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Article

Enhancing Stability in Fractional-Order Systems: Criteria and Applications

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Abstract: This study investigates the stability of fractional-order systems with infinite delay, which are prevalent in many fields due to their effectiveness in modeling complex dynamic behaviors. Recent advancements concerning the existence and various categories of stability for solutions to the given problem are also highlighted. This investigation utilizes tools such as the Picard operator approach, the Banach fixed-point theorem, an extended form of Gronwall's inequality, and several well-known special functions. We establish key stability criteria for fractional differential equations using Hadamard fractional derivatives and illustrate these concepts using a numerical example. Specifically, graphical representations of the system's responses demonstrate how fractional-order control enhances stability compared to traditional integer-order approaches. Our results emphasize the value of fractional systems in improving system performance and robustness.

Keywords: fractional calculus; stability; special functions

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1. Introduction

Stability is a fundamental concept with broad applications across multiple disciplines, playing a vital role in ensuring the resilience and reliability of systems. In civil engineering, it ensures that structures can withstand environmental stresses, thereby guaranteeing safety and durability. Similarly, aerospace engineering depends on stability to maintain aircraft control during flight, which is critical for passenger safety. In control systems and robotics, stability principles guide the design of algorithms that enable machines to respond effectively to disturbances [1].

In finance, stability concepts help investors manage risk and achieve consistent returns in fluctuating markets. Environmental scientists apply stability theories to understand and predict ecosystem behavior, which offers valuable insights for conservation efforts. In software development, stability ensures that applications perform reliably under varying conditions. Likewise, electrical engineering uses stability analyses to maintain balance in power systems and minimize outages [2].

In psychology, the stability of therapeutic frameworks supports the development of techniques that foster emotional resilience. In network design, maintaining stability ensures consistent data flows in telecommunications systems. Mechanical engineers also rely on stability principles to design safe, efficient machines capable of operating in dynamic environments. Overall, stability is critical for optimizing performance, enhancing safety, and ensuring sustainability across a wide range of fields [3].

In mathematics, stability primarily addresses the qualitative behavior of solutions to mathematical problems, particularly differential equations. It concerns how small changes in initial conditions or parameters influence the behavior of solutions over time. A foundational concept in this area is Lyapunov stability, which helps determine whether a system returns to its equilibrium state after experiencing a disturbance [4].

In this context, an equilibrium point is considered stable if solutions that start near it remain close at all future times. This principle is pivotal in control theory, where engineers examine a system's responses to inputs and perturbations to ensure its desired performance.

In numerical analysis, stability is essential for ensuring the accuracy of computational methods. Stable algorithms yield consistent and reliable results, even when inputs are slightly altered, making them indispensable in simulations and real-world modeling [5].

The insights gained from stability analyses extend across disciplines—from physics to economics—allowing researchers to predict and understand complex behaviors in dynamic systems. Mathematical tools such as stabilizing feedback mechanisms, regularization techniques, and robustness methods help reinforce stability across applications. Ultimately, the study of stability in mathematics underpins both theoretical and practical advancements in the analysis and control of dynamic systems [6,7].

To provide a recent overview of fractional differential and difference equations, we will survey the following selected studies:

D. Otrocol and V. Ilea's work [8], which explored the concepts of UH stability and generalized UH–Rassias stability for a specific delay differential equation,

$$\begin{cases} \psi'(\sigma) = \Psi(\sigma, \psi(\sigma), \psi(h(\sigma))), & \sigma \in [e, d], \\ \psi(\sigma) = \varphi(\sigma), & \sigma \in [e - h, e]. \end{cases}$$

J. Wang and Y. Zhang's [9] study, which established some findings regarding the existence, uniqueness, and UHML stability of Caputo-type fractional differential equations:

$$\begin{cases} {}^{c}D_{0^+}^{\mathfrak{P}} \psi(\sigma) = \Psi(\sigma, \psi(\sigma), \psi(h(\sigma))), & \sigma \in [0, d], \\ \psi(\sigma) = \varphi(\sigma), & \sigma \in [-h, 0]. \end{cases}$$

Liu et al.'s work, found in [10], which demonstrated the existence, uniqueness, and UHML stability of solutions for a certain class of λ-Hilfer fractional differential equations:

$$\begin{cases} {}^{H}D_{0^+}^{\mathfrak{P}_1, \mathfrak{P}_2, \lambda} \psi(\sigma) = \Psi(\sigma, \psi(\sigma), \psi(h(\sigma))), & \sigma \in [0, d], \\ I_{0^+}^{1-\mathfrak{P}_3} \psi(0)- = \psi_0 \in \mathbb{R}, \\ \psi(\sigma) = \varphi(\sigma), & \sigma \in [-h, 0]. \end{cases}$$

K.D. Kucche and P.U. Shikhare's study [11], which examined the existence and uniqueness of solutions, as well as Ulam-type stabilities, for Volterra delay integro-differential equations defined on a finite interval:

$$\begin{cases} \psi'(\sigma) = \Psi(\sigma, y(\sigma), y(g(\sigma)), \int_0^\sigma h(\sigma, s, y(s), y(g(s)))ds), & \sigma \in [0, d], \\ \psi(\sigma) = \varphi(\sigma), & \sigma \in [-r, 0], \ r \in (0, \infty). \end{cases}$$

And Ref. [12], where the authors established the existence, uniqueness, and UHML stability of solutions for a class of λ-Hilfer problems related to fractional differential equations with infinite delay:

$$\begin{cases} {}^{H}D_{0^+}^{\mathfrak{P}_1, \mathfrak{P}_2, \lambda} \psi(\sigma) = \Psi(\sigma, \psi_\sigma), & \sigma \in (0, d], \\ I_{0^+}^{1-\mathfrak{P}_3} \psi(0) = \psi_0 \in \mathbb{R}, \\ \psi(\sigma) = \varphi(\sigma), & \sigma \in (-\infty, 0]. \end{cases}$$

For more detailed information on existing research related to the stability of fractional systems and the existence and uniqueness of solutions, please refer to references [13–15]. These works offer comprehensive insights into the theoretical foundations and recent developments in this field. They also examine various analytical methods and stability criteria essential for understanding the behavior of fractional differential equations. Reviewing these sources will provide a solid background and context for the current study.

In references [16,17], the authors introduce several special functions and explore their interrelations. They use these functions to propose a new concept of stability known as multi-stability. This type of stability proves particularly useful in optimizing stability-related problems, especially within the framework of fractional calculus.

Building on the insights from these foundational sources [12,16,17], we present a study on the stability of fractional systems with infinite delay, emphasizing their effectiveness in modeling complex dynamic behavior and exploring recent advances in the existence and types of stability of solutions. Utilizing methods such as the Picard operator approach, Banach's fixed-point theorem, a generalized Gronwall inequality, and various special functions, we derive key stability conditions for systems involving Hadamard fractional derivatives. Numerical examples and graphical illustrations demonstrate that fractional-order control significantly enhances system stability and robustness compared to conventional integer-order methods.

In this paper, the deep connection between fractional calculus and special functions is clearly evident. We emphasize the important role that special functions play in optimizing our problem. These functions often serve as fundamental tools for expressing solutions to complex problems, enabling precise the modeling of intricate behaviors and enhancing the efficiency of optimization processes. Their well-established properties facilitate analytical solutions, often simplifying numerical computations and ultimately leading to more accurate and reliable results across a wide range of optimization scenarios.

2. Preliminaries

In this section, we present the definitions and auxiliary lemmas that will be used in the main results. In line with the Hadamard fractional derivatives employed in this work, we begin by introducing an associated weighted function space tailored to this framework. We then establish several essential properties of fractional derivatives and integrals involving the natural logarithm function. Finally, we introduce the key analytical tools necessary for studying the existence, uniqueness, and stability of their solutions.

2.1. Special Functions

For every $\sigma, \alpha, \beta, \gamma \in \mathbb{C}$ and $\Re(\alpha), \Re(\beta), \Re(\gamma) > 0$, the Wright function $W_{\alpha,\beta}$, the one-parameter Mittag–Leffler function Ξ_α, and the hypergeometric function $H_{\alpha,\beta,\gamma}$ are defined as follows [16]:

$$W_{\alpha,\beta}(\sigma) = \sum_{i=0}^{\infty} \frac{\sigma^i}{i!\Gamma(\alpha i + \beta)},$$

$$\Xi_\alpha(\sigma) = \sum_{i=0}^{\infty} \frac{\sigma^i}{\Gamma(\alpha i + 1)},$$

and

$$H_{\alpha,\beta,\gamma}(\sigma) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{i=0}^{\infty} \frac{\sigma^i}{i!} \frac{\Gamma(\alpha+i)\Gamma(\beta+i)}{\Gamma(\gamma+i)}.$$

2.2. Weighted Spaces

We let $0 < \mathfrak{P} < 1$, $a, b > 0$, $\epsilon = (-\infty, b]$, $\epsilon_1 = [a, b]$, $\epsilon_2 = (a, b]$, and $\epsilon^\diamond = (-\infty, a]$. Consider the space of absolutely continuous functions $AC(\epsilon_1, \mathbb{R})$ given on ϵ_1. Now, we define them for $n \in \mathbb{N}$

$$AC^n(\epsilon_1, \mathbb{R}) = \Big\{ \Lambda : \epsilon_1 \longrightarrow \mathbb{R} : \Lambda^{(n-1)}(\sigma) \in AC(\epsilon_1, \mathbb{R}) \Big\}.$$

Suppose $C_{1-\mathfrak{P},\ln(\sigma)}$ and $C^n_{1-\mathfrak{P},\ln(\sigma)}$ illustrate the weighted spaces defined as

$$C_{1-\mathfrak{P},\ln(\sigma)}(\epsilon_1, \mathbb{R}) = \Big\{ \Lambda : \epsilon_2 \longrightarrow \mathbb{R} : [\ln(\frac{\sigma}{a})]^{1-\mathfrak{P}} \Lambda(\sigma) \in C(\epsilon_1, \mathbb{R}) \Big\},$$

and

$$C^n_{1-\mathfrak{P},\ln(\sigma)}(\epsilon_1, \mathbb{R}) = \Big\{ \Lambda : \epsilon_1 \longrightarrow \mathbb{R} : \Lambda(\sigma) \in C^{n-1}(\epsilon_1, \mathbb{R}), \Lambda^{(n)}(\sigma) \in C_{1-\mathfrak{P},\ln(\sigma)}(\epsilon_1, \mathbb{R}) \Big\},$$

with the following norms

$$\|\Lambda\|_{C_{1-\mathfrak{P},\ln(\sigma)}} = \max_{\sigma \in \epsilon_1} \left| [\ln(\frac{\sigma}{a})]^{1-\mathfrak{P}} \Lambda(\sigma) \right|,$$

and

$$\|\Lambda\|_{C^n_{1-\mathfrak{P},\ln(\sigma)}} = \sum_{i=0}^{n-1} \|\Lambda^{(i)}\|_C + \|\Lambda^{(n)}\|_{C_{1-\mathfrak{P},\ln(\sigma)}},$$

respectively.

2.3. Hadamard Derivatives

Let $0 < \mathfrak{P} < 1$ and $\Lambda \in C(\epsilon_1, \mathbb{R})$. The Hadamard derivative of order $\mathfrak{P}$ is given as follows [16]:

$${}^{\mathscr{H}}D^{\mathfrak{P},\ln(\sigma)}_{a^+} \Lambda(\sigma) = \frac{\sigma d}{d\sigma} I^{1-\mathfrak{P},\ln(\sigma)}_{a^+} \Lambda(\sigma),$$

where

$$I^{\mathfrak{P},\ln(\sigma)}_{a^+} \Lambda(\sigma) = \frac{1}{\Gamma(\sigma)} \int_a^\sigma \frac{[\ln(\frac{\sigma}{s})]^{\mathfrak{P}-1}}{s} \Lambda(s) ds.$$

Below are several properties of fractional derivatives and integrals, including those of the Hadamard fractional derivative.

Lemma 1 ([9,12]). *Let $C^{\mathfrak{P}}_{1-\mathfrak{P},\ln(\sigma)}(\epsilon_1, \mathbb{R}) = \{\Lambda \in C_{1-\mathfrak{P},\ln(\sigma)}(\epsilon_1, \mathbb{R}),\ {}^{\mathscr{H}}D^{\mathfrak{P},\ln(\sigma)}_{a^+} \Lambda \in C_{1-\mathfrak{P},\ln(\sigma)}(\epsilon_1, \mathbb{R})\}$. We obtain the following:*

(R1) Let $0 < \mathfrak{P} < 1$ and $\Lambda \in C^{\mathfrak{P}}_{1-\mathfrak{P},\ln(\sigma)}(\epsilon_1, \mathbb{R})$. Then,

$${}^{\mathscr{H}}D^{\mathfrak{P},\ln(\sigma)}_{a^+} I^{\mathfrak{P},\ln(\sigma)}_{a^+} \Lambda(\sigma) = \Lambda(\sigma),$$

and

$$I^{\mathfrak{P},\ln(\sigma)}_{a^+} {}^{\mathscr{H}}D^{\mathfrak{P},\ln(\sigma)}_{a^+} \Lambda(\sigma) = \Lambda(\sigma) - \frac{I^{1-\mathfrak{P},\ln(\sigma)}_{a^+} \Lambda(a)}{\Gamma(\mathfrak{P})} [\ln(\frac{\sigma}{a})]^{\mathfrak{P}-1}.$$

(R2) *Let* $0 < \mathfrak{P} < 1$ *and* $\kappa > 0$. *Then,*

$$I_{a^+}^{\mathfrak{P},\ln(\sigma)}[\ln(\frac{\sigma}{a})]^{\kappa-1} = \frac{\Gamma(\kappa)}{\Gamma(\mathfrak{P}+\kappa)}[\ln(\frac{\sigma}{a})]^{\mathfrak{P}+\kappa-1},$$

and

$$D_{a^+}^{\mathfrak{P},\ln(\sigma)}[\ln(\frac{\sigma}{a})]^{\mathfrak{P}-1} = 0.$$

(R3) *Let* $0 < \mathfrak{P} < 1$ *and* $\Lambda \in C_{\mathfrak{P}}(\epsilon_1, \mathbb{R})$. *Then,* $I_{a^+}^{\mathfrak{P},\ln(\sigma)}\Lambda(a) = 0$.

(R4) *Let* $0 < \mathfrak{P} < 1$. *Then,* $I_{a^+}^{\mathfrak{P},\ln(\sigma)}(.)$ *is bounded on* $C_{\mathfrak{P}}(\epsilon_1, \mathbb{R})$.

(R5) *Let* $0 < \mathfrak{P} < 1$, $\Phi : \epsilon_2 \times \mathbb{R} \longrightarrow \mathbb{R}$ *be a continuous function. Then, for* $\sigma \in \epsilon_2$, *the fractional-order problem*

$$\begin{aligned} {}^{\mathscr{H}}D_{a^+}^{\mathfrak{P},\ln(\sigma)}\phi(\sigma) &= \Phi(\sigma, \phi(\sigma)), \\ I_{a^+}^{1-\mathfrak{P},\ln(\sigma)}\phi(a) &= \phi_a, \end{aligned}$$

is equivalent to

$$\phi(\sigma) = \frac{[\ln(\frac{\sigma}{a})]^{\mathfrak{P}-1}}{\Gamma(\mathfrak{P})}\phi_a + \frac{1}{\Gamma(\mathfrak{P})}\int_a^\sigma \frac{[\ln(\frac{\sigma}{s})]^{\mathfrak{P}-1}}{s}\Phi(s, \phi(s))ds.$$

(R6) *Let* A, B *be integrable functions and* C *be continuous on* $[a, b]$. *If, for all* $\sigma \in [a, b]$,

$$A(\sigma) \leq B(\sigma) + C(\sigma)\int_a^\sigma \frac{[\ln(\frac{\sigma}{s})]^{\mathfrak{P}-1}}{s}A(s)ds,$$

then,

$$A(\sigma) \leq B(\sigma) + \int_a^\sigma \sum_{i=1}^{\infty} \frac{(C(\sigma)\Gamma(\mathfrak{P}))^i}{\Gamma(\mathfrak{P}i)} \frac{[\ln(\frac{\sigma}{s})]^{\mathfrak{P}i-1}}{s}B(s)ds.$$

Furthermore, if B *is a nondecreasing function on* $[a, b]$, *then*

$$A(\sigma) \leq B(\sigma)\,\Xi_{\mathfrak{P}}\Big(C(\sigma)\Gamma(\mathfrak{P})[\ln(\frac{\sigma}{s})]^{\mathfrak{P}}\Big),$$

where $\Xi_{\mathfrak{P}}(.)$ *is the one-parameter Mittag–Leffler function.*

2.4. Picard Operators

Consider the metric space (Ω, δ). Now, $\chi : \Omega \longrightarrow \Omega$ is called a Picard operator if there is a $\phi^* \in \Omega$, s.t., $\varpi_\chi = \phi^*$ in which $\varpi_\chi = \{\phi \in \Omega : \chi(\phi) = \phi\}$, and also for $n \in \mathbb{N}$, $\lim_{n\longrightarrow\infty}\{\chi^n(\phi_a)\} = \phi^*$ for every $\phi_a \in \Omega$.

Below is an application of the Picard operator.

Lemma 2 ([9,18])**.** *Assume* $(\Omega, \delta, \leq)$ *is an ordered metric space and* $\chi : \Omega \longrightarrow \Omega$ *is an increasing Picard operator with* $\varpi_\chi = \{\phi_\chi^*\}$. *Then,* $\phi \leq \chi(\phi)$ *implies* $\phi \leq \phi_\chi^*$ *and* $\phi \geq \chi(\phi)$ *implies* $\phi \geq \phi_\chi^*$; *here,* $\phi \in \Omega$.

2.5. Admissible Phase Spaces

A linear topological space of a function from $(-\infty, a]$ to $\mathbb{R}$ with seminorm $\|.\|_B$ is called an admissible phase space if B has the following properties [19,20]:

(1) If $\phi : (-\infty, b] \longrightarrow \mathbb{R}$ is continuous on $[a, b]$ and $\phi_a \in B$, then for every $\sigma \in [a, b]$ we know that:

(1-1) $\phi_\sigma \in B$, where $\phi_\sigma(s) = \phi(\sigma + s)$, $-\infty < s \leq a$ (here, $\phi_\sigma(.)$ represents the history of the state from time $-\infty$ up to time σ).
(1-2) There exists a $\nu > 0$ with $|\phi(\sigma)| \leq \nu\|\phi_\sigma\|_B$ and $|\mu(a)| \leq \nu\|\mu\|_B$ for $\mu \in B$.
(1-3) $\|\phi_\sigma\|_B \leq \Upsilon_1(\sigma) \sup_{a\leq s\leq \sigma} |\phi(s)| + \Upsilon_2(\sigma)\|\phi_a\|_B$, where $\Upsilon_1, \Upsilon_2 : [a, \infty) \longrightarrow [a, \infty)$, Υ_1 is continuous, and Υ_2 is locally bounded. We set $\Upsilon_{ib} = \sup\{\Upsilon_i(\sigma) : \sigma \in [a, b]\}$, $i = 1, 2$.

(2) For the function $\phi(.)$ defined in (1), the function $\sigma \longrightarrow \phi_\sigma$ is continuous from $[a, b]$ to B.
(3) B is complete.

In the following, we will study the uniqueness and stability of a new fractional-order system involving the Hadamard fractional derivatives using Banach's contractive principle and Picard operators and we will give an illustrative example and examine other control functions.

3. Main Results

It is well known that fractional calculus is an emerging tool which uses fractional differential and integral equations to develop more sophisticated mathematical models that can accurately describe complex systems. There are many definitions of fractional derivatives available in the literature, such as the Riemann–Liouville derivative, which plays an important role in the development of the theory of fractional analysis. Another commonly used one is the Hadamard fractional derivative; for studies related to the existence, uniqueness, and stability of solutions for fractional boundary value problems in Hadamard differential equations see [21,22].

The Hadamard fractional derivative is distinguished by its intrinsic scale-invariant nature, which is primarily due to its logarithmic kernel. This feature allows it to effectively model processes that exhibit multiplicative or fractal behavior. Unlike the Riemann–Liouville derivative, which often complicates the specification of initial conditions due to its non-zero derivative of constants, the Hadamard derivative more naturally accommodates initial values when appropriately formulated. While the Caputo derivative also facilitates physically interpretable initial conditions, the Hadamard derivative offers a distinct advantage in modeling systems where the underlying dynamics are governed by scale transformations rather than additive processes.

In comparison to the Kilbas–Hilfer derivative—a flexible formulation that interpolates between the Riemann–Liouville and Caputo derivatives—the Hadamard derivative provides a more direct framework for problems emphasizing multiplicative scale invariance rather than varying degrees of memory effects. Although Hilfer's approach introduces adaptability through a tunable parameter, the Hadamard derivative captures the essence of multiplicative self-similarity via its explicit logarithmic kernel, avoiding the need for additional parameters and offering a more straightforward modeling approach for scale-dependent phenomena. For example, the Hadamard derivative is particularly useful in modeling phenomena such as financial markets or biological growth processes that exhibit scale invariance, as it simplifies the analysis by directly incorporating the natural logarithmic relationships inherent in these systems [16].

Moreover, compared to the Riemann–Liouville derivative, the Hadamard framework aligns more naturally with systems characterized by power-law and fractal behaviors. It offers meaningful interpretations of phenomena across diverse scientific fields such as physics, finance, and biology, where the evolution of a system depends on multiplicative—rather than additive—factors. By incorporating these features while retaining the analytical strengths of fractional calculus, such as memory effects and nonlocal interactions, the Hadamard derivative becomes a powerful tool for modeling complex multiscale systems.

In essence, the Hadamard fractional derivative's principal advantage lies in its ability to combine the benefits of fractional calculus with a structure that inherently respects scale invariance—something that is often challenging to achieve with the Riemann–Liouville or Caputo derivatives. While those are generally suited to systems with additive or linear memory dynamics, the Hadamard derivative excels in contexts where multiplicative behavior predominates, offering a more natural, accurate, and intuitive mathematical framework for such problems [23].

We consider the Banach space

$$\kappa_{B,\mathfrak{P},\ln(\sigma)} = \left\{ \eta : \epsilon \longrightarrow \mathbb{R} \;\middle|\; \eta|_{\epsilon^\diamond} \in B, \eta|_{\epsilon_1} \in C_{1-\mathfrak{P},\ln(\sigma)}(\epsilon_1, \mathbb{R}) \right\},$$

with the norm $\|\eta\|_{\kappa_{B,\mathfrak{P},\ln(\sigma)}} = \|\eta_0\|_B + \|\eta\|_{C_{1-\mathfrak{P},\ln(\sigma)}}$.

In this section, we investigate the stability of the following fractional-order system:

$$\begin{cases} {}^{\mathscr{H}}D_{a^+}^{\mathfrak{P},\ln(\sigma)}\phi(\sigma) = \Phi(\sigma, \phi_\sigma), & \sigma \in \epsilon_2, \\ I_{a^+}^{1-\mathfrak{P},\ln(\sigma)}\phi(a^+) = \phi_a, & \\ \phi(\sigma) = \omega(\sigma), & \sigma \in \epsilon^\diamond, \end{cases} \tag{1}$$

where $\Phi : \epsilon_2 \times B \to \mathbb{R}, \omega \in B, \mathfrak{P} \in (0,1)$ and $\phi_a \in \mathbb{R}$.

Theorem 1. *Assume the following:*

(A1) Let $\Phi : \epsilon_1 \times B \longrightarrow \mathbb{R}$ be a continuous function; there is a positive constant Δ_Φ, s.t., for every $\sigma \in \epsilon_2$, and $x, y \in B$, such that we have $|\Phi(\sigma, x) - \Phi(\sigma, y)| \leq \Delta_\Phi \|x - y\|_B$.

(A2) Let $\frac{[\ln(\frac{b}{a})]^{1-\mathfrak{P}}}{\Gamma(\mathfrak{P}+1)}\Delta_\Phi \Upsilon_{1b} < 1$, in which $\Upsilon_{1b} := \sup\{|\Upsilon_1(\sigma)| : \sigma \in \epsilon_1\}$, and Υ_1 is defined in Section 2.5.

(A3) Let $\sup_{a \leq \sigma \leq b}[\ln(\frac{\sigma}{a})]^{1-\mathfrak{P}} < 1$.

Then, the fractional-order system has the following properties (1):

(T1) A unique solution in $\kappa_{B,\mathfrak{P},\ln(\sigma)}$.

(T2) Stability.

Proof. (T1). Notice that $\phi \in \kappa_{B,\mathfrak{P},\ln(\sigma)}$, given by

$$\phi(\sigma) = \begin{cases} \frac{[\ln(\frac{\sigma}{a})]^{\mathfrak{P}-1}}{\Gamma(\mathfrak{P})}\phi_a + \int_a^\sigma \frac{\Phi(s,\phi_s)}{s\Gamma(\mathfrak{P})}[\ln(\frac{\sigma}{s})]^{\mathfrak{P}-1}ds, & \sigma \in \epsilon_2, \\ \omega(\sigma), & \sigma \in \epsilon^\diamond. \end{cases} \tag{2}$$

is a solution of (1) if ${}^{\mathscr{H}}D_{a^+}^{\mathfrak{P},\ln(\sigma)}\phi(\sigma) = \Phi(\sigma, \phi_\sigma), \sigma \in \epsilon_2$, with the conditions $I_{a^+}^{1-\mathfrak{P},\ln(\sigma)}\phi(a) = \phi_a$, $\phi(\sigma) = \omega(\sigma)$, for every $\sigma \in \epsilon^\diamond$ and $\phi|_{\epsilon_1} \in C_{1-\mathfrak{P},\ln(\sigma)}(\epsilon_1, \mathbb{R})$.

Let $\mathcal{L} : \kappa_{B,\mathfrak{P},\ln(\sigma)} \longrightarrow \kappa_{B,\mathfrak{P},\ln(\sigma)}$ be given by

$$\mathcal{L}(\phi(\sigma)) = \begin{cases} \frac{[\ln(\frac{\sigma}{a})]^{\mathfrak{P}-1}}{\Gamma(\mathfrak{P})}\phi_a + \int_a^\sigma \frac{\Phi(s,\phi_s)}{s\Gamma(\mathfrak{P})}[\ln(\frac{\sigma}{s})]^{\mathfrak{P}-1}ds, & \sigma \in \epsilon_2, \\ \omega(\sigma), & \sigma \in \epsilon^\diamond. \end{cases} \tag{3}$$

Let $\omega \in B$ and consider $\tilde{\omega} : \epsilon \longrightarrow \mathbb{R}$, which is given by

$$\tilde{\omega}(\sigma) = \begin{cases} 0, & \sigma \in \epsilon_2, \\ \omega(\sigma), & \sigma \in \epsilon^\diamond, \end{cases} \tag{4}$$

and so, $\tilde{\omega}(a) = \omega(a)$. For $\chi \in C_{1-\mathfrak{P},\ln(\sigma)}(\epsilon_1, \mathbb{R})$ when $\chi(a) = 0$, consider $\tilde{\chi} : I \longrightarrow \mathbb{R}$, given by

$$\tilde{\chi}(\sigma) = \begin{cases} \frac{[\ln(\frac{\sigma}{a})]^{1-\mathfrak{P}}}{\Gamma(\mathfrak{P})}\chi(\sigma), & \sigma \in \epsilon_2, \\ 0, & \sigma \in \epsilon^\diamond. \end{cases} \tag{5}$$

Suppose ϕ satisfies the following integral equation:

$$\phi(\sigma) = \frac{[\ln(\frac{\sigma}{a})]^{\mathfrak{P}-1}}{\Gamma(\mathfrak{P})}\phi_a + \int_a^\sigma \frac{\Phi(s,\phi_s)}{s\Gamma(\mathfrak{P})}[\ln(\frac{\sigma}{s})]^{\mathfrak{P}-1}ds, \qquad \sigma \in \epsilon_2,$$

and $\phi(\sigma) = \omega(\sigma)$ for $\sigma \in I^\diamond$. It is easy to see that $\phi_\sigma = \tilde{\omega}_\sigma + \tilde{\chi}_\sigma$ for $\sigma \in \epsilon_2$, iff χ satisfies $\chi_a = 0$ and

$$\chi(\sigma) = \frac{[\ln(\frac{\sigma}{a})]^{\mathfrak{P}-1}}{\Gamma(\mathfrak{P})}\omega(a) + \int_a^\sigma \frac{\Phi(s,\tilde{\omega}_\sigma + \tilde{\chi}_\sigma)}{s\Gamma(\mathfrak{P})}[\ln(\frac{\sigma}{s})]^{\mathfrak{P}-1}ds, \qquad \sigma \in \epsilon_2,$$

with $\tilde{\chi}_a = 0$. We now consider $\kappa^\diamond_{B,\mathfrak{P},\ln(\sigma)} = \{\chi \in \kappa_{B,\mathfrak{P},\ln(\sigma)}, \chi_a = 0\}$. For $\chi \in \kappa^\diamond_{B,\mathfrak{P},\ln(\sigma)}$, let

$$\|\chi\|_{\kappa^\diamond_{B,\mathfrak{P},\ln(\sigma)}} = \|\chi_0\|_B + \|\chi\|_{C_{1-\mathfrak{P}.\ln(\sigma)}} = \sup\{[\ln(\frac{\sigma}{a})]^{1-\mathfrak{P}}|\chi(\sigma)| : \sigma \in \epsilon_2\}.$$

Note that $(\kappa^\diamond_{B,\mathfrak{P},\ln(\sigma)}, \|\,.\,\|_{\kappa^\diamond_{B,\mathfrak{P},\ln(\sigma)}})$ is a Banach space and we can let $\mathcal{L}^\diamond : \kappa^\diamond_{B,\mathfrak{P},\ln(\sigma)} \longrightarrow \kappa^\diamond_{B,\mathfrak{P},\ln(\sigma)}$ be given by

$$\mathcal{L}^\diamond(\chi(\sigma)) = \frac{[\ln(\frac{\sigma}{a})]^{\mathfrak{P}-1}}{\Gamma(\mathfrak{P})}\omega(a) + \int_a^\sigma \frac{\Phi(s,\tilde{\omega}_\sigma + \tilde{\chi}_\sigma)}{s\Gamma(\mathfrak{P})}[\ln(\frac{\sigma}{s})]^{\mathfrak{P}-1}ds, \qquad \sigma \in \epsilon_2,$$

and $\mathcal{L}^\diamond(\chi(\sigma)) = 0$ for $\sigma \in \epsilon^\diamond$. We now show that $\mathcal{L}^\diamond$ has a unique fixed point. Let $\Omega = \sup_{s\in\epsilon_1}|\Phi(s,0)|$ $(< \infty)$. For $\sigma, \sigma+\delta \in \epsilon_2$, we find that

$$\begin{aligned}
& |\mathcal{L}^\diamond(\chi(\sigma+\delta)) - \mathcal{L}^\diamond(\chi(\sigma))| \\
= & \left|\frac{\omega(a)}{\Gamma(\mathfrak{P})}\left([\ln(\frac{\sigma+\delta}{a})]^{\mathfrak{P}-1} - [\ln(\frac{\sigma}{a})]^{\mathfrak{P}-1}\right)\right. \\
& + \frac{1}{\Gamma(\mathfrak{P})}\int_a^{\sigma+\delta} \frac{\Phi(s,\tilde{\omega}_\sigma + \tilde{\chi}_\sigma)}{s}[\ln(\frac{\sigma+\delta}{s})]^{\mathfrak{P}-1}ds \\
& \left. - \frac{1}{\Gamma(\mathfrak{P})}\int_a^{\sigma} \frac{\Phi(s,\tilde{\omega}_\sigma + \tilde{\chi}_\sigma)}{s}[\ln(\frac{\sigma+\delta}{s})]^{\mathfrak{P}-1}ds\right| \\
\leq & \left|\frac{\omega(a)}{\Gamma(\mathfrak{P})}\left([\ln(\frac{\sigma+\delta}{a})]^{\mathfrak{P}-1} - [\ln(\frac{\sigma}{a})]^{\mathfrak{P}-1}\right)\right. \\
& + \frac{1}{\Gamma(\mathfrak{P})}\int_a^{\sigma+\delta} \frac{1}{s}[\ln(\frac{\sigma+\delta}{s})]^{\mathfrak{P}-1}\left((|\Phi(s,\tilde{\omega}_\sigma + \tilde{\chi}_\sigma - \Phi(s,0))| + |\Phi(s,0)|)ds\right)ds \\
& \left. - \frac{1}{\Gamma(\mathfrak{P})}\int_a^{\sigma} \frac{1}{s}[\ln(\frac{\sigma+\delta}{s})]^{\mathfrak{P}-1}\left((|\Phi(s,\tilde{\omega}_\sigma + \tilde{\chi}_\sigma) - \Phi(s,0))| + |\Phi(s,0)|)ds\right)ds\right| \\
\leq & \left|\frac{\omega(a)}{\Gamma(\mathfrak{P})}\left([\ln(\frac{\sigma+\delta}{a})]^{\mathfrak{P}-1} - [\ln(\frac{\sigma}{a})]^{\mathfrak{P}-1}\right)\right. \\
& \left. + \frac{1}{\Gamma(\mathfrak{P}+1)}\left([\ln(\frac{\sigma+\delta}{a})]^{\mathfrak{P}} - [\ln(\frac{\sigma}{a})]^{\mathfrak{P}}\right)(\Delta_\Phi \Upsilon_{1b}\|\chi\|_{\kappa^\diamond_{B,\mathfrak{P},\ln(\sigma)}} + \Omega)\right|.
\end{aligned}$$

Thus, $|\mathcal{L}^\diamond(\chi(\sigma+\delta)) - \mathcal{L}^\diamond(\chi(\sigma))| \longrightarrow 0$ when $\delta \longrightarrow 0$. Assume $\chi, \chi^\diamond \in \kappa^\diamond_{B,\mathfrak{P},\ln(\sigma)}$ and $\sigma \in \epsilon_2$. Then,

$$\begin{aligned}
& \left|[\ln(\frac{\sigma}{a})]^{1-\mathfrak{P}}\mathcal{L}^\diamond(\chi(\sigma)) - [\ln(\frac{\sigma}{a})]^{1-\mathfrak{P}}\mathcal{L}^\diamond(\chi^\diamond(\sigma))\right| \\
\leq & \frac{[\ln(\frac{\sigma}{a})]^{1-\mathfrak{P}}}{\Gamma(a)}\int_a^\sigma \frac{[\ln(\frac{\sigma}{s})]^{\mathfrak{P}-1}}{s}|\Phi(s,\tilde{\omega}_s + \tilde{\chi}_s) - \Phi(s,\tilde{\omega}_s + \tilde{\chi}^\diamond_s)|ds \\
\leq & \frac{[\ln(\frac{\sigma}{a})]^{1-\mathfrak{P}}}{\Gamma(a)}\Delta_\Phi\int_a^\sigma \frac{[\ln(\frac{\sigma}{s})]^{\mathfrak{P}-1}}{s}\|\tilde{\chi}_s - \tilde{\chi}^\diamond_s\|_B ds.
\end{aligned}$$

Also, we know that

$$\begin{aligned}\|\tilde{\chi}_s - \tilde{\chi}_s^\diamond\|_B &\leq \Upsilon_1(\sigma) \sup_{a\leq\varphi\leq s} |\tilde{\chi}(\varphi) - \tilde{\chi}^\diamond(\varphi)| + \Upsilon_2(\sigma)\|\tilde{\chi}_a - \tilde{\chi}_a^\diamond\|_B \\ &\leq \Upsilon_{1b} \sup_{a\leq\varphi\leq s} [\ln(\frac{\sigma}{a})]^{1-\mathfrak{P}} |\chi(\varphi) - \chi^\diamond(\varphi)| \\ &= \Upsilon_{1b}\Big(\|\chi_a - \chi_a^\diamond\|_B + \|\chi - \chi^\diamond\|_{C_{1-\mathfrak{P},\ln(\sigma)}} \Big) \\ &= \Upsilon_{1b}\|\chi - \chi^\diamond\|_{\kappa^\diamond_{B,\mathfrak{P},\ln(\sigma)}},\end{aligned}$$

so

$$\begin{aligned}&\Big|[\ln(\frac{\sigma}{a})]^{1-\mathfrak{P}}\mathcal{L}^\diamond(\chi(\sigma)) - [\ln(\frac{\sigma}{a})]^{1-\mathfrak{P}}\mathcal{L}^\diamond(\chi^\diamond(\sigma))\Big| \\ \leq\ & \Delta_\Phi \Upsilon_{1b}\|\chi - \chi^\diamond\|_{\kappa^\diamond_{B,\mathfrak{P},\ln(\sigma)}} \frac{[\ln(\frac{\sigma}{a})]^{1-\mathfrak{P}}}{\Gamma(\mathfrak{P})} \int_0^\sigma \frac{[\ln(\frac{\sigma}{s})]^{\mathfrak{P}-1}}{s} ds \\ =\ & \frac{\ln(\frac{\sigma}{a})}{\Gamma(\mathfrak{P}+1)} \Delta_\Phi \Upsilon_{1b}\|\chi - \chi^\diamond\|_{\kappa^\diamond_{B,\mathfrak{P},\ln(\sigma)}}.\end{aligned}$$

Therefore,

$$\|\mathcal{L}^\diamond\chi - \mathcal{L}^\diamond\chi^\diamond\|_{\kappa^\diamond_{B,\mathfrak{P},\ln(\sigma)}} \leq \frac{\ln(\frac{\sigma}{a})}{\Gamma(\mathfrak{P}+1)} \Delta_\Phi \Upsilon_{1b}\|\chi - \chi^\diamond\|_{\kappa^\diamond_{B,\mathfrak{P},\ln(\sigma)}},$$

so $\mathcal{L}^\diamond$ is a contraction. The Banach fixed-point theorem guarantees that $\mathcal{L}^\diamond$ has a unique fixed point $\chi \in \kappa^\diamond_{B,\mathfrak{P},\ln(\sigma)}$. Let $\phi = \tilde{\omega} + \chi$, and note that $\mathcal{L}$ has a unique fixed point $\chi \in \kappa_{B,\mathfrak{P},\ln(\sigma)}$. Hence, we obtain the desired result.

(T2) Let $\phi(\sigma) = \tilde{\omega}(\sigma) + \chi(\sigma)$ be the solution of (1) in $\kappa_{B,\mathfrak{P},\ln(\sigma)}$ with $\phi_\sigma = \tilde{\omega}_\sigma + \tilde{\chi}_\sigma$ and let $\mathcal{E} = \tilde{\omega}(\sigma) + \mathcal{Y}(\sigma)$ satisfy the inequality $\nu > 0$,

$$|{}^{\mathscr{H}}D_{a^+}^{\mathfrak{P},\ln(\sigma)}\mathcal{E}(\sigma) - \Phi(\sigma, \mathcal{E}_\sigma)| \leq \nu W_{\alpha,\beta}([\ln(\frac{\sigma}{a})]^{\mathfrak{P}}), \qquad \sigma \in \epsilon,$$

where $\mathcal{E}_\sigma = \tilde{\omega}_\sigma + \tilde{\mathcal{Y}}_\sigma$.

Note that the function $\mathcal{Y} \in C_{1-\mathfrak{P},\ln(\sigma)}(\epsilon_1, \mathbb{R})$ is a solution of

$$|{}^{\mathscr{H}}D_{a^+}^{\mathfrak{P},\ln(\sigma)}\mathcal{Y}(\sigma) - \Phi(\sigma, \tilde{\omega}_\sigma + \tilde{\mathcal{Y}}_\sigma)| \leq \nu W_{\alpha,\beta}([\ln(\frac{\sigma}{a})]^{\mathfrak{P}}), \qquad \sigma \in \epsilon_2,$$

if there is a function $h \in C_{1-\mathfrak{P},\ln(\sigma)}(\epsilon_1, \mathbb{R})$, s.t., $|h(\sigma)| \leq \nu W_{\alpha,\beta}([\ln(\frac{\sigma}{a})]^{\mathfrak{P}})$ for $\sigma \in \epsilon_2$ and ${}^{\mathscr{H}}D_{a^+}^{\mathfrak{P},\ln(\sigma)}\mathcal{Y}(\sigma) = \Phi(\sigma, \tilde{\omega}_\sigma + \tilde{\mathcal{Y}}_\sigma) + h(\sigma)$ for $\sigma \in \epsilon_2$.

Recall the fractional-order system (1) is stable [16] with respect to $W_{\alpha,\beta}([\ln(\frac{\sigma}{a})]^{\mathfrak{P}})$ if there is a $\theta > 0$, s.t., for every $\nu > 0$, and every solution $\mathcal{Y} \in \kappa^\diamond_{B,\mathfrak{P},\ln(\sigma)} = \{\chi \in \kappa_{B,\mathfrak{P},\ln(\sigma)}, \chi_a = 0\}$ to the inequality

$$|{}^{\mathscr{H}}D_{a^+}^{\mathfrak{P},\ln(\sigma)}\mathcal{Y} - \Phi(\sigma, \tilde{\omega}_\sigma + \tilde{\mathcal{Y}}_\sigma)| \leq \nu W_{\alpha,\beta}([\ln(\frac{\sigma}{a})]^{\mathfrak{P}}), \qquad \sigma \in \epsilon_2, \tag{6}$$

there is a solution $\chi \in \kappa^\diamond_{B,\mathfrak{P},\ln(\sigma)}$ to (1) with

$$|\chi(\sigma) - \mathcal{Y}(\sigma)| \leq \theta\nu W_{\alpha,\beta}([\ln(\frac{\sigma}{a})]^{\mathfrak{P}}), \qquad \sigma \in \epsilon.$$

Observe that if $\mathcal{Y} \in C_{1-\mathfrak{P},\ln(\sigma)}(\epsilon_1, \mathbb{R})$ satisfies the inequality (6), then $\mathcal{Y}$ is a solution of the integral inequality

$$\Big|\mathcal{Y} - \frac{\omega(a)}{\Gamma(\mathfrak{P})}[\ln(\frac{\sigma}{a})]^{\mathfrak{P}-1} - \frac{1}{\Gamma(\mathfrak{P})}\int_a^\sigma \frac{[\ln(\frac{\sigma}{s})]^{\mathfrak{P}-1}}{s}\Phi(\sigma, \tilde{\omega}_s + \tilde{\mathcal{Y}}_s) ds\Big| \leq \nu W_{\alpha,\beta}([\ln(\frac{\sigma}{a})]^{\mathfrak{P}}).$$

Now, we consider $\Xi \in \kappa^\diamond_{B,\mathfrak{P},\ln(\sigma)}$, the unique solution to the fractional-order system

$$\begin{cases} {}^{\mathscr{H}}D_{a^+}^{\mathfrak{P},\ln(\sigma)}\Xi(\sigma) = \Phi(\sigma, \tilde{\omega}_\sigma + \tilde{\Xi}_\sigma), & \sigma \in \epsilon_2, \\ I_{a^+}^{1-\mathfrak{P},\ln(\sigma)}\Xi(a^+) = I_{a^+}^{1-\mathfrak{P},\ln(\sigma)}\mathcal{Y}(a^+), & \\ \Xi(\sigma) = \mathcal{Y}(\sigma), & \sigma \in \epsilon^\diamond. \end{cases} \tag{7}$$

Using (T1), we obtain

$$\Xi(\sigma) = \begin{cases} \frac{[\ln(\frac{\sigma}{a})]^{\mathfrak{P}-1}}{\Gamma(\mathfrak{P})}\phi_a + \int_a^\sigma \frac{\Phi(s,\tilde{\omega}_s+\tilde{\Xi}_s)}{s\Gamma(\mathfrak{P})}[\ln(\frac{\sigma}{s})]^{\mathfrak{P}-1}ds, & \sigma \in \epsilon_2, \\ \mathcal{Y}(\sigma), & \sigma \in \epsilon^\diamond. \end{cases}$$

Hence, we know that

$$\begin{aligned} & |\mathcal{Y}(\sigma) - \chi(\sigma)| \\ \leq \; & \left|\mathcal{Y}(\sigma) - \frac{[\ln(\frac{\sigma}{a})]^{\mathfrak{P}-1}}{\Gamma(\mathfrak{P})}\phi_a - \int_a^\sigma \frac{\Phi(s,\, \tilde{\omega}_s + \tilde{\mathcal{Y}}_s)}{s\Gamma(\mathfrak{P})}[\ln(\frac{\sigma}{s})]^{\mathfrak{P}-1}ds\right| \\ & + \int_a^\sigma \frac{1}{s\Gamma(\mathfrak{P})}[\ln(\frac{\sigma}{s})]^{\mathfrak{P}-1}|\Phi(s,\, \tilde{\omega}_s + \tilde{\mathcal{Y}}_s) - \Phi(s,\, \tilde{\omega}_s + \tilde{\Xi}_s)|ds \\ \leq \; & \nu W_{\alpha,\beta}([\ln(\frac{\sigma}{a})]^{\mathfrak{P}}) + \int_a^\sigma \frac{\Delta_\Phi}{s\Gamma(\mathfrak{P})}[\ln(\frac{\sigma}{s})]^{\mathfrak{P}-1}\|\tilde{\mathcal{Y}}_s - \tilde{\chi}_s\|_B ds. \end{aligned}$$

In a similar way to that in (T1), we obtain

$$\|\tilde{\mathcal{Y}}_s - \tilde{\chi}_s\|_B \leq \Gamma_{1b} \sup_{a\leq\sigma\leq s}[\ln(\frac{\sigma}{a})]^{1-\mathfrak{P}}|\mathcal{Y}(\sigma) - \Xi(\sigma)|.$$

Combining these, we find

$$|\mathcal{Y}(\sigma) - \chi(\sigma)| \;\; \leq \;\; \nu W_{\alpha,\beta}([\ln(\frac{\sigma}{a})]^{\mathfrak{P}}) + \frac{\Delta_\Phi}{\Gamma(\mathfrak{P})}\int_a^\sigma \frac{[\ln(\frac{\sigma}{s})]^{\mathfrak{P}-1}}{s}\Upsilon_{1b}[\ln(\frac{s}{a})]^{1-\mathfrak{P}}|\mathcal{Y}(s) - \Xi(s)|ds.$$

For every $\zeta \in \kappa^\diamond_{B,\mathfrak{P},\ln(\sigma)}$, consider $\mathcal{L} : \kappa^\diamond_{B,\mathfrak{P},\ln(\sigma)} \longrightarrow \kappa^\diamond_{B,\mathfrak{P},\ln(\sigma)}$, given by

$$\mathcal{L}(\zeta(\sigma)) = \begin{cases} \nu W_{\alpha,\beta}([\ln(\frac{\sigma}{a})]^{\mathfrak{P}}) + \frac{\Delta_\Phi \Upsilon_{1b}}{\Gamma(\mathfrak{P})}\int_a^\sigma [\ln(\frac{\sigma}{s})]^{\mathfrak{P}-1}\frac{\zeta(s)[\ln(\frac{s}{a})]^{1-\mathfrak{P}}}{s}ds, & \sigma \in \epsilon_2, \\ 0, & \sigma \in \epsilon^\diamond, \end{cases}$$

We now show that $\mathcal{L}$ is a Picard operator. For every $\sigma \in \epsilon_2$ and $\zeta, \zeta^\diamond \in C_{1-\mathfrak{P},\ln(\sigma)}(I_1, \mathbb{R})$, we know that

$$\begin{aligned} & |[\ln(\frac{\sigma}{a})]^{1-\mathfrak{P}}(\mathcal{L}(\zeta(\sigma)) - \mathcal{L}(\zeta^\diamond(\sigma)))| \\ \leq \; & \frac{\Delta_\Phi \Upsilon_{1b}[\ln(\frac{\sigma}{a})]^{1-\mathfrak{P}}}{\Gamma(\mathfrak{P})}\int_a^\sigma \frac{[\ln(\frac{\sigma}{s})]^{\mathfrak{P}-1}}{s}[\ln(\frac{s}{a})]^{1-\mathfrak{P}}|\zeta(s) - \zeta^\diamond(s)|ds \\ \leq \; & \frac{\Delta_\Phi \Upsilon_{1b}[\ln(\frac{\sigma}{a})]^{1-\mathfrak{P}}}{\Gamma(\mathfrak{P})}\int_a^\sigma \frac{[\ln(\frac{\sigma}{s})]^{\mathfrak{P}-1}}{s}\|\zeta - \zeta^\diamond\|_{C_{1-\mathfrak{P},\ln(\sigma)}}ds \\ \leq \; & \frac{\Delta_\Phi \Upsilon_{1b}[\ln(\frac{\sigma}{a})]}{\Gamma(\mathfrak{P}+1)}\|\zeta - \zeta^\diamond\|_{C_{1-\mathfrak{P},\ln(\sigma)}}. \end{aligned}$$

This implies that

$$\|\mathcal{L}\zeta - \mathcal{L}\zeta^\diamond\|_{C_{1-\mathfrak{P},\ln(\sigma)}} \leq \frac{\Delta_\Phi \Upsilon_{1b}[\ln(\frac{\sigma}{a})]}{\Gamma(\mathfrak{P}+1)}\|\zeta - \zeta^\diamond\|_{C_{1-\mathfrak{P},\ln(\sigma)}}.$$

Thus, $\mathcal{L}$ is a contraction mapping on $\kappa^{\diamond}_{B,\mathfrak{P},\ln(\sigma)}$. From the Banach fixed-point theorem applied to $\mathcal{L}$, we observe that $\mathcal{L}$ is a Picard operator and $\varpi_{\mathcal{L}} = \zeta^{\diamond}$. For every $\sigma \in \epsilon_2$,

$$\begin{aligned} \zeta^{\diamond}(\sigma) &= \mathcal{L}\zeta^{\diamond}(\sigma) \\ &= \nu W_{\alpha,\beta}([\ln(\frac{\sigma}{a})]^{\mathfrak{P}}) + \frac{\Delta_{\Phi}\Upsilon_{1b}}{\Gamma(\mathfrak{P})}\int_a^{\sigma}[\ln(\frac{\sigma}{s})]^{\mathfrak{P}-1}\frac{\zeta^{\diamond}(s)[\ln(\frac{s}{a})]^{1-\mathfrak{P}}}{s}ds. \end{aligned}$$

For $a < \sigma_1 < \sigma_2 \le b$,

$$\begin{aligned} & \zeta^{\diamond}(\sigma_2) - \zeta^{\diamond}(\sigma_1) \\ = \; & \nu(W_{\alpha,\beta}([\ln(\frac{\sigma_2}{a})]^{\mathfrak{P}}) - W_{\alpha,\beta}([\ln(\frac{\sigma_1}{a})]^{\mathfrak{P}})) \\ & +\Delta_{\Phi}\Upsilon_{1b}\frac{1}{\Gamma(\mathfrak{P})}\int_a^{\sigma_2}[\ln(\frac{\sigma_2}{s})]^{\mathfrak{P}-1}\frac{\zeta^{\diamond}(s)}{s}[\ln(\frac{s}{a})]^{1-\mathfrak{P}}ds \\ & +\frac{\Delta_{\Phi}\Upsilon_{1b}}{\Gamma(\mathfrak{P})}\int_a^{\sigma_1}\frac{1}{s}([\ln(\frac{\sigma_2}{s})]^{\mathfrak{P}-1} - [\ln(\frac{\sigma_1}{s})]^{\mathfrak{P}-1})[\ln(\frac{s}{a})]^{1-\mathfrak{P}}\zeta^{\diamond}(s)ds \\ & +\frac{\Delta_{\Phi}\Upsilon_{1b}}{\Gamma(\mathfrak{P})}\int_{\sigma_1}^{\sigma_2}\frac{1}{s}[\ln(\frac{\sigma_2}{s})]^{\mathfrak{P}-1}[\ln(\frac{s}{a})]^{1-\mathfrak{P}}\zeta^{\diamond}ds \\ \ge \; & \nu W_{\alpha,\beta}([\ln(\frac{\sigma_2}{a})]^{\mathfrak{P}}) - \nu W_{\alpha,\beta}([\ln(\frac{\sigma_1}{a})]^{\mathfrak{P}}) \\ & +([\ln(\frac{\sigma_2}{a})]^{1-\mathfrak{P}} - [\ln(\frac{\sigma_1}{a})]^{1-\mathfrak{P}})\frac{\min_{s\in[a,b]}\zeta^{\diamond}(s)\Delta_{\Phi}\Upsilon_{1b}}{\Gamma(\mathfrak{P}+1)} \\ & +([\ln(\frac{\sigma_2}{a})]^{\mathfrak{P}} - [\ln(\frac{\sigma_1}{a})]^{\mathfrak{P}})\frac{\min_{s\in[a,b]}\zeta^{\diamond}(s)\Delta_{\Phi}\Upsilon_{1b}}{\Gamma(\mathfrak{P}+1)} \\ > \; & 0. \end{aligned}$$

This implies that $\zeta^{\diamond}$ is strictly increasing and, from (A3), we obtain

$$\zeta^{\diamond} \le \nu W_{\alpha,\beta}([\ln(\frac{\sigma}{a})]^{\mathfrak{P}}) + \frac{\Delta_{\Phi}\Upsilon_{1b}}{\Gamma(\mathfrak{P})}\int_a^{\sigma}\frac{[\ln(\frac{\sigma}{s})]^{\mathfrak{P}-1}}{s}\zeta^{\diamond}(s)ds.$$

Applying Lemma 1, for $\sigma \in \epsilon_1$, we obtain

$$\begin{aligned} \zeta^{\diamond}(\sigma) &\le \nu W_{\alpha,\beta}([\ln(\frac{\sigma}{a})]^{\mathfrak{P}})\Xi_{\mathfrak{P}}(\Delta_{\Phi}\Upsilon_{1b}[\ln(\frac{\sigma}{a})]^{\mathfrak{P}}) \\ &\le \nu\rho W_{\alpha,\beta}([\ln(\frac{b}{a})]^{\mathfrak{P}}), \end{aligned}$$

in which $\rho = \Xi_{\mathfrak{P}}(\Delta_{\Phi}\Upsilon_{1b}[\ln(\frac{\sigma}{a})]^{\mathfrak{P}})$. Setting $\zeta = |\mathcal{Y} - \Xi|$, we obtain $\zeta(\sigma) \le \mathcal{L}\zeta(\sigma)$. Making use of Lemma 2, we conclude that $\zeta(\sigma) \le \zeta^{\diamond}(\sigma)$. Then, for every $\sigma \in I$,

$$\begin{aligned} |\mathcal{E}(\sigma) - \phi(\sigma)| &= |\mathcal{Y}(\sigma) - \Xi(\sigma)| \\ &\le \nu\rho W_{\alpha,\beta}([\ln(\frac{b}{a})]^{\mathfrak{P}}). \end{aligned}$$

Hence, we obtain the desired result. □

Example 1. *Employing the values provided in [12,16], we consider*

$$B_{\varepsilon} = \{\phi \in C(\epsilon^{\diamond},\mathbb{R})| \lim_{\sigma\to-\infty}\exp(\varepsilon\sigma)\phi(\sigma) \; exist \; in \; \mathbb{R}, \; \varepsilon \in \mathbb{R}^{+}\},$$

with the norm $\|\phi\|_{\varepsilon} = \sup\{\exp(\varepsilon\sigma)|\phi(\sigma)|, \; \sigma \in (-\infty, a]\}$. *Consider the fractional-order system* (1), *which is as follows:* ${}^{\mathscr{H}}D_{a+}^{0.50,\ln(\sigma)}\phi(\sigma) = \frac{\|\phi_{\sigma}\|}{\exp(\varepsilon\sigma)8+\exp((1+\varepsilon)\sigma)}$, *for* $\sigma \in (a,b]$;

$I_{a^+}^{0.50,\ln(\sigma)}\phi(a^+) = 0.60$, $\phi(\sigma) = \omega(\sigma)$, *for* $\sigma \in (-\infty, a]$; $\Phi(\sigma,\phi) = \frac{\phi}{8\exp(\varepsilon\sigma)+\exp(\sigma(1+\varepsilon))}$; $a = 0.25$; *and* $b = 0.75$. *For every* $\phi, \phi^\diamond \in B_\varepsilon$ *and every* $\sigma \in (a,b]$, *we obtain*

$$|\phi(\sigma,\psi) - \phi(\sigma,\phi)| \leq (8+\exp(\sigma))^{-1}\|\psi - \phi\|_\varepsilon.$$

Also,
$\Delta_\Phi \Upsilon_{1b}[\ln(\frac{b}{a})]^{\mathfrak{P}}(\Gamma(1.50))^{-1} < 1$. *For every* $\sigma \in (-\infty, a]$, *consider the inequality*

$$|{}^{\mathscr{H}}D_{a+}^{0.50,\ln(\sigma)}\psi(\sigma) - \frac{\psi_\sigma}{\exp(\varepsilon\sigma)(8+\exp(\sigma))}| \leq \nu W_{0.40,0.80}([\ln(\frac{b}{a})]^{0.50}).$$

Applying the previous results, we obtain $|\psi - \phi| \leq \Xi_{0.50}(\frac{1}{8})\nu W_{0.40,0.80}([\ln(\frac{b}{a})]^{0.50})$.

We will examine other special control functions, specifically the exponential function $\Psi_2 = \exp(\sigma)$, *the Mittag–Leffler function* $\Psi_3 = \Xi_\alpha(\sigma)$, *and the hypergeometric function* $\Psi_4 = H_{\alpha,\beta,\gamma}(\sigma)$ *in place of* $\Psi_1 = W_{\alpha,\beta}(\sigma)$, *for different values of* $\mathfrak{P} = 0.15, 0.45, 0.65, 0.85$.

We present the contour plots of $v_i(\sigma) = \Xi_{\mathfrak{P}}(\Delta_\Phi \Upsilon_{1b}[\ln(\frac{\sigma}{a})]^{\mathfrak{P}})\Psi_i([\ln(\frac{b}{a})]^{\mathfrak{P}})$, *for* $i = 1, \cdots, 4$, *in Figures 1–4. In addition, we propose the contour plots of* $|v_1(\sigma) - v_j(\sigma)|$, *for* $j = 3,2,4$ *and diverse values of* $\mathfrak{P}$, *in Figure 5.*

Contour lines are essential in mathematical modeling, as they provide a clear means of visualizing and analyzing complex surfaces and scalar fields. By representing points of equal value, they help identify patterns, trends, and critical regions in data without the need for 3D graphs. During optimization, they reveal gradients, minima, and maxima, thereby guiding algorithms such as gradient descent. In physics and engineering, they are used to model equipotential lines, pressure distributions, and temperature gradients. In geospatial applications they are employed as contour lines for terrain mapping and hazard prediction, while in statistics, they are used to illustrate probability densities and clustering. Their ability to simplify multidimensional problems into interpretable 2D plots makes them invaluable across scientific and computational disciplines.

In Figure 5, we systematically calculate the differences in the resulting error when using the Wright control function compared to other cases where the controllers are various special functions, as indicated. As observed, the error discrepancy is smaller when the controllers are modeled using Wright and hypergeometric functions compared to other configurations. This suggests that the choice of controller function has a significant impact on the stability of the solutions. Therefore, selecting an appropriate control function can be highly effective in optimizing various aspects of the problem, including minimizing error, enhancing stability, and ultimately identifying the optimal solution.

To facilitate a clearer understanding and provide a more detailed description of the above content, we present Table 1, which displays the use of various special functions as control functions. This table illustrates the versatility and applicability of different mathematical tools in modeling and analyzing the proposed problem. By including these special functions, we offer a comprehensive comparison and highlight the effectiveness of each in capturing complex behaviors. This approach not only enhances the interpretability of the results but also provides valuable insights into how different functions can be leveraged to improve the accuracy and robustness of models used across diverse scenarios.

For this purpose, we introduce the following notation:

$$E_i(\sigma) := argmin_\sigma\left\{|\psi(\sigma) - \phi(\sigma)| - \Xi_{\mathfrak{P}}\left(\Delta_\Phi \Upsilon_{1b}\left[\ln(\frac{\sigma}{a})\right]^{\mathfrak{P}}\right)\Psi_i\left(\left[\ln(\frac{b}{a})\right]^{\mathfrak{P}}\right)\right\},$$

where $i = 1, \cdots, 4$, $\sigma \in [a,b]$, $\Delta_\Phi \in \mathbb{R}^+$, *and* Υ_{1b} *is as defined in Section 2.5.*

The error values E_i, $i = 1,2,3,4$, *for various values of* $\mathfrak{P}$ *are presented in Table 1. By comparing the results across the two different ranges, it can be concluded that selecting hypergeometric functions, followed by Wright functions, as control functions yields better outcomes than using exponential or Mittag–Leffler functions as controllers.*

Special functions are indispensable tools in stability analyses and optimization, as they enable the expression of solutions to complex mathematical models that are otherwise difficult to manage.

Their unique analytical properties allow for a more precise characterization of system behavior, particularly in the nonlinear and boundary value problems common in control theory. These functions contribute to the development of tighter stability bounds, enhance the accuracy of predictive models, and support the design of optimal control strategies. By capturing subtle dynamics that standard functions often overlook, special functions significantly broaden the analytical framework used for addressing challenging problems related to system stability and optimality.

Table 1. The obtained errors $E_i(\sigma)$ for $i = 1, \ldots, 4$.

	$\sigma \in (0.25, 0.50)$				$\sigma \in (0.50, 0.75)$			
Errors	$E_4(\sigma)$	$E_1(\sigma)$	$E_3(\sigma)$	$E_2(\sigma)$	$E_4(\sigma)$	$E_1(\sigma)$	$E_3(\sigma)$	$E_2(\sigma)$
$\mathfrak{P} = 0.10$	0.0325	0.0561	0.0691	0.0802	0.0705	0.0869	0.0915	0.1034
$\mathfrak{P} = 0.35$	0.0407	0.0612	0.0723	0.0834	0.0763	0.0892	0.0944	0.1068
$\mathfrak{P} = 0.60$	0.0489	0.0644	0.0754	0.0873	0.0801	0.0933	0.0982	0.1092
$\mathfrak{P} = 0.85$	0.0576	0.0685	0.0790	0.0902	0.0850	0.0966	0.1017	0.1126

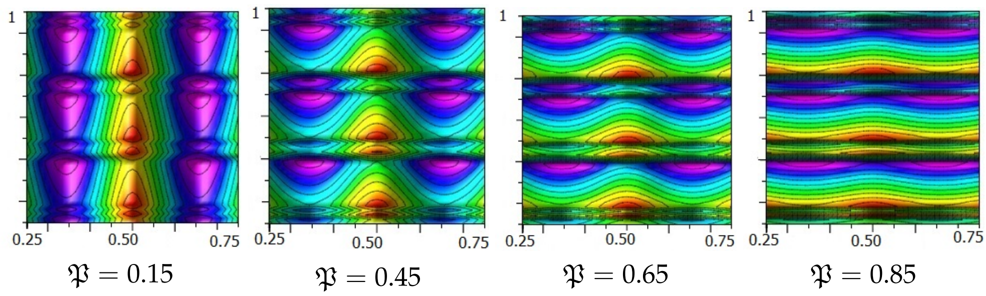

Figure 1. The plots of $v_4(\sigma)$ for different values of $\mathfrak{P}$ and $\sigma \in [0.25, 0.75]$.

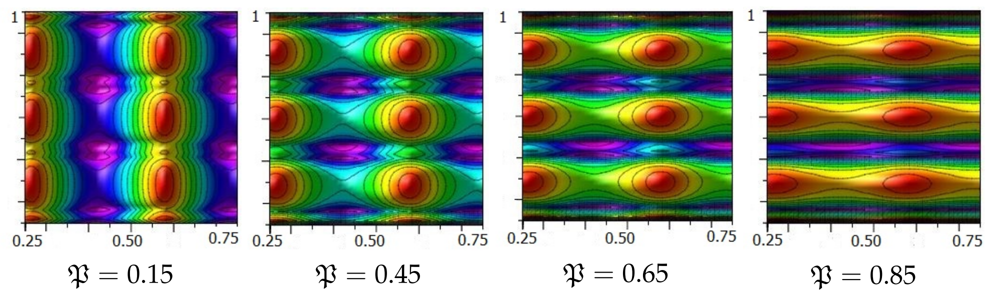

Figure 2. The plots of $v_1(\sigma)$ for different values of $\mathfrak{P}$ and $\sigma \in [0.25, 0.75]$.

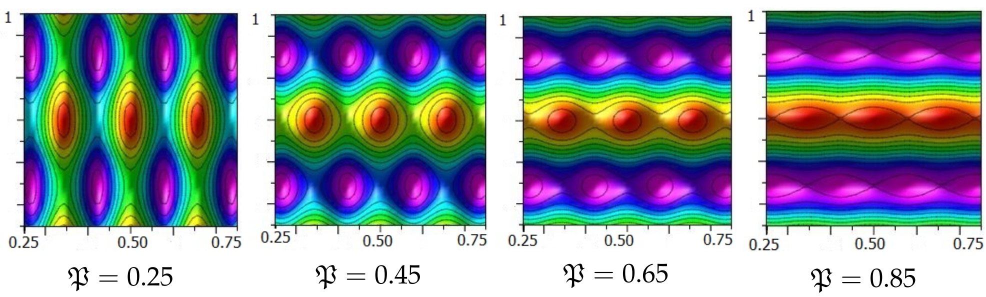

Figure 3. The plots of $v_3(\sigma)$ for different values of $\mathfrak{P}$ and $\sigma \in [0.25, 0.75]$.

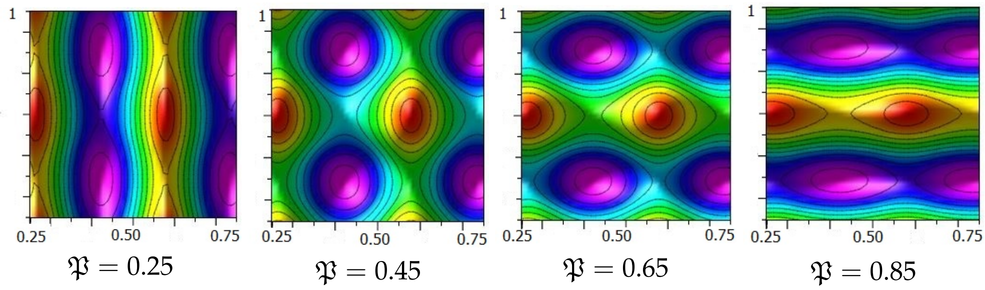

Figure 4. The plots of $v_2(\sigma)$ for different values of $\mathfrak{P}$ and $\sigma \in [0.25, 0.75]$.

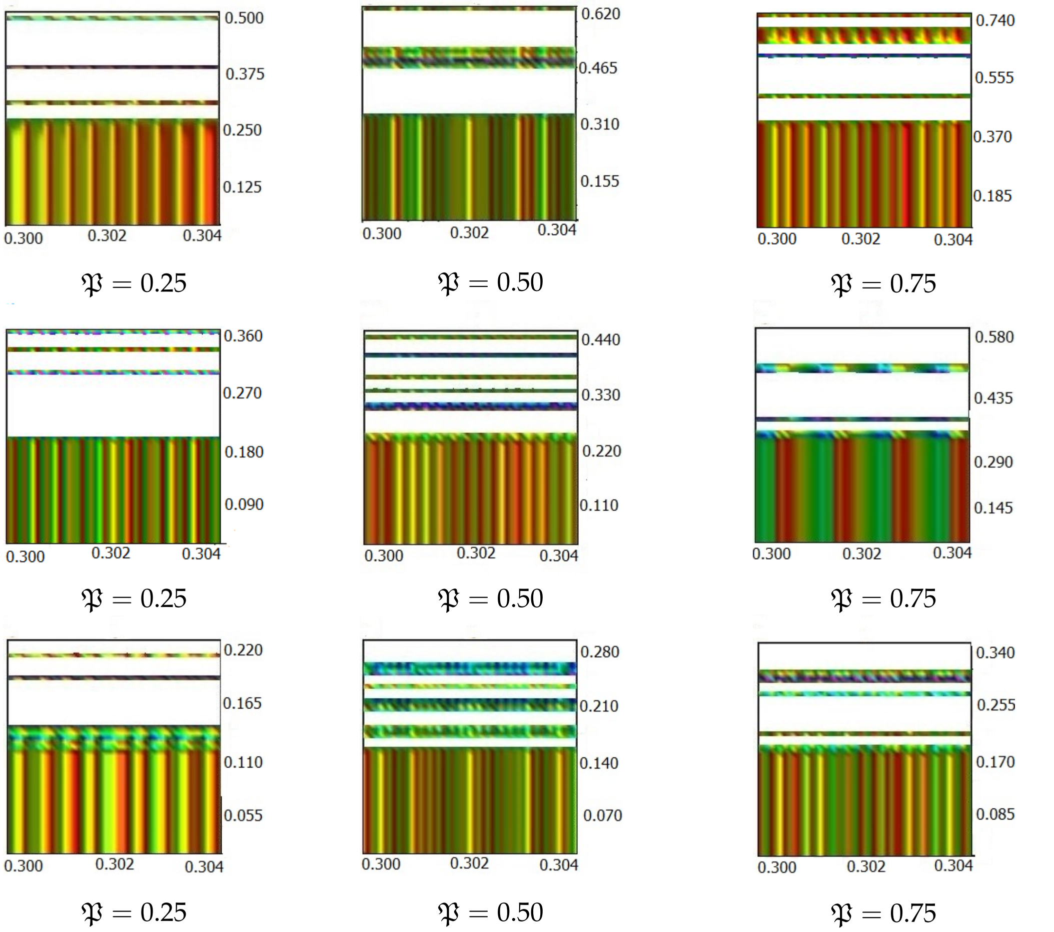

Figure 5. The contour plots of $E_j(\sigma) := |v_1(\sigma) - v_j(\sigma)|$, for $j = 2, 3, 4$, $\sigma \in [0.300, 0.304]$, and diverse values of $\mathfrak{P}$. The errors are smaller when using Wright and hypergeometric controllers, indicating that the choice of control function critically influences stability and can significantly improve error minimization and solution quality.

4. Conclusions

By applying Banach's contraction principle and Picard operators, we investigated the uniqueness and stability of the new fractional-order system described in Equation (1). We also presented a numerical example along with graphical representations of the system's responses to various control functions, such as the exponential function, the Mittag–Leffler function, and the hypergeometric function. Establishing the stability criteria for the proposed problem is a crucial step toward understanding its dynamic behavior and ensuring practical robustness. This study highlights the distinctive features of the problem and underscores the significance of the findings in advancing the current state of our knowledge. Notably, this is the first work to explore fractional differential equations with infinite delay involving the Hadamard fractional derivative, paving the way for new theoretical developments. We believe the results presented here will have a meaningful impact on future research, encouraging their further application in engineering and the applied sciences and opening up new pathways for the modeling of complex systems using advanced tools from fractional calculus. In terms of future research, we recommend exploring the use of other special functions—such as Fox's H-function, the Fox–Wright function, and the Meijer G-function [16]—as control functions and comparing the resulting errors with those obtained using the control functions employed in this study.

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References

1. Lyapunov, A.M. The general problem of the stability of motion. *Int. J. Control* **1992**, *55*, 531–534. [CrossRef]
2. Hale, J.K.; Koçak, H. *Dynamics and Bifurcations*; Springer Science and Business Media: Berlin/Heidelberg, Germany , 2012; Volume 3.
3. Sastry, S. *Nonlinear Systems: Analysis, Stability, and Control*; Springer Science and Business Media: Berlin/Heidelberg, Germany, 2013; Volume 10.
4. Blanchard, O.J.; Giavazzi, F. *Improving the Stability and Growth Pact Through Proper Accounting of Public Investment. Fiscal Policy, Stabilization, and Growth: Prudence or Abstinence*; World Bank: Washington, DC, USA, 2008; pp. 259–272.
5. Siraj, M. A Survey on routing algorithms and routing metrics for Wireless Mesh Networks. *World Appl. Sci. J.* **2014**, *30*, 870–886.
6. Okokpujie, I.P.; Tartibu, L.K.; Musa-Basheer, H.O.; Adeoye, A.O.M. Effect of coatings on mechanical, corrosion and tribological properties of industrial materials: A comprehensive review. *J. Bio-Tribo-Corros.* **2024**, *10*, 2. [CrossRef]
7. Mao, J.; Kang, S.B.; Park, J.O. Grain refinement, thermal stability and tensile properties of 2024 aluminum alloy after equal-channel angular pressing. *J. Mater. Process. Technol.* **2005**, *159*, 314–320. [CrossRef]
8. Otrocol, D.; Ilea, V. Ulam stability for a delay differential equation. *Open Math.* **2013**, *11*, 1296–1303. [CrossRef]
9. Wang, J.; Zhang, Y. Ulam–Hyers–Mittag-Leffler stability of fractional-order delay differential equations. *Optimization* **2014**, *63*, 1181–1190. [CrossRef]
10. Liu, K.; Wang, J.; O'Regan, D. Ulam–Hyers–Mittag-Leffler stability for ψ-Hilfer fractional-order delay differential equations. *Adv. Differ. Equ.* **2019**, *2019*, 50. [CrossRef]
11. Kucche, K.D.; Shikhare, P.U. Ulam stabilities for nonlinear Volterra delay integro-differential equations. *J. Contemp. Math. Anal. (Armen. Acad. Sci.)* **2019**, *54*, 276–287. [CrossRef]
12. Abdo, M.S.; Panchal, S.K.; Wahash, H.A. Ulam–Hyers–Mittag-Leffler stability for a ψ-Hilfer problem with fractional order and infinite delay. *Results Appl. Math.* **2020**, *7*, 100115. [CrossRef]
13. Wei, L.; Duan, Z. Scattering for the fractional magnetic Schrödinger operators. *Acta Math. Sci. Ser. B (Engl. Ed.)* **2024**, *44*, 2391–2410. [CrossRef]
14. Ye, M.; Jiang, H. Bifurcation control for a fractional-order delayed SEIR rumor spreading model with incommensurate orders. *Acta Math. Sci. Ser. B (Engl. Ed.)* **2023**, *43*, 2662–2682. [CrossRef]
15. Gunasekar, T.; Raghavendran, P.; Santra, S.S.; Sajid, M. Existence controllability results for neutral fractional Volterra-Fredholm integro-differential equations. *J. Math. Comput. Sci.* **2024**, *34*, 361–380. [CrossRef]
16. Aderyani, S.R.; Saadati, R.; Li, C.; Allahviranloo, T. *Towards Ulam Type Multi Stability Analysis: A Novel Approach for Fuzzy Dynamical Systems*; Springer Nature: Berlin/Heidelberg, Germany, 2024; Volume 523.
17. Aderyani, S.R.; Saadati, R.; O'Regan, D.; Abdeljawad, T. UHML stability of a class of Δ-Hilfer FDEs via CRM. *AIMS Math.* **2022**, *7*, 5910–5919. [CrossRef]
18. Sousa, J.V.D.; de Oliveira, E.C. A Gronwall inequality and the Cauchy-type problem by means of ψ-Hilfer operator. *Differ. Equ. Appl.* **2019**, *11*, 87–106.
19. Hale, J.; Kato, J. Phase space for retarded equations with infinite delay. *Funkc. Ekvac.* **1978**, *21*, 11–41.
20. Hino, Y.; Murakami, S.; Naito, T. *Functional Differential Equations with Infinite Delay*; Springer: Berlin/Heidelberg, Germany, 2006.

21. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations;* Elsevier: Amsterdam, The Netherlands, 2006.
22. Hadamard, J. Essai sur l'etude des fonctions donnees par leur developpment de Taylor. *J. Math. Pures Appl.* **1892**, *4*, 101–186.
23. Aderyani, S.R.; Saadati, R.; O'Regan, D. Sufficient Conditions for Optimal Stability in Hilfer–Hadamard Fractional Differential Equations. *Mathematics* **2025**, *13*, 1525. [CrossRef]