{s}=\left(\frac{\partial^2}{\partial\sigma_1^2}+\ldots\frac{\partial^2}{\partial\sigma_n^2}\right)^{s}=\sum_{s_1+\ldots+s_n=s}\binom{s}{s_1,\ldots,s_n}\frac{\partial^{2s_1}}{\partial\sigma_1^{2s_1}}\cdots\frac{\partial^{2s_n}}{\partial\sigma_n^{2s_n}},\\
&\sum_{s_1+\ldots+s_n=s}\binom{s}{s_1,\ldots,s_n}=n^s,
\end{aligned}
$$

we have, for any $g(t,\sigma)\in S_0$,

$$
\begin{aligned}
\|\mathbb{V}g\|\le&\sum_{s=0}^{\infty}\sum_{s_1+\ldots+s_{m+1}=s}\binom{s}{s_1,s_2,\ldots,s_{m+1}}\\
&\cdot\frac{|\lambda_1|^{s_1}\ldots|\lambda_m|^{s_m}(n^l)^{s_{m+1}}}{\Gamma((\rho-\rho_1)s_1+\ldots+(\rho-\rho_m)s_m+\rho s_{m+1}+1)}\\
&\cdot\sup_{(t,\sigma)\in[0,1]\times[0,1]^n,\, i_1+s_2\ldots+i_n=ls_{m+1}}\left|\frac{\partial^{2i_1}}{\partial\sigma_1^{2i_1}}\cdots\frac{\partial^{2i_n}}{\partial\sigma_n^{2i_n}}g(t,\sigma)\right|\\
&=E_{(\rho-\rho_1,\ldots,\rho-\rho_m,\rho),\,1}(|\lambda_1|,\ldots,|\lambda_m|,n^l M_g^l)<+\infty.
\end{aligned}
$$

Thus, $\mathbb{V}$ is a continuous mapping over S_0 under the norm of $C([0,1]\times[0,1]^n)$.

In addition,

$$
\mathbb{V}\left(1+\sum_{j=1}^{m}\lambda_j\, I_t^{\rho-\rho_j}-I_t^{\rho}\triangle^{l}\right)=\left(1+\sum_{j=1}^{m}\lambda_j\, I_t^{\rho-\rho_j}-I_t^{\rho}\triangle^{l}\right)\mathbb{V}=1.
$$

It follows that

$$\begin{aligned}
&\mathbb{V}\left(1+\sum_{j=1}^{m}\lambda_j\, I_t^{\rho-\rho_j}-I_t^{\rho}\triangle^l\right)\\
&=1+\sum_{s=1}^{\infty}(-1)^s\left(\sum_{j=1}^{m}\lambda_j\, I_t^{\rho-\rho_j}-I_t^{\rho}\triangle^l\right)^s+\sum_{s=0}^{\infty}(-1)^s\left(\sum_{j=1}^{m}\lambda_j\, I_t^{\rho-\rho_j}-I_t^{\rho}\triangle^l\right)^{s+1}\\
&=1+\sum_{s=0}^{\infty}(-1)^{s+1}\left(\sum_{j=1}^{m}\lambda_j\, I_t^{\rho-\rho_j}-I_t^{\rho}\triangle^l\right)^{s+1}+\sum_{s=0}^{\infty}(-1)^s\left(\sum_{j=1}^{m}\lambda_j\, I_t^{\rho-\rho_j}-I_t^{\rho}\triangle^l\right)^{s+1}\\
&=1.
\end{aligned}$$

Clearly, such $\mathbb{V}$ is unique.

From Equation (16), we obtain

$$\begin{aligned}
&M(t,\sigma)=\\
&\sum_{s=0}^{\infty}(-1)^s\sum_{s_1+\ldots+s_{m+1}=s}\binom{s}{s_1,s_2,\ldots,s_{m+1}}\lambda_1^{s_1}\ldots\lambda_m^{s_m}(-1)^{s_{m+1}}\\
&\cdot I_t^{(\rho-\rho_1)s_1+\ldots+(\rho-\rho_m)s_m+\rho s_{m+1}+\rho}\triangle^{ls_{m+1}}g(t,\sigma)\\
&+\sum_{s=0}^{\infty}(-1)^s\sum_{s_1+\ldots+s_{m+1}=s}\binom{s}{s_1,s_2,\ldots,s_{m+1}}\lambda_1^{s_1}\ldots\lambda_m^{s_m}(-1)^{s_{m+1}}\\
&\quad\cdot\frac{t^{(\rho-\rho_1)s_1+\ldots+(\rho-\rho_m)s_m+\rho s_{m+1}}}{\Gamma((\rho-\rho_1)s_1+\ldots+(\rho-\rho_m)s_m+\rho s_{m+1}+1)}\triangle^{ls_{m+1}}\theta(\sigma)\\
&+\sum_{s=0}^{\infty}(-1)^s\sum_{s_1+\ldots+s_{m+1}=s}\binom{s}{s_1,s_2,\ldots,s_{m+1}}\lambda_1^{s_1}\ldots\lambda_m^{s_m}(-1)^{s_{m+1}}\\
&\quad\cdot\frac{t^{(\rho-\rho_1)s_1+\ldots+(\rho-\rho_m)s_m+\rho s_{m+1}+1}}{\Gamma((\rho-\rho_1)s_1+\ldots+(\rho-\rho_m)s_m+\rho s_{m+1}+2)}\triangle^{ls_{m+1}}\beta(\sigma)\\
&+\sum_{j=1}^{m}\frac{\lambda_j}{\Gamma(\rho-\rho_j+1)}\sum_{s=0}^{\infty}(-1)^s\sum_{s_1+\ldots+s_{m+1}=s}\binom{s}{s_1,s_2,\ldots,s_{m+1}}\lambda_1^{s_1}\ldots\lambda_m^{s_m}(-1)^{s_{m+1}}\\
&\quad\cdot\frac{t^{(\rho-\rho_1)s_1+\ldots+(\rho-\rho_m)s_m+\rho s_{m+1}+\rho-\rho_j}}{\Gamma((\rho-\rho_1)s_1+\ldots+(\rho-\rho_m)s_m+\rho s_{m+1}+\rho-\rho_j+1)}\triangle^{ls_{m+1}}\theta(\sigma)\\
&+\sum_{j=1}^{m}\frac{\lambda_j}{\Gamma(\rho-\rho_j+2)}\sum_{s=0}^{\infty}(-1)^s\sum_{s_1+\ldots+s_{m+1}=s}\binom{s}{s_1,s_2,\ldots,s_{m+1}}\lambda_1^{s_1}\ldots\lambda_m^{s_m}(-1)^{s_{m+1}}\\
&\quad\cdot\frac{t^{(\rho-\rho_1)s_1+\ldots+(\rho-\rho_m)s_m+\rho s_{m+1}+\rho-\rho_j+1}}{\Gamma((\rho-\rho_1)s_1+\ldots+(\rho-\rho_m)s_m+\rho s_{m+1}+\rho-\rho_j+2)}\triangle^{ls_{m+1}}\beta(\sigma),
\end{aligned}$$

which is a well-defined solution of Equation (6) since g, θ, and β are all in S_0. The uniqueness follows similarly. We finish the proof. □

Remark 5. *If $\lambda_1 = \ldots = \lambda_m = 0$, then we can easily change the domain $(t,\sigma) \in [0,1] \times [0,1]^n$ to $(t,\sigma) \in \mathbb{R}^+ \times \mathbb{R}^n$ by using the inverse operator $1 - I_t\rho\triangle^l$ directly and set*

$$\begin{aligned}
S_0=\Big\{&g\in C(\mathbb{R}^+\times\mathbb{R}^n):\exists\text{ a constant } M_g>0 \text{ and a positive function } \theta(t,x)\\
&\text{in } C(\mathbb{R}^+\times\mathbb{R}^n)\text{ such that }\left|\frac{\partial^{2s_1}}{\partial\sigma_1^{2s_1}}\cdots\frac{\partial^{2s_n}}{\partial\sigma_n^{2i_n}}g(t,\sigma)\right|\le\theta(t,x)M_g^{s_1+\ldots+s_n}\Big\}.
\end{aligned}$$

If

$$\rho=2,\quad \lambda_1=\ldots=\lambda_m=0,\quad n=l=1,\quad g(t,\sigma)=0,$$

then Equation (6) becomes the wave equation in $\mathbb{R}$ given below:

$$\begin{cases} \frac{\partial^2}{\partial t^2} M(t,\sigma) = \frac{\partial^2}{\partial \sigma^2} M(t,\sigma), \\ M(0,\sigma) = \theta(\sigma), \quad M_t'(0,\sigma) = \beta(\sigma), \quad (t,\sigma) \in \mathbb{R}^+ \times \mathbb{R}. \end{cases} \tag{17}$$

It follows from Theorem 7 that it has the solution

$$\begin{aligned} M(t,\sigma) &= \sum_{s=0}^{\infty} (-1)^s \sum_{s_1+\ldots+s_{m+1}=s} \binom{s}{s_1, s_2, \ldots, s_{m+1}} \lambda_1^{s_1} \ldots \lambda_m^{s_m} (-1)^{s_{m+1}} \\ &\quad \cdot \frac{t^{(\rho-\rho_1)s_1+\ldots+(\rho-\rho_m)s_m+\rho s_{m+1}}}{\Gamma((\rho-\rho_1)s_1+\ldots+(\rho-\rho_m)s_m+\rho s_{m+1}+1)} \triangle^{l s_{m+1}} \theta(\sigma) \\ &\quad + \sum_{s=0}^{\infty} (-1)^s \sum_{s_1+\ldots+s_{m+1}=s} \binom{s}{s_1, s_2, \ldots, s_{m+1}} \lambda_1^{s_1} \ldots \lambda_m^{s_m} (-1)^{s_{m+1}} \\ &\quad \cdot \frac{t^{(\rho-\rho_1)s_1+\ldots+(\rho-\rho_m)s_m+\rho s_{m+1}+1}}{\Gamma((\rho-\rho_1)s_1+\ldots+(\rho-\rho_m)s_m+\rho s_{m+1}+2)} \triangle^{l s_{m+1}} \beta(\sigma) \\ &= \sum_{s=0}^{\infty} \frac{t^{2s}}{(2s)!} \frac{d^{2s}}{d\sigma^{2s}} \theta(\sigma) + \sum_{s=0}^{\infty} \frac{t^{2s+1}}{(2s+1)!} \frac{d^{2s}}{d\sigma^{2s}} \beta(\sigma), \end{aligned}$$

by noting that $\lambda_1 = \ldots = \lambda_m = 0$. We are going to prove that this solution can be reduced to

$$M(t,\sigma) = \frac{\theta(\sigma+t)+\theta(\sigma-t)}{2} + \frac{1}{2}\int_{\sigma-t}^{\sigma+t} \beta(\zeta) d\zeta,$$

which is the classical solution to Equation (17) (d'Alembert's formula). Since $\theta \in S_0$, we have Taylor's expansion at the point σ:

$$\theta(\sigma+t) = \sum_{s=0}^{\infty} \frac{\theta^{(s)}(\sigma)}{s!} t^s,$$

which implies that

$$\frac{\theta(\sigma+t)+\theta(\sigma-t)}{2} = \sum_{s=0}^{\infty} \frac{t^{2s}}{(2s)!} \frac{d^{2s}}{d\sigma^{2s}} \theta(\sigma).$$

On the other hand,

$$\beta(\zeta) = \sum_{s=0}^{\infty} \frac{\beta^{(s)}(\sigma)}{s!} (\zeta-\sigma)^s,$$

since $\beta \in S_0$. This claims that

$$\frac{1}{2}\int_{\sigma-t}^{\sigma+t} \beta(\zeta) d\zeta = \sum_{s=0}^{\infty} \frac{t^{2s+1}}{(2s+1)!} \frac{d^{2s}}{d\sigma^{2s}} \beta(\sigma).$$

Evidently, if

$$\rho = 2, \ \ \lambda_1 = \ldots = \lambda_m = 0, \ \ l = 1, \ \ g(t,\sigma) = 0,$$

then Equation (6) turns out to the wave equation in $\mathbb{R}^n$,

$$\begin{cases} \frac{\partial^2}{\partial t^2} M(t,\sigma) = \triangle M(t,\sigma), \\ M(0,\sigma) = \theta(\sigma), \quad M_t'(0,\sigma) = \beta(\sigma), \quad (t,\sigma) \in \mathbb{R}^+ \times \mathbb{R}^n, \end{cases} \tag{18}$$

with the solution by Theorem 7:

$$M(t,\sigma) = \sum_{s=0}^{\infty} \frac{t^{2s}}{(2s)!} \triangle^s \theta(\sigma) + \sum_{s=0}^{\infty} \frac{t^{2s+1}}{(2s+1)!} \triangle^s \beta(\sigma). \tag{19}$$

We will show that it can be converted into Kirchoff's formula [21] for $n = 3$. Let $B_n(\sigma,t)$ be the ball of the radius t about $\sigma \in \mathbb{R}^n$ and $\partial B_n(\sigma,t)$ be the boundary of $B_n(\sigma,t)$. We define the average of ϕ over $\partial B_n(\sigma,t)$ as

$$\mathcal{A}_t\phi(\sigma) = \frac{1}{SA(B_n(\sigma,t))} \int_{\partial B_n(\sigma,t)} \phi(y)ds(y) = \frac{1}{SA(B_n(0,1))} \int_{\partial B_n(0,1)} \phi(\sigma + t\theta)ds(\theta),$$

where $SA(B_n(\sigma,t))$ denotes the surface area of $B_n(\sigma,t)$ and $ds(y)$ is the surface measure of $B_n(\sigma,t)$.

Assuming that $\phi \in S_0$, we have Taylor's expansion:

$$\begin{aligned}\phi(\sigma + t\theta) =& \sigma(\sigma) + \sum_{|i|=1} \frac{\partial^i \sigma(\sigma)}{i!} (t\theta)^i + \ldots + \sum_{|i|=2j} \frac{\partial^i \sigma(\sigma)}{i!} (t\theta)^i \\ &+ \sum_{|i|=2j+1} \frac{\partial^i \sigma(\sigma)}{i!} (t\theta)^i + \ldots,\end{aligned}$$

where

$$\begin{aligned}&|i| = i_1 + i_2 + \ldots + i_n, \quad i! = i_1!i_2!\ldots i_n!, \\ &\sigma^i = \sigma_1^{i_1}\sigma_2^{i_2}\ldots\sigma_n^{i_n}, \\ &\partial^i\phi = \partial_1^{i_1}\ldots\partial_n^{i_n}\phi = \frac{\partial^{|i|}\phi}{\partial\sigma_1^{i_1}\ldots\partial\sigma_n^{i_n}}.\end{aligned}$$

Clearly,

$$t^{2j+1} \sum_{|i|=2j+1} \frac{\partial^i \phi(\sigma)}{i!} \int_{\partial B_n(0,1)} \theta^i ds(\theta) = 0, \quad j = 0, 1, \ldots,$$

due to the cancellations over the unit sphere $\partial B_n(0,1)$. Therefore,

$$\begin{aligned}\mathcal{A}_t\phi(\sigma) =& \phi(\sigma) + t^2 \sum_{|i|=2} \frac{\partial^i \phi(\sigma)}{i!} \frac{1}{SA(B_n(0,1))} \int_{\partial B_n(0,1)} \theta^i ds(\theta) \\ &+ \ldots + t^{2j} \sum_{|i|=2j} \frac{\partial^i \phi(\sigma)}{i!} \frac{1}{SA(B_n(0,1))} \int_{\partial B_n(0,1)} \theta^i ds(\theta) + \ldots.\end{aligned}$$

Thus,

$$\begin{aligned}\mathcal{A}_t\phi(\sigma) =& \phi(\sigma) + t^2 \sum_{|i|=1} \frac{\partial^{2i} \phi(\sigma)}{(2i)!} \frac{1}{SA(B_n(0,1))} \int_{\partial B_n(0,1)} \theta^{2i} ds(\theta) \\ &+ \ldots + t^{2j} \sum_{|i|=j} \frac{\partial^{2i} \phi(\sigma)}{(2i)!} \frac{1}{SA(B_n(0,1))} \int_{\partial B_n(0,1)} \theta^{2i} ds(\theta) + \ldots.\end{aligned}$$

Applying the following formulas from [22],

$$\int_{\partial B_n(0,1)} \theta^{2i} ds(\theta) = \frac{2\Gamma\left(\frac{1}{2}+i_1\right)\dots\Gamma\left(\frac{1}{2}+i_n\right)}{\Gamma\left(|i|+\frac{n}{2}\right)},$$

$$\Gamma\left(\frac{1}{2}+i_1\right) = \frac{(2i_1)!\sqrt{\pi}}{4^{i_1} i_1!},$$

we arrive at

$$\begin{aligned}
&t^{2j}\sum_{|i|=j}\frac{\partial^{2i}\phi(\sigma)}{(2i)!}\frac{1}{SA(B_n(0,1))}\int_{\partial B_n(0,1)}\theta^i ds(\theta) \\
&= t^{2j}\sum_{|i|=j}\frac{\partial^{2i}\phi(\sigma)}{(2i)!}\frac{1}{SA(B_n(0,1))}\frac{2\Gamma\left(\frac{1}{2}+i_1\right)\dots\Gamma\left(\frac{1}{2}+i_n\right)}{\Gamma\left(|i|+\frac{n}{2}\right)} \\
&= \frac{2\pi^{n/2}}{2^{2j}j!SA(B_n(0,1))\Gamma\left(j+\frac{n}{2}\right)}\triangle^j\phi(\sigma)t^{2j} \\
&= \frac{\Gamma(n/2)}{2^{2j}j!\Gamma\left(j+\frac{n}{2}\right)}\triangle^j\phi(\sigma)t^{2j},
\end{aligned}$$

where

$$SA(B_n(0,1)) = \frac{2\pi^{n/2}}{\Gamma(n/2)}.$$

This implies that

$$\mathcal{A}_t\phi(\sigma) = \Gamma(n/2)\sum_{j=0}^{\infty}\frac{1}{2^{2j}j!\Gamma\left(j+\frac{n}{2}\right)}\triangle^j\phi(\sigma)t^{2j}.$$

For $n=3$, we are going to prove that the solution given in Formula (19) is

$$\begin{aligned}
&M(t,\sigma) = \frac{\partial}{\partial t}(t\mathcal{A}_t\theta(\sigma)) + t\mathcal{A}_t\beta(\sigma) \\
&= \frac{\partial}{\partial t}\frac{1}{SA(B_n(0,1))}\left(t\int_{\partial B_n(0,1)}\phi(\sigma+t\theta)ds(\theta)\right) + \frac{t}{SA(B_n(0,1))}\int_{\partial B_n(0,1)}\beta(\sigma+t\theta)ds(\theta),
\end{aligned}$$

which is the well-known Kirchoff formula. Indeed,

$$\begin{aligned}
&\frac{\partial}{\partial t}(t\mathcal{A}_t\theta(\sigma)) + t\mathcal{A}_t\beta(\sigma) \\
&= \Gamma(3/2)\sum_{j=0}^{\infty}\frac{2j+1}{2^{2j}j!\Gamma\left(j+\frac{3}{2}\right)}\triangle^j\phi(\sigma)t^{2j} + \Gamma(3/2)\sum_{j=0}^{\infty}\frac{1}{2^{2j}j!\Gamma\left(j+\frac{3}{2}\right)}\triangle^j\phi(\sigma)t^{2j+1} \\
&= \sum_{j=0}^{\infty}\frac{t^{2j}}{(2j)!}\triangle^j\phi(\sigma) + \sum_{j=0}^{\infty}\frac{t^{2j+1}}{(2j+1)!}\triangle^j\beta(\sigma),
\end{aligned}$$

by

$$\Gamma(3/2)\frac{2j+1}{2^{2j}j!\Gamma\left(j+\frac{3}{2}\right)}=\frac{1}{(2j)!},$$

$$\Gamma(3/2)\frac{1}{2^{2j}j!\Gamma\left(j+\frac{3}{2}\right)}=\frac{1}{(2j+1)!}.$$

We can use Kirchoff's formula for the solution of the wave equation in three dimensions to derive the solution of the wave equation in two dimensions. This technique is known as the method of descent. A similar result also follows for $n=2$.

Moreover, if $n>3$ and n is odd, then the solution given in Formula (19) is

$$M(t,\sigma)=\frac{1}{c_n}\left(\frac{\partial}{\partial t}\right)\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(t^{n-2}\mathcal{A}_t\theta(\sigma)\right)+\frac{1}{c_n}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(t^{n-2}\mathcal{A}_t\beta(\sigma)\right),$$

where

$$c_n=1\cdot 3\ldots(n-2).$$

In fact, we have

$$\left(\frac{\partial}{\partial t}\right)\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}t^{2j+n-2}=(2j+n-2)\ldots(2j+3)(2j+1)t^{2j},$$

and

$$\begin{aligned}&\frac{\Gamma(n/2)(2j+n-2)\ldots(2j+3)(2j+1)}{c_n2^{2j}j!\Gamma(j+n/2)}\\&=\frac{\Gamma(n/2)(2j+n-2)\ldots(2j+3)(2j+1)}{2^jj!(2j+n-2)\ldots n\cdot(n-2)\ldots 3\cdot 1\cdot\Gamma(n/2)}\\&=\frac{1}{2^jj!\,1\cdot 3\ldots(2j-1)}=\frac{1}{(2j)!},\end{aligned}$$

which implies that

$$\frac{1}{c_n}\left(\frac{\partial}{\partial t}\right)\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(t^{n-2}\mathcal{A}_t\theta(\sigma)\right)=\sum_{s=0}^{\infty}\frac{t^{2s}}{(2s)!}\triangle^s\theta(\sigma).$$

Similarly,

$$\frac{1}{c_n}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(t^{n-2}\mathcal{A}_t\beta(\sigma)\right)=\sum_{s=0}^{\infty}\frac{t^{2s+1}}{(2s+1)!}\triangle^s\beta(\sigma).$$

If $n>3$ and n is even, then a similar conclusion follows.

Furthermore, if

$$\rho=2,\ \ \lambda_1=\ldots=\lambda_m=0,\ \ l=1,$$

then Equation (6) turns out to the non-homogenous wave equation in $\mathbb{R}^n$,

$$\begin{cases}\dfrac{\partial^2}{\partial t^2}M(t,\sigma)=\triangle M(t,\sigma)+g(t,\sigma),\\ M(0,\sigma)=\theta(\sigma),\quad M_t'(0,\sigma)=\beta(\sigma),\ \ (t,\sigma)\in\mathbb{R}^+\times\mathbb{R}^n,\end{cases}$$

with the uniform solution by Theorem 7 for all $n \geq 1$:

$$M(t,\sigma) = \sum_{s=0}^{\infty} I_t^{2s+2} \triangle^s g(t,\sigma) + \sum_{s=0}^{\infty} \frac{t^{2s}}{(2s)!} \triangle^s \theta(\sigma) + \sum_{s=0}^{\infty} \frac{t^{2s+1}}{(2s+1)!} \triangle^s \beta(\sigma). \tag{20}$$

Example 5. *The following wave equation,*

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} M(t,\sigma) = \triangle M(t,\sigma) + \sigma_1 \sigma_2 t^2, \\ M(0,\sigma) = \sigma_n, \quad M_t'(0,\sigma) = \sigma_3, \ \ (t,\sigma) \in \mathbb{R}^+ \times \mathbb{R}^n, \end{cases}$$

has the solution

$$M(t,\sigma) = \frac{t^4}{12} \sigma_1 \sigma_2 + t\sigma_3 + \sigma_n,$$

where $n \geq 3$.

It follows from Formula (20) that

$$\begin{aligned} M(t,\sigma) &= \sum_{s=0}^{\infty} I_t^{2s+2} \triangle^s (\sigma_1 \sigma_2 t^2) + \sum_{s=0}^{\infty} \frac{t^{2s}}{(2s)!} \triangle^s \sigma_n + \sum_{s=0}^{\infty} \frac{t^{2s+1}}{(2s+1)!} \triangle^s \sigma_3 \\ &= \frac{\Gamma(3)}{\Gamma(5)} t^4 \sigma_1 \sigma_2 + \sigma_n + t\sigma_3 = \frac{t^4}{12} \sigma_1 \sigma_2 + t\sigma_3 + \sigma_n. \end{aligned}$$

by noting that $\sigma_1 \sigma_2 t^2$ and σ_n are in S_0.

For $n = 3$, this approach is much simpler than the following classical one based on Kirchoff's formula:

$$\begin{aligned} &M(t,\sigma) \\ &= \frac{\partial}{\partial t} \frac{1}{SA(B_3(0,1))} \left(t \int_{\partial B_3(0,1)} \phi(\sigma + t\theta) ds(\theta) \right) + \frac{t}{SA(B_3(0,1))} \int_{\partial B_3(0,1)} \beta(\sigma + t\theta) ds(\theta) \\ &+ \frac{1}{4\pi} \int_{B_3(\sigma,t)} \frac{g(t - |y-\sigma|, y)}{|y-\sigma|} dy. \end{aligned}$$

The Laplacian appears in many well-known differential equations describing physical phenomena, such as Poisson's equation, the diffusion equation, the wave equation, and the Schrödinger equation. The inverse operator method mentioned above clearly goes in a new direction in studying these important equations under certain initial or boundary conditions.

Generally speaking, there are analytic approaches [2] (fractional Green's function, separation of variables, integral transforms, adomian decomposition method, and homotopy analysis method) and numerical methods [10] (finite difference methods, finite element methods, spectral methods, and meshless methods) dealing with fractional partial deferential equations. Section 4 introduces a novel technique of inverse operators which is also powerful in studying fractional differential equations, which are seen from the above examples.

5. Conclusions

We studied the uniqueness, existence, and stability of Equation (1) in $\mathbb{R}^n$ with three-point conditions and variable coefficients in $C([0,1] \times [0,1]^n)$ based on the inverse operator containing a multi-variable function, the new generalized two-parameter Mittag-Leffler function, Banach's contractive principle, and Leray–Schauder's fixed-point theorem. Several examples were presented to demonstrate applications of key theorems obtained. The technique used has a wide range of applications to various fractional nonlinear partial differential or integro-differential equations with initial or boundary conditions. In addition, we provided series solutions to a few well-known partial differential equations, such as the multi-term time-fractional convection problem and the generalized time-fractional diffusion-wave equation. Especially, we obtained the uniform and simple solution to the non-homogeneous wave equation in n dimensions for all $n \geq 1$, which is consistent with classical results such as d'Alembert's and Kirchoff's formulas but more powerful in finding solutions for some wave equations. As future research, it is worth considering the following time-fractional convection–diffusion equation with an initial condition and source term for the constants $a, b, \gamma \in \mathbb{R}$ by an inverse operator and the multivariate Mittage-Leffler function:

$$\begin{cases} \frac{{}_c\partial^\alpha}{\partial t^\alpha} u(t,x) + a \frac{{}_c\partial^\beta}{\partial t^\beta} u(t,x) = b \triangle_{\lambda_1,\ldots,\lambda_n} u(t,x) + \gamma \nabla u(t,x) + \phi(t,x), \\ u(0,x) = \psi(x), \end{cases}$$

where $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n, \ \ 0 < \beta < \alpha \leq 1$,

$$\triangle_{\lambda_1,\ldots,\lambda_n} = \lambda_1(x_1)\frac{\partial^2}{\partial x_1^2} + \ldots + \lambda(x_n)\frac{\partial^2}{\partial x_n^2}, \quad \nabla = \frac{\partial}{\partial x_1} + \ldots + \frac{\partial}{\partial x_n},$$

and the partial Liouville–Caputo fractional derivative $\frac{{}_c\partial^\alpha}{\partial t^\alpha}$ of the order $0 < \alpha \leq 1$ with respect to t is defined as

$$\left(\frac{{}_c\partial^\alpha}{\partial t^\alpha} u\right)(t,x) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} u'_t(\tau,x) d\tau.$$

Applications of such convection–diffusion equations span numerous scientific and engineering disciplines, such as fluid dynamics and heat transfer.

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