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Numerical Solutions of Weakly Singular Nonlinear Volterra Integral Equations With Smooth and Nonsmooth Solutions Over Large Time Intervals

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ABSTRACT

This paper proposes a spectral collocation method for solving weakly singular nonlinear Volterra integral equations over large time intervals, addressing both smooth and nonsmooth solutions. For smooth solutions, classical scaled Laguerre polynomials (SLPs) are employed. For nonsmooth solutions, a new set of generalized Laguerre functions (GLFs) is introduced. To efficiently approximate weakly singular integrals, the interval $[0, T]$ is transformed into a half-line using a suitable mapping. The Gauss points of the GLFs are then used as the collocation points to approximate the resulting integral. Also, the convergence of the proposed method is analyzed in the weighted L^2-norm. Finally, the accuracy and reliability of the new approach are demonstrated through several examples.

1 | Introduction

Many problems in applied mathematics, physics, engineering, and other scientific fields lead to mathematical models described by weakly singular Volterra equations. These equations arise in diverse applications, including seismology, heat conduction, chemical reactions, electrochemistry, and astrophysics.nce and regularity of solutions under appropriate conditions in various functional spaces. For the existence of solutions, the authors in [1] examined the smoothness of solutions to systems of singular Volterra equations under suitable smoothness conditions, while in [2], the existence of solutions to generalized Abel's integral equations was discussed using Babenko's approach. Regarding regularity, the work in [3] studied Volterra integral equations with oscillatory kernels and noncompact operators, the smoothness and singularities of solutions for both Fredholm and Volterra equations were investigated in [4]. However, due to the singular nature of these equations, obtaining analytical solutions is often limited to specific cases [5]. Consequently, numerical methods have become a fundamental approach for solving such equations, with significant advances in spectral techniques. In this research, the numerical solution of weakly singular nonlinear Volterra integral equations (WSNVIEs) of the form

$$u(t) = f(t) + \int_0^t (t-s)^{-\alpha} k(t,s) g(s,u(s)) ds, \ t \in [0,T], \ 0 < \alpha < 1 \tag{1}$$

is considered, where f, k, and g are known functions and u is the unknown function to be approximated.

Abbreviations: GLFs, generalized Laguerre functions; SLPs, scaled Laguerre polynomials; WSNVIEs, weakly singular nonlinear Volterra integral equations.

Generally, the existence of the solution of the Volterra integral equation is typically guaranteed only within a finite interval due to the potential for blow-up or singular behavior beyond this range. Specifically, the behavior of the nonlinear function $g(t, u(t))$ plays a crucial role in determining the solution's domain. If $g(t, u(t))$ is a positive nonlinear function and satisfies certain conditions, the solution interval can often be extended to $[0, \infty)$. Conversely, if $g(t, u(t)) < 0$, the solution may exhibit blow-up at a finite t, highlighting the limitations of existence beyond this point. This interplay between the nonlinearity and the solution's boundedness underlines the importance of carefully analyzing the structure of the problem. For more details, see [6, 7].

Over the last few decades, the numerical solution of weakly singular Volterra integral equations for both linear and nonlinear cases has been extensively studied using methods such as Galerkin and collocation. In the linear case, the authors [8] employed a piecewise polynomial collocation method using both uniform and graded meshes. They also established the corresponding convergence theory in the infinite norm. In [9], a Müntz polynomial spectral collocation method with graded meshes was introduced. Convergence analyses were conducted in both L^∞- and weighted L^2-norms. The authors [10] developed a spectral method based on piecewise and discontinuous polynomials, achieving optimal algebraic convergence rates in the L^2-space. Additionally, [11] utilized piecewise discontinuous polynomials as a collocation method. In [12], the authors studied two types of singular kernels, namely, algebraic and logarithmic, using the Galerkin and multi-Galerkin methods. Recently, in [13], a Galerkin method based on shifted Legendre polynomials (SLPs) for large intervals with smooth solutions was proposed. The convergence analysis in the weighted L^2-space was also established. While the nonlinear Volterra integral Equation (1) has recently received considerable attention, the authors [14] proposed a new numerical technique based on hybrid orthonormal Bernstein and block-pulse functions, including Abel's equations. A Galerkin method utilizing Hermite cubic spline multiwavelets, along with its convergence analysis, was introduced in [15]. Very recently, Galerkin and multi-Galerkin Kumar–Sloan methods based on piecewise polynomials with uniform meshes for smooth kernels and graded meshes have been used for weakly singular kernels [16]. Furthermore, Volterra integral equations are widely used in load leveling problems. For instance, some studies have focused on accuracy control in the Taylor-collocation method for weakly regular first-kind Volterra integral equations [17], while others have introduced novel techniques for solving second-kind equations with discontinuous kernels [18, 19].

The study of weakly singular nonlinear Volterra integral equations over large time intervals is highly significant due to their applications in modeling the dynamics of infectious disease transmission. For example, Equation (1) effectively represents compartmental models of disease spread [20–23]. Analyzing these equations over extended intervals allows for deeper insights into the long-term behavior of infections within a population. Furthermore, understanding the compartmental dynamics can help in predicting the spread of diseases and devising strategies to mitigate outbreaks, thereby avoiding severe public health crises.

To the best of our knowledge, existing studies have primarily focused on the numerical solutions of weakly singular Volterra integral equations on the unit interval. The work in [13] is the sole investigation addressing weakly singular linear Volterra integral equations over large intervals, which has yielded promising results. This success has motivated the extension of the research scope to include the nonlinear Equation (1), with both smooth and nonsmooth solutions. To handle smooth solutions, the collocation method based on scaled Laguerre polynomials is used. For nonsmooth solutions, a novel set of functions called generalized Laguerre functions is introduced. These functions are mutually orthogonal with respect to the nonuniform weight function $w_\beta(t) = e^{-\beta t}$ on $L^2_{w_\beta}(\mathbb{R}_+)$. They are obtained by multiplying $t^{\frac{\lambda}{2}}$ with generalized Laguerre polynomials. To avoid singularities in the integral parts, a smoothing exponential transformation is applied, as proposed and developed in [13]. This transformation converts the singular integral operator in Equation (1) into an equivalent nonsingular kernel operator defined on the half-line. Moreover, an error analysis is conducted, and the order of convergence for the approximated and iterated solutions is determined. As stated in [13], this method has the primary advantage of being effective for solving problems over large intervals, particularly in the context of weakly singular nonlinear Volterra integral equations. One of the key challenges in solving such equations over extended time intervals is the potential loss of accuracy and stability. The proposed approach effectively addresses these challenges by leveraging the SLPs and SLFs, which enhance computational efficiency and solution accuracy. Moreover, this method demonstrates strong performance for both smooth and certain types of nonsmooth solutions, offering a notable advantage over alternative techniques.

This paper is organized as follows. In Section 2, some useful notations and properties of generalized Laguerre functions (GLFs) are introduced, and a few fundamental results of generalized Laguerre polynomials (GLPs) are recalled. Section 3 presents natural conditions on the kernel as well as the nonlinear function and discusses the collocation methods to solve Equation (1). In Section 4, convergence results in the weighted L^2 space are obtained. Finally, in Section 5, numerical results are provided to illustrate the theoretical findings.

2 | Theoretical Framework

In this section, we review recent findings concerning GLPs and introduce a novel set of orthogonal functions known as GLFs.

For $\lambda > -1$ and $\beta > 0$, the GLPs of degree $n \in \mathbb{N}$ were given in [24] as

$$\mathcal{L}_n^{\lambda,\beta}(t) = \frac{1}{n!} t^{-\lambda} e^{\beta t} \partial_t^n \left(t^{n+\lambda} e^{-\beta t} \right), \; t \in \mathbb{R}_+ \tag{2}$$

They fulfill the following recurrence relations:

$$\begin{aligned} &\mathcal{L}_0^{\lambda,\beta}(t) = 1, \quad \mathcal{L}_1^{\beta}(t) = (1 + \lambda - \beta t); \\ &(n+1)\mathcal{L}_{n+1}^{\lambda,\beta}(t) = (2n + 1 + \lambda - \beta t)\mathcal{L}_n^{\lambda,\beta}(t) - (n+\lambda)\mathcal{L}_{n-1}^{\lambda,\beta}(t), \; n \geq 1, \end{aligned} \tag{3}$$

which are orthogonal with respect to the weight function $w_{\lambda,\beta}(t) = t^{\lambda} e^{-\beta t}$, meaning that

$$\int_0^{\infty} \mathcal{L}_n^{\lambda,\beta}(t)\mathcal{L}_m^{\lambda,\beta}(t)w_{\lambda,\beta}(t)dt = \frac{\Gamma(n+\lambda+1)}{\beta^{\lambda+1}n!}\delta_{n,m} \tag{4}$$

where $\delta_{n,m}$ is the Kronecker delta function, ensuring orthogonality between the polynomials. Furthermore, this set of GLPs constitutes a complete $L^2_{w_{\lambda,\beta}}(\mathbb{R}_+)$-orthogonal basis. Consider the weighted Hilbert space $L^2_{w_\beta}(\mathbb{R}_+)$, where $\beta > 0$ and $w_\beta(t) = e^{-\beta t} > 0$. This space is defined as follows:

$$L^2_{w_\beta}(\mathbb{R}+) = \{u \mid u \text{ is measurable on } \mathbb{R}+ \text{ and } \|u\|_{w_\beta} < \infty\} \tag{5}$$

where the norm and inner product are given by

$$\|u\|_{w_\beta} = (u,u)^{\frac{1}{2}}_{w_\beta}, \ (u,v)_{w_\beta} = \int_0^{\infty} u(t)v(t)w_\beta(t)dt \tag{6}$$

Now, we introduce the new generalized Laguerre functions with a real parameter $\lambda \geq 0$ of degree $n \geq 0$, namely,

$$\hat{\mathcal{L}}_n^{\lambda,\beta}(t) = t^{\frac{\lambda}{2}}\mathcal{L}_n^{\lambda,\beta}(t), \quad t \in \mathbb{R}_+ \tag{7}$$

The GLFs $\{\hat{\mathcal{L}}_n^{\lambda,\beta}\}$ are orthogonal in $L^2_{w_\beta}(\mathbb{R}_+)$ space, namely,

$$\begin{aligned}&\int_0^{\infty} \hat{\mathcal{L}}_n^{\lambda,\beta}(t)\hat{\mathcal{L}}_m^{\lambda,\beta}(t)w_\beta(t)dt \\ &= \int_0^{\infty} \mathcal{L}_n^{\lambda,\beta}(t)\mathcal{L}_m^{\lambda,\beta}(t)w_{\lambda,\beta}(t)dt \\ &= \frac{\Gamma(n+\lambda+1)}{\beta^{\lambda+1}n!}\delta_{n,m}, \quad n,m \in \mathbb{N}.\end{aligned} \tag{8}$$

It can be easily demonstrated that the collection of all GLFs given in (7) constitutes a complete orthogonal system in the space $L^2_{w_\beta}(\mathbb{R}_+)$. In the specific instance where $\lambda = 0$, we derive the SLPs denoted as $\mathcal{L}_n^{\beta}(t)$.

For any positive integer N, let $\mathbb{X}_N^{\lambda}$ denote the finite-dimensional approximation subspace formed by the set of GLFs, defined as

$$\mathbb{X}_N^{\lambda} := \{v \mid v(t) = t^{\frac{\lambda}{2}}\phi(t), \text{for all } \phi \in \mathbb{P}^N\},$$

where $\mathbb{P}^N$ represents the collection of all GLPs on $\mathbb{R}_+$ with degrees up to N. Let us introduce the weighted orthogonal projection operator $P_N^{\lambda,\beta} : L^2_{w_\beta}(\mathbb{R}_+) \to \mathbb{X}_N^{\lambda}$, defined by the condition

$$(P_N^{\lambda,\beta}u - u, \phi)_{w_\beta} = 0, \quad \forall \phi \in \mathbb{X}_N^{\lambda} \tag{9}$$

The operator $P_N^{\lambda,\beta}$ is explicitly given by

$$P_N^{\lambda,\beta}u(t) = \sum_{n=0}^{N} u_{n,N}^{\lambda,\beta}\hat{\mathcal{L}}_n^{\lambda,\beta}(t) \tag{10}$$

where $u_{n,N}^{\lambda,\beta}$ are the coefficients of the expansion. We denote the generalized Laguerre functions-Gauss weight for a given positive integer N as $\hat{\omega}_{N,j}^{\lambda,\beta}$, where $0 \leq j \leq N$. These weights are formally defined as follows:

$$\hat{\omega}_{N,j}^{\lambda,\beta} = \left(\xi_{N,j}^{\lambda,\beta}\right)^{-\lambda}\omega_{N,j}^{\lambda,\beta} \tag{11}$$

where $\xi_{N,j}^{\lambda,\beta}$, $0 \leq j \leq N$, represent the roots of $\mathcal{L}_{N+1}^{\lambda,\beta}(t)$. While $\omega_{N,j}^{\lambda,\beta}$ is defined as (see [25]),

$$\omega_{N,j}^{\lambda,\beta} = \frac{\Gamma(N+\lambda+2)}{\beta^{\lambda}\Gamma(N+2)\xi_{N,j}^{\lambda,\beta}\left[\partial_t \mathcal{L}_{N+1}^{\lambda,\beta}\left(\xi_{N,j}^{\lambda,\beta}\right)\right]^2}, \ 0 \leq j \leq N \tag{12}$$

Next, we introduce the discrete inner product and the discrete norm associated with $\{\xi_{N,j}^{\lambda,\beta}, \hat{\omega}_{N,j}^{\lambda,\beta}\}_{j=0}^{N}$ as follows:

$$(u,v)_{w_\beta,N} = \sum_{j=0}^{N} u\left(\xi_{N,j}^{\lambda,\beta}\right)v\left(\xi_{N,j}^{\lambda,\beta}\right)\hat{\omega}_{N,j}^{\lambda,\beta}, \quad \|u\|_{w_\beta,N} = (u,u)^{1/2}_{w_\beta,N} \tag{13}$$

Indeed, for any $u \in \mathbb{X}_{N+1}^{\lambda}$ and $v \in \mathbb{X}_{N+2}^{\lambda}$, we express them as follows:

$$u(t) = t^{\frac{\lambda}{2}}\phi_N(t), \quad v(t) = t^{\frac{\lambda}{2}}\phi_{N+1}(t) \tag{14}$$

where $\phi_{N+1} \in \mathcal{P}^{N+2}$ and $\phi_N \in \mathcal{P}^{N+1}$. Using the quadrature formula GLFs, we obtain

$$\begin{aligned}(u,v)_{w_\beta} &= \int_0^{\infty} u(t)v(t)w_\beta(t)dt = \int_0^{\infty} \phi_N(t)\phi_{N+1}(t)t^{\lambda}e^{-\beta t}dt \\ &= \sum_{j=0}^{N} \phi_N(\xi_{N,j}^{\lambda,\beta})\phi_{N+1}(\xi_{N,j}^{\lambda,\beta})\omega_{N,j}^{\lambda,\beta} \\ &= \sum_{j=0}^{N} u\left(\xi_{N,j}^{\lambda,\beta}\right)v\left(\xi_{N,j}^{\lambda,\beta}\right)\hat{\omega}_{N,j}^{\lambda,\beta} \\ &= (u,v)_{w_\beta,N}, \quad \forall u \in \mathbb{X}_{N+1}^{\lambda}, \forall v \in \mathbb{X}_{N+2}^{\lambda}.\end{aligned} \tag{15}$$

In particular, this leads to

$$\|u\|_{w_\beta} = \|u\|_{w_\beta,N}, \forall \ u \in \mathbb{X}_{N+\lambda_Z}^{\lambda} \tag{16}$$

Moreover, we have

$$\int_0^{\infty} \phi(t)w_\beta(t)dt = \sum_{j=0}^{N} \phi\left(\xi_{N,j}^{\lambda,\beta}\right)\hat{\omega}_{N,j}^{\lambda,\beta} \quad \forall \phi \in \mathbb{X}_{2N+1} \tag{17}$$

To accurately estimate the error caused by truncation in the $L^2_{w_\beta}$-norm, it is necessary to incorporate the differential operator into the analysis

$$\partial_{t,\lambda} = \partial_t - \frac{\lambda}{2t}, \tag{18}$$

$$\vdots$$

$$\partial_{t,\lambda}^{k} = \sum_{l=0}^{k}\left(\frac{-1}{t}\right)^{k-l}\binom{k}{l}\prod_{i=0}^{k-l}\left(\frac{\lambda}{2}+i\right)\partial_t^{l}. \tag{19}$$

In the following, we introduce the Sobolev space $H^m_{\lambda,\beta}(\mathbb{R}_+)$ defined by

$$H^m_{\lambda,\beta}(\mathbb{R}_+) = \left\{ u | t^{\frac{k}{2}} \partial^k_{t,\lambda} u \in L^2_{w_\beta}(\mathbb{R}_+),\ 0 \leq k \leq m \right\},$$

equipped with the seminorm and norm as follows:

$$|u|_{H^m_{\lambda,\beta}(\mathbb{R}_+)} = \left\| t^{\frac{m}{2}} \partial^m_{t,\lambda} u \right\|_{L^2_{w_\beta}(\mathbb{R}_+)}, \quad \|u\|_{H^m_{\lambda,\beta}(\mathbb{R}_+)} = \left(\sum_{k=0}^{m} |u|^2_{H^m_{\lambda,\beta}(\mathbb{R}_+)} \right)^{1/2}.$$

This lemma establishes the convergence and error analysis of GLFs basis.

Lemma 1. *Let $m \in \mathbb{N}^*$. For any $u \in H^m_{\lambda,\beta}(\mathbb{R}_+)$, we have*

$$\|P^{\lambda,\beta}_N u - u\|_{w_\beta} \leq c(\beta N)^{-\frac{m}{2}} |u|_{H^m_{\lambda,\beta}(\mathbb{R}_+)} \tag{20}$$

where c is a positive constant independent of N and u.

Proof. This proof quotes from theorem 7.9 in [26]. Let $v = ut^{-\lambda/2}$. Recall the Laguerre–Gauss interpolation operator $\Pi^{\lambda,\beta}_N : L^2_{w_{\beta,\lambda}}(\mathbb{R}+) \to \mathbb{P}^N$, of the form

$$\Pi^{\lambda,\beta}_N v(t) = \sum_{n=0}^{N} v^{\lambda,\beta}_{n,N} \mathcal{L}^{\lambda,\beta}_n(t) \tag{21}$$

In fact, we can write

$$\Pi^{\lambda,\beta}_N v(t) = t^{-\lambda/2} P^{\lambda,\beta}_N u(t) \tag{22}$$

It is clear that

$$\partial^m_t \left(\Pi^{\lambda,\beta}_N v - v \right) = \partial^m_t \left(t^{-\lambda/2} \left(P^{\lambda,\beta}_N u - u \right) \right) = t^{-\lambda/2} \partial^m_{t,\lambda} \left(P^{\lambda,\beta}_N u - u \right),$$

and likewise,

$$\partial^m_{t,\lambda} u = t^{\lambda/2} \partial^m_t v.$$

Now, we can write

$$\begin{aligned} \|P^{\lambda,\beta}_N u - u\|^2_{w_\beta} &= \int_0^\infty |P^{\lambda,\beta}_N u(t) - u(t)|^2 e^{-\beta t} dt \\ &= \int_0^\infty \left| t^{-\lambda/2} \left(P^{\lambda,\beta}_N u(t) - u(t) \right) \right|^2 t^\lambda e^{-\beta t} dt \\ &= \int_0^\infty |\Pi^{\lambda,\beta}_N v(t) - v(t)|^2 t^\lambda e^{-\beta t} dt \\ &= \|\Pi^{\lambda,\beta}_N v - v\|^2_{w_{\beta,\lambda}}. \end{aligned} \tag{23}$$

Hence,

$$\|P^{\lambda,\beta}_N u - u\|_{w_\beta} = \|\Pi^{\lambda,\beta}_N v - v\|_{w_{\beta,\lambda}} \tag{24}$$

Finally, according to theorem 2.1 of [24], we have

$$\|\Pi^{\lambda,\beta}_N v - v\|_{w_{\beta,\lambda}} \leqslant c(\beta N)^{-\frac{m}{2}} \|\partial^m_t v\|_{w_{\beta,\lambda+m}} \tag{25}$$

which implies

$$\|P^{\lambda,\beta}_N u - u\|_{w_\beta} \leq c(\beta N)^{-\frac{m}{2}} |u|_{H^m_{\lambda,\beta}(\mathbb{R}_+)} \tag{26}$$

□

3 | Wsnvies

Consider the integral equation form (1). Throughout this paper, we give the following assumptions on the functions k and g:

A1. $\max_{t,s\in[0,T]} |k(t,s)| = C_0 < \infty$.

A2. The function $G(t, u(t))$ is Lipschitz continuous respect to u i.e., for any $u, v \in L^2_{w_\beta}(\mathbb{R}_+)$, there exist a constant C_1 such that

$$|g(t, u(t)) - g(t, v(t))| \leqslant C_1 |u(t) - v(t)|,$$

and

$$\int_0^{+\infty} |g(t, u(t))|^2 e^{-\beta t} dx \leq C_2 < \infty.$$

A3. The first order partial derivative $g^{(0,1)}(., u(.))$ of $g(., u(.))$ is bounded and Lipschitz continuous in u i.e., for any $u, v \in L^2_{w_\beta}(\mathbb{R}_+)$, there exist a constant C_1 such that

$$|g^{(0,1)}(t, u(t)) - g^{(0,1)}(t, v(t))| \leqslant C_3 |u(t) - v(t)|.$$

Also for the weakly singular kernel, we have that

$$\begin{aligned} \max_{0\leq t\leq T} \int_0^t (t-s)^{-\alpha} |k(t,s)|^2 ds &\leq C_0^2 \max_{0\leq t\leq T} \int_0^t (t-s)^{-\alpha} ds \\ &\leq C_0^2 \frac{T^{1-\alpha}}{1-\alpha} = C_4 < \infty. \end{aligned} \tag{27}$$

First, to achieve good convergence results, we apply a transformation to the singular integral parts, defined as $s(.,.) : [0,T] \times \mathbb{R}_+ \to [0,t]$ by

$$s = \theta_t(x) = t(1 - e^{-x}), \quad x = \theta_t^{-1}(s) = \ln\left(\frac{t}{t-s}\right).$$

Hence, the integral Equation (1) is transformed into

$$u(t) = f(t) + \int_0^\infty k(t, \theta_t(x)) t^{1-\alpha} e^{-(1-\alpha)x} g(\theta_t(x), u(\theta_t(x))) dx, \quad t \in [0,T],\ 0 < \alpha < 1, \tag{28}$$

Now consider the nonlinear integral operator defined in $L^2_{w_\beta}(\mathbb{R}_+)$ as follows:

$$\begin{aligned} \mathcal{K}u(t) &= \int_0^t (t-s)^{-\alpha} k(t,s) g(s, u(s)) ds \\ &= \int_0^\infty k(t, \theta_t(x)) t^{1-\alpha} e^{-(1-\alpha)x} g(\theta_t(x), u(\theta_t(x))) dx. \end{aligned}$$

Then Equation (28) can be written as

$$u = f + \mathcal{K}u \tag{29}$$

Again, let us define the operator $\mathcal{T}$ as

$$\mathcal{T}(u) = f + \mathcal{K}(u), \quad u \in L^2_{w_\beta}(\mathbb{R}_+) \tag{30}$$

From Equations (29) and (30), we have

$$u = \mathcal{T}(u) \tag{31}$$

Clearly, based on assumption A2, for any $u, v \in L^2_{w_\beta}(\mathbb{R}_+)$, we have

$$\begin{aligned} |\mathcal{T}(u)(t) - \mathcal{T}(v)(t)| &= |\mathcal{K}(u)(t) - \mathcal{K}(v)(t)| \\ &= \left| \int_0^t (t-s)^{-\alpha} k(t,s)(g(s,u(s)) - g(s,v(s)))ds \right| \\ &\le C_1 \int_0^t (t-s)^{-\alpha} |k(t,s)||u(s) - v(s)|ds. \end{aligned} \tag{32}$$

Using the Cauchy–Schwarz inequality along with assumption A1, we obtain

$$\begin{aligned} |\mathcal{T}(u)(t) - \mathcal{T}(v)(t)|^2 e^{-\beta t} &\le C_1 \left(\int_0^t (t-s)^{-\alpha} |k(t,s)|^2 ds \right) \\ &\quad \left(\int_0^t (t-s)^{-\alpha} e^{-\beta(t-s)} |u(s) - v(s)|^2 e^{-\beta s} ds \right) \\ &\le C_4 C_1 \left(\int_{-\infty}^{\infty} (t-s)^{-\alpha} e^{-\beta(t-s)} |u(s) \right. \\ &\quad \left. - v(s)|^2 \chi_{[0,t]}(s) e^{-\beta s} ds \right), \end{aligned} \tag{33}$$

where

$$\chi_{[0,t]}(s) = \begin{cases} 1, & \text{if } s \in [0,t], \\ 0, & \text{otherwise.} \end{cases}$$

Let us introduce the following definitions:

$$g(t) = t^{-\alpha} e^{-\beta t} H(t) \tag{34}$$

where H represents the Heaviside function, and

$$h(s) = |u(s) - v(s)|^2 \chi_{[0,t]}(s) e^{-\beta s} \tag{35}$$

With these, we can express the convolution as

$$(g \star h)(t) := \int_{-\infty}^{\infty} g(t-s) h(s) ds.$$

It is evident that $\|h\|_{1,(-\infty,\infty)} = \|u - v\|_{w_\beta}$. Furthermore, we can easily show that

$$\|g\|_{1,(-\infty,\infty)} = \beta^{\alpha-1}\Gamma(1-\alpha).$$

Indeed,

$$\int_0^\infty t^{-\alpha} e^{-\beta t} dt = \beta^{\alpha-1} \int_0^\infty t^{-\alpha} e^{-t} dt = \beta^{\alpha-1}\Gamma(1-\alpha).$$

Therefore, by applying Young's theorem 4.15 [27 , pp. 104], we get

$$\|g \star h\|_1 \le \|g\|_{1,(-\infty,\infty)} \|h\|_{1,(-\infty,\infty)} = \beta^{\alpha-1}\Gamma(1-\alpha)\|u - v\|_{w_\beta} \tag{36}$$

This implies

$$\begin{aligned} \|\mathcal{T}u - \mathcal{T}v\|_{w_\beta} &\le C_3 C_1 \|g \star h\|_{1,(-\infty,\infty)} \\ &\le C_4 C_1 \beta^{\alpha-1}\Gamma(1-\alpha)\|u - v\|_{w_\beta}. \end{aligned} \tag{37}$$

Therefore, if $C_3 C_1 \beta^{\alpha-1}\Gamma(1-\alpha) < 1$, then by applying the Banach contraction principle, we deduce that the operator $\mathcal{T}$ possesses a unique fixed point, denoted as $u_0 \in L^2_{w_\beta}(\mathbb{R}_+)$, such that

$$u_0 = \mathcal{T}(u_0) \tag{38}$$

Now, we consider the following operators, respectively,

$$\begin{gathered} \mathcal{L}(v)(t) := \int_0^t (t-s)^{-\alpha} k(t,s) v(s) ds, \\ t \in [0,T], \text{ where } v(t) = g(t, u(t)). \end{gathered} \tag{39}$$

and

$$\mathcal{G}(u)(t) := g(t, u(t)), \quad t \in [0,T] \tag{40}$$

Applying the same argument to prove (37), we obtain

$$\|\mathcal{L}v\|_{w_\beta} \le C_4 \beta^{\alpha-1}\Gamma(1-\alpha)\|v\|_{w_\beta} \tag{41}$$

Clearly,

$$\mathcal{L}(v) + f = u, \tag{42}$$

and

$$\mathcal{G}(u) = v \tag{43}$$

For ease of analysis, we introduce a nonlinear operator $\mathcal{F}(v) := \mathcal{G}(\mathcal{L}(v) + f)$. This leads to the equation:

$$\mathcal{F}(v_0) = v_0 \tag{44}$$

where $v_0 = \mathcal{G}(u_0)$. This technique was originally proposed by the author of [28] to facilitate the study of nonlinear integral equations. It has since been widely applied in various studies involving nonlinear integral equations. Recently, it was employed by the author of [29] to solve the Hammerstein integral equation on the half-line. Additionally, this technique was used in [30] to solve the generalized Hammerstein integral equation on the entire real line, simplifying numerical simulations and proving the convergence of the method. This motivated us to adopt this technique for solving WSNVIEs.

Let us also introduce the Fréchet derivative of the nonlinear operator $\mathcal{F}$ at v_0 from $L^2_{w_\beta}(\mathbb{R}_+)$ into itself defined by

$$(\mathcal{F}\prime(v_0)w) := \mathcal{G}'(\mathcal{L}(v_0) + f)(\mathcal{L}(w)), \quad w \in L^2_{w_\beta}(\mathbb{R}_+) \tag{45}$$

where

$$\mathcal{G}'(u)(t) := g^{(0,1)}(t, u(t)).$$

We demonstrate in the following Lemma that $\mathcal{F}'(v_0)$ is an bounded operator from $L^2_{w_\beta}(\mathbb{R}_+)$ into $L^2_{w_\beta}(\mathbb{R}_+)$.

Lemma 2. *If the condition below holds*

$$\int_0^{+\infty} |g^{(0,1)}(t, u_0(t))|^2 e^{-\beta t} dx \le C_5^2 \tag{46}$$

Then, we have

$$\|\mathcal{F}'(v_0)w\|_{w_\beta} \le C_5 C_4 \beta^{\alpha-1}\Gamma(1-\alpha)\|w\|_{w_\beta}. \tag{47}$$

Proof. For all $w \in L^2_{w_\beta}(\mathbb{R}_+)$, we have

$$\|\mathcal{F}'(v_0)w\|_{w_\beta} \le \|\mathcal{G}'(u_0)\|_{w_\beta}\|\mathcal{L}(w)\|_{w_\beta}.$$

Due to condition (46) and (41), there holds

$$\|\mathcal{F}'(v_0)w\|_{w_\beta} \le C_5 C_4 \beta^{\alpha-1}\Gamma(1-\alpha)\|w\|_{w_\beta}.$$

Hence, the theorem is proved. □

3.1 | Glfs Collocation Method

In this section, we introduce a collocation method associated with SGLFs for solving nonlinear second-kind Volterra integral equations featuring weakly singular kernels. To achieve this, we implement this approach to consider Equation (44), where $v_N^{\lambda,\beta}(t)$ defined as follows:

$$v_N^{\lambda,\beta}(t) = \sum_{n=0}^{N} v_{n,N}^{\lambda,\beta}\hat{\mathcal{L}}_n^{\lambda,\beta}(t) = \mathbf{V}^T L_N^{\lambda,\beta}(t) \tag{48}$$

where

$$\mathbf{V} = \left(v_{0,N}^{\lambda,\beta}, \dots, v_{N,N}^{\lambda,\beta}\right)^{\mathrm{T}},$$

$$L_N^{\lambda,\beta}(t) = \left(\hat{\mathcal{L}}_0^{\lambda,\beta}(t), \dots, \hat{\mathcal{L}}_N^{\lambda,\beta}(t)\right)^{\mathrm{T}}.$$

Upon replacing Equation (48) in the Equation (44), we derive the residual function

$$\begin{aligned} R_N^{\lambda,\beta}(t) &= v_N^{\lambda,\beta}(t) - \mathcal{F}\left(v_N^{\lambda,\beta}(t)\right) \\ &= v_N^{\lambda,\beta}(t) - g\left(t, \int_0^\infty k\left(t, \theta_t(x)\right) t^{1-\alpha} e^{-(1-\alpha)x} v_N^{\lambda,\beta}\left(\theta_t(x)\right) dx + f(t)\right) \\ &= \mathbf{V}^T L_N^{\lambda,\beta}(t) - g\left(t, \int_0^\infty k\left(t, \theta_t(x)\right) t^{1-\alpha} e^{-(1-\alpha)x} \mathbf{V}^T L_N^{\lambda,\beta}\left(\theta_t(x)\right) dx + f(t)\right), \end{aligned} \tag{49}$$

where

$$R_N^{\lambda,\beta}\left(\xi_{N,j}^{\lambda,\beta}\right) = 0, \quad j = 0, 1, \dots, N \tag{50}$$

equivalently,

$$\mathbf{V}^T L_N^{\lambda,\beta}\left(\xi_{N,j}^{\lambda,\beta}\right) - g\left(\xi_{N,j}^{\lambda,\beta}, \mathbf{V}^T\left(\xi_{N,j}^{\lambda,\beta}\right)^{1-\alpha}\int_0^\infty k\left(\xi_{N,j}^{\lambda,\beta}, \theta_{\xi_{N,j}^{\lambda,\beta}}(x)\right) e^{-(1-\alpha)x} L_N^{\lambda,\beta}\left(\theta_{\xi_{N,j}^{\lambda,\beta}}(x)\right) dx + f\left(\xi_{N,j}^{\lambda,\beta}\right)\right) = 0. \tag{51}$$

The integral parts in (51) can be calculated approximately by using the generalized Laguerre–Gauss quadrature set $\left\{\xi_{M,j}^{\lambda,\beta}, \omega_{M,j}^{\lambda,\beta}\right\}_{j=0}^{M}$, as follows:

$$\begin{aligned} &\int_0^\infty k\left(_{N,j}^{\lambda,\beta}, \theta_{\xi_{N,j}^{\lambda,\beta}}(x)\right) e^{-(1-\alpha)x}\mathbf{V}^T L_N^{\lambda,\beta}\left(\theta_{\xi_{N,j}^{\lambda,\beta}}(x)\right) dx \\ &\simeq \sum_{i=1}^{M}\left(k\left(\xi_{N,j}^{\lambda,\beta}, \theta_{\xi_{N,j}^{\lambda,\beta}}\left(\xi_{M,j}^{\lambda,1-\alpha}\right)\right)\mathbf{V}^T L_N^{\lambda,\beta}\left(\theta_{\xi_{N,j}^{\lambda,\beta}}\left(\xi_{M,j}^{\lambda,1-\alpha}\right)\right)\right)\omega_{M,i}^{\lambda,1-\alpha}. \end{aligned}$$

Let us denote

$$\mathbf{V} = \left[v_{0,N}^{\lambda,\beta}, \dots, v_{N,N}^{\lambda,\beta}\right]^{\mathrm{T}}, \; \mathbf{f} = \left[f\left(\xi_{N,j}^{\lambda,\beta}\right), \dots, f\left(\xi_{N,j}^{\lambda,\beta}\right)\right]^{\mathrm{T}},$$

and

$$\hat{\mathbf{D}}_j = \left\{L_N^{\lambda,\beta}\left(\theta_{\xi_{N,j}^{\lambda,\beta}}\left(\xi_{M,i}^{\lambda,1-\alpha}\right)\right)\right\}_{i=0}^{N}, \quad \hat{\mathbf{D}} = \left\{\hat{\mathbf{D}}_j\right\}_{j=0}^{N}, \quad \mathbf{G}(\mathbf{V}) = \left(g\left(\xi_{N,j}^{\lambda,\beta}, \mathbf{M}\hat{\mathbf{D}}\mathbf{W}\mathbf{V}\right)\right)_{j=0}^{N} + \mathbf{f}.$$

Then, Equation (51) is transformed as

$$\mathbf{V}^T\mathbf{D} - \mathbf{G}(\mathbf{V}) = 0 \tag{52}$$

To solve (52), we use Newton's method.

4 | Convergence Analysis

In this subsection, we delve into the convergence analysis of the Collocation solution, beginning with the presentation of the following equation:

$$v_N^{\lambda,\beta} = P_N^{\lambda,\beta}\mathcal{F}_M\left(v_N^{\lambda,\beta}\right) \tag{53}$$

where

$$\mathcal{F}_M(v) = \mathcal{G}\left(\mathcal{L}_M(v) + f\right),$$

and

$$\mathcal{L}_M(v)(t) = \sum_{i=0}^{M} t^{1-\alpha}k\left(t, \theta_t\left(\xi_{Z,i}^{\lambda,\beta,M}\right)\right)v\left(\theta_t\left(\xi_{Z,i}^{\lambda,\beta,M}\right)\right)\omega_{Z,i}^{\lambda,1-\alpha,M}.$$

To initiate our analysis, let us define the iterated solution as

$$\tilde{v}_N^{\lambda,\beta} = \mathcal{F}_M\left(v_N^{\lambda,\beta}\right) \tag{54}$$

Applying $P_N^{\lambda,\beta}$ to both sides of Equation (54), we obtain

$$P_N^{\lambda,\beta}\tilde{v}_N^{\lambda,\beta}=P_N^{\lambda,\beta}\mathcal{F}_M\left(v_N^{\lambda,\beta}\right) \tag{55}$$

From Equations (53) and (55), we deduce $P_N^{\lambda,\beta}\tilde{v}_N^{\lambda,\beta}=v_N^{\lambda,\beta}$, indicating that the iterated solution $\tilde{v}_N^{\lambda,\beta}$ satisfies the equation:

$$\tilde{v}_N^{\lambda,\beta}=\mathcal{F}_M\left(P_N^{\lambda,\beta}\tilde{v}_N^{\lambda,\beta}\right) \tag{56}$$

Let us define the operators $\mathcal{F}_{M,N}^{\lambda,\beta}$ and $\widetilde{\mathcal{F}}_{M,N}^{\lambda,\beta}$ as follows:

$$\mathcal{F}_{M,N}^{\lambda,\beta}(v):=P_N^{\lambda,\beta}\mathcal{F}_M(v), \tag{57}$$

$$\widetilde{\mathcal{F}}_{M,N}^{\lambda,\beta}(v):=\mathcal{F}_M\left(P_N^{\lambda,\beta}v\right). \tag{58}$$

So we can write Equation (53) and Equation (54) as follows, respectively,

$$\mathcal{F}_{M,N}^{\lambda,\beta}\left(v_N^{\lambda,\beta}\right)=v_N^{\lambda,\beta}, \tag{59}$$

$$\widetilde{\mathcal{F}}_{M,N}^{\lambda,\beta}\left(\tilde{v}_N^{\lambda,\beta}\right)=\tilde{v}_N^{\lambda,\beta}. \tag{60}$$

To establish the convergence analysis, we introduce the Fréchet derivative of the operator $\widetilde{\mathcal{F}}_{M,N}^{\lambda,\beta}$ at v_0, as follows:

$$\widetilde{\mathcal{F}}_{M,N}^{\lambda,\beta}{}'\left(v_0\right)w=\mathcal{F}_M{}'\left(P_N^{\lambda,\beta}v_0\right)w \tag{61}$$

In the following lemma, we demonstrate the uniform boundedness of operators $\left(I-\widetilde{\mathcal{F}}_{M,N}^{\lambda,\beta}{}'\left(v_0\right)\right)^{-1}$.

Lemma 3. *Let v_0 is the solution of (44) and 1 is not an eigenvalue of $\mathcal{F}'\left(v_0\right)$. If the following condition holds:*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}_+}\left|\partial_x^M\left(k\left(t,\theta_t(x)\right)v_0\left(\theta_t(x)\right)\right)\right|\le C_6<\infty.$$

Then, we have

$$\|\widetilde{\mathcal{F}}_{M,N}^{\lambda,\beta}{}'\left(v_0\right)-\mathcal{F}'\left(v_0\right)\|_{w_\beta}\to 0 \text{ as } N\to\infty,\ (N\le M), \tag{62}$$

$$\|\left(I-\widetilde{\mathcal{F}}_{M,N}^{\lambda,\beta}{}'\left(v_0\right)\right)^{-1}\|_{w_\beta}<\infty. \tag{63}$$

Proof. Using (41) and A3, for all $w\in L^2_{w_\beta}\left(\mathbb{R}_+\right)$, we have

$$\begin{aligned}
&\|\left(\widetilde{\mathcal{F}}_{M,N}^{\lambda,\beta}{}'\left(v_0\right)-\mathcal{F}'\left(v_0\right)\right)w\|_{w_\beta}\\
&\le\|\mathcal{G}'\left(\mathcal{L}_M\left(P_N^{\lambda,\beta}v_0\right)+f\right)-\mathcal{G}'\left(\mathcal{L}\left(v_0\right)+f\right)\|_{w_\beta}\|\mathcal{L}(w)\|_{w_\beta}\\
&\le C_3\|\mathcal{L}_M\left(P_N^{\lambda,\beta}v_0\right)-\mathcal{L}\left(v_0\right)\|_{w_\beta}\|\mathcal{L}(w)\|_{w_\beta}\\
&\le C_3\left(\|\mathcal{L}_M\left(P_N^{\lambda,\beta}v_0\right)-\mathcal{L}_M\left(v_0\right)\|_{w_\beta}+\|\mathcal{L}_M\left(v_0\right)-\mathcal{L}\left(v_0\right)\|_{w_\beta}\right)\\
&\|\mathcal{L}(w)\|_{w_\beta}\\
&\le C_3C_4\beta^{(\alpha-1)}\Gamma(1-\alpha)\\
&\left(\|\mathcal{L}_M\left(P_N^{\lambda,\beta}v_0-v_0\right)\|_{w_\beta}+\|\mathcal{L}_M\left(v_0\right)-\mathcal{L}\left(v_0\right)\|_{w_\beta}\right)\|w\|_{w_\beta}.
\end{aligned} \tag{64}$$

First, we estimate $\left|\mathcal{L}_M\left(P_N^{\lambda,\beta}v_0-v_0\right)(t)\right|$ as follows:

$$\begin{aligned}
&\left|\mathcal{L}_M\left(P_N^{\lambda,\beta}v_0-v_0\right)(t)\right|\\
&=\left|\sum_{i=0}^{M}t^{1-\alpha}k\left(t,\theta_t\left(\xi_{M,i}^{\lambda,\beta}\right)\right)\left(P_N^{\lambda,\beta}v_0\left(\theta_t\left(\xi_{M,i}^{\lambda,\beta}\right)\right)-v_0\left(\theta_t\left(\xi_{M,i}^{\lambda,\beta}\right)\right)\right)\omega_{M,i}^{\lambda,1-\alpha}\right|\\
&\le\sum_{i=0}^{+\infty}t^{1-\alpha}\left|k\left(t,\theta_t\left(\xi_{M,i}^{\lambda,\beta}\right)\right)\right|\left|P_N^{\lambda,\beta}v_0\left(\theta_t\left(\xi_{M,i}^{\lambda,\beta}\right)\right)-v_0\left(\theta_t\left(\xi_{M,i}^{\lambda,\beta}\right)\right)\right|\omega_{M,i}^{\lambda,1-\alpha}\\
&\le\int_0^{\infty}t^{1-\alpha}|k\left(t,\theta_t(x)\right)||P_N^{\lambda,\beta}v_0\left(\theta_t(x)\right)-v_0\left(\theta_t(x)\right)|e^{-(1-\alpha)x}dx\\
&\le\int_0^{t}(t-s)^{-\alpha}|k(t,s)||P_N^{\lambda,\beta}v_0(s)-v_0(s)|ds.
\end{aligned} \tag{65}$$

Using (20) and (41), we get

$$\begin{aligned}
\|\mathcal{L}_M\left(P_N^{\lambda,\beta}v_0-v_0\right)\|_{w_\beta}&\le C_4\beta^{(\alpha-1)}\Gamma(1-\alpha)\|P_N^{\lambda,\beta}v_0-v_0\|_{w_\beta}\\
&\le cC_4\beta^{(\alpha-1)}\Gamma(1-\alpha)(\beta N)^{-\frac{m}{2}}|v_0|_{H^m_{\lambda,\beta}(\mathbb{R}_+)}.
\end{aligned} \tag{66}$$

From theorem 1 in [31], we have

$$\begin{aligned}
\left|\mathcal{L}_M(v_0)(t)-\mathcal{L}(v_0)(t)\right|&\le c((\alpha-1)M)^{-\frac{m}{2}}\int_0^{\infty}\\
&\left|t^{1-\alpha}\partial_x^M\left(k(t,\theta_t(x))v_0(\theta_t(x))\right)\right|e^{-(1-\alpha)x}dx.\\
&\le cM_6((\alpha-1)M)^{-\frac{m}{2}}t^{1-\alpha}\int_0^{\infty}e^{-(1-\alpha)x}dx.\\
&\le\frac{cM_6}{1-\alpha}((\alpha-1)M)^{-\frac{m}{2}}t^{1-\alpha}.
\end{aligned}$$

Then

$$\|\mathcal{L}_M(v_0)-\mathcal{L}(v_0)\|_{w_\beta}\le\frac{cM_6\Gamma(\alpha)}{(1-\alpha)\beta^\alpha}((\alpha-1)M)^{-\frac{m}{2}} \tag{67}$$

Set $C_\beta^\alpha=C_4\beta^{(\alpha-1)}\Gamma(1-\alpha)$ and $C_6^{\beta,\alpha}=\frac{cM_6\Gamma(\alpha)}{(1-\alpha)\beta^\alpha}$. From (64), (66), and (67), we get

$$\begin{aligned}
\|\left(\widetilde{\mathcal{F}}_{M,N}^{\lambda,\beta}{}'(v_0)-\mathcal{F}'(v_0)\right)w\|_{w_\beta}\le C_3C_\beta^\alpha\Big(&cC_\beta^\alpha(\beta N)^{-\frac{m}{2}}|v_0|_{H^m_{\lambda,\beta}(\mathbb{R}_+)}\\
&+C_6^{\beta,\alpha}((\alpha-1)M)^{-\frac{m}{2}}\Big)\|w\|_{w_\beta}.
\end{aligned}$$

This implies

$$\begin{aligned}
&\|\widetilde{\mathcal{F}}_{M,N}^{\lambda,\beta}{}'(v_0)-\mathcal{F}'(v_0)\|_{w_\beta}\\
&\le C_3C_\beta^\alpha\left(cC_\beta^\alpha(\beta N)^{-\frac{m}{2}}|v_0|_{H^m_{\lambda,\beta}(\mathbb{R}_+)}+C_6^{\beta,\alpha}((\alpha-1)M)^{-\frac{m}{2}}\right).
\end{aligned} \tag{68}$$

Given that $\|\widetilde{\mathcal{F}}_{M,N}^{\lambda,\beta}{}'(v_0)-\mathcal{F}'(v_0)\|_{w_\beta}$ tends to zero as N approaches infinity, and since the operator $(I-\mathcal{F}'(v_0))^{-1}$ exists and is bounded, we can apply theorem 2.3.5 from [32 , pp. 66] to conclude that for sufficiently large N, the inverse $\left(I-\widetilde{\mathcal{F}}_{M,N}^{\lambda,\beta}{}'(v_0)\right)^{-1}$ exists and is uniformly bounded. In other words, there exists a constant $C_7>0$ such that $\|\left(I-\widetilde{\mathcal{F}}_{M,N}^{\lambda,\beta}{}'(v_0)\right)^{-1}\|_{w_\beta}\le C_7<\infty$. □

Theorem 1. (Existence of iterated and approximate solutions with their convergence). *Let $v_0 \in L^2_{w_\beta}(\mathbb{R}_+)$ be an isolated solution of operator (44), and let $\mathcal{F}'(v_0)$ denote the Fréchet derivative of $\mathcal{F}$ at v_0. Then, the operator Equations (59) and (60) admit unique solutions $v_N^{\lambda,\beta}$ and $\tilde{v}_N^{\lambda,\beta}$, respectively, belonging to $\mathcal{B}(v_0,\delta)$, where $\mathcal{B}(v_0,\delta) = \{v \mid \|v - v_0\|_{w_\beta} \le \delta\}$ for some $\delta > 0$, and for sufficiently large N. Furthermore, there exists a constant $0 < q < 1$, independent of N, such that*

$$\|\tilde{v}_N^{\lambda,\beta} - v_0\|_{w_\beta} \leqslant \frac{\gamma_{M,N}^{\lambda,\beta}}{(1-q)}, \tag{69}$$

$$\|v_N^{\lambda,\beta} - v_0\|_{w_\beta} \leqslant \frac{\gamma_{M,N}^{\lambda,\beta}}{(1-q)} + c(\beta N)^{-\frac{m}{2}} |v_0|_{H^m_{\lambda,\beta}(\mathbb{R}_+)}, \tag{70}$$

where $\gamma_{M,N}^{\lambda,\beta} = \|\left(I - \widetilde{\mathcal{F}}_{M,N}^{\lambda,\beta\prime}(v_0)\right)^{-1}\left(\widetilde{\mathcal{F}}_{M,N}^{\lambda,\beta}(v_0) - \mathcal{F}(v_0)\right)\|_{w_\beta}$.

Proof. For any $v \in \mathcal{B}(v_0,\delta)$ and $v \in L^2_{w_\beta}(\mathbb{R}_+)$, we have

$$\begin{aligned}&\|\left(\widetilde{\mathcal{F}}_{M,N}^{\lambda,\beta\prime}(v_0) - \widetilde{\mathcal{F}}_{M,N}^{\lambda,\beta\prime}(v)\right)w\|_{w_\beta}\\ &= \|\left(\mathcal{F}'(P_N^{\lambda,\beta}v_0) - \mathcal{F}'(P_N^{\lambda,\beta}v)\right)w\|_{w_\beta}.\end{aligned} \tag{71}$$

According to (66), we have

$$\begin{aligned}&\|\left(\widetilde{\mathcal{F}}_{M,N}^{\lambda,\beta\prime}(v_0) - \widetilde{\mathcal{F}}_{M,N}^{\lambda,\beta\prime}(v)\right)w\|_{w_\beta}\\ &\le \|\mathcal{L}_M(P_N^{\lambda,\beta}v_0 - P_N^{\lambda,\beta}v)\|_{w_\beta}\|\mathcal{L}(w)\|_{w_\beta}\\ &\le C_4^2\beta^{2(\alpha-1)}\Gamma^2(1-\alpha)\|P_N^{\lambda,\beta}v_0 - P_N^{\lambda,\beta}v\|_{w_\beta}\|w\|_{w_\beta}\\ &\le C_4^2\beta^{2(\alpha-1)}\Gamma^2(1-\alpha)\|P_N^{\lambda,\beta}\|_{w_\beta}\|v_0 - v\|_{w_\beta}\|w\|_{w_\beta}\\ &\le C_4^2\beta^{2(\alpha-1)}\Gamma^2(1-\alpha)\|P_N^{\lambda,\beta}\|_{w_\beta}\delta\|w\|_{w_\beta}.\end{aligned} \tag{72}$$

This implies

$$\|\left(\widetilde{\mathcal{F}}_{M,N}^{\lambda,\beta\prime}(v_0) - \widetilde{\mathcal{F}}_{M,N}^{\lambda,\beta\prime}(v)\right)\|_{w_\beta} \le C_4^2\beta^{2(\alpha-1)}\Gamma^2(1-\alpha)\|P_N^{\lambda,\beta}\|_{w_\beta}\delta. \tag{73}$$

From (73), we have

$$\begin{aligned}&\sup_{\|v-v_0\|_{w_\beta}\leqslant\delta}\|\left(I - \widetilde{\mathcal{F}}_{M,N}^{\lambda,\beta\prime}(v_0)\right)^{-1}\left(\widetilde{\mathcal{F}}_{M,N}^{\lambda,\beta\prime}(v_0) - \widetilde{\mathcal{F}}_{M,N}^{\lambda,\beta\prime}(v)\right)\|_{w_\beta}\\ &\leqslant C_7C_4^2\beta^{2(\alpha-1)}\Gamma^2(1-\alpha)\|P_N^{\lambda,\beta}\|_{w_\beta}\delta.\end{aligned}$$

Let us define $q = C_7C_4^2\beta^{2(\alpha-1)}\Gamma^2(1-\alpha)\|P_N^{\lambda,\beta}\|_{w_\beta}\delta$. We choose δ to be sufficiently small such that $0 < q < 1$, thereby proving of eq. (4.4) from theorem 2 in [33]. By using (20) and A2, we can derive

$$\begin{aligned}\gamma_{M,N}^{\lambda,\beta} &= \|\left(I - \widetilde{\mathcal{F}}_{M,N}^{\lambda,\beta\prime}(v_0)\right)^{-1}\left(\widetilde{\mathcal{F}}_{M,N}^{\lambda,\beta}(v_0) - \mathcal{F}(v_0)\right)\|_{w_\beta}\\ &\leqslant C_7\|\widetilde{\mathcal{F}}_{M,N}^{\lambda,\beta}(v_0) - \mathcal{F}(v_0)\|_{w_\beta}\\ &= C_7\|\mathcal{F}_M(P_N^{\lambda,\beta}v_0) - \mathcal{F}(v_0)\|_{w_\beta}\\ &\leqslant C_7\|\mathcal{G}(\mathcal{L}_M(P_N^{\lambda,\beta}v_0) + f) - \mathcal{G}(\mathcal{L}(v_0) + f)\|_{w_\beta}\\ &\leqslant C_7C_1\|\mathcal{L}_M(P_N^{\lambda,\beta}v_0) - \mathcal{L}(v_0)\|_{w_\beta}\\ &\leqslant C_7C_1\|\mathcal{L}_M(P_N^{\lambda,\beta}v_0) - \mathcal{L}_M(v_0) + \mathcal{L}_M(v_0) - \mathcal{L}(v_0)\|_{w_\beta}\\ &\leqslant C_7C_1\left(\|\mathcal{L}_M(P_N^{\lambda,\beta}v_0) - \mathcal{L}_M(v_0)\|_{w_\beta} + \|\mathcal{L}_M(v_0) - \mathcal{L}(v_0)\|_{w_\beta}\right).\end{aligned} \tag{74}$$

From (67) and (66), we get

$$\begin{aligned}\gamma_{M,N}^{\lambda,\beta} \leqslant C_7C_1\Big(&cC_4\beta^{(\alpha-1)}\Gamma(1-\alpha)(\beta N)^{-\frac{m}{2}}|v_0|_{H^m_{\lambda,\beta}(\mathbb{R}_+)}\\ &+C_6^{\beta,\alpha}((\alpha-1)M)^{-\frac{m}{2}}\Big).\end{aligned} \tag{75}$$

Choosing a sufficiently large value for N, denoted as N large enough, ensures that $\gamma_{M,N}^{\lambda,\beta} \le \delta(1-q)$ holds. This condition satisfies eq. (4.6) of theorem 2 in [33]. Therefore, using theorem 2 from [33], we derive the inequality:

$$\frac{\gamma_{M,N}^{\lambda,\beta}}{(1+q)} \leqslant \|\tilde{v}_N^{\lambda,\beta} - v_0\|_{w_\beta} \leqslant \frac{\gamma_{M,N}^{\lambda,\beta}}{(1-q)} \tag{76}$$

For (70), we use the relation between approximate and iterated solutions as $v_N^{\lambda,\beta} = P_N^{\lambda,\beta}\tilde{v}_N^{\lambda,\beta}$. Then, we have

$$v_0 - v_N^{\lambda,\beta} = v_0 - P_N^{\lambda,\beta}\tilde{v}_N^{\lambda,\beta} = v_0 - p_N^{\lambda,\beta}v_0 + P_N^{\lambda,\beta}v_0 - P_N^{\lambda,\beta}\tilde{v}_N^{\lambda,\beta} \tag{77}$$

This is implies

$$\|v_0 - v_N^{\lambda,\beta}\|_{w_\beta} \leqslant \|v_0 - P_N^{\lambda,\beta}v_0\|_{w_\beta} + \|P_N^{\lambda,\beta}\|_{w_\beta}\|v_0 - \tilde{v}_N^{\lambda,\beta}\|_{w_\beta} \tag{78}$$

Using (20) and (76), we get

$$\|v_0 - v_N^{\lambda,\beta}\|_{w_\beta} \leqslant \frac{\gamma_{M,N}^{\lambda,\beta}}{(1-q)} + c(\beta N)^{-\frac{m}{2}}|v_0|_{H^m_{\lambda,\beta}(\mathbb{R}_+)}. \tag{79}$$

This completes the proof. ◻

Now, we establish the convergence of iterated and approximate solutions $u_N^{\lambda,\beta}$ and $\tilde{u}_N^{\lambda,\beta}$, corresponding to the approximate solutions $v_N^{\lambda,\beta}$ and $\tilde{v}_N^{\lambda,\beta}$, respectively,

$$u_N^{\lambda,\beta} = \mathcal{L}_M(v_N^{\lambda,\beta}) + f, \tag{80}$$

$$\tilde{u}_N^{\lambda,\beta} = \mathcal{L}_M(\tilde{v}_N^{\lambda,\beta}) + f. \tag{81}$$

Theorem 2. *If u_0 and v_0 are the solutions of the operator Equations (38) and (44), respectively, then the errors between $u_N^{\lambda,\beta}$, $\tilde{u}_N^{\lambda,\beta}$, and u_0 are bounded as follows:*

$$\begin{aligned}&\|u_0 - u_N^{\lambda,\beta}\|_{w_\beta}\\ &\le C_4\beta^{\alpha-1}\Gamma(1-\alpha)\left(\frac{\gamma_{M,N}^{\lambda,\beta}}{(1-q)} + c(\beta N)^{-\frac{m}{2}}|v_0|_{H^m_{\lambda,\beta}(\mathbb{R}_+)}\right),\end{aligned} \tag{82}$$

and

$$\|u_0 - \tilde{u}_N^{\lambda,\beta}\|_{w_\beta} \le C_4\beta^{\alpha-1}\Gamma(1-\alpha)\frac{\gamma_{M,N}^{\lambda,\beta}}{(1-q)} \tag{83}$$

respectively.

Proof. Using Equations (64),(80), and (81), we can write

$$\begin{aligned}|u_0(t) - \tilde{u}_N^{\lambda,\beta}(t)| &= |\mathcal{L}(v_0)(t) - \mathcal{L}_M(\tilde{v}_N^{\lambda,\beta})(t)|\\ &\le \int_0^t (t-s)^{-\alpha}|k(t,s)||v_0(s) - v_N^{\lambda,\beta}(s)|ds.\end{aligned} \tag{84}$$

From Equation (41), we get

$$\|u_0 - u_N^{s,\alpha}\|_{w_\beta} \le C_4\beta^{\alpha-1}\Gamma(1-\alpha)\|v_0 - v_N^{\lambda,\beta}\|_{w_\beta}. \tag{85}$$

Then using (70), we obtain

$$\|u_0 - u_N^{s,\alpha}\|_{w_\beta} \le C_4\beta^{\alpha-1}\Gamma(1-\alpha) \left(\frac{\gamma_{M,N}^{\lambda,\beta}}{(1-q)} + c(\beta N)^{-\frac{m}{2}}|v_0|_{H^m_{\lambda,\beta}(\mathbb{R}_+)}\right). \tag{86}$$

To estimate the error between $\tilde{u}_N^{\lambda,\beta}$ and u_0, we apply the same argument used to prove (82). Thus, we obtain

$$\|u_0 - \tilde{u}_N^{\lambda,\beta}\|_{w_\beta} \le C_4\beta^{\alpha-1}\Gamma(1-\alpha)\|v_0 - \tilde{v}_N^{\lambda,\beta}\|_{w_\beta} \tag{87}$$

From (69), we get

$$\|u_0 - \tilde{u}_N^{\lambda,\beta}\|_{w_\beta} \le C_4\beta^{\alpha-1}\Gamma(1-\alpha)\frac{\gamma_{M,N}^{\lambda,\beta}}{(1-q)} \tag{88}$$

□

5 | Numerical Experiments

This section is presented to validate the theoretical results. This is achieved by providing several test problems, which are solved using the collocation method with uniform degrees N and M for various scaling factors β on large time partitions of $[0,T]$, where M, $(M \ge N)$ represents the number of collocation points and weight functions for the generalized Laguerre functions-Gauss quadrature set. The algorithms are implemented using Matlab.

Example 1. We consider the following nonlinear WSNVIEs with smooth and nonsmooth solutions:

$$u(t) = b(t) - \int_0^t (t-s)^{-\alpha}u^2(s)ds, \quad t \in [0,T] \tag{89}$$

with

$$b(t) = t^\eta + t^{2\eta+1-\alpha}B(2\eta+1, 1-\alpha),$$

where $B(v,\mu)$ represents the Beta function, defined as

$$B(v,\mu) = \int_0^1 x^{v-1}(1-x)^{\mu-1}dx \text{ for } v,\mu > 0.$$

The analytical solution is $u(t) = t^\eta$. This example serves as a generalization of the case presented in [13], extending it to the nonlinear scenario. Here, we introduce the η-parameter, residing in $\mathbb{R}_+$, to examine the sensitivity of the current method with SLGSG scheme when confronted with nonlinear problems.

- $u(x) = x$, which is a smooth solution. We set $\lambda = 0$. In Tables 1 and 2, we present the values of the $L^2_{w_\beta}$ and L^∞-errors for $N = 1$ and various values of M. The $L^2_{w_\beta}$-errors are evaluated at $\beta = 1$, while the L^∞-errors are evaluated at $\beta = 0.5$, with $T = 1, 10, 100, 1000$, respectively. Additionally, Figure 1a,b illustrates the graphs of $\log_{10}$ of the $L^2_{w_\beta}$-errors for both the approximate and iterate solutions with $N = 1$ at different values of M, where $\beta = 1$, and $\alpha = 0.2, 0.4, 0.6, 0.8$, respectively. Meanwhile, Figure 1c,d depicts the graphs of $\log_{10}$ of the $L^2_{w_\beta}$-errors at the same values of $N, M, \alpha, T =$ 10,000, and $\beta = 0.001$. Clearly, both the approximate and iterate solutions converge at exponential rates as predicted by Theorem 2.
- $u(x) = x^{0.5}$. We set $\lambda = 1$ and $N = 1$. In Tables 3 and 4, we report the values of the $L^2_{w_\beta}$- and L^∞-errors with $\beta = 1$ and 0.5, respectively. While the graphs of the log_{10} of the $L^2_{w_\beta}$- and L^∞-errors versus M for $(T,\beta) = (10{,}000, 0.001)$ at $\alpha = 0.2, 0.4, 0.6, 0.8$ are displayed in Figure 2. Clearly, both the approximate and iterate solutions converge at exponential rates as predicted by Theorem 2.

Based on these results, it is evident that our methods exhibit exponential convergence for both iterative and approximate solutions over large intervals in both norms. This observation aligns with findings in [13], where the GLFs collocation method demonstrated accuracy over large intervals for nonlinear problems.

Example 2. Consider the nonlinear Volterra integral equation with singular kernel [34]

$$u(t) = t^3 - \frac{4096}{6435}t^{\frac{17}{2}} + \int_0^t ts(t-s)^{-\frac{1}{2}}u(s)^2ds \tag{90}$$

The exact solution is $u(t) = t^3$. We set $\lambda = 0$ and employ the SLPs method for various of N and M to solve the singular kernel nonlinear Volterra integral Equation (90). The $L^2_{w_\beta}$-errors are listed in Table 5, while the L_∞-errors are provided in Table 6 for $T = 1, 10, 100$, and 1000. Also, we compare the performance of the proposed method with the methods developed by the authors of [14, 15, 34] in Table 7, where we list the absolute errors at some mesh points for $N = 4, 5$ and $M = 64$ and $\beta = 4$. Based on the presented results in Table 6, the iterated solution demonstrates good convergence for small intervals but exhibits poor convergence for larger ones. Conversely, the approximate solution showcases good convergence across both small and large intervals. From Table 7, we can see that the convergence rate of the SLP method for the iterated solution is better than that obtained using the methods described in [14, 15, 34].

Example 3. Consider the nonlinear VIE [16]:

$$u(t) + \int_0^t s(t-s)^{-0.5}u(s)^3ds = t^{0.5} + \frac{5}{16}\pi t^3 \tag{91}$$

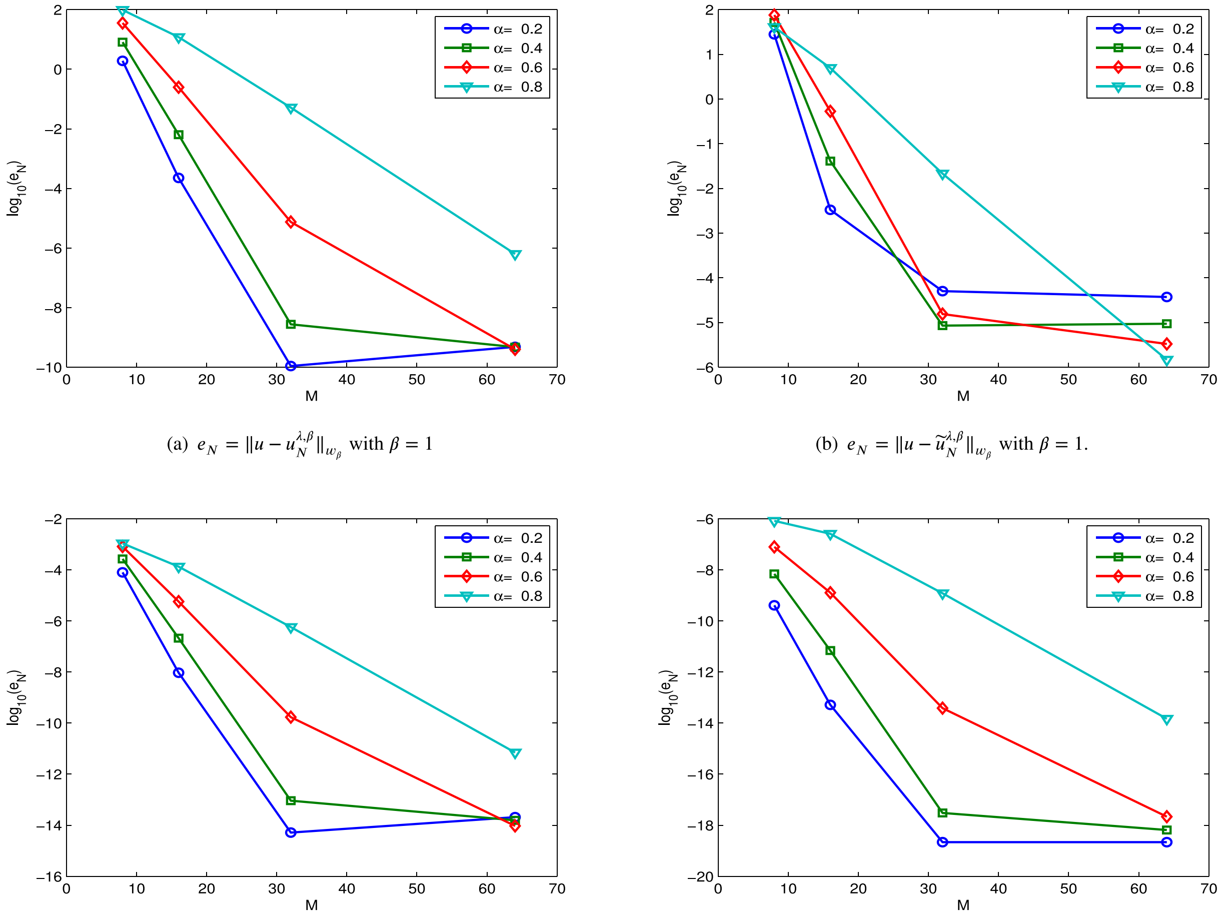
The exact solution is given by $u(t) = t^{0.5}$. We choose $\lambda = 1$ and apply the SLFs method to solve the singular kernel nonlinear Volterra integral Equation (91). We vary parameters N and M across different settings. The $L^2_{w_\beta}$-errors are detailed in Table 8, and L_∞-errors are presented in Table 9 for $T = 1, 10, 100$, and 1000. Additionally, we compare our method's performance with that of the methods proposed by [16] in Table 10. The maximum errors for $N = M$ and various β values are listed. From Table 9, it is evident that the iterated solution shows good convergence for small and larger intervals. In contrast, the approximate solution

TABLE 1 | The $L^2_{w_\beta}$ errors for Example 1 with $\eta = 1$ for $N = 1$ at $\alpha = 0.5$.

M	4	8	16	32	64
$\|u - u_N^{\lambda,\beta}\|_{w_\beta}$	8.32e−03	6.75e−04	1.51e−06	4.97e−12	1.74e−14
$\|u - \tilde{u}_N^{\lambda,\beta}\|_{w_\beta}$	8.66e−03	6.05e−04	1.58e−06	5.19e−12	1.89e−14

TABLE 2 | Comparison of errors for Example 1 at $\eta = 1$ with $\alpha = 0.5$ and $\beta = 0.5$.

	$T = 1$		$T = 10$		$T = 100$		$T = 1000$	
M	$\|u - u_N^{\lambda,\beta}\|_\infty$	$\|u - \tilde{u}_N^{\lambda,\beta}\|_\infty$	$\|u - u_N^{\lambda,\beta}\|_\infty$	$\|u - \tilde{u}_N^{\beta}\|_\infty$	$\|u - u_N^{\lambda,\beta}\|_\infty$	$\|u - \tilde{u}_N^{\lambda,\beta}\|_\infty$	$\|u - u_N^{\lambda,\beta}\|_\infty$	$\|u - \tilde{u}_N^{\lambda,\beta}\|_\infty$
16	2.79e−06	2.77e−06	3.80e−05	2.64e−05	3.90e−04	1.99e−02	3.91e−03	7.14e−00
32	9.18e−12	9.11e−12	1.25e−10	8.70e−11	1.28e−09	6.45e−08	1.29e−08	2.35e−05
64	3.22e−14	3.18e−14	4.37e−13	4.30e−13	4.49e−12	2.11e−10	4.51e−11	2.80e−08



(a) $e_N = \|u - u_N^{\lambda,\beta}\|_{w_\beta}$ with $\beta = 1$

(b) $e_N = \|u - \tilde{u}_N^{\lambda,\beta}\|_{w_\beta}$ with $\beta = 1$.

(c) $e_N = \|u - u_N^{\lambda,\beta}\|_\infty$ with $\beta = 0.001$

(d) $e_N = \|u - \tilde{u}_N^{\lambda,\beta}\|_\infty$ with $\beta = 0.001$.

FIGURE 1 | Graphs the $L^2_{w_\beta}$- and L^∞-errors for $N = 1$ and $\eta = 1$ at $\lambda = 0$. [Colour figure can be viewed at wileyonlinelibrary.com]

exhibits consistent convergence across both small and large intervals. Table 10 indicates that the MSLP method achieves a superior convergence rate compared to the method described in [16].

Example 4. Consider the fractional susceptible-infectious-susceptible (SIS) epidemic model [20, 21]

$$D_t^\xi S(t) = \mu - \psi S(t)u(t) - \mu S(t), \;\; D_t^\xi u(t) \quad = \psi S(t)u(t) - \mu u(t),$$

with the constraint

$$S(t) + u(t) = 1 \text{ and } S(0) = S_0, \quad u(0) = u_0.$$

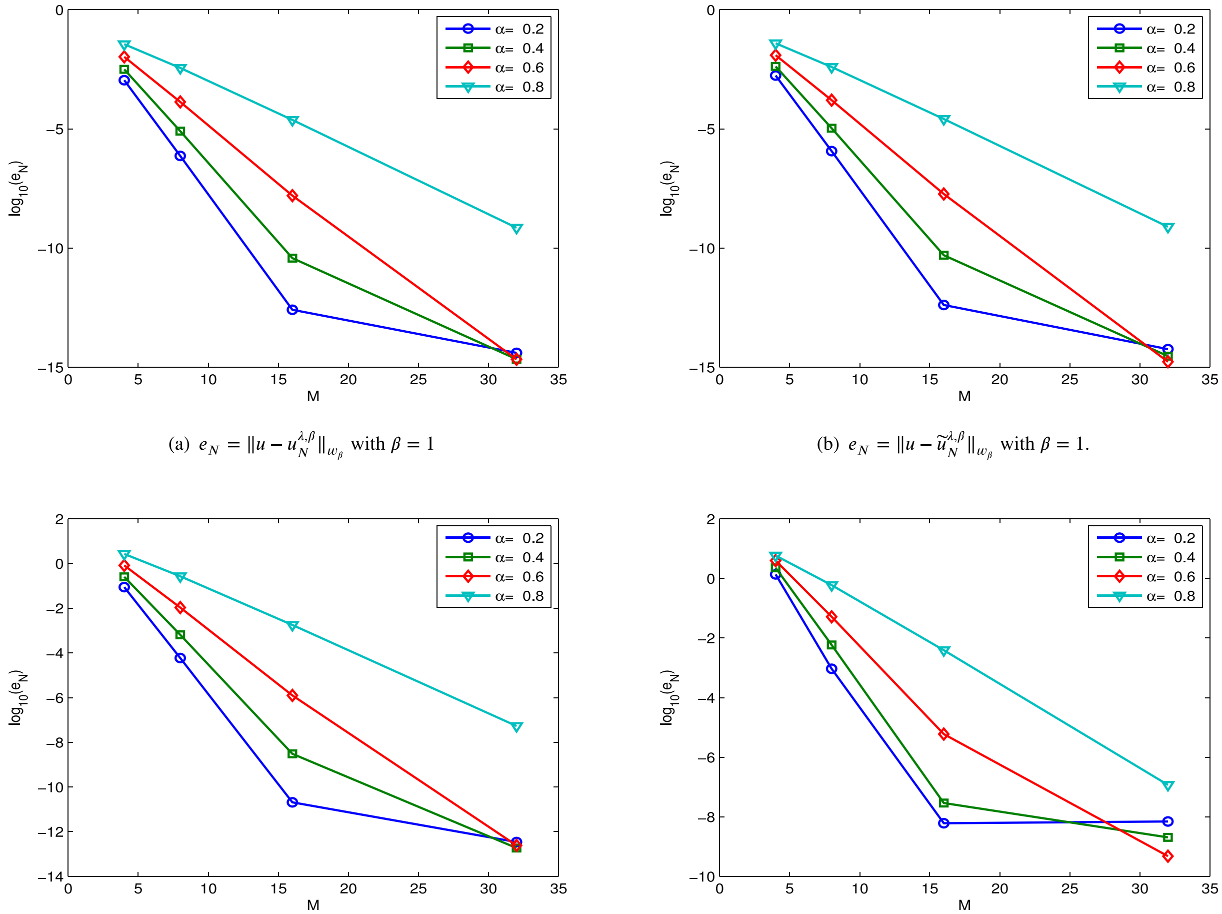
where

TABLE 3 | The $L^2_{w_\beta}$-errors for Example 1 with $\alpha = 0.5$ and $\eta = 0.5$.

M	2	4	8	16	32
$\|u - u_N^{\lambda,\beta}\|_{w_\beta}$	6.07e−02	5.76e−03	3.23e−05	6.98e−10	2.19e−16
$\|u - \tilde{u}_N^{\lambda,\beta}\|_{w_\beta}$	7.38e−02	6.99e−03	3.91e−05	8.43e−10	1.71e−15

TABLE 4 | Comparison of errors for Example 1 with $\alpha = 0.5$ and $\eta = 0.5$.

	$T = 1$		$T = 10$		$T = 100$		$T = 1000$	
M	$\|u - u_N^{\lambda,\beta}\|_\infty$	$\|u - \tilde{u}_N^{\lambda,\beta}\|_\infty$	$\|u - u_N^{\lambda,\beta}\|_\infty$	$\|u - \tilde{u}_N^{\lambda,\beta}\|_\infty$	$\|u - u_N^{\lambda,\beta}\|_\infty$	$\|u - \tilde{u}_N^{\lambda,\beta}\|_\infty$	$\|u - u_N^{\lambda,\beta}\|_\infty$	$\|u - \tilde{u}_N^{\lambda,\beta}\|_\infty$
8	2.97e−05	1.33e−05	7.72e−05	1.06e−04	2.91e−04	1.12e−01	1.78e−02	3.95e+01
16	6.41e−10	2.87e−10	1.66e−09	2.29e−09	6.24e−09	2.41e−08	3.83e−07	8.52e−04
32	3.33e−16	4.44e−16	1.78e−15	2.26e−14	2.13e−14	3.95e−12	3.95e−11	1.27e−09



(a) $e_N = \|u - u_N^{\lambda,\beta}\|_{w_\beta}$ with $\beta = 1$

(b) $e_N = \|u - \tilde{u}_N^{\lambda,\beta}\|_{w_\beta}$ with $\beta = 1$.

(c) $e_N = \|u - u_N^{\lambda,\beta}\|_\infty$ with $\beta = 0.001$

(d) $e_N = \|u - \tilde{u}_N^{\lambda,\beta}\|_\infty$ with $\beta = 0.001$.

FIGURE 2 | Graphs the $L^2_{w_\beta}$- and L^∞-errors for $N = 1$ and $\eta = 0.5$ at $\lambda = 1$. [Colour figure can be viewed at wileyonlinelibrary.com]

- $0 < \xi < 1$: controls the strength of the singularity, modeling memory effects.
- The unknown functions $S(t)$ and $u(t)$ represent the percentage of susceptible and infected people at time t with initial data S_0 and u_0.
- μ is the birth rate and the death removal rate, ψ is the contact rate.
- $D_t^\xi u(t) := \frac{1}{\Gamma(1-\xi)} \int_0^t \frac{u'(s)}{(t-s)^\xi}\, ds, \quad t \geq 0.$
- $I_t^\xi u(t) := \frac{1}{\Gamma(\xi)} \int_0^t \frac{u(s)}{(t-s)^{1-\xi}}\, ds, \quad t \geq 0.$

TABLE 5 | The $L^2_{w_\beta}$-errors for Example 2 with $N = 3$ and $\beta = 4$.

M	8	16	32	64	128
$\|u - u_N^\beta\|_{w_\beta}$	7.55e−04	8.11e−05	1.52e−06	2.54e−09	4.78e−14
$\|u - \tilde{u}_N^\beta\|_{w_\beta}$	1.33e−01	1.43e−02	2.68e−04	4.48e−07	8.44e−12

TABLE 6 | Comparison of errors for Example 2 with $N = 3$ and $\beta = 4$.

	$T = 1$		$T = 10$		$T = 100$		$T = 1000$	
M	$\|u - u_N^\beta\|_\infty$	$\|u - \tilde{u}_N^{\lambda,\beta}\|_\infty$	$\|u - u_N^{\lambda,\beta}\|_\infty$	$\|u - \tilde{u}_N^{\lambda,\beta}\|_\infty$	$\|u - u_N^{\lambda,\beta}\|_\infty$	$\|u - \tilde{u}_N^{\lambda,\beta}\|_\infty$	$\|u - u_N^{\lambda,\beta}\|_\infty$	$\|u - \tilde{u}_N^{\lambda,\beta}\|_\infty$
64	1.57e−08	2.00e−08	6.33e−07	2.77e−01	2.48e−03	Non-conv	2.79e−00	Non-conv
128	2.94e−13	2.10e−13	1.19e−11	1.53e−05	4.69e−08	Non-conv	5.28e−05	Non-conv

TABLE 7 | Comparison results for Example 2.

	$\|u(x) - u_N^{\lambda,\beta}(x)\|$		$\|u(x) - \tilde{u}_N^{\lambda,\beta}(x)\|$		**Method in [15]**	**Method in [14]**	**Method in [34]**
x	$N = 5$	$N = 4$	$N = 5$	$N = 4$	$N = 5$	$N = 4$	$N = 4$
0.1	3.51e−10	1.35e−09	1.49e−15	5.19e−15	2.02e−11	1.07e−06	9.66e−05
0.2	4.61e−10	2.88e−09	1.11e−13	6.42e−13	5.88e−10	1.14e−06	4.71e−04
0.3	1.77e−11	1.55e−09	2.74e−13	4.34e−12	4.17e−09	8.90e−06	7.62e−04
0.4	7.13e−10	1.69e−09	4.91e−12	5.39e−12	1.57e−08	2.09e−05	5.29e−04
0.5	1.32e−09	6.02e−09	3.77e−11	1.48e−10	3.98e−08	2.22e−04	4.70e−03
0.6	1.61e−09	1.07e−08	1.40e−10	8.04e−10	8.91e−08	2.26e−04	4.34e−04
0.7	1.39e−09	1.50e−08	3.31e−10	2.80e−09	2.11e−07	1.33e−03	2.70e−05
0.8	5.47e−10	1.84e−08	4.71e−10	7.50e−09	5.07e−07	4.53e−03	5.00e−05
0.9	9.97e−10	2.04e−08	1.63e−11	1.67e−08	1.33e−06	1.50e−03	7.32e−03

TABLE 8 | The $L^2_{w_\beta}$-errors for Example 3 with $N = 1$ and $\beta = 5$.

M	16	32	64	128	256
$\|u - u_N^{\lambda,\beta}\|_{w_\beta}$	8.22e−06	3.79e−07	2.68e−08	2.16e−09	1.82e−10
$\|u - \tilde{u}_N^{\lambda,\beta}\|_{w_\beta}$	1.50e−05	6.90e−07	4.89e−08	3.93e−09	3.32e−10

TABLE 9 | Comparison of errors for Example 3 with $N = 1$, $\lambda = 1$ and $\beta = 0.001$.

	$T = 1$		$T = 10$		$T = 100$		$T = 1000$	
M	$\|u - u_N^{\lambda,\beta}\|_\infty$	$\|u - \tilde{u}_N^{\lambda,\beta}\|_\infty$	$\|u - u_N^{\lambda,\beta}\|_\infty$	$\|u - \tilde{u}_N^{\lambda,\beta}\|_\infty$	$\|u - u_N^{\lambda,\beta}\|_\infty$	$\|u - \tilde{u}_N^{\lambda,\beta}\|_\infty$	$\|u - u_N^{\lambda,\beta}\|_\infty$	$\|u - \tilde{u}_N^{\lambda,\beta}\|_\infty$
128	2.84e−08	1.11e−15	8.99e−08	1.14e−12	2.84e−07	1.28e−09	8.99e−07	9.97e−07
256	2.40e−09	3.33e−16	7.60e−09	2.63e−13	2.40e−08	2.86e−10	7.60e−08	2.80e−07

TABLE 10 | Comparison of errors for Example 3 with $M = N$ and $\lambda = 1$.

	$\beta = 0.001$		$\beta = 0.0001$		$\beta = 0.00001$		**Method in [16]**	
N	$\|u - u_N^{\lambda,\beta}\|_\infty$	$\|u - \tilde{u}_N^{\lambda,\beta}\|_\infty$	$\|u - u_N^{\lambda,\beta}\|_\infty$	$\|u - \tilde{u}_N^{\lambda,\beta}\|_\infty$	$\|u - u_N^{\lambda,\beta}\|_\infty$	$\|u - \tilde{u}_N^{\lambda,\beta}\|_\infty$	$\|u - u_N\|_\infty$	$\|u - \tilde{u}_N\|_\infty$
2	2.77e−02	1.61e−09	2.77e−02	5.09e−12	2.77e−02	1.62e−14	1.99e−01	1.99e−01
4	9.08e−03	1.71e−09	9.08e−03	5.42e−12	9.08e−03	1.73e−14	9.85e−02	6.34e−02
8	5.57e−04	4.26e−10	5.57e−04	1.35e−12	5.57e−04	8.22e−15	3.65e−02	1.17e−02
16	8.11e−05	2.91e−10	8.11e−05	9.23e−13	8.11e−05	2.78e−15	1.07e−02	1.74e−03

TABLE 11 | Comparison results for Example 4 with $\alpha = 1 - \xi$, $N = 14$, $M = 34$, and $\beta = 6$.

t	$\xi = 0.7$ Method in [21]	$u_N^{\lambda,\beta}$	$\tilde{u}_N^{\lambda,\beta}$	$\xi = 0.8$ Method in [21]	$u_N^{\lambda,\beta}$	$\tilde{u}_N^{\lambda,\beta}$	
0.01	0.68834	0.69058	0.68823	0.69266	0.69361	0.69264	
0.02	0.68163	0.68374	0.68140	0.68749	0.68842	0.68743	
0.03	0.67619	0.67768	0.67593	0.68302	0.68368	0.68294	
0.04	0.67150	0.67230	0.67137	0.67898	0.67933	0.67890	
0.05	0.66731	0.66750	0.66727	0.67526	0.67533	0.67520	
	$\xi = 0.95$			$\xi = 1$			
t	Method in [21]	$u_N^{\lambda,\beta}$	$\tilde{u}_N^{\lambda,\beta}$	Method in [21]	$u_N^{\lambda,\beta}$	$\tilde{u}_N^{\lambda,\beta}$	$u(t)$
0.01	0.69645	0.69656	0.69644	0.69722	0.69722	0.69722	0.69722
0.02	0.69321	0.69333	0.69321	0.69450	0.69450	0.69450	0.69450
0.03	0.69013	0.69022	0.69012	0.69182	0.69182	0.69182	0.69182
0.04	0.68716	0.68721	0.68715	0.68919	0.68919	0.68919	0.68919
0.05	0.68429	0.68430	0.68428	0.68661	0.68661	0.68661	0.68661

TABLE 12 | Comparison results for Example 4 with $M = N + 20$ and $\beta = 2.5$.

N	$\alpha = 0.2$, $t = 0.5$ $u_N^{\lambda,\beta}$	$t = 1$ $u_N^{\lambda,\beta}$	$t = 2$ $u_N^{\lambda,\beta}$	$t = 3$ $u_N^{\lambda,\beta}$	$\alpha = 0.05$, $t = 0.5$ $u_N^{\lambda,\beta}$	$t = 1$ $u_N^{\lambda,\beta}$	$t = 2$ $u_N^{\lambda,\beta}$	$t = 3$ $u_N^{\lambda,\beta}$
4	0.59893	0.55525	0.55088	0.55633	0.60372	0.55399	0.53351	0.53383
8	0.59536	0.56817	0.53423	0.53328	0.60206	0.56107	0.52375	0.51452
12	0.59819	0.56631	0.54007	0.52289	0.60279	0.56056	0.52546	0.51129
14	0.59885	0.56542	0.53976	0.52495	0.60239	0.56297	0.50105	0.59352
	$\alpha = 0.2$				$\alpha = 0.05$			
N	$t = 0.5$ $\tilde{u}_N^{\lambda,\beta}$	$t = 1$ $\tilde{u}_N^{\lambda,\beta}$	$t = 2$ $\tilde{u}_N^{\lambda,\beta}$	$t = 3$ $\tilde{u}_N^{\lambda,\beta}$	$t = 0.5$ $\tilde{u}_N^{\lambda,\beta}$	$t = 1$ $\tilde{u}_N^{\lambda,\beta}$	$t = 2$ $\tilde{u}_N^{\lambda,\beta}$	$t = 3$ $\tilde{u}_N^{\lambda,\beta}$
4	0.59319	0.56757	0.54039	0.49515	0.60049	0.56068	0.52709	0.49336
8	0.59855	0.56623	0.53889	0.52751	0.60282	0.56058	0.52508	0.51279
12	0.59863	0.56558	0.53863	0.52670	0.60289	0.56040	0.52515	0.51226
14	0.59845	0.56585	0.53802	0.52786	0.60214	0.56179	0.53136	0.45510

TABLE 13 | The $L^2_{w_\beta}$-errors for Example 5 with $N = 16$, $\lambda = 0$, and $\beta = 16$.

M	16	32	64	128	256
$\|u - u_N^{\lambda,\beta}\|_{w_\beta}$	8.87e−07	4.50e−10	1.07e−13	1.41e−13	2.27e−14
$\|u - \tilde{u}_N^{\lambda,\beta}\|_{w_\beta}$	8.87e−07	4.50e−10	1.07e−13	1.41e−13	2.27e−14

From [23], this problem reduces to

$$u(t) = u_0 - \frac{1}{\Gamma(\xi)} \int_0^t \frac{u(s)(\psi - \mu - \psi u(t))}{(t-s)^{1-\xi}} ds \qquad (92)$$

From [21], we consider the following:

$$D_t^{\xi} S(t) = 1 - 2S(t)u(t) - S(t), \quad S_0 = 0.7 \qquad (93)$$

$$D_t^{\xi} u(t) = 2S(t)u(t) - u(t), \quad u_0 = 0.3 \qquad (94)$$

For $\xi = 1$, from [35] the exact solutions are

$$S(t) = 1 - \frac{1}{2 - \frac{4}{7}e^{-t}},$$

$$u(t) = \frac{1}{2 - \frac{4}{7}e^{-t}}.$$

TABLE 14 | Comparison of errors for Example 5 with $N = 16$, $\lambda = 0$, and $\beta = 16$.

	$T = 1$		$T = 3$		$T = 4$		$T = 5$	
M	$\|u - u_N^{\lambda,\beta}\|_\infty$	$\|u - \tilde{u}_N^{\lambda,\beta}\|_\infty$	$\|u - u_N^{\lambda,\beta}\|_\infty$	$\|u - \tilde{u}_N^{\lambda,\beta}\|_\infty$	$\|u - u_N^{\lambda,\beta}\|_\infty$	$\|u - \tilde{u}_N^{\lambda,\beta}\|_\infty$	$\|u - u_N^{\lambda,\beta}\|_\infty$	$\|u - \tilde{u}_N^{\lambda,\beta}\|_\infty$
16	8.66e−04	8.66e−04	6.50e−01	5.96e−01	Non-conv	Non-conv	Non-conv	Non-conv
32	2.78e−07	2.77e−07	1.18e−01	9.97e−02	Non-conv	Non-conv	Non-conv	Non-conv
64	1.70e−10	3.38e−08	1.92e−07	1.14e−12	3.36e−04	2.39e−04	5.29e−02	1.17e−02
128	4.68e−11	4.68e−11	4.31e−08	4.34e−08	1.04e−04	7.49e−05	1.48e−02	3.71e−03
256	6.79e−12	6.79e−12	2.32e−09	2.36e−09	4.58e−06	3.34e−06	5.52e−04	1.35e−04

This example demonstrates the application of our method to a real-world problem involving a fractional susceptible-infectious-susceptible (SIS) epidemic model. By applying the technique introduced in Section 3.1, we obtained the numerical solution for Equation (92) with the initial condition (93). The values of the iterated and approximate solutions are displayed for $t = 0.01, 0.02, 0.03, 0.04, 0.5$ in Table 11, with parameters $N = 14$, $M = 34$, $\beta = 6$, and $\xi = 0.7, 0.8, 0.95$ or $\alpha = 0.3, 0.2, 0.05$. The obtained results are compared with those generated using the method described in [21]. The results in Table 11 indicate that our method converges to the exact solution $u(t) = \frac{1}{2-\frac{4}{7}e^{-t}}$ as $\alpha \to 0$. In Table 12, we present the values of the iterated and approximate solutions for $t = 0.5, 1, 2, 3$ at $N = 4, 8, 12, 14$ with $\alpha = 0.2, 0.05$, and $\beta = 2.5$.

Example 5. Consider the nonlinear VIE [36]:

$$u(t) - \int_0^t (t-s)^{-0.6} \sin(u(s))\,ds = t - 25\frac{8^{1/5}\pi^{1/2}t^{7/5}\Gamma(2/5)}{14\Gamma(1/5)\Gamma(7/10)} F_1\left(1; \frac{6}{5}, \frac{17}{10}; -\frac{t^2}{4}\right), \tag{95}$$

where F_1 is a hypergeometric function, and the exact solution is given by $u(t) = t$. We choose $\lambda = 0$, $\beta = 16$, and apply the SLPs method to solve the singular kernel nonlinear Volterra integral Equation (95) with $N = 16$ and $M = 16, 32, 64, 128, 252$ (Table 13). The $L^2_{w_\beta}$-errors are detailed in Table 13, and the L_∞-errors are presented in Table 14 for $T = 1, 3, 4$, and 5.

6 | Conclusion

This research discussed the numerical solution of weakly singular nonlinear Volterra integral equations on large intervals, encompassing both smooth and nonsmooth solutions, using generalized Laguerre functions with a suitable variable transformation. The error estimate in the weighted L^2-norm is established with respect to N, M, and m, where N is the highest degree of the generalized Laguerre polynomials employed in the approximation, M is the highest degree of the Gauss formula, and m is the minimum between the smoothness degree of the solution $v_0(t)$ for $t \in [0, T]$ and $k(t, \phi_t(x))v_0(\phi_t(x))$ with respect to x for $x \in \mathbb{R}_+$. Moreover, it is important to note that the accuracy can be significantly enhanced by selecting appropriate values for the parameter β and a large M. Specifically, setting $\lambda = 0$ for smooth functions and $\lambda \neq 0$ for nonsmooth functions.

Author Contributions

Walid Remili: conceptualization, investigation, writing – original draft, methodology, validation, visualization, writing – review and editing, project administration, formal analysis, software, data curation, supervision, resources. **Azedine Rahmoune:** conceptualization, investigation, funding acquisition, writing – original draft, methodology, validation, visualization, software, project administration, formal analysis, data curation, writing – review and editing. **Chenkuan Li:** conceptualization, investigation, methodology, validation, formal analysis, writing – review and editing, visualization, resources, project administration, writing – original draft.

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Conflicts of Interest

The authors declare no conflicts of interest.

Data Availability Statement

The authors have nothing to report.

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